Scattering from Yukawa potential in Quantum Mechanics

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We consider the Yukawa or screened Coulomb potential (short-range order). This potential was proposed by Yukawa as a model for the nucleon -nucleon interaction. Here, we discuss the scattering of particle by the Yukawa potential by using the Born approximation. The scattering amplitude in the first order Born approximation ($f^{(1)}$) is real. According to the optical theorem the forward scattering amplitude has an imaginary part proportional to total cross section. In order to overcome such a difficulty, it is natural to try calculating the second order Born term ($f^{(2)}$). Using the Mathematica, we will get the exact expression for $f^{(2)}$ and will confirm that the optical theorem is valid for the Yukawa potential up to the order of Born second-order approximation. We also discuss the partial shift at low energy limit. Note that this article is a part of the lecture notes of Phys.422 (Quantum Mechanics II). You can find all the mathematics on the theory of Born approximation in my lecture note on Physics 422 (Quantum Mechanics II), Chapter 13.
http://bingweb.binghamton.edu/~suzuki/QuantumMechanics2.html

For the first time, I encountered the problem of scattering from the Yukawa potential in a book of quantum mechanics problems and solutions, edited by Masao Kotani and Hiroomi Umezawa (Shokabo, 1968, in Japanese), when I was a undergraduate student. I remembered that I struggled to solve this problem since the calculation of the second Born approximation includes the Cauchy theorem, Jordan's lemma, and complicated integrals.

Note that the present article does not include any topics which are something new. I just collect interesting topics related to the scattering from the Yukawa potential, from many textbooks of scattering, which were published long time ago.

1. Hideki Yukawa and the meson theory

Hideki Yukawa (湯川 秀樹, *Yukawa Hideki*, 23 January 1907 – 8 September 1981) was a Japanese theoretical physicist and the first Japanese Nobel laureate for his prediction of the pi meson, or pion.

https://en.wikipedia.org/wiki/Hideki_Yukawa

((**Meson theory**)) **The Story of Spin (S. Tomonaga)**

The meson theory of Yukawa was briefly explained by Prof. S. Tomonaga in his book.

Yukawa arrived at the idea that there should exist a yet-to-be discovered charged boson, the heavy quantum, and that this charged particle shuttles between the neutron and proton. He concluded that if this particle has a mass about 100 times as large as that of the electron, then the effective range of the nuclear force is on the order of 10^{-13} cm (=1 fm). From the analogy that the force between charged particles is mediated by the electromagnetic field, he thought that the nuclear force is mediated by an unknown field which might be called the nuclear force field. When he adopted the Klein-Gordon equation as the equation of this field, then instead of the Coulomb potential e^2/r for the electromagnetic field, $g^2 e^{-\mu r}/r$ appeared, and from the de Broglie-Einstein relation, in place of the zero-mass boson (photon), which is the quantized electromagnetic field, he obtained a boson of mass $\mu\hbar / c$ for the nuclear force field. As e^2 / r is the potential for the Coulomb force, he thought, and calculated the boson mass by putting 10^{-13} cm for the force range $1/\mu$, obtaining 100 times the electron mass that I ("Tomonaga") mentioned above. Thus, this boson is lighter than the proton but heavier that the electron and much heavier than the photon. He therefore, name this particle the heavy quantum as opposed to the light quantum.

((**Note**)) **Serway et al. Modern Physics**

The mass of pion is roughly estimated from the Heisenberg's principle of uncertainty (**Serway et al.**)

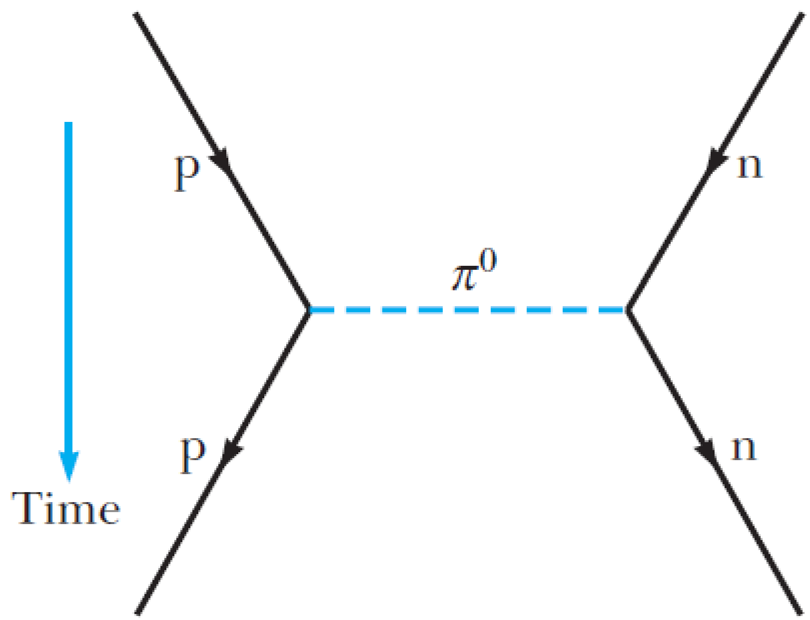


Fig.1 Feynman diagram representing a proton interacting with a neutron via the strong force. The pion mediates the strong force. The blue arrow shows the direction of increasing time.

Now consider the pion exchange between a proton and a neutron that transmits the nuclear force according to Yukawa. We can reason that the energy ΔE needed to create a pion of mass m_π is given by Einstein's equation $\Delta E = m_\pi c^2$. Again, the very existence of the pion would violate conservation of energy if it lasted for a time greater than $\Delta t \simeq \dfrac{\hbar}{2\Delta E}$, where ΔE is the energy of the pion and Δt is the time it takes the pion to travel from one nucleon to the other. Therefore,

$$\Delta t = \frac{\hbar}{2\Delta E} = \frac{\hbar}{2m_\pi c^2}. \tag{1}$$

Because the pion cannot travel faster than the speed of light, the maximum distance d it can travel in a time is $c\Delta t$. Using Eq.(1) and $d = c\Delta t$, we find this maximum distance to be

$$d = \frac{\hbar}{2m_\pi c}. \tag{2}$$

We know that the range of the nuclear force is approximately 1 fm = 10^{-15} m. Using this value for *d* in Eq.(2).we calculate the rest energy of the pion to be

$$m_\pi c^2 = \frac{\hbar c}{2d} = 100 \text{ MeV}$$

This corresponds to a mass of 100 MeV/c^2 (approximately 250 times the mass of the electron), a value in reasonable agreement with the observed pion mass.

2. Yukawa potential

Yukawa potential (also called a screened Coulomb potential) is of the form:

$$V(r) = \frac{V_0}{\mu r} e^{-\mu r}.$$

where V_0 = constant and attractive for positive V_0 and repulsive for negative V_0. The range of the potential is given by a=$1/\mu$. For a Coulomb potential, we see that $\mu \to 0$ and the concept of a range has no meaning since the range becomes infinite. For convenience, here we make a plot of V/V_0 as a function of kr, where k is the wave number of a particle and μ/k is changed as a parameter;

$$\frac{V(r)}{V_0} = \frac{1}{\frac{\mu}{k}(kr)} e^{-\frac{\mu}{k}kr},$$

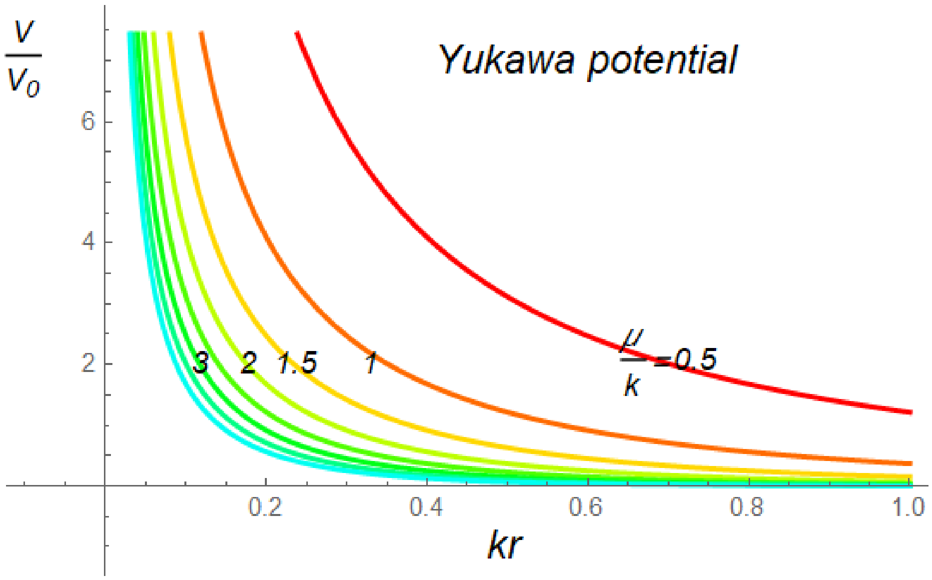


Fig.2 Plot of the Yukawa potential as a function of the distance r. For convenience, we make a plot of V/V_0 as a function of kr, where k is the wave number of a particle, the parameter μ/k is changed between 0.5 and 4.5, $\Delta(\mu/k)=0.5$. V_0 = constant. The range of the potential is given by $1/\mu$. For a Coulomb potential, we see that $\mu \to 0$ and the concept of a range has no meaning since the range becomes infinite.

3. Scattering amplitude (the first order Born approximation)

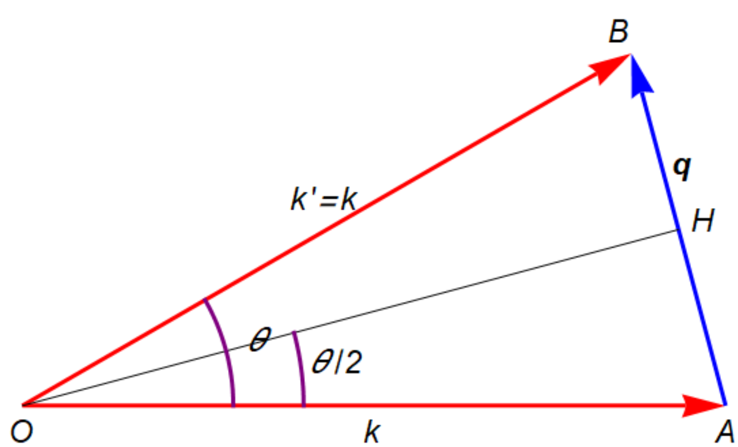


Fig.3 Elastic scattering. $|\mathbf{k}| = |\mathbf{k}'| = k$. $\mathbf{q} = \mathbf{k}' - \mathbf{k}$. $q = 2k\sin\frac{\theta}{2}$.

We now calculate the first-order Born approximation for the scattering amplitude (Yukawa potential),

$$
\begin{aligned}
f^{(1)} &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} V(\mathbf{r}) \\
&= -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3 \langle \mathbf{k}'|\hat{V}|\mathbf{k}\rangle \\
&= -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3 \frac{1}{(2\pi)^3}\frac{4\pi V_0}{\mu}\frac{1}{\mu^2+q^2} \\
&= -\frac{2mV_0}{\hbar^2\mu}\frac{1}{\mu^2+q^2} \\
&= -\frac{\lambda\mu}{4\pi}\frac{1}{\mu^2+q^2}
\end{aligned}
$$

or

$$f^{(1)} = -\frac{\lambda\mu}{4\pi}\frac{1}{\mu^2+q^2},$$

where

$$V(r) = \frac{V_0}{\mu r}e^{-\mu r}, \qquad \mathbf{q} = \mathbf{k}' - \mathbf{k}, \qquad q = 2k\sin\frac{\theta}{2}$$

with

$$\lambda = 4\pi\left(\frac{2mV_0}{\hbar^2\mu^2}\right)$$

Evidently, the optical theorem is not valid only for the first order approximation since

$$\mathrm{Im}[f^{(1)}]=0, \qquad \sigma_{tot}=\frac{4\pi}{k}\mathrm{Im}[f^{(1)}(0)]\to 0$$

So, we have to calculate the Born second order in order to check the validity of the optical theorem.

