Landau level of conduction electron in the presence of magnetic field

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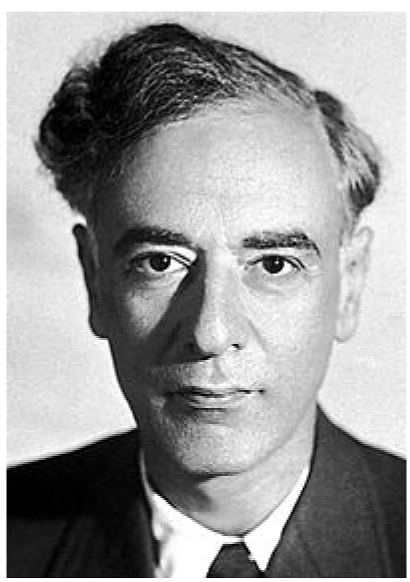
We discuss the motion of electrons in an external magnetic field. First we consider the classical motion of electron. The electrons undergo a circular motion in the plane perpendicular to the magnetic field, with the cyclotron angular frequency. Next we discuss the motion of electron in terms of quantum mechanics. The wavefunction depends on the choice of gauges such as Landau gauge, Coulomb gauge, and symmetric gauge. The energy of the electrons is quantized. Each quantized energy level is called the Landau level.

Lev Davidovich Landau (January 22, 1908– April 1, 1968) was a prominent Soviet physicist who made fundamental contributions to many areas of theoretical physics. His accomplishments include the independent co-discovery of the density matrix method in quantum mechanics (alongside John von Neumann), the quantum mechanical theory of diamagnetism, the theory of superfluidity, the theory of second-order phase transitions, the Ginzburg–Landau theory of superconductivity, the theory of Fermi liquid, the explanation of Landau damping in plasma physics, the Landau pole in quantum electrodynamics, the two-component theory of neutrinos, and Landau's equations for *S* matrix singularities. He received the 1962 Nobel Prize in Physics for his development of a mathematical theory of superfluidity that accounts for the properties of liquid helium II at a temperature below 2.17 K.



http://en.wikipedia.org/wiki/Lev_Landau

1. Cyclotron motion of electron in the presence of magnetic field (classical theory)

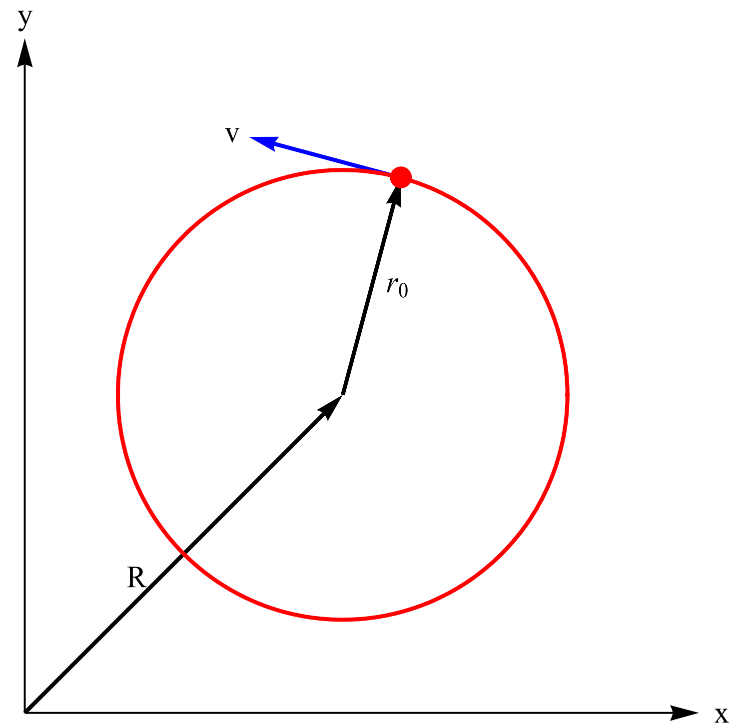


Fig. Cyclotron motion in classical mechanics.

We consider the motion of the electron in the presence of a magnetic field $\boldsymbol{B}$. The Lorentz force is given by

$$\boldsymbol{F} = m_e \dot{\boldsymbol{r}} = -\frac{e}{c}\dot{r} \times \boldsymbol{B} \qquad \text{(Lorentz force)}$$

where m_e is the mass of electron, $-e$ is the charge of electron ($e>0$), $\boldsymbol{B}$ is an external magnetic field along the z axis. The solution of this differential equation is as follows.

$$\boldsymbol{r} = \boldsymbol{R} + r_0(\cos(\omega_c t), \sin(\omega_c t), 0),$$

$$\boldsymbol{v} = r_0\omega_c(-\sin(\omega_c t), \cos(\omega_c t), 0),$$

where the cyclotron angular frequency ω_c is given by

$$\omega_c = \frac{eB}{m_e c}.$$

The trajectory of an electron is a circle with a radius r_0 centered at the position vector ***R***= $\boldsymbol{R} = (X, Y, 0)$. Suppose that the relative co-ordinate from the center is given by the posion vector $(\xi, \eta, 0)$. Then we have

$$\boldsymbol{r} = (x, y) = (X + \xi, Y + \eta, 0), \qquad \boldsymbol{v} = \omega_c(-\eta, \xi, 0),$$

where

$$\xi = r_0 \cos(\omega_c t), \qquad \eta = r_0 \sin(\omega_c t).$$

2. Gauge transformation (classical theory)

The Hamiltonian H [q = -e (e>0)] is given by

$$H = \frac{1}{2m_e}(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A})^2 + q\phi = \frac{1}{2m_e}(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A})^2 - e\phi,$$

Where q = -e (e>0) ***A*** is the vector potential and ϕ is the scalar potential. The electric field and the magnetic field are expressed by

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \qquad \boldsymbol{E} = -\nabla\phi - \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}.$$

The canonical momentum is defined by

$$\boldsymbol{\pi} = m_e \boldsymbol{v} = \boldsymbol{p} - \frac{q}{c}\boldsymbol{A} = \boldsymbol{p} + \frac{e}{c}\boldsymbol{A}.$$

The gauge transformation is defined by

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi, \qquad \phi' = \phi - \frac{1}{c}\frac{\partial\chi}{\partial t},$$

where χ is an arbitrary function of the position vector ***r***. The orbital angular momentum is defined by

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times (\boldsymbol{\pi} - \frac{e}{c}\boldsymbol{A}) = \boldsymbol{r} \times \boldsymbol{\pi} - \frac{e}{c}\boldsymbol{r} \times \boldsymbol{A}$$

since

$$\boldsymbol{p} = \boldsymbol{\pi} - \frac{e}{c}\boldsymbol{A}$$

(i) Symmetric gauge

In the presence of the magnetic field $\boldsymbol{B}$ (constant), we can choose the vector potential as

$$\boldsymbol{A} = \frac{1}{2}(\boldsymbol{B}\times\boldsymbol{r}) = \frac{1}{2}\begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ 0 & 0 & B \\ x & y & z \end{vmatrix} = \frac{1}{2}(-By, Bx, 0)$$

where

$$\nabla\times\boldsymbol{A} = \frac{1}{2}\nabla\times(\boldsymbol{B}\times\boldsymbol{r}) = \boldsymbol{B}$$

(ii) Landau gauge

In the gauge transformation:

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi$$

we choose $\chi = \frac{1}{2}Bxy$. Then we have

$$\nabla\chi = \frac{1}{2}B(y, x, 0)$$

Thus the new vector potential $\boldsymbol{A}'$ is obtained as

$$\boldsymbol{A}' = (0, Bx, 0) \qquad \text{(Landau gauge)}$$

3. Commutation relations in quantum mechanics (general gauge)

We discuss the commutation relations in quantum mechanics. Since the gauge is not specified, the discussion below is applicable for any gauge. We start with the quantum mechanical operator,

$$\hat{\boldsymbol{\pi}} = \hat{\boldsymbol{p}} + \frac{e}{c}\hat{\boldsymbol{A}}.$$

where $e>0$, and $\hat{\boldsymbol{A}}$ is a vector potential depending on the position operator $\hat{\boldsymbol{r}}$. The commutation relation is given by

$$[\hat{x}, \hat{\pi}_x] = [\hat{x}, \hat{p}_x + \frac{e}{c}\hat{A}_x] = i\hbar\hat{1},$$

$$[\hat{y}, \hat{\pi}_y] = [\hat{y}, \hat{p}_y + \frac{e}{c}\hat{A}_y] = i\hbar\hat{1}$$

$$[\hat{x}, \hat{\pi}_y] = [\hat{x}, \hat{p}_y + \frac{e}{c}\hat{A}_y] = 0$$

$$[\hat{y}, \hat{\pi}_x] = [\hat{y}, \hat{p}_x + \frac{e}{c}\hat{A}_x] = 0$$

where we use the relation

$$[\hat{x}, \hat{A}_x] = [\hat{y}, \hat{A}_x] = 0, \quad [\hat{x}, \hat{A}_y] = [\hat{y}, \hat{A}_y] = 0$$

We also get the commutation relation,

$$\begin{aligned}
[\hat{\pi}_x, \hat{\pi}_y] &= [\hat{p}_x + \frac{e}{c}\hat{A}_x, \hat{p}_y + \frac{e}{c}\hat{A}_y] \\
&= \frac{e}{c}[\hat{p}_x, \hat{A}_y] - \frac{e}{c}[\hat{p}_y, \hat{A}_x] \\
&= \frac{e\hbar}{ic}\frac{\partial \hat{A}_y}{\partial \hat{x}} - \frac{e\hbar}{ic}\frac{\partial \hat{A}_x}{\partial \hat{y}} \\
&= \frac{e\hbar}{ic}B_z
\end{aligned}$$

or

$$[\hat{\pi}_x, \hat{\pi}_y] = \frac{e\hbar}{ic}B_z$$

where

$$\frac{\partial \hat{A}_y}{\partial \hat{x}} - \frac{\partial \hat{A}_x}{\partial \hat{y}} = B_z .$$

We have the commutation relations,

$$[\hat{\pi}_y, \hat{\pi}_z] = \frac{e\hbar}{ic} B_x , \qquad \text{and} \qquad [\hat{\pi}_z, \hat{\pi}_x] = \frac{e\hbar}{ic} B_y .$$

Suppose that $\boldsymbol{B} = (0,0,B)$ or $B_z = B$. Then we get

$$[\hat{\pi}_x, \hat{\pi}_y] = \frac{e\hbar B}{ic} , \qquad [\hat{\pi}_y, \hat{\pi}_z] = 0 , \qquad [\hat{\pi}_z, \hat{\pi}_x] = 0 .$$

Note that

$$[\hat{\pi}_x, \hat{\pi}_y] = \frac{e\hbar^2 B}{ic\hbar} = -i\frac{\hbar^2}{\ell^2} ,$$

where the characteristic length l is defined by

$$\ell^2 = \frac{c\hbar}{eB} .$$

4. Hamiltonian (general gauge)

The Hamiltonian $\hat{H}$ is given by

$$\hat{H} = \frac{1}{2m_e}(\hat{\boldsymbol{p}} + \frac{e}{c}\boldsymbol{A})^2 = \frac{1}{2m_e}(\hat{\pi}_x{}^2 + \hat{\pi}_y{}^2) .$$