((**Note**)) **Calculation of $f^{(1)}$ (detail)**

$$\begin{aligned}
\langle \mathbf{k}'|\hat{V}|\mathbf{k}\rangle &= \frac{1}{(2\pi)^3}\int d\mathbf{r}e^{-i\mathbf{k}'\cdot\mathbf{r}}V(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}} \\
&= \frac{1}{(2\pi)^3}\int d\mathbf{r}e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}V(\mathbf{r}) \\
&= \frac{1}{(2\pi)^3}\int d\mathbf{r}e^{-i\mathbf{q}\cdot\mathbf{r}}\frac{V_0}{\mu r}e^{-\mu r}
\end{aligned}$$

$$\mathbf{q}=\mathbf{k}'-\mathbf{k}$$

$$\begin{aligned}
\langle \mathbf{k}'|\hat{V}|\mathbf{k}\rangle &= \frac{1}{(2\pi)^3}\frac{V_0}{\mu}\int_0^\infty 2\pi r^2 dr\int_0^\pi \sin\theta d\theta\frac{e^{-iqr\cos\theta}}{r}e^{-\mu r} \\
&= \frac{1}{(2\pi)^3}(2\pi)\frac{V_0}{\mu}\int_0^\infty e^{-\mu r}rdr\int_0^\pi \sin\theta d\theta e^{-iqr\cos\theta}
\end{aligned}$$

Here we note that

$$\begin{aligned}
\int_0^\pi \sin\theta d\theta e^{-iqr\cos\theta} &= \left[\frac{e^{-iqr\cos\theta}}{iqr}\right]_0^\pi \\
&= \frac{e^{iqr}-e^{-iqr}}{iqr} \\
&= \frac{2\sin(qr)}{qr}
\end{aligned}$$

$$\langle \mathbf{k}'|\hat{V}|\mathbf{k}\rangle = \frac{1}{(2\pi)^3}(2\pi)\frac{V_0}{\mu}\int_0^\infty e^{-\mu r} r dr \frac{2\sin(qr)}{qr}$$
$$= \frac{1}{(2\pi)^3}\frac{V_0}{\mu}\frac{4\pi}{q}\int_0^\infty dr e^{-\mu r}\sin(qr)$$

Using the formula of the Laplace transformation, we get

$$\langle \mathbf{k}'|\hat{V}|\mathbf{k}\rangle = \frac{1}{(2\pi)^3}\frac{V_0}{\mu}\frac{4\pi}{q}\frac{q}{\mu^2+q^2}$$
$$= \frac{1}{(2\pi)^3}\frac{4\pi V_0}{\mu}\frac{1}{\mu^2+q^2}$$

The differential cross section:

$$\frac{d\sigma^{(1)}}{d\Omega} = \left|f^{(1)}\right|^2 = (\frac{2mV_0}{\hbar^2\mu})^2\frac{1}{(\mu^2+q^2)^2} = \frac{\lambda^2\mu^2}{16\pi^2}\frac{1}{(\mu^2+q^2)^2}$$

$$\sigma^{(1)} = \frac{\lambda^2\mu^2}{16\pi^2}\int d\Omega\frac{1}{(\mu^2+q^2)^2}$$

with $d\Omega = 2\pi\sin\theta d\theta$; and $q - 2k\sin\frac{\theta}{2}$

$$\sigma^{(1)} = 2\pi\frac{\lambda^2\mu^2}{16\pi^2}\int_0^\pi \sin\theta d\theta\frac{1}{(\mu^2+4k^2\sin^2\frac{\theta}{2})^2}$$
$$= 2\pi\frac{\lambda^2\mu^2}{16\pi^2}\int_0^\pi\frac{\sin\theta d\theta}{[\mu^2+2k^2(1-\cos\theta)]^2}$$
$$= 2\pi\frac{\lambda^2\mu^2}{16\pi^2}\frac{2}{\mu^2(\mu^2+4k^2)}$$
$$= \frac{\lambda^2}{4\pi}\frac{1}{\mu^2+4k^2}$$

In the limit of $k \to \infty$, we have

$$\sigma^{(1)} = \frac{1}{16\pi}\frac{\lambda^2}{k^2} \propto \frac{1}{E}.$$

((**Mathematica**)) **Calculation of the integral**

```
g3 = Sin[θ] / (μ^2 + 2 k^2 (1 - Cos[θ]))^2;
Integrate[g3, {θ, 0, π},
 Assumptions → {μ > 0, k > 0}]
```

$$\frac{2}{4\,k^2\mu^2 + \mu^4}$$

4. Rutherford scattering

Here we derive the differential cross section for the Rutherford scattering. There is a Coulomb interaction between the α-particle (Z_1e = +2e charge; $e > 0$) and the nucleus with positive charge (+Z_2e). In the expression of the differential cross section

$$\frac{d\sigma^{(1)}}{d\Omega} = \frac{\lambda^2\mu^2}{16\pi^2}\frac{1}{(\mu^2+q^2)^2},$$

we put $\frac{V_0}{\mu} = Z_1Z_2e^2$, we get

$$\frac{d\sigma^{(1)}}{d\Omega} = (Z_1Z_2e^2)^2(\frac{2m}{\hbar^2})^2\frac{1}{(\mu^2+q^2)^2}.$$

In the limit of $\mu \to 0$,

$$\frac{d\sigma^{(1)}}{d\Omega} = \frac{1}{16}(Z_1 Z_2)^2 e^4 (\frac{2m}{\hbar^2 k^2})^2 \frac{1}{\sin^4 \frac{\theta}{2}}$$

$$= \frac{(Z_1 Z_2)^2 e^4}{16E^2} \frac{1}{\sin^4 \frac{\theta}{2}}$$

where

$$q = 2k \sin\frac{\theta}{2}, \qquad E = \frac{\hbar^2 k^2}{2m}.$$

This is just the famous Rutherford scattering of the alpha particle ($+Z_1 e$) by a Coulomb potential of charge $+Z_2 e$. Note that this differential cross section using quantum mechanics is in complete agreement with that obtained from a classical analysis of Coulomb scattering.

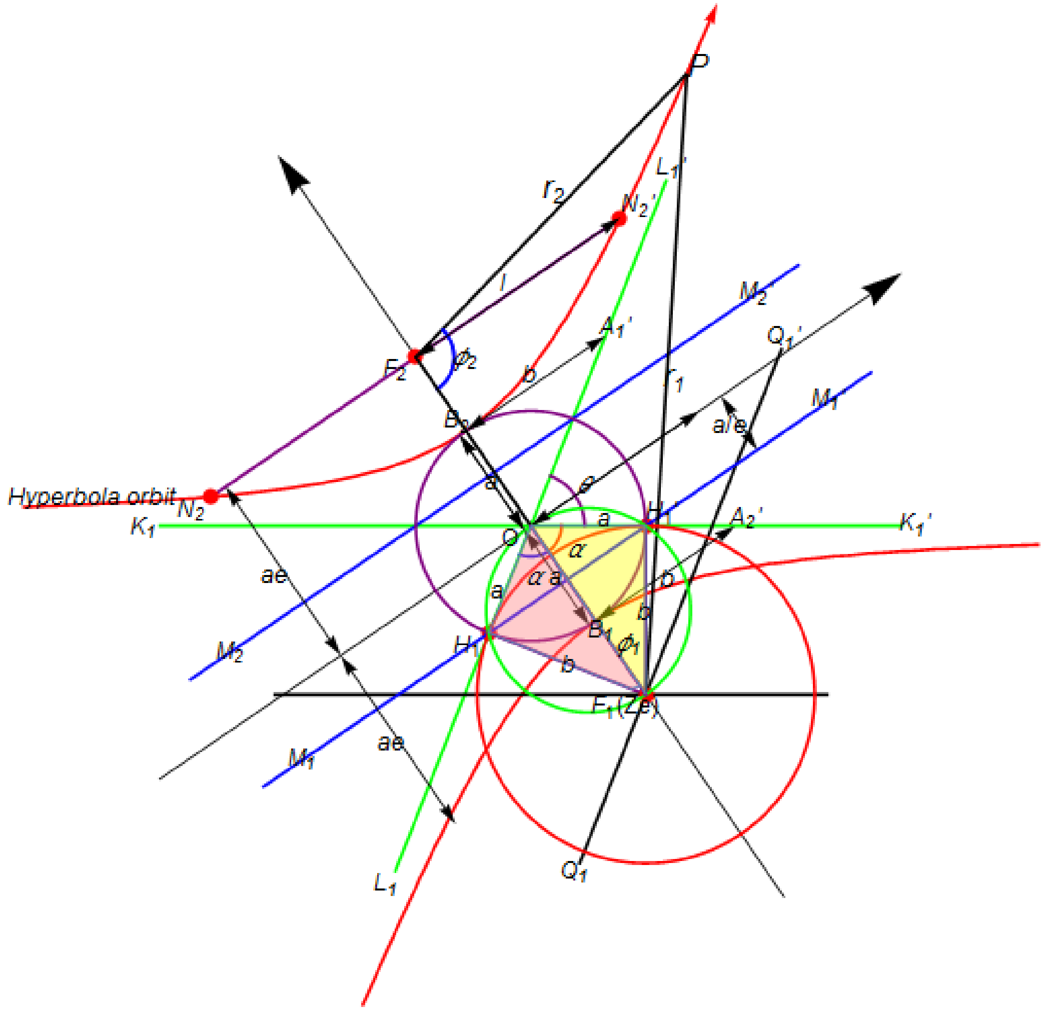


Fig.4 Rutherford scattering. Hyperbolic trajectory. $r_1 - r_2 = 2a$. The target nucleus with Ze is located at the focal point F1. The alpha particle is at the point P on the hyperbola $ae = \sqrt{a^2 + b^2}$. See further detail in http://bingweb.binghamton.edu/~suzuki/QuantumMechanicsII/13-2_Rutherford_scattering.pdf

5. Second-order Born approximation for the Yukawa potential

We now study the second-order Born approximation for the Yukawa potential'

$$f^{(2)} = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3 \langle \mathbf{k}' | \hat{V}\hat{G}_0(E_{\mathbf{k}} + i\varepsilon)\hat{V} | \mathbf{k} \rangle$$

with

$$\langle \mathbf{k}'|\hat{V}\hat{G}_0(E_\mathbf{k}+i\varepsilon)\hat{V}|\mathbf{k}\rangle = \int d\mathbf{k}''\langle \mathbf{k}'|\hat{V}|\mathbf{k}''\rangle\langle \mathbf{k}''|\hat{G}_0(E_\mathbf{k}+i\varepsilon)|\mathbf{k}''\rangle\langle \mathbf{k}''|\hat{V}|\mathbf{k}\rangle,$$

$$\begin{aligned}
\langle \mathbf{k}''|\hat{G}_0(E_\mathbf{k}+i\varepsilon)|\mathbf{k}''\rangle &= \langle \mathbf{k}''|\frac{1}{E_\mathbf{k}-\hat{H}+i\varepsilon}|\mathbf{k}''\rangle \\
&= \langle \mathbf{k}''|\frac{1}{E_\mathbf{k}-E_{\mathbf{k}''}+i\varepsilon}|\mathbf{k}''\rangle \\
&= \frac{1}{E_\mathbf{k}-E_{\mathbf{k}''}+i\varepsilon} \\
&= \frac{2m}{\hbar^2}\frac{1}{\mathbf{k}^2-\mathbf{k}''^2+i\varepsilon}
\end{aligned}$$

$$\langle \mathbf{k}'|\hat{V}|\mathbf{k}\rangle = \frac{1}{(2\pi)^3}\frac{4\pi V_0}{\mu}\frac{1}{\mu^2+(\mathbf{k}'-\mathbf{k})^2},$$

$$\begin{aligned}
f^{(2)} &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\int d\mathbf{Q}\langle \mathbf{k}'|\hat{V}|\mathbf{Q}\rangle\langle \mathbf{Q}|\hat{G}_0(E_\mathbf{k}+i\varepsilon)|\mathbf{Q}\rangle\langle \mathbf{Q}|\hat{V}|\mathbf{k}\rangle \\
&= -\frac{1}{4\pi}\left(\frac{2m}{\hbar^2}\right)^2(2\pi)^3\frac{1}{(2\pi)^6}\left(\frac{4\pi V_0}{\mu}\right)^2\int d\mathbf{Q}\frac{1}{\mu^2+(\mathbf{Q}-\mathbf{k}')^2}\frac{1}{\mu^2+(\mathbf{Q}-\mathbf{k})^2}\frac{1}{\mathbf{k}^2-\mathbf{Q}^2+i\varepsilon} \\
&= \frac{\lambda^2\mu^2}{32\pi^4}\int d\mathbf{Q}\frac{1}{\mu^2+(\mathbf{Q}-\mathbf{k}')^2}\frac{1}{\mu^2+(\mathbf{Q}-\mathbf{k})^2}\frac{1}{\mathbf{Q}^2-\mathbf{k}^2-i\varepsilon}
\end{aligned}$$

where ε is a positive small constant. We now calculate the integral M (Dalitz integral) defined by

$$M = \int d\mathbf{Q}\frac{1}{\mu^2+(\mathbf{Q}-\mathbf{k}')^2}\frac{1}{\mu^2+(\mathbf{Q}-\mathbf{k})^2}\frac{1}{\mathbf{Q}^2-\mathbf{k}^2-i\varepsilon}.$$

We use the formula (Feynman integral representation)

$$\int_0^1 dz \frac{1}{[az+b(1-z)]^2} = \frac{1}{ab},$$

In general;

$$\frac{1}{a^m b^n} = \frac{(m+n-1)!}{(m-1)!(n-1)!}\int_0^1 \frac{t^{m-1}(1-t)^{n-1}}{[at+b(1-t)]^{m+n}}dt \qquad \text{(Feynman, 1949)}$$

((**Mathematica**))

```
Clear["Global`*"];
g1 = 1/(a z + b (1 - z))^2;
Integrate[g1, {z, 0, 1},
 Assumptions → Assumptions → {μ > 0, q > 0, k > 0}]

1/(a b)

g11 = (z^(m-1) (1 - z)^(n-1))/(a z + b (1 - z))^(m+n);
Integrate[g11, {z, 0, 1},
 Assumptions → Assumptions → {a > 0, b > 0, m > 0}]

((a/b)^-m b^(-m-n) Gamma[m] Gamma[n])/Gamma[m + n]
```

Here we also use the notations

$$a = \mu^2 + (\mathbf{Q}-\mathbf{k}')^2, \qquad b = \mu^2 + (\mathbf{Q}-\mathbf{k})^2$$

and

$$
\begin{aligned}
az + b(1-z) &= [\mu^2 + (\mathbf{Q}-\mathbf{k}')^2]z + [\mu^2 + (\mathbf{Q}-\mathbf{k})^2](1-z) \\
&= \mu^2 + (\mathbf{Q}^2 - 2\mathbf{Q}\cdot\mathbf{k}' + \mathbf{k}'^2)z + (\mathbf{Q}^2 - 2\mathbf{Q}\cdot\mathbf{k} + \mathbf{k}^2)(1-z) \\
&= \mu^2 + \mathbf{k}'^2 z + \mathbf{k}^2(1-z) + \mathbf{Q}^2 - 2\mathbf{Q}\cdot[\mathbf{k}'z + \mathbf{k}(1-z)] \\
&= \mu^2 + k^2 + \mathbf{Q}^2 - 2\mathbf{Q}\cdot[\mathbf{k}'z + \mathbf{k}(1-z)]
\end{aligned}
$$

where

$$\mathbf{q} = \mathbf{k}' - \mathbf{k},$$

$$|\mathbf{k}'| = |\mathbf{k}| = k, \qquad \mathbf{k}'\cdot\mathbf{k} = k^2\cos\theta$$

Thus, we have

$$
\begin{aligned}
az + b(1-z) &= [\mathbf{Q} - \mathbf{k}'z - \mathbf{k}(1-z)]^2 + \mu^2 + k^2 - [\mathbf{k}'z + \mathbf{k}(1-z)]^2 \\
&= [\mathbf{Q} - \mathbf{k}'z - \mathbf{k}(1-z)]^2 + \mu^2 + k^2 - \mathbf{g}^2 \\
&= (\mathbf{Q}-\mathbf{g})^2 + \tau^2
\end{aligned}
$$

For convenience, we use the parameters.

$$\mathbf{g} = \mathbf{k}'z + \mathbf{k}(1-z) = \mathbf{q}z + \mathbf{k},$$

$$\tau^2 = \mu^2 + k^2 - g^2.$$

$$
\begin{aligned}
g^2 &= [\mathbf{k}z + \mathbf{k}'(1-z)]\cdot[\mathbf{k}z + \mathbf{k}'(1-z)] \\
&= \mathbf{k}^2 z^2 + \mathbf{k}'^2(1-z)^2 + 2\mathbf{k}\cdot\mathbf{k}'z(1-z) \\
&= k^2[2z^2 - 2z + 1 + 2z(1-z)\cos\theta] \\
&= k^2[1 - 2z(1-z) + 2z(1-z)(1 - 2\sin^2\frac{\theta}{2})] \\
&= k^2[1 - 4z(1-z)\sin^2\frac{\theta}{2}]
\end{aligned}
$$

$$\mathbf{g}^2 + \tau^2 = \mu^2 + k^2 .$$

So that, *M* can be rewritten as

$$M = \int_0^1 dz \int d\mathbf{Q} \frac{1}{4[(\mathbf{Q}-\mathbf{g})^2 + \tau^2]^2} \frac{1}{\mathbf{Q}^2 - \mathbf{k}^2 - i\varepsilon}$$
$$= \int_0^1 dz \int d\mathbf{Q} \frac{1}{[(\mathbf{Q}-\mathbf{g})^2 + \tau^2]^2} \frac{1}{\mathbf{Q}^2 - \mathbf{k}^2 - i\varepsilon}$$

and

$$M = \int_0^1 dz L(\mathbf{g}, \boldsymbol{\tau}) ,$$

with

$$L(\mathbf{g}, \tau) = \int d\mathbf{Q} \frac{1}{[(\mathbf{Q}-\mathbf{g})^2 + \tau^2]^2 [\mathbf{Q}^2 - k^2 - i\varepsilon]} ,$$

$$J(\mathbf{g}, \tau) = \int d\mathbf{Q} \frac{1}{[(\mathbf{Q}-\mathbf{g})^2 + \tau^2][\mathbf{Q}^2 - k^2 - i\varepsilon]} ,$$

and

$$M = -\frac{1}{2\tau} \frac{\partial}{\partial \tau} J(\mathbf{g}, \tau) .$$

First, we need to calculate $J(\mathbf{g}, \tau)$. Suppose that the vector $\boldsymbol{g}$ is directed along the Q_z axis. When the angle between the vector $\boldsymbol{Q}$ and $\boldsymbol{g}$ is χ, we have a scalar product as

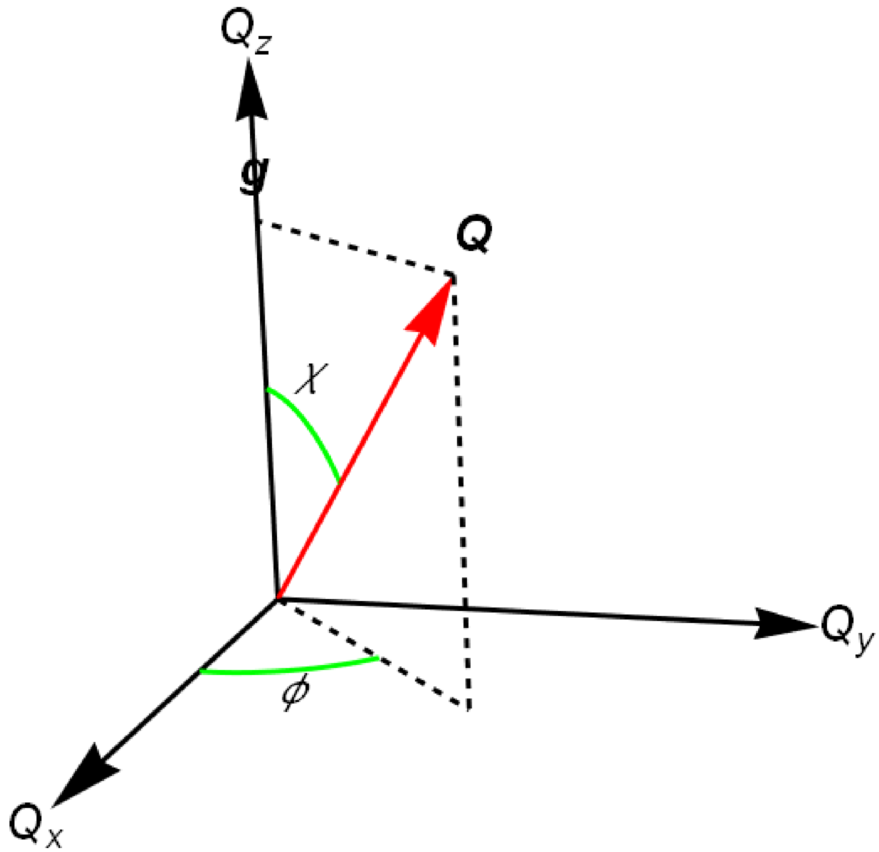


Fig.5 3D $\boldsymbol{Q}$ space. The vector $\boldsymbol{g}$ is directed along the Q_z axis. The angle between $\boldsymbol{Q}$ and g is χ.

The scalar product:

$$\mathbf{Q} \cdot \mathbf{g} = Qg \cos \chi$$

$$(\mathbf{Q} - \mathbf{g})^2 = \mathbf{Q}^2 - 2\mathbf{Q} \cdot \mathbf{g} + \mathbf{g}^2 = \mathbf{Q}^2 - 2Qg \cos \chi + \mathbf{g}^2$$

$$J(\mathbf{g},\tau)=\int_0^\infty Q^2 dQ\int_0^\pi 2\pi\sin\chi d\chi\frac{1}{[Q^2-2Qg\cos\chi+g^2+\tau^2][Q^2-k^2-i\varepsilon]}$$
$$=\int_0^\pi 2\pi\sin\chi d\chi\int_0^\infty dQ\frac{Q^2}{[Q^2-2Qg\cos\chi+g^2+\tau^2][Q^2-k^2-i\varepsilon]}$$
$$=\int_0^\infty\frac{Q^2dQ}{Q^2-k^2-i\varepsilon}\int_0^\pi\frac{2\pi\sin\chi d\chi}{[Q^2-2Qg\cos\chi+g^2+\tau^2]}$$

We define the function $F(Q)$ as

$$F(Q)=\int_0^\pi\frac{2\pi\sin\chi d\chi}{Q^2+g^2+\tau^2-2Qg\cos\chi}$$
$$=\frac{\pi}{gQ}\ln(\frac{g^2+\tau^2+2gQ+Q^2}{g^2+\tau^2-2gQ+Q^2})$$