We define the creation and annihilation operators such that

$$\hat{a} = \frac{l}{\sqrt{2}\hbar}(\hat{\pi}_x - i\hat{\pi}_y) , \qquad \hat{a}^+ = \frac{l}{\sqrt{2}\hbar}(\hat{\pi}_x + i\hat{\pi}_y) ,$$

or

$$\hat{\pi}_x = \frac{\hbar}{\sqrt{2}l}(\hat{a} + \hat{a}^+) , \qquad \hat{\pi}_y = \frac{\hbar}{i\sqrt{2}l}(\hat{a}^+ - \hat{a}) ,$$

$$[\hat{a},\hat{a}^+] = \frac{l^2}{2\hbar^2}[\hat{\pi}_x - i\hat{\pi}_y, \hat{\pi}_x + i\hat{\pi}_y] = \frac{l^2}{\hbar^2} i[\hat{\pi}_x, \hat{\pi}_y] = \frac{l^2}{\hbar^2} i(-i\frac{\hbar^2}{l^2}) = \hat{1},$$

$$\hat{\pi}_x{}^2 + \hat{\pi}_y{}^2 = \frac{\hbar^2}{2l^2}[(\hat{a}+\hat{a}^+)^2 - (\hat{a}^+ - \hat{a})^2] = \frac{\hbar^2}{l^2}(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}) = \frac{\hbar^2}{l^2}(2\hat{a}^+\hat{a}+1).$$

Thus we have

$$\hat{H} = \frac{\hbar^2}{m_e l^2}(\hat{a}^+\hat{a} + \frac{1}{2}) = \hbar\omega_c(\hat{a}^+\hat{a} + \frac{1}{2})$$

where

$$\hbar\omega_c = \frac{\hbar^2}{m_e l^2} = \frac{\hbar^2}{m_e \frac{c\hbar}{eB}} = \frac{\hbar eB}{m_e c}$$

When $\hat{a}^+\hat{a} = \hat{N}$, the Hamiltonian is described by

$$\hat{H} = \hbar\omega_c(\hat{N} + \frac{1}{2}).$$

We find the energy levels for the free electrons in a homogeneous magnetic field, which is known as Landau levels. We mote that

$$\hat{H}|n\rangle = \hbar\omega_c(\hat{N} + \frac{1}{2})|n\rangle = \hbar\omega_c(n + \frac{1}{2})|n\rangle$$

with the energy eigenvalue $E_n = (n + \frac{1}{2})\hbar\omega_c$ (n = 0, 1, 2, 3, ...).

5. Guiding centers (general case)

Here we derive the expression for the guiding centers in quantum mechanics (general gauge). The Hamiltonian $\hat{H}$ is given by

$$\hat{H} = \frac{1}{2m_e}(\hat{\boldsymbol{p}} + \frac{e}{c}\boldsymbol{A})^2 = \frac{1}{2m_e}(\hat{\pi}_x{}^2 + \hat{\pi}_y{}^2) = \frac{1}{2m_e}[(\hat{p}_x + \frac{e}{c}A_x)^2 + \frac{1}{2m_e}(\hat{p}_x + \frac{e}{c}A_y)^2]$$

where e>0, and

$$\hat{\pi}_x = \hat{p}_x + \frac{e}{c} A_x, \qquad \hat{\pi}_y = \hat{p}_y + \frac{e}{c} A_y .$$

(i) The commutation relation (I)

We have the commutation relations,

$$[\hat{x}, 2m_e \hat{H}] = [\hat{x}, \hat{\pi}_x{}^2 + \hat{\pi}_y{}^2] = [\hat{x}, \hat{\pi}_x{}^2] = 2i\hbar \hat{\pi}_x \tag{1}$$

$$[\hat{y}, 2m_e \hat{H}] = [\hat{y}, \hat{\pi}_x{}^2 + \hat{\pi}_y{}^2] = [\hat{y}, \hat{\pi}_y{}^2] = 2i\hbar \hat{\pi}_y \tag{2}$$

where

$$[\hat{x}, \hat{\pi}_x] = i\hbar \hat{1},$$
$$[\hat{y}, \hat{\pi}_y] = i\hbar \hat{1},$$
$$[\hat{x}, \hat{\pi}_y] = 0,$$
$$[\hat{y}, \hat{\pi}_x] = 0,$$
$$[\hat{\pi}_x, \hat{\pi}_y] = -i\frac{\hbar^2}{l^2}, \qquad [\hat{\pi}_y, \hat{\pi}_z] = 0, \quad [\hat{\pi}_z, \hat{\pi}_x] = 0$$

(ii) The commutation relation (II):

We also have the commutation relations,

$$[\hat{\pi}_x, 2m_e \hat{H}] = [\hat{\pi}_x, \hat{\pi}_x{}^2 + \hat{\pi}_y{}^2] = [\hat{\pi}_x, \hat{\pi}_y{}^2] = (-2i)\frac{\hbar^2}{l^2}\hat{\pi}_y$$

$$[\hat{\pi}_y, 2m_e \hat{H}] = [\hat{\pi}_y, \hat{\pi}_x{}^2 + \hat{\pi}_y{}^2] = [\hat{\pi}_y, \hat{\pi}_x{}^2] = 2i\frac{\hbar^2}{l^2}\hat{\pi}_x, \tag{3}$$

From Eqs,(1)-(4), we have

$$[\hat{x} - \frac{l^2}{\hbar}\hat{\pi}_y, 2m_e \hat{H}] = 2i\hbar \hat{\pi}_x - \frac{l^2}{\hbar} 2i \frac{\hbar^2}{l^2}\hat{\pi}_x = 0 .$$

$$[\hat{y} + \frac{l^2}{\hbar}\hat{\pi}_x, 2m_e \hat{H}] = 2i\hbar \hat{\pi}_y + \frac{l^2}{\hbar}(-2i)\frac{\hbar^2}{l^2}\hat{\pi}_y = 0$$

So we can define the guiding center as

$$\hat{X} = \hat{x} - \frac{l^2}{\hbar}\hat{\pi}_y, \qquad \hat{Y} = \hat{y} + \frac{l^2}{\hbar}\hat{\pi}_x \qquad \text{(general gauge)}$$

with $[\hat{X}, \hat{H}] = 0$ and $[\hat{Y}, \hat{H}] = 0$. Thus we have

$$\frac{d}{dt}\langle X \rangle = 0 \qquad \frac{d}{dt}\langle Y \rangle = 0, \qquad \text{(constant motion).}$$

We note that

$$\begin{aligned}
[\hat{X}, \hat{Y}] &= [\hat{x} - \frac{l^2}{\hbar}\hat{\pi}_y, \hat{y} + \frac{l^2}{\hbar}\hat{\pi}_x] \\
&= \frac{l^2}{\hbar}[\hat{x}, \hat{\pi}_x] + \frac{l^2}{\hbar}[\hat{y}, \hat{\pi}_y] + \frac{l^4}{\hbar^2}[\hat{\pi}_x, \hat{\pi}_y] \\
&= 2\frac{l^2}{\hbar}i\hbar\hat{1} + \frac{l^4}{\hbar^2}(-i\frac{\hbar^2}{l^2})\hat{1} = il^2\hat{1}
\end{aligned}$$

We introduce the new operators as

$$\hat{b} = \frac{1}{\sqrt{2}l}(\hat{X} + i\hat{Y}), \qquad \hat{b}^+ = \frac{1}{\sqrt{2}l}(\hat{X} - i\hat{Y})$$

or

$$\hat{X} = \frac{l}{\sqrt{2}}(\hat{b} + \hat{b}^+), \qquad \hat{Y} = \frac{l}{\sqrt{2}i}(\hat{b} - \hat{b}^+)$$

obeying

$$[\hat{b}, \hat{b}^+] = \frac{1}{2l^2}[\hat{X} + i\hat{Y}, \hat{X} - i\hat{Y}] = -\frac{2}{2l^2}i[\hat{X}, \hat{Y}] = \hat{1}$$

Then we have

$$\hat{X}^2 + \hat{Y}^2 = l^2(2\hat{b}^+\hat{b} + \hat{1})$$

(i) Landau gauge

$$\hat{\pi}_x = \hat{p}_x + \frac{e}{c}\hat{A}_x = \hat{p}_x,$$

$$\hat{\pi}_y = \hat{p}_y + \frac{e}{c}\hat{A}_x = \hat{p}_y + \frac{eB}{c}\hat{x} = \hat{p}_y + \frac{\hbar}{l^2}\hat{x}$$

Then the guiding center is given by

$$\hat{X} = \hat{x} - \frac{l^2}{\hbar}\hat{\pi}_y = \hat{x} - \frac{l^2}{\hbar}(\hat{p}_y + \frac{\hbar}{l^2}\hat{x}) = -\frac{l^2}{\hbar}\hat{p}_y$$

$$\hat{Y} = \hat{y} + \frac{l^2}{\hbar}\hat{\pi}_x = \hat{y} + \frac{l^2}{\hbar}\hat{p}_x$$

(ii) Symmetric gauge

$$\hat{\pi}_x = \hat{p}_x + \frac{e}{c}A_x = \hat{p}_x - \frac{eB}{2c}\hat{y} = \hat{p}_x - \frac{\hbar}{2l^2}\hat{y},$$

$$\hat{\pi}_y = \hat{p}_y + \frac{e}{c}A_y = \hat{p}_y + \frac{eB}{2c}\hat{x} = \hat{p}_y + \frac{\hbar}{2l^2}\hat{x}$$

Then the guiding center is given by

$$\hat{X} = \hat{x} - \frac{l^2}{\hbar}\hat{\pi}_y = \hat{x} - \frac{l^2}{\hbar}(\hat{p}_y + \frac{\hbar}{2l^2}\hat{x}) = \frac{1}{2}\hat{x} - \frac{l^2}{\hbar}\hat{p}_y$$

$$\hat{Y} = \hat{y} + \frac{l^2}{\hbar}\hat{\pi}_x = \hat{y} + \frac{l^2}{\hbar}(\hat{p}_x - \frac{\hbar}{2l^2}\hat{y}) = \frac{1}{2}\hat{y} + \frac{l^2}{\hbar}\hat{p}_x$$

6. Relative co-ordinate (general gauge)

The guiding center is given by

$$\hat{x} = \hat{X} + \frac{l^2}{\hbar}\hat{\pi}_y, \qquad \hat{y} = \hat{Y} - \frac{l^2}{\hbar}\hat{\pi}_x$$

The operators for the relative co-ordinates are defined by

$$\hat{\xi} = \hat{x} - \hat{X} = \frac{l^2}{\hbar}\hat{\pi}_y, \qquad \hat{\eta} = \hat{y} - \hat{Y} = -\frac{l^2}{\hbar}\hat{\pi}_x$$

Then the Hamiltonian is expressed by only the relative co-ordinates

$$\hat{H} = \frac{1}{2m_e}(\hat{\pi}_x{}^2 + \hat{\pi}_y{}^2) = \frac{\hbar^2}{2m_e l^4}(\hat{\xi}^2 + \hat{\eta}^2)$$

where

$$[\hat{\xi}, \hat{\eta}] = -il^2, \ [\hat{X}, \hat{Y}] = il^2$$

7. Heisenberg's principle of uncertainty (general gauge)

We consider the commutation relation of $\hat{X}$ and $\hat{Y}$,

$$\left[\hat{X}, \hat{Y}\right] = [\hat{x} + \frac{l^2}{\hbar}\hat{\pi}_y, \hat{y} - \frac{l^2}{\hbar}\hat{\pi}_x] = \frac{l^4}{\hbar^2}[\hat{\pi}_x, \hat{\pi}_y] = \frac{l^4}{\hbar^2}(-i)\frac{\hbar^2}{\ell^2} = -il^2$$

since

$$[\hat{\pi}_x, \hat{\pi}_y] = -i\frac{\hbar^2}{\ell^2}$$

Using the Schwarz inequality, we get

$$\Delta X \Delta Y \geq \frac{1}{2}l^2, \qquad \Delta\pi_x \Delta\pi_y \geq \frac{1}{2}\frac{\hbar^2}{\ell^2}$$

In fact we have

$$\Delta X \Delta Y = 2\pi l^2 > \frac{1}{2}l^2.$$

((**Note**)) **Schwartz inequality**

$\hat{A}$ and $\hat{B}$ are two Hermitian operators with the condition

$$[\hat{A}, \hat{B}] = i\hat{C}.$$

Then we have a Heisenberg's principle of uncertainty:

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle C \rangle\right|.$$

This principle is a direct consequence of the non-commutability between two observables.