((**Mathematica**))

```
g2 = 2 π Sin[χ] / (Q^2 + g^2 + τ^2 - 2 Q g Cos[χ]);
Integrate[g2, {χ, 0, π},
 Assumptions → {Q > 0, g > 0, τ > 0}]
```

$$\frac{\pi\,\mathrm{Log}\left[1+\frac{4\,g\,Q}{(g-Q)^2+\tau^2}\right]}{g\,Q}$$

So that, we get

$$
\begin{aligned}
J(\mathbf{g},\tau) &= \int_0^\infty \frac{Q^2 dQ}{Q^2 - k^2 - i\varepsilon} F(Q) \\
&= \int_0^\infty \frac{Q^2 dQ}{Q^2 - k^2 - i\varepsilon} [\frac{\pi}{gQ} \ln(\frac{g^2 + \tau^2 + 2gQ + Q^2}{g^2 + \tau^2 - 2gQ + Q^2})] \\
&= \frac{\pi}{g} \int_0^\infty \frac{Q dQ}{Q^2 - k^2 - i\varepsilon} \ln(\frac{g^2 + \tau^2 + 2gQ + Q^2}{g^2 + \tau^2 - 2gQ + Q^2})
\end{aligned}
$$

We note that the integrand of this integral is an even function of *Q*. This integral can be rewritten as

$$
J(\mathbf{g},\tau) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{Q^2 dQ}{Q^2 - k^2 - i\varepsilon} F(Q) .
$$

Using the form of $F(Q)$; $F(Q) = \int_0^\pi \frac{2\pi \sin\chi d\chi}{Q^2 + g^2 + \tau^2 - 2Qg\cos\chi}$, $J(\mathbf{g},\tau)$ can be obtained as

$$
\begin{aligned}
J(\mathbf{g},\tau) &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{Q^2 dQ}{Q^2 - k^2 - i\varepsilon} \int_0^\pi \frac{2\pi \sin\chi d\chi}{Q^2 + g^2 + \tau^2 - 2Qg\cos\chi} \\
&= \pi \int_0^\pi \sin\chi d\chi \int_{-\infty}^{\infty} \frac{Q^2 dQ}{(Q^2 - k^2 - i\varepsilon)[(Q - g\cos\chi)^2 + g^2\sin^2\chi + \tau^2]}
\end{aligned}
$$

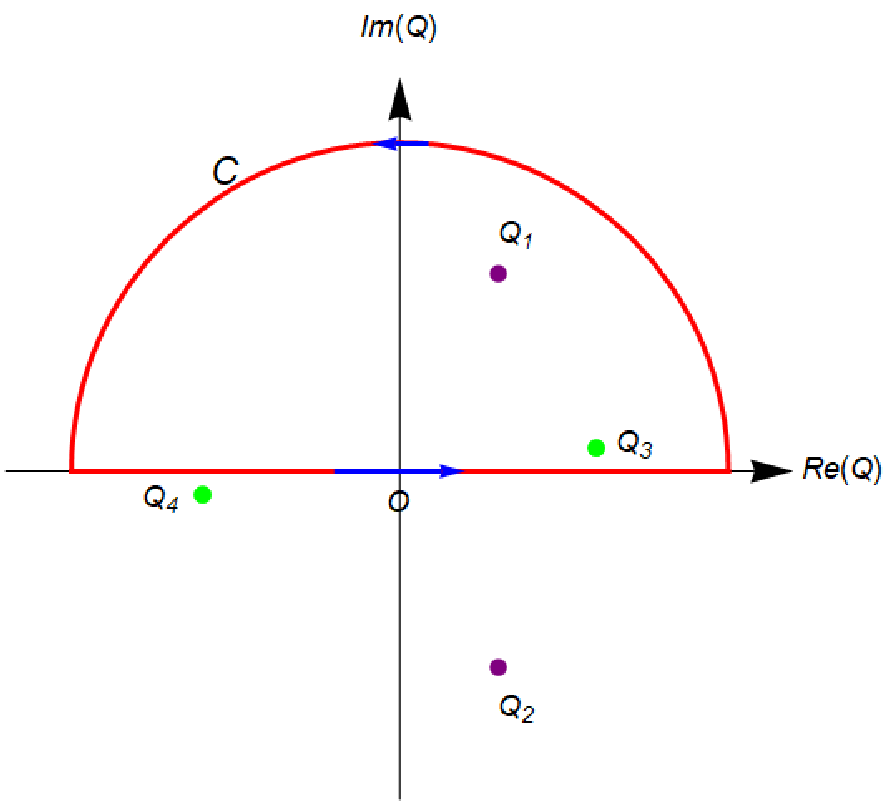


Fig.6 Complex plane of 2D-***Q*** space. There are two single poles at $Q = Q_1$ and $Q = Q_3$ inside the contour C (counterclockwise). Two single poles ($Q = Q_2$ and $Q = Q_4$) outside the contour C.

Using the **Jordan's lemma** (the absolute value of the integrand reduces to zero in the limit of $|Q| \to \infty$ in the complex Q-plane), we extend the region of the integral over the upper half of complex plane. Using the **Cauchy theorem**, we get

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty} \frac{Q^2 dQ}{(Q^2 - k^2 - i\varepsilon)[(Q - g\cos\chi)^2 + g^2\sin^2\chi + \tau^2]} \\
&= \oint_C \frac{Q^2 dQ}{(Q^2 - k^2 - i\varepsilon)[(Q - g\cos\chi)^2 + g^2\sin^2\chi + \tau^2]} \\
&= \oint_C \frac{Q^2 dQ}{(Q - Q_1)(Q - Q_2)(Q - Q_3)(Q - Q_4)} \\
&= 2\pi i[\mathrm{Re}\, s(Q = Q_1) + \mathrm{Re}\, s(Q = Q_3)]
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Re} s(Q=Q_1) &= \lim_{Q\to Q_1} \frac{Q^2(Q-Q_1)}{(Q-Q_1)(Q-Q_2)(Q-Q_3)(Q-Q_4)} \\
&= \lim_{Q\to Q_1} \frac{Q^2}{(Q-Q_2)(Q-Q_3)(Q-Q_4)} \\
&= \frac{Q_1{}^2}{(Q_1-Q_2)(Q_1-Q_3)(Q_1-Q_4)} \\
&= \frac{k^3}{2} \; \frac{1}{k^2+g^2+\tau^2-2g\cos\chi} \\
&= \frac{k^3}{2} \; \frac{1}{k^2+\mu^2+k^2-2g\cos\chi}
\end{aligned}
$$

where

$$
g^2+\tau^2=\mu^2+k^2,
$$

$$
\begin{aligned}
\operatorname{Re} s(Q=Q_3) &= \lim_{Q\to Q_3} \frac{Q^2(Q-Q_3)}{(Q_1-Q_1)(Q-Q_2)(Q-Q_3)(Q-Q_4)} \\
&= \lim_{Q\to Q_3} \frac{Q^2}{(Q-Q_1)(Q-Q_2)(Q-Q_4)} \\
&= \frac{Q_3{}^2}{(Q_3-Q_1)(Q_3-Q_2)(Q_3-Q_4)} \\
&= \frac{(g\cos\chi+i\alpha)^2}{2i\alpha(g\cos\chi+i\alpha+k)(g\cos\chi+i\alpha-k)} \\
&= \frac{(g\cos\chi+i\alpha)^2}{2i\alpha[(g\cos\chi+i\alpha)^2-k^2)]}
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{Re} s(Q=Q_3) &= \frac{k^2}{2k} \frac{1}{(k-g\cos\chi)^2+g^2\sin^2\chi+\tau^2} \\
&= \frac{k}{2} \; \frac{1}{k^2+g^2+\tau^2-2kg\cos\chi}
\end{aligned}
$$

Where we define α as

$$\alpha = \sqrt{g^2 \sin^2 \chi + \tau^2}$$

We now calculate

$$\begin{aligned} J(\mathbf{g},\tau) &= 2\pi^2 i \int_0^\pi \sin \chi d\chi [\mathrm{Res}(Q = Q_1) + \mathrm{Res}(Q = Q_3)) \\ &= J_1 + J_3 \end{aligned}$$

where

$$\begin{aligned} J_1 &= 2\pi^2 i \int_0^\pi \sin \chi d\chi [\mathrm{Res}(Q = Q_1)] \\ &= 2\pi^2 i \int_0^\pi \sin \chi d\chi \frac{(g \cos \chi + i\alpha)^2}{2i\alpha[(g \cos \chi + i\alpha)^2 - k^2]} \\ &= \pi^2 \int_0^\pi \sin \chi d\chi \frac{(g \cos \chi + i\alpha)^2}{\alpha[(g \cos \chi + i\alpha)^2 - k^2]} \end{aligned}$$

For simplicity, we choose a variable,

$$x = g \cos \chi + i\alpha = g \cos \chi + i\sqrt{g^2 \sin^2 \chi + \tau^2}$$

and

$$dx = [-g\sin\chi + i\frac{g^2\sin\chi\cos\chi}{\sqrt{g^2\sin^2\chi+\tau^2}}]d\chi$$

$$= \frac{ig\sin\chi}{\sqrt{g^2\sin^2\chi+\tau^2}}[i\sqrt{g^2\sin^2\chi+\tau^2} + g\cos\chi]d\chi$$

$$= \frac{ig\sin\chi}{\sqrt{g^2\sin^2\chi+\tau^2}}(g\cos\chi + i\alpha)d\chi$$

$$= \frac{ig\sin\chi}{\alpha}xd\chi$$

$$\sin\chi d\chi = \frac{\alpha}{ig}\frac{dx}{x}$$

Then we have

$$J_1 = \pi^2\int_{g+i\tau}^{-g+i\tau}\frac{\alpha dx}{igx}\frac{x^2}{\alpha(x^2-k^2)}$$

$$= \frac{i\pi^2}{g}\int_{-g+i\tau}^{g+i\tau}\frac{xdx}{x^2-k^2}$$

$$= \frac{i\pi^2}{2g}\int_{-g+i\tau}^{g+i\tau}(\frac{1}{x-k}+\frac{1}{x+k})dx$$

$$= \frac{i\pi^2}{2g}\ln\frac{(i\tau-k+g)(i\tau+k+g)}{(i\tau-k-g)(i\tau+k-g)}$$

We also calculate the integral J_3

$$J_3 = 2\pi^2 i\int_0^\pi \sin\chi d\chi[\mathrm{Res}(Q=Q_3)]$$

$$= 2\pi^2 i\int_0^\pi \sin\chi d\chi\frac{k}{2}\ \frac{1}{k^2+g^2+\tau^2-2kg\cos\chi}$$

$$= k\pi^2 i\int_0^\pi\ \frac{\sin\chi d\chi}{k^2+g^2+\tau^2-2kg\cos\chi}$$

$$= \frac{\pi^2 i}{2g}\ln\frac{(k+g)^2+\tau^2}{(k-g)^2+\tau^2}$$

Thus, we have

$$
\begin{aligned}
J &= J_1 + J_3 \\
&= \frac{\pi^2 i}{2g}[\ln\frac{(i\tau+k+g)(i\tau-k-g)}{(i\tau+k-g)(i\tau-k+g)} + \ln\frac{(i\tau-k+g)(i\tau+k+g)}{(i\tau-k-g)(i\tau+k-g)}] \\
&= \frac{\pi^2 i}{g}\ln\frac{(i\tau+k+g)}{(i\tau+k-g)}
\end{aligned}
$$

M is defined by

$$
M = -\frac{1}{2\tau}\frac{\partial}{\partial\tau}J(\mathbf{g},\tau) = -\frac{\pi^2}{\tau}\frac{1}{k^2-g^2-\tau^2+2i\tau k}.
$$

The scattering amplitude in the second order Born approximation is

$$
\begin{aligned}
f^{(2)} &= \frac{1}{32\pi^2}\lambda^2\mu^2\int_0^1 M dz \\
&= -\frac{1}{32\pi^2}\lambda^2\mu^2\int_0^1\frac{dz}{\tau(k^2-g^2-\tau^2+2i\tau k)} \\
&= -\frac{1}{32\pi^2}\lambda^2\mu^2\int_0^1\frac{dz}{\tau(-\mu^2+2i\tau k)}
\end{aligned}
$$

where

$$
g^2+\tau^2 = k^2+\mu^2,
$$

and

$$
\begin{aligned}
\tau^2 &= \mu^2 + k^2 - g^2 \\
&= \mu^2 + 4k^2 \sin^2 \frac{\theta}{2} z(1-z)\,, \\
&= \mu^2 + q^2 z(1-z)
\end{aligned}
$$

Using the separation of the real part and imaginary part of the integrand

$$
\begin{aligned}
\frac{1}{\tau(-\mu^2 + 2i\tau k)} &= \frac{(-\mu^2 - 2i\tau k)}{\tau(\mu^4 + 4\tau^2 k^2)} \\
&= -\frac{\mu^2}{\tau(\mu^4 + 4\tau^2 k^2)} + \frac{-2ik}{(\mu^4 + 4\tau^2 k^2)}
\end{aligned}
$$

(a) Imaginary part $\mathrm{Im}[f^{(2)}]$

We have the imaginary part,

$$
\begin{aligned}
\mathrm{Im}[f^{(2)}] &= -\frac{1}{32\pi^2}\lambda^2\mu^2 \int_0^1 \frac{-2k dz}{\mu^4 + 4\tau^2 k^2} \\
&= \frac{1}{16\pi^2} k\lambda^2\mu^2 \int_0^1 \frac{dz}{\mu^4 + 4\tau^2 k^2} \\
&= \frac{1}{16\pi^2} k\lambda^2\mu^2 \int_0^1 \frac{dz}{\mu^4 + 4\mu^2 k^2 + 4k^2 q^2 z(1-z)} \\
&= \frac{1}{16\pi^2} k\lambda^2\mu^2 \frac{\ln\left(\dfrac{\sqrt{\mu^4 + k^2 q^2 + 4\mu^2 k^2} + kq}{\sqrt{\mu^4 + k^2 q^2 + 4\mu^2 k^2} - kq}\right)}{2kq\sqrt{\mu^4 + k^2 q^2 + 4\mu^2 k^2}}
\end{aligned}
$$

or

$$
\mathrm{Im}[f^{(2)}] = \frac{1}{32\pi^2}\frac{\lambda^2\mu^2}{q} \frac{\ln\left(\dfrac{\sqrt{\mu^4 + k^2 q^2 + 4\mu^2 k^2} + kq}{\sqrt{\mu^4 + k^2 q^2 + 4\mu^2 k^2} - kq}\right)}{\sqrt{\mu^4 + k^2 q^2 + 4\mu^2 k^2}}\,,
$$

where the integral is calculated by Mathematica

$$I_{IM} = \int_0^1 \frac{dz}{\mu^4 + 4\mu^2 k^2 + 4k^2 q^2 z(1-z)} = \frac{\ln(\frac{\sqrt{\mu^4 + k^2 q^2 + 4\mu^2 k^2} + kq}{\sqrt{\mu^4 + k^2 q^2 + 4\mu^2 k^2} - kq})}{2kq\sqrt{\mu^4 + k^2 q^2 + 4\mu^2 k^2}}$$

Note that in the limit of $q \to 0$, we get

$$\lim_{q\to 0} I_{IM} = \frac{1}{\mu^2}\frac{1}{4k^2 + \mu^2},$$

leading to the relation

$$\mathrm{Im}[f^{(2)}(0)] = \frac{\lambda^2}{16\pi^2} k \frac{1}{4k^2 + \mu^2}.$$

((**Mathematica**)) Calculation of I_{IM}

```
Clear["Global`*"];
```

Calculation of Integral of the real part

```
eq = 1 / (μ^4 + 4 k^2 (μ^2 + q^2 z (1 - z)))
```

$$\frac{1}{\mu^4 + 4 k^2 \left(q^2 (1-z) z + \mu^2\right)}$$

```
f1 = Integrate[eq, {z, 0, 1},
  Assumptions → {μ > 0, q > 0, k > 0}]
```

$$\frac{\mathrm{Log}\left[\frac{k q + \sqrt{\mu^4 + k^2 (q^2 + 4\mu^2)}}{-k q + \sqrt{\mu^4 + k^2 (q^2 + 4\mu^2)}}\right]}{2 k q \sqrt{\mu^4 + k^2 \left(q^2 + 4 \mu^2\right)}}$$

Limit of the real part at q → 0

at Limit of part q real the → 0

```
Limit[f1, q → 0]
```

$$\frac{1}{4 k^2 \mu^2 + \mu^4}$$

(b) Real part $\mathrm{Re}[f^{(2)}]$

Next, we calculate the real part,

$$\mathrm{Re}[f^{(2)}] = \frac{1}{32\pi^2}\lambda^2\mu^2\int_0^1 \frac{\mu^2 dz}{\tau(\mu^4 + 4\tau^2 k^2)}$$
$$= \frac{1}{32\pi^2}\mu^4\lambda^2\int_0^1 \frac{dz}{\sqrt{\mu^2 + q^2 z(1-z)}[\mu^4 + 4\mu^2 k^2 + 4k^2 q^2 z(1-z)]}$$

where the solution of this integral is also obtained using the Mathematica

$$I_R = \int_0^1 \frac{dz}{\sqrt{\mu^2 + q^2 z(1-z)}[\mu^4 + 4\mu^2 k^2 + 4k^2 q^2 z(1-z)]}$$

$$= \frac{1}{q\mu^2\sqrt{\mu^4 + k^2(q^2+4\mu^2)}}[\text{Arctan}\frac{k(q^2+4\mu^2) + q\sqrt{\mu^4 + k^2(q^2+4\mu^2)}}{2\mu^3}$$

$$-\text{Arctan}\frac{k(q^2+4\mu^2) - q\sqrt{\mu^4 + k^2(q^2+4\mu^2)}}{2\mu^3}]$$

and

$$\lim_{q\to 0} I_R = \frac{1}{\mu^3}\frac{1}{4k^2+\mu^2}.$$

Thus, we have

$$\text{Re}[f^{(2)}] = \frac{1}{32\pi^2}\mu^4\lambda^2 \frac{1}{q\mu^2\sqrt{\mu^4 + k^2(q^2+4\mu^2)}}$$

$$\times[\text{Arctan}\frac{k(q^2+4\mu^2) + q\sqrt{\mu^4 + k^2(q^2+4\mu^2)}}{2\mu^3}$$

$$-\text{Arctan}\frac{k(q^2+4\mu^2) - q\sqrt{\mu^4 + k^2(q^2+4\mu^2)}}{2\mu^3}]$$

We also have simpler expression as

$$\text{Re}[f^{(2)}] = \frac{1}{32\pi^2}\frac{\lambda^2\mu^2}{q}\frac{1}{\sqrt{\mu^4 + k^2(q^2+4\mu^2)}}$$

$$\text{Arctan}[\frac{q\mu\sqrt{\mu^4 + k^2(q^2+4\mu^2)}}{\mu^4 + k^2(q^2+4\mu^2) - \frac{q^2\mu^2}{4}}]$$

(see the note for this derivation)

We note that

$$\lim_{q\to 0}\frac{1}{q}\,\mathrm{Arctan}[\frac{q\mu}{\sqrt{\mu^4+4k^2 4\mu^2}}]=\frac{\mu}{\sqrt{\mu^2+4k^2}}$$

Using this relation, we have

$$\mathrm{Re}[f^{(2)}(0)]=\frac{1}{32\pi^2}\lambda^2\mu\frac{1}{\mu^2+4k^2},$$

at $q=0$.

((**Mathematica**)) **Calculation of I_R.**

```
Clear["Global`*"];
```

Calculation of the integral for real part

```
eq =
  1/
   (Sqrt[μ^2 + q^2 z (1 - z)]
     (μ^4 + 4 k^2 (μ^2 + q^2 z (1 - z))));

f1 = Integrate[eq, {z, 0, 1},
  Assumptions → {μ > 0, q > 0, k > 0}]
```

$$\left(-\text{ArcTan}\left[\frac{k\left(q^2+4\mu^2\right)-q\sqrt{\mu^4+k^2\left(q^2+4\mu^2\right)}}{2\mu^3}\right]+\text{ArcTan}\left[\frac{k\left(q^2+4\mu^2\right)+q\sqrt{\mu^4+k^2\left(q^2+4\mu^2\right)}}{2\mu^3}\right]\right)\Big/\left(q\mu^2\sqrt{\mu^4+k^2\left(q^2+4\mu^2\right)}\right)$$

Limit value of integral IR at q→0

```
Limit[f1, q → 0]
```

$$\frac{1}{4k^2\mu^3+\mu^5}$$

((**Note**)) **Simplification of the expression of** $\mathrm{Re}[f^{(2)}]$

We now simplify the expression of $\mathrm{Re}[f^{(2)}]$ using simple trigonometry.

$$
\begin{aligned}
A &= \operatorname{Arctan}\frac{k(q^2+4\mu^2)+q\sqrt{\mu^4+k^2(q^2+4\mu^2)}}{2\mu^3} \\
&\quad -\operatorname{Arctan}\frac{k(q^2+4\mu^2)-q\sqrt{\mu^4+k^2(q^2+4\mu^2)}}{2\mu^3} \\
&= \alpha-\beta
\end{aligned}
$$

where we use α and β as

$$
\tan\alpha = \frac{k(q^2+4\mu^2)+q\sqrt{\mu^4+k^2(q^2+4\mu^2)}}{2\mu^3},
$$

$$
\tan\beta = \frac{k(q^2+4\mu^2)-q\sqrt{\mu^4+k^2(q^2+4\mu^2)}}{2\mu^3}.
$$

We note that

$$
\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}.
$$

Then, we get

$$
\tan\alpha-\tan\beta = \frac{q}{\mu^3}\sqrt{\mu^4+k^2(q^2+4\mu^2)},
$$

$$1+\tan\alpha\tan\beta = 1+[\frac{k(q^2+4\mu^2)+q\sqrt{\mu^4+k^2(q^2+4\mu^2)}}{2\mu^3}]$$

$$\times[\frac{k(q^2+4\mu^2)-q\sqrt{\mu^4+k^2(q^2+4\mu^2)}}{2\mu^3}]$$

$$=1+\frac{1}{4\mu^6}[k^2(q^2+4\mu^2)^2-(q^2\mu^4+q^4k^2+4k^2q^2\mu^2)]$$

$$=1+\frac{1}{4\mu^6}[k^2(q^4+8q^2\mu^2+16\mu^4)-(q^2\mu^4+q^4k^2+4k^2q^2\mu^2)]$$

$$=1+\frac{1}{4\mu^4}[4k^2(q^2+4\mu^2)-q^2\mu^2]$$

$$=\frac{1}{\mu^4}[\mu^4+k^2(q^2+4\mu^2)]-\frac{q^2}{4\mu^2}$$

Thus, we have

$$\tan(\alpha-\beta)=\frac{\frac{q}{\mu^3}\sqrt{\mu^4+k^2(q^2+4\mu^2)}}{\frac{1}{\mu^4}[\mu^4+k^2(q^2+4\mu^2)]-\frac{q^2}{4\mu^2}},$$

$$=\frac{q\mu\sqrt{\mu^4+k^2(q^2+4\mu^2)}}{[\mu^4+k^2(q^2+4\mu^2)]-\frac{q^2\mu^2}{4}}$$

or

$$\alpha-\beta=\mathrm{Arctan}[\frac{q\mu\sqrt{\mu^4+k^2(q^2+4\mu^2)}}{[\mu^4+k^2(q^2+4\mu^2)]-\frac{q^2\mu^2}{4}}].$$