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8. Translation operator derived from the commutation relations

We have the commutation relations

$$[\hat{x}, \hat{H}] = \frac{i\hbar}{m_e}\hat{\pi}_x \tag{1}$$

$$[\hat{y}, \hat{H}] = \frac{i\hbar}{m_e}\hat{\pi}_y \tag{2}$$

We also have the commutation relations:

$$[\hat{\pi}_x, \hat{H}] = -i\frac{\hbar^2}{m_e l^2}\hat{\pi}_y \tag{3}$$

$$[\hat{\pi}_y, \hat{H}] = i\frac{\hbar^2}{m_e l^2}\hat{\pi}_x, \tag{4}$$

From these equations, we get

$$[\hat{\pi}_x, \hat{H}] = -i\frac{\hbar^2}{m_e l^2}\hat{\pi}_y = -i\frac{\hbar^2}{m_e l^2}\frac{m_e}{i\hbar}[\hat{y}, \hat{H}] = -\frac{\hbar}{l^2}[\hat{y}, \hat{H}]$$

or

$$[\hat{\pi}_x + \frac{\hbar}{l^2}\hat{y}, \hat{H}] = 0$$

We also have

$$[\hat{\pi}_y, \hat{H}] = i\frac{\hbar^2}{m_e l^2}\hat{\pi}_x = i\frac{\hbar^2}{m_e l^2}\frac{m_e}{i\hbar}[\hat{x}, \hat{H}] = \frac{\hbar}{l^2}[\hat{x}, \hat{H}]$$

or

$$[\hat{\pi}_y - \frac{\hbar}{l^2}\hat{x}, \hat{H}] = 0$$

The pseudo-momentum for the translation operator as a generator can be defined by

$$\hat{\boldsymbol{K}} = \hat{\boldsymbol{\pi}} + \frac{\hbar}{l^2}(\hat{y}, -\hat{x}) = \hat{\boldsymbol{\pi}} + \frac{eB}{c}(\hat{y}, -\hat{x}) = \hat{\boldsymbol{\pi}} - \frac{e}{c}\boldsymbol{B} \times \hat{\boldsymbol{r}}$$

or

$$\hat{\boldsymbol{K}} = \hat{\boldsymbol{\pi}} + \frac{\hbar}{l^2}(\hat{y}, -\hat{x}) = (\hat{\pi}_x + \frac{\hbar}{l^2}\hat{y}, \hat{\pi}_y - \frac{\hbar}{l^2}\hat{x})$$

Then the translation operator can be defined by

$$\hat{T}(\boldsymbol{a}) = \exp(-\frac{i}{\hbar}\hat{\boldsymbol{K}} \cdot \boldsymbol{a})$$

Clearly the translation operator $\hat{\boldsymbol{K}}$ commutes with the Hamiltonian $\hat{H}$. $\hat{\boldsymbol{K}}$ commutes with $\hat{H}$. However, $\hat{K}_x$ and $\hat{K}_y$ do not commute;

$$[\hat{K}_x, \hat{K}_x] = i\frac{\hbar^2}{l^2}.$$

The translation operators do not commute,

$$\hat{T}(\boldsymbol{a})\hat{T}(\boldsymbol{b}) = \hat{T}(\boldsymbol{b})\hat{T}(\boldsymbol{a})\exp[-\frac{i}{l^2}(\boldsymbol{a} \times \boldsymbol{b})_z]$$

9. Translation operator (II) derived from the gauge transformation (Cohen Tannoudji et al)

We start with the definition of the translation operator $\hat{T}(\boldsymbol{a})$;

$$|\psi_T\rangle = \hat{T}(\boldsymbol{a})|\psi\rangle$$

$$\langle \boldsymbol{r}|\psi_T\rangle = \langle \boldsymbol{r}|\hat{T}(\boldsymbol{a})|\psi\rangle = \langle \boldsymbol{r} - \boldsymbol{a}|\psi\rangle = \psi(\boldsymbol{r} - \boldsymbol{a})$$

The Hamiltonian commutes with the translation operator.

$$[\hat{H},\hat{T}(\boldsymbol{a})]=0$$

where

$$\hat{T}(\boldsymbol{a})=\exp(-\frac{i}{\hbar}\hat{\boldsymbol{p}}\cdot\boldsymbol{a})$$

The Schrödinger equation for $\langle\boldsymbol{r}|\psi\rangle$ is given by

$$H(\boldsymbol{r})\psi(\boldsymbol{r})=\frac{1}{2m_e}[\boldsymbol{p}+\frac{e}{c}\boldsymbol{A}(\boldsymbol{r})]^2\psi(\boldsymbol{r})=E\psi(\boldsymbol{r})$$

The Schrödinger equation for $\langle\boldsymbol{r}-\boldsymbol{a}|\psi\rangle=\psi(\boldsymbol{r}-\boldsymbol{a})$ is given by

$$\frac{1}{2m_e}[\boldsymbol{p}+\frac{e}{c}\boldsymbol{A}(\boldsymbol{r}-\boldsymbol{a})]^2\psi_T(\boldsymbol{r})=E\psi_T(\boldsymbol{r})$$

Suppose that the vector potential is given by

$$\boldsymbol{A}_T(\boldsymbol{r})=\boldsymbol{A}(\boldsymbol{r}-\boldsymbol{a})=\frac{1}{2}\boldsymbol{B}\times(\boldsymbol{r}-\boldsymbol{a})=\frac{1}{2}\boldsymbol{B}\times\boldsymbol{r}-\frac{1}{2}\boldsymbol{B}\times\boldsymbol{a}=\boldsymbol{A}(\boldsymbol{r})-\frac{1}{2}\boldsymbol{B}\times\boldsymbol{a}$$

or

$$\boldsymbol{A}=\boldsymbol{A}_T+\frac{1}{2}\boldsymbol{B}\times\boldsymbol{a}=\boldsymbol{A}_T+\nabla\chi \qquad \text{(Gauge transformation)}$$

Then we have

$$\chi=\frac{1}{2}\boldsymbol{r}\cdot(\boldsymbol{B}\times\boldsymbol{a})=\frac{1}{2}\boldsymbol{a}\cdot(\boldsymbol{r}\times\boldsymbol{B})=-\frac{1}{2}\boldsymbol{a}\cdot(\boldsymbol{B}\times\boldsymbol{r})$$

As a result, we get the Schrödinger equation for $\psi_T'(\boldsymbol{r})$ as

$$\frac{1}{2m_e}[\boldsymbol{p} + \frac{e}{c}\boldsymbol{A}(\boldsymbol{r})]^2 \psi_T'(\boldsymbol{r}) = E\psi_T'(\boldsymbol{r})$$

Under such a gauge transformation, the wavefunction changes as

$$|\psi_T'\rangle = \exp(-\frac{ie}{\hbar c}\chi)|\psi_T\rangle = \exp(-\frac{ie}{\hbar c}\chi)\exp(-\frac{i}{\hbar}\hat{\boldsymbol{p}}\cdot\boldsymbol{a})|\psi\rangle$$

When the vector a is in the x-y plane,

$$\begin{aligned}
|\psi_T'\rangle &= \exp(-\frac{ie}{2\hbar c}\hat{\boldsymbol{r}}\cdot(\boldsymbol{B}\times\boldsymbol{a}))\exp(-\frac{i}{\hbar}\hat{\boldsymbol{p}}\cdot\boldsymbol{a})|\psi\rangle \\
&= \exp(\frac{i}{\hbar}\boldsymbol{a}\cdot\frac{e(\boldsymbol{B}\times\hat{\boldsymbol{r}})}{2c})\exp(-\frac{i}{\hbar}\hat{\boldsymbol{p}}\cdot\boldsymbol{a})|\psi\rangle \\
&= \exp[-\frac{i}{\hbar}(\hat{\boldsymbol{p}} - \frac{e(\boldsymbol{B}\times\boldsymbol{r})}{2c})\cdot\boldsymbol{a}]|\psi\rangle \\
&= \exp[-\frac{i}{\hbar}\hat{\boldsymbol{K}}\cdot\boldsymbol{a}]|\psi\rangle
\end{aligned}$$

since the two operators in front of $|\psi\rangle$ are commutable. The generator of the translation operator is given by

$$\hat{\boldsymbol{K}} = \hat{\boldsymbol{p}} - \frac{e}{2c}\boldsymbol{B}\times\boldsymbol{r} = \hat{\boldsymbol{p}} - \frac{eB}{2c}(-\hat{y},\hat{z},0) \qquad (1)$$

From the previouse discussion we get

$$\hat{\boldsymbol{K}} = \hat{\boldsymbol{\pi}} - \frac{e}{c}\boldsymbol{B}\times\hat{\boldsymbol{r}} = \hat{\boldsymbol{p}} + \frac{e}{c}\boldsymbol{A} - \frac{e}{c}\boldsymbol{B}\times\hat{\boldsymbol{r}} = \hat{\boldsymbol{p}} + \frac{e}{2c}\boldsymbol{B}\times\hat{\boldsymbol{r}} - \frac{e}{c}\boldsymbol{B}\times\hat{\boldsymbol{r}}$$

or

$$\hat{\boldsymbol{K}} = \hat{\boldsymbol{p}} - \frac{e}{2c}\boldsymbol{B}\times\hat{\boldsymbol{r}}$$

which is the same as Eq.(1).

((Note)) Gauge transformation

Under the gauge transformation,

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi \,, \qquad \phi' = \phi - \frac{1}{c}\frac{\partial}{\partial t}\chi$$

the wave function changes as

$$\psi' = \exp(-\frac{ie}{\hbar c}\chi)\psi \,,$$

10. Angular momentum (symmetric gauge)

The angular momentum is defined by

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \,,$$

where

$$\hat{\boldsymbol{p}} = \hat{\boldsymbol{\pi}} - \frac{e}{c}\hat{\boldsymbol{A}} \,,$$

$$\hat{\pi}_x = \hat{p}_x + \frac{e}{c}\hat{A}_x = \hat{p}_x - \frac{\hbar}{2l^2}\hat{y} \,, \qquad \hat{\pi}_y = \hat{p}_y + \frac{e}{c}\hat{A}_y = \hat{p}_y + \frac{\hbar}{2l^2}\hat{x}$$

$$\hat{X} = \hat{x} - \frac{l^2}{\hbar}\hat{\pi}_y = \hat{x} - \frac{l^2}{\hbar}(\hat{p}_y + \frac{\hbar}{2l^2}\hat{x}) = \frac{1}{2}\hat{x} - \frac{l^2}{\hbar}\hat{p}_y$$

$$\hat{Y} = \hat{y} + \frac{l^2}{\hbar}\hat{\pi}_x = \hat{y} + \frac{l^2}{\hbar}(\hat{p}_x - \frac{\hbar}{2l^2}\hat{y}) = \frac{1}{2}\hat{y} + \frac{l^2}{\hbar}\hat{p}_x$$