6. Optical theorem

The scattering amplitude up to the second order, is

$$f=f^{(1)}+f^{(2)}.$$

At $q = 0$, we get the scattering amplitude as

$$f^{(1)} = -\frac{\lambda}{4\pi}\mu\frac{1}{\mu^2} = -\frac{\lambda}{4\pi}\frac{1}{\mu},$$

and

$$\begin{aligned} f^{(2)} &= \frac{\lambda^2}{32\pi^2}\frac{\mu}{\mu^2+4k^2} + i\frac{1}{16}\lambda^2\left(\frac{k}{\mu^2+4k^2}\right) \\ &= \frac{\lambda^2}{32\pi^2}\left(\frac{\mu+2ik}{\mu^2+4k^2}\right) \\ &= \frac{\lambda^2}{32\pi^2}\frac{1}{\mu-2ik} \end{aligned}$$

If the Born approximation is to be reliable, we must certainly have the condition

$$\frac{\left|f^{(2)}\right|}{\left|f^{(1)}\right|} \ll 1.$$

For the forward scattering ($q = 0$), we gave

$$\frac{\left|f^{(2)}\right|}{\left|f^{(1)}\right|} = \frac{\frac{\lambda^2}{32\pi^2}\frac{1}{\sqrt{\mu^2+4k^2}}}{\frac{\lambda}{4\pi\mu}} = \frac{\lambda}{8\pi}\frac{\mu}{\sqrt{\mu^2+4k^2}} \ll 1.$$

When $k \gg \mu$ (high energy)

$$\frac{\left|f^{(2)}\right|}{\left|f^{(1)}\right|} = \frac{\lambda}{16\pi}\frac{\mu}{k} \ll 1.$$

The condition steadily improved as k increases. When $k \ll \mu$ (low energy, $k = 0$),

$$\frac{|f^{(2)}|}{|f^{(1)}|} = \frac{\lambda}{8\pi} \ll 1.$$

By taking into account of the second order $\mathrm{Im}[f^{(2)}(0)]$, we confirm that the optical theorem is valid such that

$$\begin{aligned}
\frac{4\pi}{k}\mathrm{Im}[f(0)] &= \frac{4\pi}{k}\mathrm{Im}[f^{(1)}(0) + f^{(2)}(0)] \\
&= \frac{4\pi}{k}\mathrm{Im}[f^{(2)}(0)] \\
&= \frac{\lambda^2}{4\pi}\frac{1}{\mu^2 + 4k^2} \\
&= \sigma^{(1)}
\end{aligned}$$

where

$$\sigma_{tot}{}^{(1)} = \frac{\lambda^2}{4\pi}\frac{1}{\mu^2 + 4k^2}.$$

We note that the differential cross section is

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= |f|^2 \\
&= \left| f^{(1)} + \mathrm{Re}[f^{(2)}] + \mathrm{Im}[f^{(2)}] \right|^2 \\
&= (f^{(1)} + \mathrm{Re}[f^{(2)}])^2 + (\mathrm{Im}[f^{(2)}])^2
\end{aligned}$$

$$f^{(1)} = -\frac{\lambda\mu}{4\pi}\frac{1}{\mu^2 + q^2},$$

$$\mathrm{Re}[f^{(2)}]=\frac{1}{32\pi^2}\frac{\lambda^2\mu^2}{q}\frac{1}{\sqrt{\mu^4+k^2(q^2+4\mu^2)}}$$
$$\mathrm{Arctan}[\frac{q\mu\sqrt{\mu^4+k^2(q^2+4\mu^2)}}{\mu^4+k^2(q^2+4\mu^2)-\frac{q^2\mu^2}{4}}]$$

and

$$\mathrm{Im}[f^{(2)}]=\frac{1}{32\pi^2}\frac{\lambda^2\mu^2}{q}\frac{\ln(\frac{\sqrt{\mu^4+k^2(q^2+4\mu)}+kq}{\sqrt{\mu^4+k^2(q^2+4\mu^2)}-kq})}{\sqrt{\mu^4+k^2(q^2+4\mu^2)}}$$

So that, we have the final form

$$\begin{aligned}
f &= f^{(1)}+f^{(2)} \\
&= -\frac{\lambda\mu}{4\pi}\frac{1}{\mu^2+q^2} \\
&\quad +\frac{1}{32\pi^2}\frac{\lambda^2\mu^2}{q}[\frac{1}{\sqrt{\mu^4+k^2(q^2+4\mu^2)}}\mathrm{Arctan}[\frac{q\mu\sqrt{\mu^4+k^2(q^2+4\mu^2)}}{\mu^4+k^2(q^2+4\mu^2)-\frac{q^2\mu^2}{4}}] \\
&\quad +i\frac{\ln(\frac{\sqrt{\mu^4+k^2(q^2+4\mu^2)}+kq}{\sqrt{\mu^4+k^2(q^2+4\mu^2)}-kq})}{\sqrt{\mu^4+k^2(q^2+4\mu^2)}}]
\end{aligned}$$

7. Validity of the Born approximation (another approach); direct calculation

The validity for the Born approximation is satisfied under the condition

$$\left|\frac{m}{2\pi\hbar^2}\int d\mathbf{r}\frac{e^{ikr}}{r}V(r)\right|<<1\,,$$

$$\left|\frac{m}{2\pi\hbar^2}\int d\mathbf{r}\frac{e^{ikr}}{r}\frac{V_0}{\mu r}e^{-\mu r}\right| << 1.$$

We note that

$$\begin{aligned}\int d\mathbf{r}\frac{e^{ikr}}{r}\frac{V_0}{\mu r}e^{-\mu r} &= 4\pi\int_0^\infty drr^2\frac{e^{ikr}}{r}\frac{V_0}{\mu r}e^{-\mu r}\\ &= 4\pi\frac{V_0}{\mu}\int_0^\infty dre^{(ik-\mu)r}\\ &= 4\pi\frac{V_0}{\mu}\frac{1}{\mu-ik}\end{aligned}$$

$$\left|\frac{m}{2\pi\hbar^2}4\pi\frac{V_0}{\mu}\frac{1}{\mu-ik}\right| = \frac{2m}{\hbar^2}\frac{V_0}{\mu}\frac{1}{\sqrt{\mu^2+k^2}} << 1.$$

(a) Low energy limit ($k << \mu$ **)**

$$\frac{2m}{\hbar^2}\frac{V_0}{\mu^2} << 1 \qquad\qquad |\lambda| << 4\pi,$$

(b) High energy limit ($k >> \mu$ **)**

$$\frac{2m}{\hbar^2}\frac{V_0}{\mu k} << 1 \qquad\qquad |\lambda| << 4\pi\frac{k}{\mu},$$

The inequality becomes easier to satisfy to satisfy as k increases, implying that the Born approximation becomes much accurate at high incident particle energy.

((**Note**)) **Scattering length *a***

The scattering length a is defined by

$$a = \frac{\beta}{\mu},$$

as will be derived later, with $\beta = \frac{4\pi}{\lambda} = \frac{2mV_0}{\hbar^2 \mu^2}$. The condition for the scattering for the high energy limit (as described above) is

$$|\lambda| << 4\pi \frac{k}{\mu}.$$

This condition can be rewritten as

$$ka \gg 1.$$

The de Broglie wavelength Λ is related as the wave number as

$$\Lambda = \frac{2\pi}{k}.$$

So that, the condition is also expressed as

$$a \gg \frac{\Lambda}{2\pi}.$$

The de Broglie wavelength Λ is much shorter than the scattering length *a*.

8. Partial phase shift

The Born approximation is a good approximation for the high energy incident particle, but is not so for the low energy incident particle.

$$f(E,q) = \sum_{l=0}^{\infty} (2l+1) f_l(E) P_l(\cos\theta),$$

$$f_l(E) = \frac{1}{2} \int_{-1}^{1} dz P_l(z) f(E,q),$$

$$z=\cos\theta\,,$$

$$1-\cos\theta=\frac{q^2}{2k^2}=\frac{\hbar^2 q^2}{4mE}\,,$$

$$\begin{aligned}
f^{(1)}&=-\frac{\lambda\mu}{4\pi}\frac{1}{\mu^2+q^2}\\
&=-\frac{\lambda\mu}{4\pi}\frac{1}{\mu^2+2k^2(1-\cos\theta)}\\
&=-\frac{\lambda\mu}{4\pi}\frac{1}{\mu^2+2k^2(1-z)}\\
&=-\frac{\lambda\mu}{4\pi}\frac{1}{\mu^2+\frac{4mE}{\hbar^2}(1-z)}
\end{aligned}$$

For $l=0$ (s-wave)

$$\begin{aligned}
f_0(E)&=\frac{1}{2}\int_{-1}^{1}dzf^{(1)}(E,q)\\
&=\frac{-\lambda\mu}{8\pi}\int_{-1}^{1}dz\frac{1}{\mu^2+\frac{4mE}{\hbar^2}(1-z)}\\
&=\frac{-\lambda\mu}{8\pi}\frac{\hbar^2}{4mE}\ln[\mu^2+\frac{4mE}{\hbar^2}(1-z)]\,|_{-1}^{1}\\
&=\frac{\lambda\mu\hbar^2}{32\pi mE}\ln[\frac{\mu^2}{\mu^2+\frac{8mE}{\hbar^2}}]\\
&=\frac{\gamma}{4E}\ln[\frac{\mu^2}{\mu^2+\frac{8mE}{\hbar^2}}]
\end{aligned}$$

where

$$\gamma=\frac{V_0}{\mu}$$

which has the expected branch point at $E=-\frac{\hbar^2\mu^2}{8m}$. Thus, while the scattering amplitude $f^{(1)}$ has no branch points, the partial amplitude does. This illustrates (which is, in fact, true of all partial waves) that the left-hand branch cut is introduced into the partial-wave amplitude in the process of making the projection (Taylor).

9. Phase shift analysis using the Heine expansion

Using the Heine expansion,

$$\frac{1}{x-\cos\theta}=\sum_{l=0}^{\infty}(2l+1)Q_l(x)P_l(\cos\theta)$$

we get the expansion of $f^{(1)}$ as

$$\begin{aligned}
f^{(1)}&=-\frac{2mV_0}{\hbar^2\mu}\frac{1}{\mu^2+4k^2\sin^2\frac{\theta}{2}}\\
&=-\frac{2mV_0}{\hbar^2\mu}\frac{1}{\mu^2+2k^2(1-\cos\theta)}\\
&=-\frac{mV_0}{\hbar^2\mu k^2}\frac{1}{1+\frac{\mu^2}{2k^2}-\cos\theta}\\
&=-\frac{mV_0}{\hbar^2\mu k^2}\frac{1}{x-\cos\theta}
\end{aligned}$$

$$x=1+\frac{\mu^2}{2k^2}$$

$$f^{(1)} = -\frac{mV_0}{\hbar^2 \mu k^2} \frac{1}{x - \cos\theta}$$

$$= -\frac{mV_0}{\hbar^2 \mu k^2} \sum_{l=0}^{\infty} (2l+1) Q_l(x) P_l(\cos\theta)$$

$$= \sum_{l=0}^{\infty} (2l+1) f_l(k) P_l(\cos\theta)$$

$$f_l(k) = -\frac{mV_0}{\hbar^2 \mu k^2} Q_l(x) .$$

10. Problem and Solution (Sakurai and Napolitano)

I found a very interesting problem on the phase shift analysis of the scattering due to the Yukawa potential, in famous textbook on quantum mechanics.

Problem 6-5

J.J. Sakurai and J. Napolitano

Modern Quantum mechanics, 3rd edition (Cambridge, 2021)

((**Problem**))

6.5 A spinless particle is scattered by a weak Yukawa potential

$$V = \frac{V_0 e^{-\mu r}}{\mu r},$$

where $\mu > 0$ but V_0 can be positive or negative. It was shown in the text that the first-order Born amplitude is given by

$$f^{(1)}(\theta) = -\frac{2mV_0}{\hbar^2 \mu} \frac{1}{[2k^2(1-\cos\theta)+\mu^2]}.$$

(a) Using $f^{(1)}(\theta)$ and assuming $|\delta_l| \ll 1$, obtain an expression for δ_l in terms of a Legendre function of the second kind,

$$Q_l(\zeta) = \frac{1}{2}\int_{-1}^{1} \frac{P_l(\zeta')}{\zeta - \zeta'} d\zeta'.$$

(b) Use the expansion formula

$$Q_l(\zeta) = \frac{l!}{1\cdot 3\cdot 5\cdots(2l+1)} \times \left\{ \frac{1}{\zeta^{l+1}} + \frac{(l+1)(l+2)}{2(2l+3)} \frac{1}{\zeta^{l+3}} + \frac{(l+1)(l+2)(l+3)(l+4)}{2\cdot 4\cdot (2l+3)(2l+5)} \frac{1}{\zeta^{l+5}} + \cdots \right\} \quad (|\zeta| > 1)$$

to prove each assertion.

(i) δ_l is negative (positive) when the potential is repulsive (attractive).

(ii) When the de Broglie wavelength is much longer than the range of the potential, δ_l is proportional to k^{2l+1}. Find the proportionality constant.

((**Solution**))

Yukawa potential

$$V(r) = \frac{V_0}{\mu r} e^{-\mu r}.$$

The first Born approximation

$$f^{(1)}(\theta) = -\frac{2mV_0}{\hbar^2 \mu} \frac{1}{q^2 + \mu^2},$$

where

$$\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{k}',$$

$$q^2 = k^2 + k'^2 - 2kk'\cos\theta$$
$$= 2k^2(1 - \cos\theta)$$

where the angle between the vctors $\boldsymbol{k}$ and $\boldsymbol{k}'$ is θ. $k' = k$

(b)

The scattering amplitude $f(\theta)$ can be expanded in terms of the phase shift δ_l as

$$f(\theta) = \sum_{l'=0}^{\infty}(2l'+1)f_{l'}(k)P_{l'}(\cos\theta).$$

Note that

$$I = \int_{-1}^{1} d(\cos\theta)P_l(\cos\theta)f(\theta)$$
$$= \int_{-1}^{1} d(\cos\theta)\sum_{l'=0}^{\infty}(2l'+1)f_{l'}(k)P_l(\cos\theta)P_{l'}(\cos\theta)$$

with the use of formula

$$\int_{-1}^{1} d(\cos\theta)P_l(\cos\theta)P_{l'}(\cos\theta) = \frac{2}{2l+1}\delta_{l,l'}.$$

Then, we get

$$I = \sum_{l'=0}^{\infty}(2l'+1)f_{l'}(k)\frac{2}{2l+1}\delta_{l,l'}$$
$$= (2l+1)f_l(k)\frac{2}{2l+1}$$
$$= 2f_l(k)$$

Thus, we have

$$
\begin{aligned}
f_l(k) &= \frac{1}{2}\int_{-1}^{1} d(\cos\theta) P_l(\cos\theta) f(\theta) \\
&= \frac{1}{2}\int_{-1}^{1} d(\cos\theta) P_l(\cos\theta)\left(-\frac{2mV_0}{\hbar^2\mu}\right)\frac{1}{2k^2(1-\cos\theta)+\mu^2} \\
&= -\frac{mV_0}{\mu\hbar^2 k^2}\frac{1}{2}\int_{-1}^{1} dz \frac{P_l(z)}{1+\frac{\mu^2}{2k^2}-z} \\
&= -\frac{mV_0}{\mu\hbar^2 k^2} Q_l(1+\frac{\mu^2}{2k^2})
\end{aligned} \qquad (1)
$$

where $Q_l(\varsigma)$ is the Legendre function of the second kind and is defined by

$$
Q_l(x) = \frac{1}{2}\int_{-1}^{1} dz \frac{P_l(z)}{x-z},
$$

with

$$
x = 1 + \frac{\mu^2}{2k^2}.
$$

We make a plot of $Q_l(x)$ as a function of x for $x>1$, where l is changed as a parameter. (numerical integration is done by using the Mathematica).

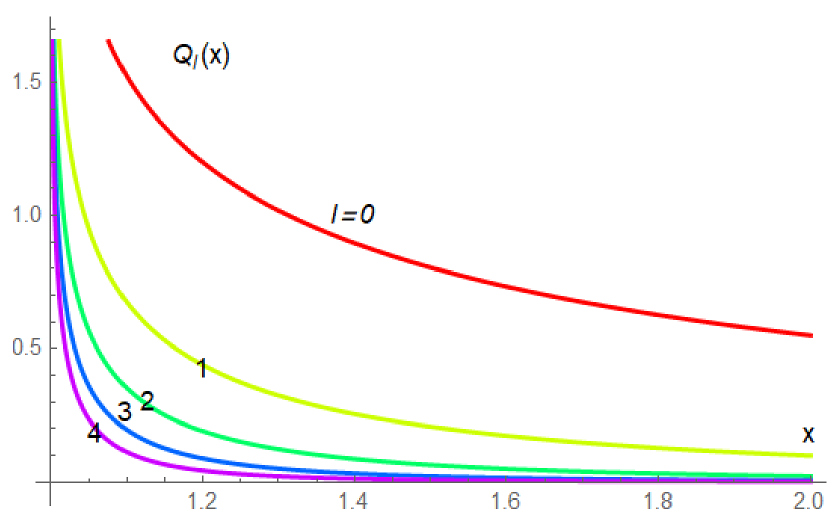


Fig.8 $Q_l(x)$ as a function of *x*. $x = 1 + \frac{\mu^2}{k^2}$. l = 0, 1, 2, 3,….

(b)

(i)

As is shown in the above figure, we have

$Q_l(x) > 0$ for $x > 1$.

$$x = 1 + \frac{\mu^2}{2k^2}$$

(ii) de Broglie wave length $\lambda = \frac{2\pi}{k} >> \frac{1}{\mu}$ (the range of potential)

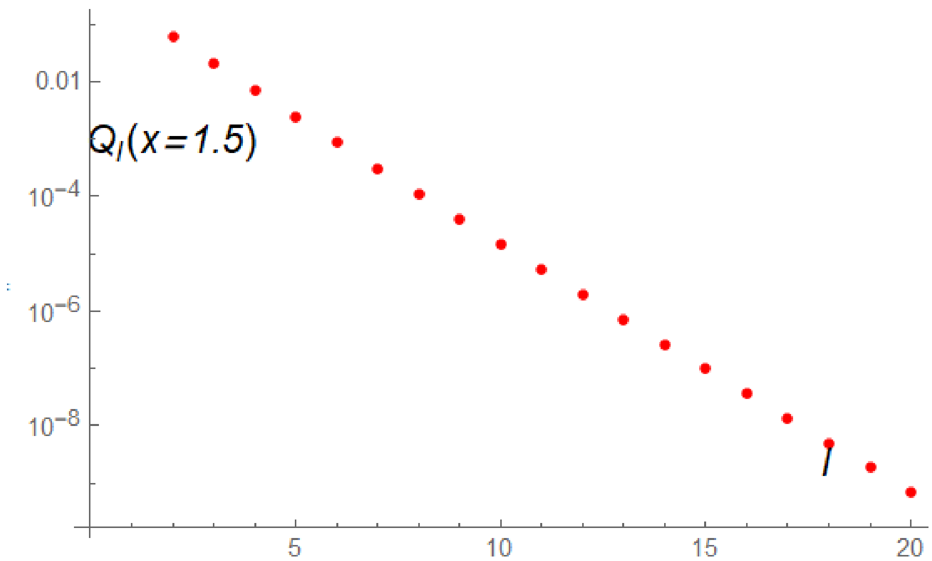
or $$\frac{\mu}{k} >> 1$$

$$\begin{aligned}
f_l(k) &= -\frac{mV_0}{\mu\hbar^2 k^2} Q_l(1 + \frac{\mu^2}{2k^2}) \\
&\approx -\frac{mV_0}{\mu\hbar^2 k^2} Q_l(\frac{\mu^2}{2k^2}) \\
&= -\frac{mV_0}{\mu\hbar^2 k^2} \frac{l!}{(2l+1)!!} \left(\frac{2k^2}{\mu^2}\right)^{l+1} \\
&= -\frac{mV_0}{\hbar^2 \mu^{2l+3}} \frac{2^{l+1} l!}{(2l+1)!!} k^{2l+1}
\end{aligned}$$

or

$$f_l(k) = -\frac{mV_0}{\hbar^2 \mu^2} \frac{2^{l+1} l!}{(2l+1)!!} (\frac{k}{\mu})^{2l+1} = -\frac{\lambda}{8\pi} \frac{2^{l+1} l!}{(2l+1)!!} (\frac{k}{\mu})^{2l+1}$$

which is proportional to k^{2l+1}.



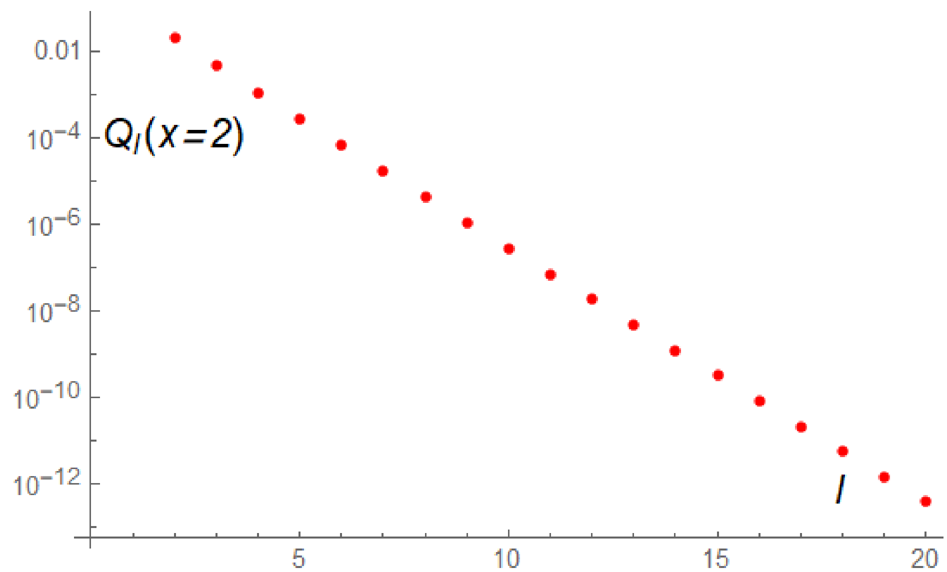


Fig.9 $Q_l(x)$ as a function of l with $x=1.5$ and 2. $x=1+\dfrac{\mu^2}{2k^2}$.