Then we have

$$\begin{aligned}\hat{X}^2 + \hat{Y}^2 &= (\frac{1}{2}\hat{x} - \frac{l^2}{\hbar}\hat{p}_y)(\frac{1}{2}\hat{x} - \frac{l^2}{\hbar}\hat{p}_y) + (\frac{1}{2}\hat{y} + \frac{l^2}{\hbar}\hat{p}_x)(\frac{1}{2}\hat{y} + \frac{l^2}{\hbar}\hat{p}_x) \\ &= \frac{1}{4}(\hat{x}^2 + \hat{y}^2) + \frac{l^4}{\hbar^2}(\hat{p}_x{}^2 + \hat{p}_y{}^2) - \frac{l^2}{\hbar}(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) \qquad (1) \\ &= \frac{1}{4}(\hat{x}^2 + \hat{y}^2) + \frac{l^4}{\hbar^2}(\hat{p}_x{}^2 + \hat{p}_y{}^2) - \frac{l^2}{\hbar}\hat{L}_z\end{aligned}$$

$$\hat{\pi}_x{}^2 + \hat{\pi}_y{}^2 = (\hat{p}_x - \frac{\hbar}{2l^2}\hat{y})(\hat{p}_x - \frac{\hbar}{2l^2}\hat{y}) + (\hat{p}_y + \frac{\hbar}{2l^2}\hat{x})(\hat{p}_y + \frac{\hbar}{2l^2}\hat{x})$$

$$= (\hat{p}_x{}^2 + \hat{p}_y{}^2) + \frac{\hbar^2}{4l^4}(\hat{x}^2 + \hat{y}^2) + \frac{\hbar}{l^2}(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)$$

$$= (\hat{p}_x{}^2 + \hat{p}_y{}^2) + \frac{\hbar^2}{4l^4}(\hat{x}^2 + \hat{y}^2) + \frac{\hbar}{l^2}\hat{L}_z$$

$$= \frac{\hbar^2}{l^4}[\frac{1}{4}(\hat{x}^2 + \hat{y}^2) + \frac{l^4}{\hbar^2}(\hat{p}_x{}^2 + \hat{p}_y{}^2) + \frac{l^2}{\hbar}\hat{L}_z]$$

or

$$\frac{l^4}{\hbar^2}(\hat{\pi}_x{}^2 + \hat{\pi}_y{}^2) = \frac{1}{4}(\hat{x}^2 + \hat{y}^2) + \frac{l^4}{\hbar^2}(\hat{p}_x{}^2 + \hat{p}_y{}^2) + \frac{l^2}{\hbar}\hat{L}_z \qquad (2)$$

From Eqs.(1) and (2) we have

$$\hat{L}_z = \frac{l^2}{2\hbar}(\hat{\pi}_x{}^2 + \hat{\pi}_y{}^2) - \frac{\hbar}{2l^2}(\hat{X}^2 + \hat{Y}^2)$$

or

$$\hat{X}^2 + \hat{Y}^2 = \frac{l^4}{\hbar^2}(\hat{\pi}_x{}^2 + \hat{\pi}_y{}^2) - \frac{2l^2}{\hbar}\hat{L}_z$$

In this case, he Hamiltonian is rewitten as

$$\hat{H} = \frac{1}{2m_e}(\hat{\pi}_x{}^2 + \hat{\pi}_y{}^2) = \frac{\hbar^2}{2m_e l^4}(\hat{X}^2 + \hat{Y}^2) + \frac{\hbar}{m_e l^2}\hat{L}_z$$

Note that

$$\frac{\hbar}{m_e l^2}\hat{L}_z = \frac{e\hbar B}{m_e c}\frac{\hat{L}_z}{\hbar} = 2\mu_B B\frac{\hat{L}_z}{\hbar}$$

where $\mu_B = \dfrac{e\hbar}{2m_e c}$ is the Bohr magneton.

In general gauge, we have

$$\hat{\pi}_x{}^2 + \hat{\pi}_y{}^2 = \frac{\hbar^2}{l^2}(2\hat{a}^+\hat{a} + 1)$$

and

$$\hat{X}^2 + \hat{Y}^2 = l^2(2\hat{b}^+\hat{b} + \hat{1})$$

with

$$[\hat{b}, \hat{b}^+] = \hat{1}$$

The angular momentum in the symmetric gauge is given by

$$\begin{aligned}
\hat{L}_z &= \frac{l^2}{2\hbar}(\hat{\pi}_x{}^2 + \hat{\pi}_y{}^2) - \frac{\hbar}{2l^2}(\hat{X}^2 + \hat{Y}^2) \\
&= \frac{l^2}{2\hbar}\frac{\hbar^2}{l^2}(2\hat{a}^+\hat{a} + 1) - \frac{\hbar}{2l^2}l^2(2\hat{b}^+\hat{b} + \hat{1}) \\
&= \hbar(\hat{a}^+\hat{a} - \hat{b}^+\hat{b})
\end{aligned}$$

11. Schrödinger equation (Landau gauge)

In the absence of an electric field, the H

$$\hat{H} = \frac{1}{2m_e}[\hat{p}_x{}^2 + (\hat{p}_y + \frac{eB}{c}\hat{x})^2 + \hat{p}_z{}^2] = \frac{1}{2m_e}[\hat{p}_x{}^2 + (\hat{p}_y + \frac{\hbar}{l^2}\hat{x})^2 + \hat{p}_z{}^2]$$

when we use the Landau gauge and $l^2 = \frac{c\hbar}{eB}$ (e>0). The guiding center is given by

$$\hat{X} = \hat{x} - \frac{l^2}{\hbar}\hat{\pi}_y = \hat{x} - \frac{l^2}{\hbar}(\hat{p}_y + \frac{\hbar}{l^2}\hat{x}) = -\frac{l^2}{\hbar}\hat{p}_y .$$

We note that this Hamiltonian $\hat{H}$ commutes with $\hat{p}_y$ and $\hat{p}_z$.

$$[\hat{H}, \hat{p}_y] = 0 \qquad \text{and} \qquad [\hat{H}, \hat{p}_z] = 0$$

Then $\left|n, k_y, k_z\right\rangle$ is a simultaneous eigenket of $\hat{H}$, $\hat{p}_y$ and $\hat{p}_z$.

$$\hat{H}\left|n,k_y,k_z\right\rangle = E_n\left|n,k_y,k_z\right\rangle$$

$$\hat{p}_y\left|n,k_y,k_z\right\rangle = \hbar k_y\left|n,k_y,k_z\right\rangle,$$

and

$$\hat{p}_z\left|n,k_y,k_z\right\rangle = \hbar k_z\left|n,k_y,k_z\right\rangle$$

We also note that

$$\hat{X}\left|n,k_y,k_z\right\rangle = -\frac{l^2}{\hbar}\hat{p}_y\left|n,k_y,k_z\right\rangle = -\frac{l^2}{\hbar}\hbar k_y\left|n,k_y,k_z\right\rangle = -l^2 k_y\left|n,k_y,k_z\right\rangle$$

In other words, $\left|n,k_y,k_z\right\rangle$ is the eigenket of $\hat{X}$ with the eigenvalue $X = -l^2 k_y$

In the position representation, we have

$$\left\langle y\right|\hat{p}_y\left|n,k_y,k_z\right\rangle = \hbar k_y\left\langle y\middle|n,k_y,k_z\right\rangle,$$

$$\left\langle z\right|\hat{p}_y\left|n,k_y,k_z\right\rangle = \hbar k_y\left\langle y\middle|n,k_y,k_z\right\rangle$$

or

$$\frac{\hbar}{i}\frac{\partial}{\partial y}\left\langle y\middle|n,k_y,k_z\right\rangle = \hbar k_y\left\langle y\middle|n,k_y,k_z\right\rangle,$$

$$\frac{\hbar}{i}\frac{\partial}{\partial z}\left\langle z\middle|n,k_y,k_z\right\rangle = \hbar k_z\left\langle z\middle|n,k_y,k_z\right\rangle.$$

Then the wave function has the form of

$$\psi(x,y,z) = e^{ik_y y + ik_z z}\phi(x).$$

where

$$X = -l^2 k_y.$$

We assume the periodic boundary condition along the y axis.

$$\psi(x, y + L_y, z) = \psi(x, y, z)$$

or

$$e^{ik_y L_y} = 1$$

or

$$k_y = -\frac{X}{l^2} = \frac{2\pi}{L_y} n_y \qquad (n_y\text{: intergers})$$

The Schrödinger equation for the wavefunction $\psi(x, y, z)$ is given by

$$\frac{1}{2m_e}[(\frac{\hbar}{i}\frac{\partial}{\partial x})^2 + (\frac{\hbar}{i}\frac{\partial}{\partial y} + \frac{\hbar}{l^2}x)^2 + (\frac{\hbar}{i}\frac{\partial}{\partial z})^2]\psi(x, y, z) = E\psi(x, y, z)$$

with

$$\omega_c = \frac{eB}{m_e c}.$$

The Schrödinger equation for the wavefunction $\phi(x)$ is given by

$$\frac{d^2}{dx^2}\phi(x) = \frac{1}{l^4}(x - X)^2 + (-\frac{2m_e E}{\hbar^2} + k_z{}^2)\phi(x)$$

The energy eigenvalue is

$$E = E(n, k_z) = \hbar\omega_c(n + \frac{1}{2}) + \frac{\hbar^2 k_z{}^2}{2m_e}$$

$$\phi''(x) = [\frac{1}{l^4}(x - X)^2 - \frac{1}{l^2}(2n + 1)]\phi(x)$$

12. Mathematica

((**Mathematica**))

```
Clear["Global`*"]; πx := ℏ/ⅈ D[#, x] &;
πy := (ℏ/ⅈ D[#, y] + e1 B/c x #) &; πz := ℏ/ⅈ D[#, z] &;
H1 :=
  1/(2 m) (Nest[πx, #, 2] + Nest[πy, #, 2] +
      Nest[πz, #, 2]) &;

eq1 = H1 [ψ[x, y, z]] - E1 ψ[x, y, z] // Simplify;
rule1 = {ψ → (Exp[ⅈ (ky #2 + kz #3)] ϕ[#1] &)};
eq2 = eq1 /. rule1 // Simplify;
eq3 = eq2 /. {B → m c ω/e1} // FullSimplify
```

$$\frac{1}{2\,\mathrm{m}} e^{i\,(\mathrm{ky}\,\mathrm{y}+\mathrm{kz}\,\mathrm{z})} \left(\left(-2\,\mathrm{E1}\,\mathrm{m}+\mathrm{m}^2\,\mathrm{x}^2\,\omega^2+2\,\mathrm{ky}\,\mathrm{m}\,\mathrm{x}\,\omega\,\hbar+\left(\mathrm{ky}^2+\mathrm{kz}^2\right)\hbar^2\right)\phi[\mathrm{x}]-\hbar^2\,\phi''[\mathrm{x}]\right)$$

13. Landau level and the wave function

We put $\zeta = \dfrac{x}{l}$, $\qquad \varsigma_0 = \dfrac{X}{l}$

$$\phi'(x) = \frac{d\phi}{dx} = \frac{d\varsigma}{dx}\frac{d\phi}{d\varsigma} = \frac{1}{l}\frac{d\phi}{d\varsigma}, \qquad \phi''(x) = \frac{d\varsigma}{dx}\frac{d}{d\varsigma}\frac{d\varsigma}{dx}\frac{d\phi}{d\varsigma} = \frac{1}{l^2}\frac{d^2\phi}{d\varsigma^2}$$

Then we get a differential equation

$$\phi''(\varsigma) + [(2n+1) - (\varsigma - \varsigma_0)^2]\phi(\varsigma)$$

The solution of this differential equation is

$$\phi_n(\varsigma) = (\sqrt{\pi}\,2^n n!)^{-1/2} e^{-\frac{(\varsigma-\varsigma_0)^2}{2}} H_n(\varsigma - \varsigma_0)$$

or

$$\phi_n(x) = (\sqrt{\pi}2^n n!)^{-1/2} e^{-\frac{(x-X)^2}{2l^2}} H_n(\frac{x-X}{l})$$

The coordinate X is the center of orbits. Suppose that the size of the system along the x axis is L_x. The coordinate X should satisfy the condition, $0<X<L_x$. Since the energy of the system is independent of x_0, this state is degenerate.

$$0 < X = \ell^2 k_y < L_x$$

or

$$X = \ell^2 k_y = \frac{2\pi}{L_y}\ell^2 n_y < L_x$$

or

$$n_y < \frac{L_x L_y}{2\pi\ell^2}$$

Note that

$$\Delta X = \frac{2\pi l^2}{L_y}$$

The area of each Landau state is

$$\Delta A = L_y \Delta X = 2\pi l^2.$$

Thus the degeneracy is given by the number of allowed k_y values for the system.

$$g = \frac{L_x L_y}{2\pi\ell^2} = \frac{A}{2\pi\ell^2} = \frac{A}{2\pi\frac{c\hbar}{eB}} = \frac{BA}{2\Phi_0} = \frac{\Phi}{2\Phi_0}$$

where

$$2\Phi_0 = \frac{2\pi\hbar c}{e} = 4.13563\times 10^{-7}\ \text{Gauss cm}^2$$

The value of g is the total magnetic flux. There is one state per a quantum magnetic flux $2\Phi_0$. Φ_0 is the quantum fluxoid.