11. The effective range: Phase shift (Capri)

The phase shift for $l = 0$ (s-wave) is given by

$$\sin\delta_0 = -\frac{2m}{\hbar^2} k \int_0^\infty dr r^2 V(r) \left[\frac{\sin(kr)}{kr}\right]^2$$
$$= -\frac{2m}{\hbar^2} k \int_0^\infty dr r^2 \frac{V_0 e^{-\mu r}}{\mu r} \left[\frac{\sin(kr)}{kr}\right]^2$$
$$= -\frac{2m}{\hbar^2} \frac{kV_0}{\mu} \int_0^\infty dr r^2 \frac{e^{-\mu r}}{r} \left[\frac{\sin(kr)}{kr}\right]^2$$
$$= -\frac{\lambda}{16\pi} \frac{\mu}{k} \ln(1+\frac{4k^2}{\mu^2})$$
$$= -\frac{\beta}{4} \frac{\mu}{k} \ln(1+\frac{4k^2}{\mu^2})$$

where

$$\lambda = 4\pi \frac{2mV_0}{\hbar^2 \mu^2}, \qquad \beta = \frac{2mV_0}{\hbar^2 \mu^2} = \frac{\lambda}{4\pi},$$

and

$$\int_0^\infty dr r^2 \frac{e^{-\mu r}}{r} \left[\frac{\sin(kr)}{kr}\right]^2 = \frac{1}{4k^2} \ln(1+\frac{4k^2}{\mu^2})$$

Thus, we get

$$\sin\delta_0 = -\frac{\beta}{4} \frac{\mu}{k} \ln(1+\frac{4k^2}{\mu^2})$$
$$\simeq -\frac{\beta}{4} \frac{\mu}{k} (\frac{4k^2}{\mu^2} - \frac{8k^4}{\mu^4})$$
$$= -\beta \frac{k}{\mu} (1 - \frac{2k^2}{\mu^2})$$

Note that

$$k\cot\delta_0 = k\frac{\cos\delta_0}{\sin\delta_0},$$

and

$$\begin{aligned}
\cos\delta_0 &= \sqrt{1-\sin^2\delta_0} \\
&= \sqrt{1-\frac{\beta^2 k^2}{\mu^2}\left(1-\frac{2k^2}{\mu^2}\right)^2} \\
&\simeq \sqrt{1-\frac{\beta^2 k^2}{\mu^2}} \\
&\simeq 1-\frac{\beta^2 k^2}{2\mu^2}
\end{aligned}$$

$$\begin{aligned}
k\cot\delta_0 &= k\frac{\cos\delta_0}{\sin\delta_0} \\
&= -k\frac{1-\frac{\beta^2 k^2}{2\mu^2}}{\beta\frac{k}{\mu}\left(1-\frac{2k^2}{\mu^2}\right)} \\
&\simeq -\frac{\mu}{\beta}\left(1-\frac{\beta^2 k^2}{2\mu^2}\right)\left(1+\frac{2k^2}{\mu^2}\right) \\
&= -\frac{\mu}{\beta}+\frac{1}{2}k^2\left(\beta-\frac{4}{\beta}\right)\frac{1}{\mu} \\
&= -\frac{1}{a}+\frac{1}{2}r_0 k^2
\end{aligned}$$

where a is the scattering length and r_0 is the effective range.

$$a = \frac{\beta}{\mu} = \frac{\lambda}{4\pi\mu}, \quad r_0 = \frac{1}{\mu}\left(\beta-\frac{4}{\beta}\right).$$

Note that our result of r_0 is a little different from that reported by Capri. The parameter β is dimensionless. For the low energy limit, the condition for the Born approximation is $\lambda \ll 4\pi$, leading to $a \ll \frac{1}{\mu}$. For the high energy limit, the condition for the Born approximation is $|\lambda| << 4\pi \frac{k}{\mu}$, leading to $a \ll \frac{1}{\mu}\frac{k}{\mu}$.

We note that the total cross section (the first order Born approximation) is given by

$$\sigma^{(1)} = \frac{\lambda^2}{4\pi} \frac{1}{\mu^2 + 4k^2}$$

In the limit of $k \to 0$, we have

$$\boxed{\sigma^{(1)}(k=0) = \frac{1}{4\pi} \frac{\lambda^2}{\mu^2} = 4\pi a^2}$$

using the scattering length a. Thus, the scattering length determines the low-energy scattering cross section. We see that

$$\begin{aligned}
f_0(k) &= S_0(k) - 1 \\
&= e^{2i\delta_0(k)} - 1 \\
&= \frac{2i}{\cot\delta_0(k) - 1} \\
&= \frac{2i \sin\delta_0(k)}{\cos\delta_0(k) - \sin\delta_0(k)} \\
&\simeq \frac{2i \sin\delta_0(k)}{1 - \sin\delta_0(k)} \\
&= \frac{2ka}{i - ka}
\end{aligned}$$

where $\sin[\delta_0(k)] = -k\dfrac{\beta}{\mu} = -ka$. Thus, $f_0(k)$ has a single pole at $k = \dfrac{i}{a}$. This pole corresponds to a bound state of energy $E_b = -\dfrac{\hbar^2\kappa^2}{2m}$ and $\kappa = \dfrac{1}{a}$. Since the bound state is of the form $\exp(-\kappa r)/r$, its extension is a.

((**Mathematica**)) Calculation

$$g4 = \frac{1}{\mu}\,\frac{e^{-\mu r}}{k^2 r}\,\mathrm{Sin}[k\, r]^2;$$

```
s1 = Integrate[g4, {r, 0, ∞},
  Assumptions → {μ > 0,  k > 0}]
```

$$\frac{\mathrm{Log}\left[1 + \frac{4 k^2}{\mu^2}\right]}{4 k^2 \mu}$$

12. Optical theorem (Capri)

The scattering amplitude for low energy s-waves may be written as

$$\begin{aligned}
f_0(k) &= \frac{e^{2i\delta_0} - 1}{2ik} \\
&= \frac{1}{2ik} e^{i\delta_0}(e^{i\delta_0} - e^{-i\delta_0}) \\
&= \frac{1}{2ik} 2i \sin\delta_0 e^{i\delta_0} \\
&= \frac{\sin\delta_0}{ke^{-i\delta_0}} \\
&= \frac{\sin\delta_0}{k\cos\delta_0 - ik\sin\delta_0} \\
&= \frac{1}{k\cot\delta_0 - ik}
\end{aligned}$$

We note that

$$k\cot\delta_0 = -\frac{1}{a} + \frac{1}{2} r_0 k^2,$$

where a is the scattering length and r_0 is the effective range. Thus, we get

$$\begin{aligned}
f_0(k) &= \frac{1}{-\frac{1}{a} + \frac{1}{2} r_0 k^2 - ik} \\
&= -\frac{a}{1 + iak - \frac{1}{2} r_0 a k^2}
\end{aligned}$$

The optical theorem states that the total cross-section σ is given by

$$\begin{aligned}
\sigma &= \frac{4\pi}{k}\,\mathrm{Im}[f_0(k)] \\
&= \frac{4\pi}{k}\,\mathrm{Im}[\frac{\sin\delta_0}{ke^{-i\delta_0}}] \\
&= \frac{4\pi}{k}\frac{1}{k}\sin^2\delta_0
\end{aligned}$$

On the other hand, the total cross section is given by

$$\begin{aligned}
\sigma &= \int \frac{d\sigma}{d\Omega} d\Omega \\
&= \int \left| f_0(k) \right|^2 d\Omega \\
&= 4\pi \left| f_0(k) \right|^2 \\
&= 4\pi \left| \frac{\sin\delta_0}{ke^{-i\delta_0}} \right|^2 \\
&= \frac{4\pi}{k^2}\sin^2\delta_0
\end{aligned}$$

So, in this, the optical theorem is proved. We now use the expression of

$$f_0(k) = -\frac{a}{1 + iak - \frac{1}{2} r_0 a k^2}.$$

Thus, we have

$$\begin{aligned}
\sigma &= \frac{4\pi}{k} \operatorname{Im}[f_0(k)] \\
&= \frac{4\pi}{k} \frac{a^2 k}{(1 - \frac{1}{2} r_0 a k^2)^2 + a^2 k^2} \\
&= \frac{4\pi a^2}{(1 - \frac{1}{2} r_0 a k^2)^2 + a^2 k^2}
\end{aligned}$$

On the other hands by direct calculation, we find that

$$\begin{aligned}
\sigma &= \int \frac{d\sigma}{d\Omega} d\Omega \\
&= \int \left| f_0(k) \right|^2 d\Omega \\
&= 4\pi \left| f_0(k) \right|^2 \\
&= \frac{4\pi a^2}{(1 - \frac{1}{2} r_0 a k^2)^2 + a^2 k^2}
\end{aligned}$$

Thus, in this case, we also verify the optical theorem.

13. Conclusion

There are so many excellent textbooks on the scattering due to the Yukawa potential. Before I wrote the present article, I read these textbooks, which were mainly written before 1970's. I tried to calculate the scattering amplitude for the Yukawa potential up to the second-order Born approximation. I encountered several integrals which seemed to me to be so complicated. Thanks to Mathematica, I succeeded in getting exact results within seconds. It was amazing to me. After that, I checked the validity of my results by comparing with the results reported in the standard textbooks (such as textbook by J.R. Taylor). The reporting of this article in part is motivated in part by our success in obtaining the exact results by Mathematica.

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APPENDIX-1 The notations used in this article (summary)

$$V(r) = \frac{V_0}{\mu r} e^{-\mu r}$$ (Yukawa potential)

$$\lambda = 4\pi \frac{2mV_0}{\hbar^2 \mu^2}.$$

$$f^{(1)} = -\frac{2mV_0}{\hbar^2 \mu} \frac{1}{\mu^2 + q^2} = -\frac{\lambda}{4\pi} \frac{\mu}{\mu^2 + q^2}.$$ (the first order Born approximation)

$$\sigma^{(1)} = \frac{\lambda^2}{4\pi} \frac{1}{\mu^2 + 4k^2} = \frac{4\pi}{k} \mathrm{Im}[f^{(2)}].$$ (Optical theorem)

$$q = 2k \sin\frac{\theta}{2}.$$

$$\beta = \frac{\lambda}{4\pi}.$$

$$\mathrm{Re}[f^{(2)}(0)] = \frac{\lambda^2}{32\pi^2} \frac{\mu}{\mu^2 + 4k^2}.$$

$$\mathrm{Im}[f^{(2)}(0)] = \frac{\lambda^2}{32\pi^2} \frac{2k}{\mu^2 + 4k^2}.$$

$$\mathrm{Re}[f^{(2)}] = \frac{1}{32\pi^2} \frac{\lambda^2 \mu^2}{q} \frac{1}{\sqrt{\mu^4 + k^2(q^2 + 4\mu^2)}} \mathrm{Arctan}\left[\frac{q\mu\sqrt{\mu^4 + k^2(q^2 + 4\mu^2)}}{\mu^4 + k^2(q^2 + 4\mu^2) - \frac{q^2\mu^2}{4}}\right].$$

$$\mathrm{Im}[f^{(2)}] = \frac{1}{32\pi^2} \frac{\lambda^2 \mu^2}{q} \frac{\ln\left(\frac{\sqrt{\mu^4 + k^2(q^2 + 4\mu^2)} + kq}{\sqrt{\mu^4 + k^2(q^2 + 4\mu^2)} - kq}\right)}{\sqrt{\mu^4 + k^2(q^2 + 4\mu^2)}}.$$

APPENDIX-II Mathematical formula

$$P_n(x) = {}_2F_1(-n, n+1; 1; \frac{1}{2}(1-x))$$

LegendreP[n, x] for $|x| \le 1$ (Mathematica)

The Legendre function of the second kind is

$$Q_\nu(z) = \frac{\sqrt{\pi}\Gamma(\nu+1)}{\Gamma(\nu+\frac{3}{2})(2z)^{\nu+1}} {}_2F_1(\frac{\nu}{2}+1, \frac{\nu}{2}+\frac{1}{2}; \nu+\frac{3}{2}; z^{-2})$$

using the hypergeometric function, where $|z| > 1$, $\arg(z) < \pi$, and $\nu \neq$ -1, -2, -3,…

$$Q_l(x) = \frac{1}{2}\int_{-1}^{1} \frac{P_l(y)}{x-y}$$ for $|x| > 1$.

LegendreQ[n,x] for $|x| \le 1$ (Mathematica)

((**Asymptotics**))

Asymptotically for $l \to \infty$,

$$\begin{aligned} P_l(\cos\theta) &= \sqrt{\frac{\theta}{\sin\theta}} J_0[(l+\frac{1}{2})\theta] + O(l^{-1}) \\ &= \frac{2}{\sqrt{2\pi l \sin\theta}} \cos[(l+\frac{1}{2})\theta - \frac{\pi}{4}] + O(l^{-3/2}) \end{aligned}$$

$0 \le \theta \le \pi$ and for argument of magnitude greater than 1,

$$\begin{aligned} P_l(\frac{1}{\sqrt{1-e^2}}) &= I_0(le) + O(l^{-1}) \\ &= \frac{1}{\sqrt{2\pi l e}} \frac{(1+e)^{\frac{l+1}{2}}}{(1-e)^{\frac{l}{2}}} + O(l^{-1}) \end{aligned}$$

where J_0 and I_0 are Bessel functions.