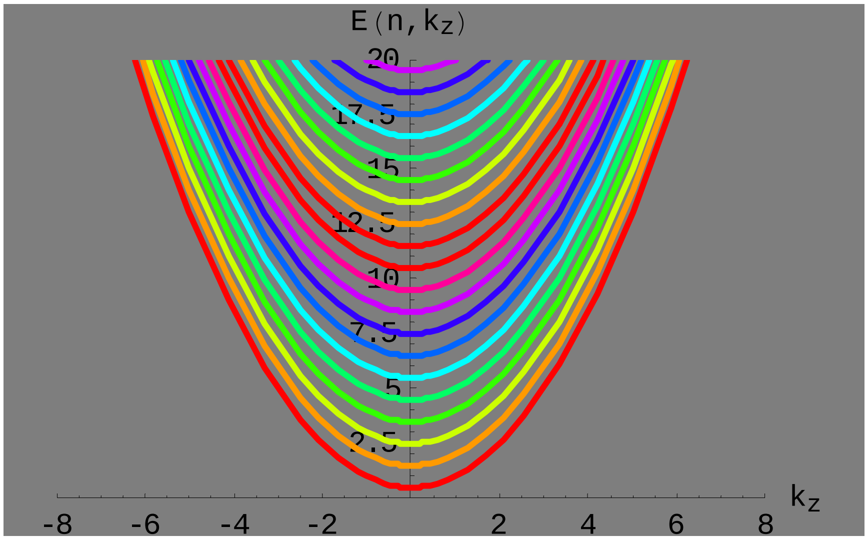


Fig. Energy dispersion relation for the Landau level.

14. Creation and annihilation operator (Landau gauge)

We define the creation and annihilation operators such that

$$\hat{a} = \frac{l}{\sqrt{2}\hbar}(\hat{\pi}_x - i\hat{\pi}_y), \qquad \hat{a}^+ = \frac{l}{\sqrt{2}\hbar}(\hat{\pi}_x + i\hat{\pi}_y)$$

with

$$[\hat{a}, \hat{a}^+] = \hat{1}$$

In the Landau gauge, we have

$$\hat{\pi}_x = \hat{p}_x, \qquad \hat{\pi}_y = \frac{\hbar}{l^2}(\hat{x} - \hat{X})$$

Then we have

$$\hat{a} = \frac{l}{\sqrt{2}\hbar}[\hat{p}_x - i\frac{\hbar}{l^2}(\hat{x} - \hat{X})],$$

$$\hat{a}^{+} = \frac{l}{\sqrt{2}\hbar}[\hat{p}_x + i\frac{\hbar}{l^2}(\hat{x} - \hat{X})]$$

$$\hat{a}|n, X\rangle = \sqrt{n}|n-1, X\rangle, \qquad \hat{a}^{+}|n, X\rangle = \sqrt{n+1}|n+1, X\rangle$$

(i) Wavefunction of the ground state

$$\hat{a}|n = 0, X\rangle = 0$$

$$\langle x|\hat{a}|n = 0, X\rangle = -i\frac{1}{\sqrt{2}}[l\frac{\partial}{\partial x} + \frac{(x - X)}{l}]\langle x|n = 0, X\rangle = 0$$

The normalized wavefunction can be solved by using Mathematica

$$\langle x|n = 0, X\rangle = \frac{1}{\sqrt{\sqrt{\pi}l}}\exp[-\frac{(x - X)^2}{2l^2}] \qquad \text{(Gaussian)}$$

In general we have

$$\begin{aligned}\langle x|N, X\rangle &= (-i)^N \frac{1}{\sqrt{N!}}\langle x|\left(\hat{a}^{+}\right)^N|n = 0, X\rangle \\ &= \frac{1}{\sqrt{2^N N!\sqrt{\pi}l}}(-l\frac{\partial}{\partial x} + \frac{x - X}{l})^N \exp[-\frac{(x - X)^2}{2l^2}] \\ &= \frac{1}{\sqrt{2^N N!\sqrt{\pi}l}} H_N(\frac{x - X}{l})\exp[-\frac{(x - X)^2}{2l^2}]\end{aligned}$$

with

$$\hat{a}^{+} = \frac{l}{\sqrt{2}\hbar}[\hat{p}_x + i\frac{\hbar}{l^2}(\hat{x} - \hat{X})]$$

For simplicity we put the factor $(-i)^N$.

((**Mathematica-1**))

Wavefunction of the ground state

```
Clear["Global`*"];
An := -ⅈ/√2 (L1 D[#, x] + (x - X)/L1 #) &;

eq1 = An[ψ[x]];
eq2 = DSolve[eq1 == 0, ψ[x], x] // Simplify
```

$$\left\{\left\{\psi[x] \to e^{-\frac{x (x-2 X)}{2 L1^2}} C[1]\right\}\right\}$$

```
ψ1[x_] = ψ[x] /. eq2[[1]] /. {C[1] → C1};

eq3 = Integrate[ψ1[x]^2, {x, -∞, ∞}] //
    Simplify[#, L1 > 0] &;

eq4 = Solve[eq3 == 1, C1];

ψ0[x_] := ψ1[x] /. eq4[[2]] // Simplify;
ψ0[x]
```

$$\frac{e^{-\frac{(x-X)^2}{2 L1^2}}}{\sqrt{L1}\,\pi^{1/4}}$$

((Mathematica-II))
Wavefunction of excited state
We calculate the two functions by using the Mathematica.

$$\psi(x,N) = \frac{1}{\sqrt{2^N N!\sqrt{\pi}l}}(-l\frac{\partial}{\partial x} + \frac{x-X}{l})^N \exp[-\frac{(x-X)^2}{2l^2}]$$

$$\chi(x,N) == \frac{1}{\sqrt{2^N N!\sqrt{\pi}l}} H_N(\frac{x-X}{l})\exp[-\frac{(x-X)^2}{2l^2}]$$

where $H_N(x)$ is a Hermite polynomial.

15. Schrödinger equation (symmetric gauge)

In the absence of an electric field, the H

$$\hat{H} = \frac{1}{2m_e}[(\hat{p}_x - \frac{\hbar}{2l^2}\hat{y})^2 + (\hat{p}_y + \frac{\hbar}{2l^2}\hat{x})^2 + \hat{p}_z{}^2]$$

when we use the symmetric gauge and $l^2 = \frac{c\hbar}{eB}$ (e>0). $\hat{H}$ can be rewritten as

$$\begin{aligned}\hat{H} &= \frac{1}{2m_e}(\hat{p}_x{}^2 + \hat{p}_y{}^2 + \hat{p}_z{}^2) + \frac{\hbar^2}{8m_e l^4}(\hat{x}^2 + \hat{y}^2) + \frac{\hbar^2}{2m_e l^2}\frac{\hat{L}_z}{\hbar} \\ &= \frac{1}{2m_e}(\hat{p}_x{}^2 + \hat{p}_y{}^2 + \hat{p}_z{}^2) + \frac{1}{8}m_e\omega_c{}^2(\hat{x}^2 + \hat{y}^2) + \mu_B B\frac{\hat{L}_z}{\hbar}\end{aligned}$$

where $\hat{L}_z$ is the orbital angular momentum.

$$\frac{\hbar^2}{2m_e l^2} = \frac{e\hbar B}{2m_e c} = \mu_B B$$

μ_B is the Bohr magneton and is defined by

$$\mu_B = \frac{e\hbar}{2m_e c}$$

The third term of $\hat{H}$ is the Zeeman energy. The orbital magnetic moment is

$$\hat{\boldsymbol{\mu}}_L = -\mu_B \frac{\hat{\boldsymbol{L}}}{\hbar}.$$

So the Zeeman energy is given by

$$-\hat{\boldsymbol{\mu}}_L \cdot \boldsymbol{B} = -(-\mu_B \frac{\hat{\boldsymbol{L}}}{\hbar}) \cdot \boldsymbol{B} = \mu_B B \frac{\hat{L}_z}{\hbar}.$$

We note that

$$[\hat{H}, \hat{L}_z] = 0, \quad [\hat{H}, \hat{p}_z] = 0, \quad [\hat{L}_z, \hat{p}_z] = 0$$

Therefore, there is a simultaneous eigenket of $\hat{H}$, $\hat{L}_z$, and $\hat{p}_z$.

$$\hat{H}|\psi\rangle = E|\psi\rangle, \qquad \hat{L}_z|\psi\rangle = m\hbar|\psi\rangle, \qquad \hat{p}_z|\psi\rangle = \hbar k_z|\psi\rangle$$

((**Note**))

We note that

$$(\hat{X}^2+\hat{Y}^2)=\frac{2m_e l^4}{\hbar^2}(\hat{H}-\frac{\hbar}{m_e l^2}\hat{L}_z)$$

Thus this operator is related to the square of radius the guiding center commutes with $\hat{H}$ and $\hat{L}_z$.

$$\begin{aligned}(\hat{X}^2+\hat{Y}^2)|\psi\rangle &= \frac{2m_e l^4}{\hbar^2}(\hat{H}-\frac{\hbar}{m_e l^2}\hat{L}_z)|\psi\rangle \\ &= \frac{2m_e l^4}{\hbar^2}(E-\frac{m\hbar^2}{m_e l^2})|\psi\rangle \\ &= \frac{2m_e l^4}{\hbar^2}(E-m\hbar\omega_c)|\psi\rangle\end{aligned}$$

Suppose that $E=(n+\frac{1}{2})\hbar\omega_c$, we have

$$\begin{aligned}\frac{2m_e l^4}{\hbar^2}(E-m\hbar\omega_c) &= \frac{2m_e l^4}{\hbar^2}\hbar\omega_c(n-m+\frac{1}{2}) \\ &= 2l^2(n-m+\frac{1}{2})\end{aligned}$$

or

$$\langle\psi|(\hat{X}^2+\hat{Y}^2)|\psi\rangle = 2l^2(n-m+1)>0$$

where

$m = n$, n-1, n-2, ……,-∞.

16. Mathematica: Proof of $[\hat{H},\hat{L}_z]=0$.

```
Clear["Global`*"];

px := ℏ/ⅈ D[#, x] &; py := ℏ/ⅈ D[#, y] &;

H1 :=
  (1/(2 m) (Nest[px, #, 2] + Nest[py, #, 2]) +
     1/2 m ω^2 (x^2 + y^2) #) &;

Lz := (x py [#] - y px[#]) &;

Lz[ψ[x, y]]
```

$-i x \hbar \psi^{(0,1)}[x, y] + i y \hbar \psi^{(1,0)}[x, y]$

```
H1[Lz[ψ[x, y]]] - Lz[H1[ψ[x, y]]] // Simplify
```

0

```
H1[Lz[ψ[x, y]]] // Simplify
```

$$\frac{1}{2m} i \hbar \left(-m^2 x \left(x^2+y^2\right) \omega^2 \psi^{(0,1)}[x,y] + x \hbar^2 \psi^{(0,3)}[x,y] + m^2 x^2 y \omega^2 \psi^{(1,0)}[x,y] + m^2 y^3 \omega^2 \psi^{(1,0)}[x,y] - y \hbar^2 \psi^{(1,2)}[x,y] + x \hbar^2 \psi^{(2,1)}[x,y] - y \hbar^2 \psi^{(3,0)}[x,y]\right)$$

```
Lz[H1[ψ[x, y]]] // Simplify
```

$$\frac{1}{2m} i \hbar \left(-m^2 x \left(x^2+y^2\right) \omega^2 \psi^{(0,1)}[x,y] + x \hbar^2 \psi^{(0,3)}[x,y] + m^2 x^2 y \omega^2 \psi^{(1,0)}[x,y] + m^2 y^3 \omega^2 \psi^{(1,0)}[x,y] - y \hbar^2 \psi^{(1,2)}[x,y] + x \hbar^2 \psi^{(2,1)}[x,y] - y \hbar^2 \psi^{(3,0)}[x,y]\right)$$

17. Cylindrical co-ordinates (symmetric gauge)

The Schrödinger equation is given by

$$-\frac{\hbar^2}{2m_e}[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho})+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2}]\psi - i\mu_B B\frac{\partial}{\partial\phi}\psi+\frac{1}{8}m_e\omega_c{}^2\rho^2\psi = E\psi$$

where

$$\langle \rho,\phi,z|\hat{L}_z|\psi\rangle = \frac{\hbar}{i}\frac{\partial}{\partial\phi}\psi(\rho,\phi,z) = m\hbar\psi(\rho,\phi,z)\,, \qquad \psi(\rho,\phi,z) \approx e^{im\phi}$$

$$\langle \rho,\phi,z|\hat{p}_z|\psi\rangle = \frac{\hbar}{i}\frac{\partial}{\partial z}\psi(\rho,\phi,z) = \hbar k_z\psi(\rho,\phi,z) \qquad \psi(\rho,\phi,z) \approx e^{ik_z z}$$

Then the wavefunction has the form of

$$\langle \rho,\phi,z|\psi\rangle = \psi(\rho,\phi,z) = e^{ik_z z}e^{im\phi}\chi(\rho)$$

Then we have

$$\chi''(\rho) + \frac{1}{\rho}\chi'(\rho) + f(\rho)\chi(\rho) = 0$$

or

$$\chi''(\rho) + \frac{1}{\rho}\chi'(\rho) + (\frac{2\eta}{l^2} - \frac{\rho^2}{4l^4} - \frac{m^2}{\rho^2})\chi(\rho) = 0$$

with

$$\begin{aligned} f(\rho) &= \frac{2m_e E}{\hbar^2} - \frac{2\mu_B B}{\hbar^2}mm_e - \frac{m_e^{\,2}\rho^2\omega_c^{\,2}}{4\hbar^2} - \frac{m^2}{\rho^2} - k_z^{\,2} \\ &= \frac{2m_e}{\hbar^2}(E - \frac{\hbar^2}{2m_e}k_z^{\,2} - \mu_B Bm) - (\frac{m_e^{\,2}\rho^2\omega_c^{\,2}}{4\hbar^2} + \frac{m^2}{\rho^2}) \\ &= \frac{2\eta}{l^2} - \frac{\rho^2}{4l^4} - \frac{m^2}{\rho^2} \end{aligned}$$

where

$$l^2 = \frac{c\hbar}{eB} \qquad \omega_c = \frac{eB}{m_e c}, \qquad \mu_B = \frac{e\hbar}{2m_e c},$$

$$\eta = \frac{1}{\hbar\omega_c}(E - \frac{\hbar^2}{2m_e}k_z^{\,2} - \mu_B Bm) = \frac{1}{\hbar\omega_c}(E - \frac{\hbar^2}{2m_e}k_z^{\,2}) - \frac{m}{2}$$

((**Matheamtica**))

```
Clear["Global`"]; Clear[B]; r1 = Sqrt[z^2 + ρ^2];
ux = {Cos[ϕ], -Sin[ϕ], 0}; uy = { Sin[ϕ], Cos[ϕ], 0};
uz = {0, 0, 1}; r = {ρ, 0, z}; ur = 1/r1 {ρ, 0, z}; x = ux.r;
y = uy.r;
z = uz.r;

P := (-ⅈ ℏ Grad[#, {ρ, ϕ, z}, "Cylindrical"]) &;
Px = (ux.P[#]) & // Simplify; Py = (uy.P[#]) & // Simplify;
Pz = (uz.P[#]) & // Simplify;
L := (-ⅈ ℏ Cross[r, Grad[#, {ρ, ϕ, z}, "Cylindrical"]]) &;
Lx := (ux.L[#]) & // Simplify; Ly := (uy.L[#]) & // Simplify;
Lz := (uz.L[#]) & // Simplify;

Lz[ψ[ρ, ϕ, z]]
```

$-i\hbar\,\psi^{(0,1,0)}[\rho, \phi, z]$

```
Pz[ψ[ρ, ϕ, z]]
```

$-i\hbar\,\psi^{(0,0,1)}[\rho, \phi, z]$

```
H1 =
 1/(2 me) (Nest[Px, ψ[ρ, ϕ, z], 2] + Nest[Py, ψ[ρ, ϕ, z], 2] +
      Nest[Pz, ψ[ρ, ϕ, z], 2]) + 1/8 me ωc^2 ρ^2 ψ[ρ, ϕ, z] +
   (μB B)/ℏ Lz[ψ[ρ, ϕ, z]] // FullSimplify
```

$$\frac{1}{8\,\text{me}\,\rho^2}\left(\text{me}^2\,\rho^4\,\omega c^2\,\psi[\rho,\phi,z]-4\left(\rho^2\,\hbar^2\,\psi^{(0,0,2)}[\rho,\phi,z]+2\,i\,B\,\text{me}\,\mu B\,\rho^2\,\psi^{(0,1,0)}[\rho,\phi,z]+\hbar^2\left(\psi^{(0,2,0)}[\rho,\phi,z]+\rho\left(\psi^{(1,0,0)}[\rho,\phi,z]+\rho\,\psi^{(2,0,0)}[\rho,\phi,z]\right)\right)\right)\right)$$

```
rule1 = {ψ → (Exp[ⅈ m #2] Exp[ⅈ kz #3] χ[#1] &)};
```

```
eq1 = (H1 - E1 ψ[ρ, ϕ, z]) /. rule1 // Simplify
```

$$\frac{1}{8\,\text{me}\,\rho^2}e^{i\,(kz\,z+m\,\phi)}\left(\left(-8\,E1\,\text{me}\,\rho^2+8\,B\,m\,\text{me}\,\mu B\,\rho^2+\text{me}^2\,\rho^4\,\omega c^2+4\,m^2\,\hbar^2+4\,kz^2\,\rho^2\,\hbar^2\right)\chi[\rho]-4\,\rho\,\hbar^2\,(\chi'[\rho]+\rho\,\chi''[\rho])\right)$$

$$\frac{1}{4\,\rho^2\,\hbar^2}\left(\left(-8\,E1\,\text{me}\,\rho^2+8\,B\,m\,\text{me}\,\mu B\,\rho^2+\text{me}^2\,\rho^4\,\omega c^2+4\,m^2\,\hbar^2+4\,kz^2\,\rho^2\,\hbar^2\right)\chi[\rho]-4\,\rho\,\hbar^2\,(\chi'[\rho]+\rho\,\chi''[\rho])\right)\ //\ \text{Expand}$$

$$kz^2\,\chi[\rho]+\frac{m^2\,\chi[\rho]}{\rho^2}-\frac{2\,E1\,\text{me}\,\chi[\rho]}{\hbar^2}+\frac{2\,B\,m\,\text{me}\,\mu B\,\chi[\rho]}{\hbar^2}+\frac{\text{me}^2\,\rho^2\,\omega c^2\,\chi[\rho]}{4\,\hbar^2}-\frac{\chi'[\rho]}{\rho}-\chi''[\rho]$$

We use a new variable,

$$\varsigma=\frac{\rho^2}{2l^2}.$$

(Landau and Lifshitz, Quantum Mechanics). Then we have

$$\frac{2}{l^2}\varsigma\chi''(\varsigma)+\frac{2}{l^2}\chi'(\varsigma)+[\frac{2}{l^2}\eta-(\frac{\varsigma}{2l^2}+\frac{m^2}{2\varsigma l^2})]\chi(\varsigma)=0$$

or

$$\varsigma\chi''(\varsigma)+\chi'(\varsigma)+(-\frac{\varsigma}{4}+\eta-\frac{m^2}{4\varsigma})\chi(\varsigma)=0$$

where

$$\frac{1}{\rho}\chi'(\rho) = \frac{1}{\rho}\frac{\partial\varsigma}{d\rho}\chi'(\varsigma) = \frac{1}{\rho}\frac{\rho}{l^2}\chi'(\varsigma) = \frac{1}{l^2}\chi'(\varsigma)$$

$$\begin{aligned}
\chi''(\rho) &= \frac{\partial\varsigma}{d\rho}\frac{\partial}{\partial\varsigma}\frac{\partial\varsigma}{d\rho}\frac{\partial}{\partial\varsigma}\chi(\varsigma) \\
&= \frac{\rho}{l^2}\frac{\partial}{\partial\varsigma}\frac{\rho}{l^2}\frac{\partial}{\partial\varsigma}\chi(\varsigma) \\
&= \frac{\rho}{l^4}\frac{\partial}{\partial\varsigma}\rho\frac{\partial}{\partial\varsigma}\chi(\varsigma) \\
&= \frac{\rho}{l^4}[\frac{\partial\rho}{\partial\varsigma}\chi'(\varsigma) + \rho\chi''(\varsigma)] \\
&= \frac{\rho}{l^4}[\frac{l^2}{\rho}\chi'(\varsigma) + \rho\chi''(\varsigma)] \\
&= \frac{1}{l^2}\chi'(\varsigma) + \frac{2\varsigma}{l^2}\chi''(\varsigma)
\end{aligned}$$

We solve this differential equation using by the series expansion.

(i) In the limit of $\varsigma \to \infty$,

$$\chi''(\varsigma) - \frac{1}{4}\chi(\varsigma) = 0$$

The solution of $\chi(\varsigma)$ is approximately given by

$$\chi(\varsigma) \approx \exp(-\frac{1}{2}\varsigma)$$

(ii) In the limit of $\varsigma \to 0$

$$\varsigma\chi''(\varsigma) + \chi'(\varsigma) - \frac{m^2}{4\varsigma}\chi(\varsigma) = 0$$

The asymptotic form of $\chi(\varsigma)$ is given by

$$\chi(\varsigma) \approx \varsigma^{|m|/2}$$

So that we seek a solution in the form

$$\chi(\varsigma) = \exp(-\frac{\varsigma}{2})\varsigma^{|m|/2}u(\varsigma)$$

The differential equation for $u(\varsigma)$ is given by

$$\varsigma u''(\varsigma) + (1+|m|-\varsigma)u'(\varsigma) - \frac{1}{2}(1+|m|-2\eta)u(\varsigma) = 0 .$$

18. Solving of the differential equation

For convenience, we use a new variable x instead of ζ.

$$xu''(x) + (1+|m|-x)u'(x) - \frac{1}{2}(1+|m|-2\eta)u(x) = 0$$

We assume that $u(x)$ can be expressed by

$$u(x) = x^p \sum_{k=0}^{\infty} C(k)x^k$$

We get the coefficients for the first several terms

$$p(p+|m|)C(0) = 0 ,$$

$$-\frac{1}{2}(1+|m|+2p-2\eta)C(0) + (1+p)(1+|m|+p)C(1) = 0 ,$$

$$-\frac{1}{2}(3+|m|+2p-2\eta)C(1) + (2+p)(2+|m|+p)C(2) = 0$$

From the first equation, we get $p = 0$ or $p = -|m|$. We need to choose $p = 0$, so that $\chi(x)$ is finite at $x = 0$. Then we get the recursion relation

$$C(k+1) = \frac{(1+|m|+2k-2\eta)}{2(1+k)(1+|m|+k)}C(k) .$$

Suppose that

$$(1+|m|+2n_r-2\eta)=0 . \qquad \text{or} \qquad \eta=\frac{1+|m|}{2}+n_r .$$

Then we have $C(n_r+1)=C(n_r+2)=...=0$. $u(x)$ is a polynomial of x with the order of n_r,

$$u_n(x)=\sum_{k=0}^{n_r} C(k)x^k = C(0)+C(1)x+C(2)x^2+...+C(n_r)x^{n_r}$$

with the recursion formula

$$\begin{aligned} C(k+1) &= \frac{\{(1+|m|+2k-(1+|m|+2n_r)\}}{2(1+k)(1+|m|+k)}C(k) \\ &= \frac{(k-n_r)}{(k+1)(k+|m|+1)}C(k) \end{aligned}$$

Note that $u(x)$ satisfies the differential equation,

$$xu''(x)+(1+|m|-x)u'(x)+n_r u(x)=0 .$$

The solution is given by the generalized Laguerrel polynomial

$$u(x)=L_{n_r}^{|m|}(x)$$

Note that in the confluent hypergeometric representation (Arfken), we have the relation

$$L_{n_r}^{|m|}(x)=\frac{(n_r+|m|)!}{n_r!|m|!}M(-n_r,|m|+1,x)$$

Then we get the solution of the above differential equation

$$\chi(x)=A\exp(-\frac{x}{2})x^{|m|/2}L_{n_r}^{|m|}(x)$$

with

$$x=\frac{\rho^2}{2l^2}$$

The normalized wavefunction is

$$\psi(x) = \frac{1}{\sqrt{2\pi l}} e^{im\phi} e^{ik_z z} \sqrt{\frac{n_r!}{(n_r + |m|)!}} \exp\left(-\frac{\rho^2}{4l^2}\right) \left(\frac{\rho^2}{\sqrt{2}l}\right)^{|m|/2} L_{n_r}^{|m|}\left(\frac{\rho^2}{2l^2}\right)$$

The energy eigenvalue is obtained as

$$\eta = \frac{1}{\hbar\omega_c}\left(E_n - \frac{\hbar^2}{2m_e}k_z^{\ 2}\right) - \frac{m}{2} = n_r + \frac{1}{2} + \frac{|m|}{2}$$

or

$$E_{n_r} = \hbar\omega_c\left(n_r + \frac{1}{2} + \frac{m + |m|}{2}\right) + \frac{\hbar^2}{2m_e}k_z^{\ 2}$$

It is useful to introduce a new quantum number *n* as

$$n = n_r + \frac{1}{2}(m + |m|) \qquad \text{for } n_r = 0, 1, 2, 3, \ldots.$$

When $m>0$,

$$n = n_r + \frac{1}{2}(m + |m|) = n_r + m$$

when $m<0$,

$$n = n_r$$

The standard form of the Landau spectrum is

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right) + \frac{\hbar^2}{2m_e}k_z^{\ 2}$$

Finally we get the expression for the wavefunction

$$\psi(x)=\frac{1}{\sqrt{2\pi}l}e^{im\phi}e^{ik_z z}\sqrt{\frac{n_r!}{(n_r+|m|)!}}\exp(-\frac{\rho^2}{4l^2})\left(\frac{\rho^2}{\sqrt{2}l}\right)^{|m|/2}L_{n_r}^{|m|}(\frac{\rho^2}{2l^2})$$

$$=\frac{1}{\sqrt{2\pi}l}e^{im\phi}e^{ik_z z}\sqrt{\frac{[n-\frac{(m+|m|)}{2}]!}{[n-\frac{(m-|m|)}{2}]!}}\exp(-\frac{\rho^2}{4l^2})\left(\frac{\rho^2}{\sqrt{2}l}\right)^{|m|/2}L_{n-\frac{1}{2}(m+|m|)}^{|m|}(\frac{\rho^2}{2l^2})$$

19. The quantum numbers *n* and *m*

We get the relation from the above discussion,

$$n=n_r+\frac{1}{2}(m+|m|) \qquad \text{for } n_r = 0, 1, 2, 3,\ldots.$$

where n = 0, 1, 2, 3, …. From these conditions we have the values of (n, m) as follows

For $n_r = 0$,

$n = 0$, m = 0, -1, -2, ...,-∞
$n = 1$, $m = 1$,
$n = 2$, $m = 2$,
$n = 3$, $m = 3$,

For $n_r = 1$,

$n = 1$, m = 0, -1, -2, ...,-∞
$n = 2$, $m = 1$,
$n = 3$, $m = 2$,
$n = 4$, $m = 3$,

For $n_r = 2$,

$n = 2$, m = 0, -1, -2,..., -∞
$n = 3$, $m = 1$,
$n = 4$, $m = 2$,
$n = 5$, $m = 3$,

For $n_r = 3$

$n = 3$, m = 0,-1, -2, ... -∞
$n = 4$, $m = 1$,
$n = 5$, $m = 2$,
$n = 6$, $m = 3$,

In other words,

(i) The ground state:
$n = 0$, $m = 0, -1, -2, ...-\infty$ (the degeneracy $g_0 = \infty$)

(ii) The first excited state:
$n = 1$ $m = 1, 0, -1, -2, ..., -\infty$ (the degeneracy $g_1 = \infty$).

(ii) The second excited state:
$n = 2$ $m = 2, 1, 0, -1, -2, ..., -\infty$ (the degeneracy $g_2 = \infty$).

(ii) The third excited state:
$n = 3$ $m = 3, 2, 1, 0, -1, -2, ..., -\infty$ (the degeneracy $g_2 =$).

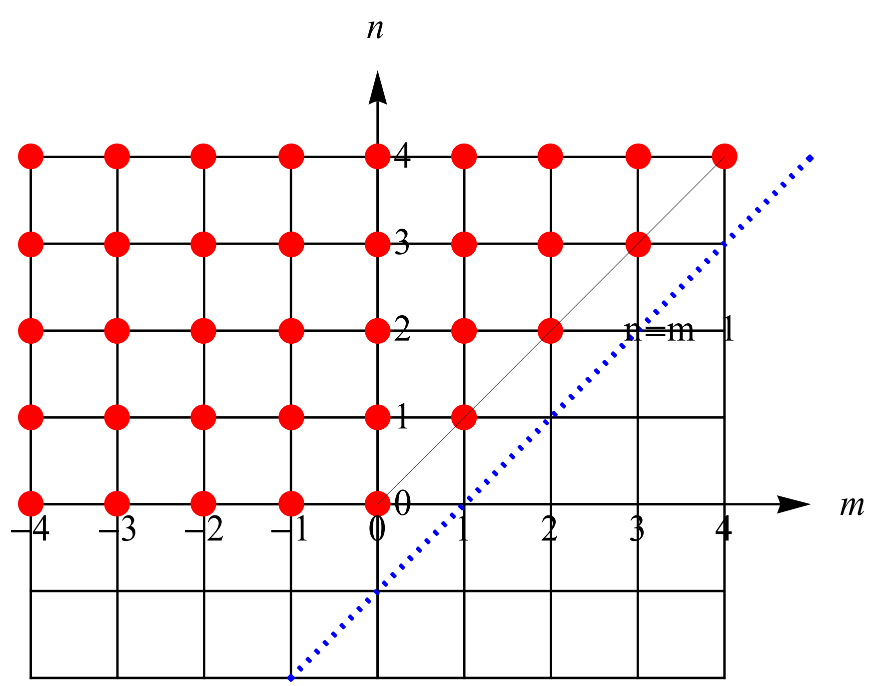


Fig. m-n plane. $n > m - -1$. n is a positive integer.

$$\psi_{n,m}(x) = \frac{1}{\sqrt{2\pi}l} e^{im\phi} e^{ik_z z} \sqrt{\frac{[n - \frac{(m+|m|)}{2}]!}{[n - \frac{(m-|m|)}{2}]!}} \exp(-\frac{\rho^2}{4l^2}) \left(\frac{\rho^2}{\sqrt{2}l}\right)^{|m|/2} L^{|m|}_{n-\frac{1}{2}(m+|m|)}(\frac{\rho^2}{2l^2})$$

The probability density is given by

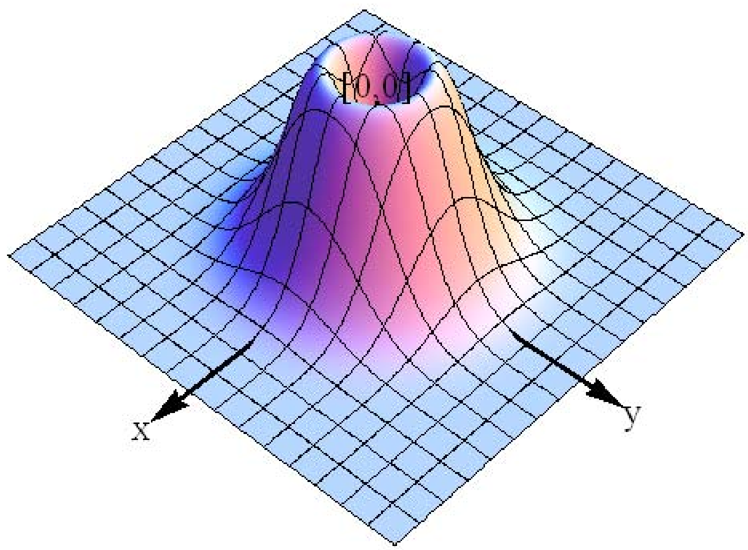
$$P_{n,m}(\rho) = 2\pi\rho \left|\psi_{n,m}(\rho)\right|^2$$

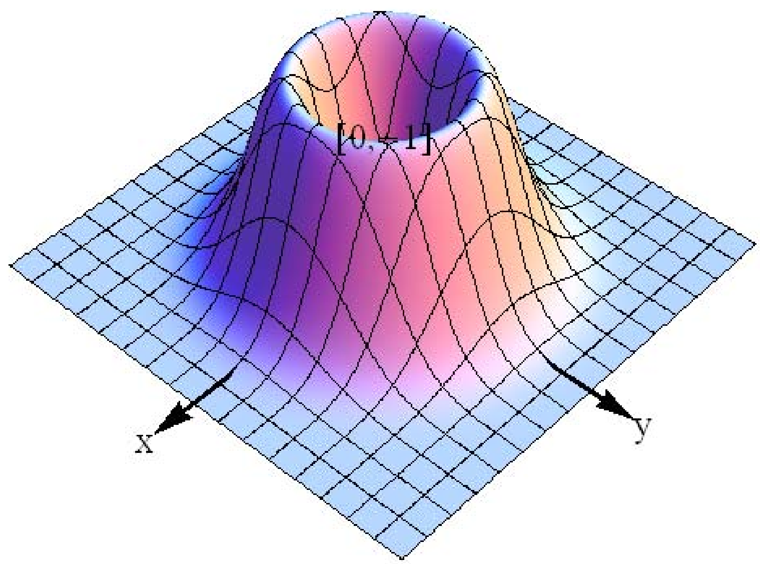
where

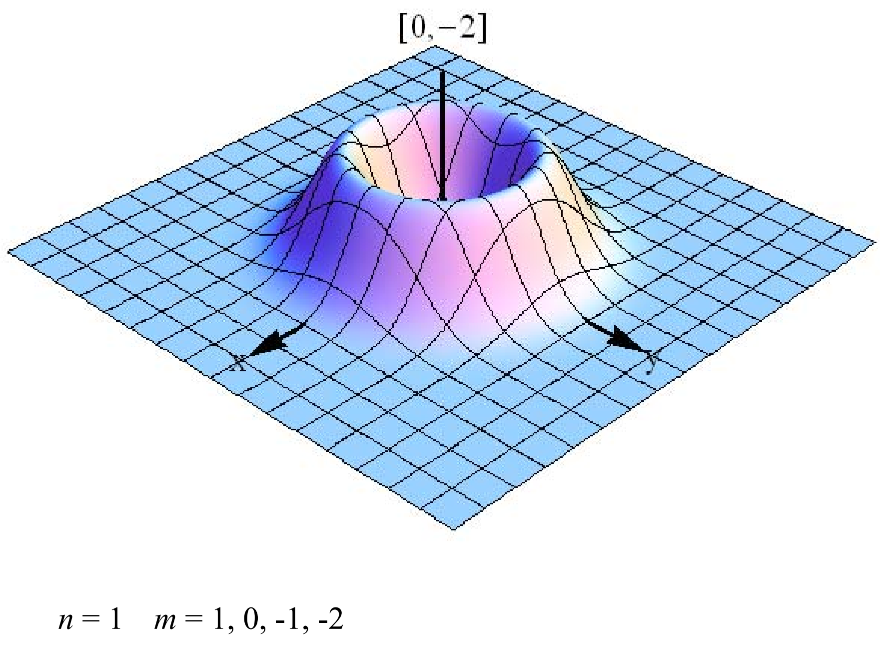
$$\int_0^\infty P(\rho)d\rho = 1$$

We make a plot of $P_{n,m}(\rho)$ by using the Mathematica (Plot3D) for several cases

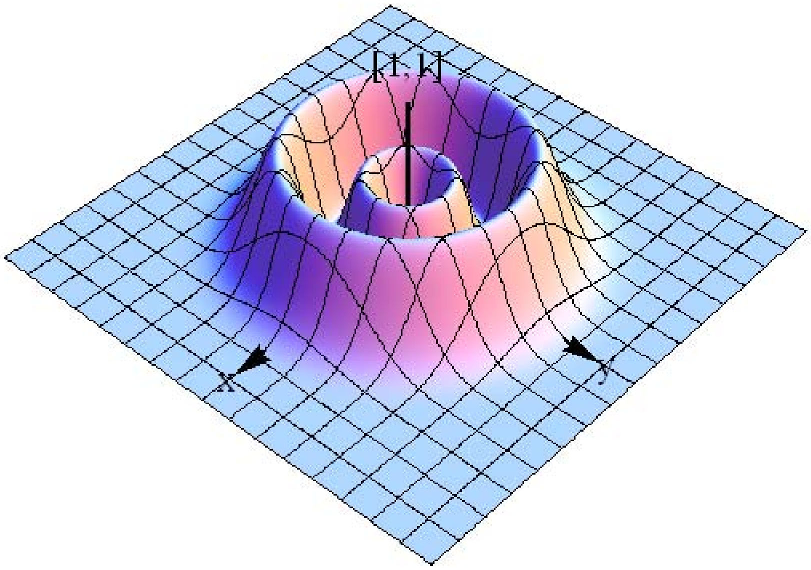
(i) $n = 0$, $m = 0$, -1, -2

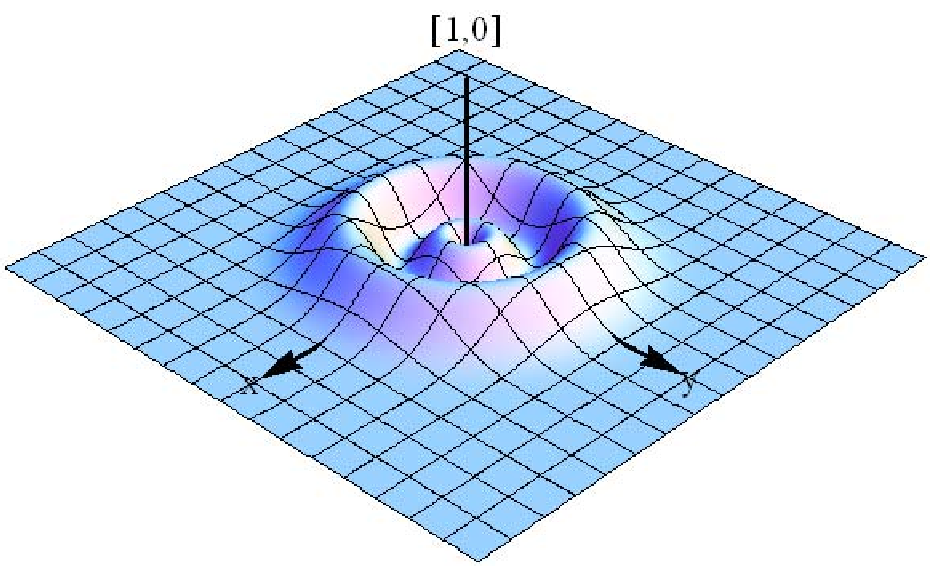


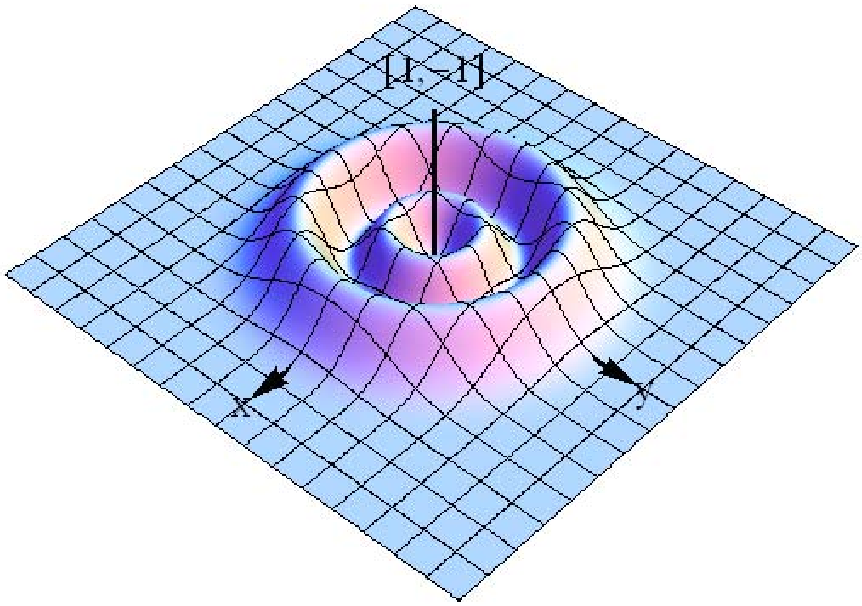


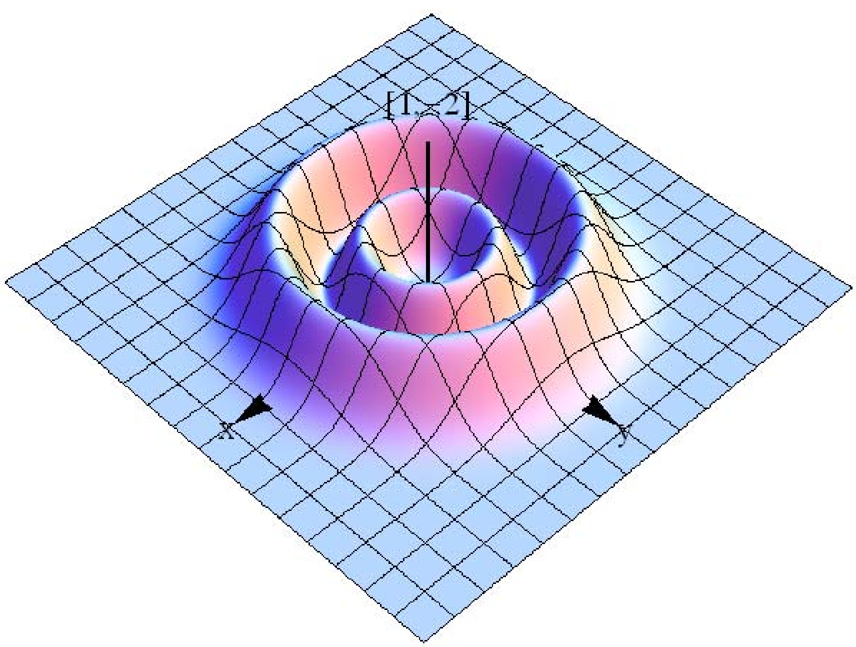


(ii) $n = 1$ $m = 1, 0, -1, -2$

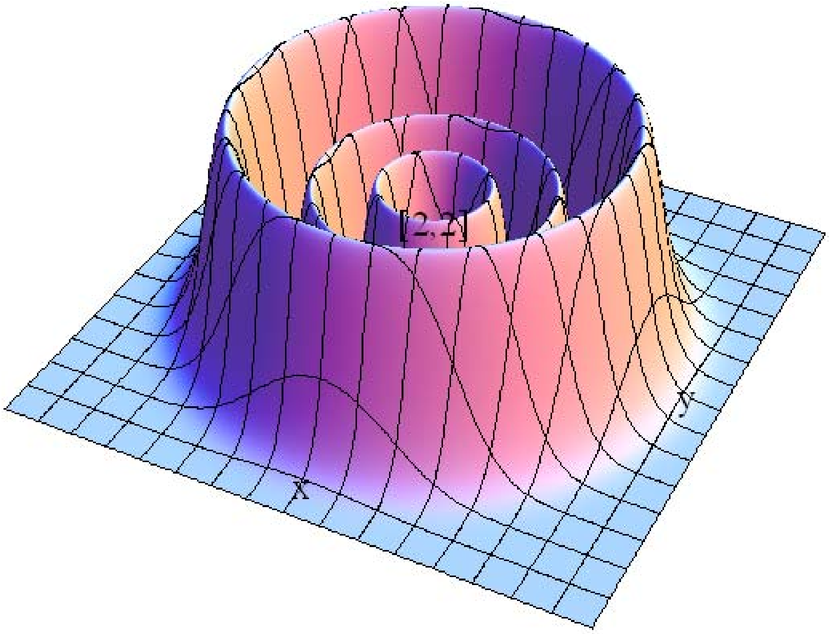

[1, 1]
x
y

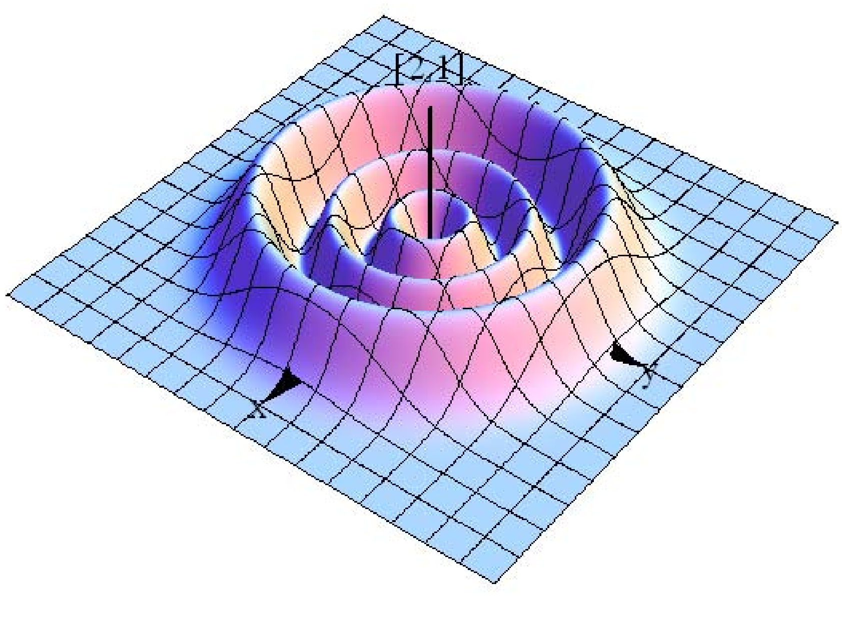

[1,0]
x
y

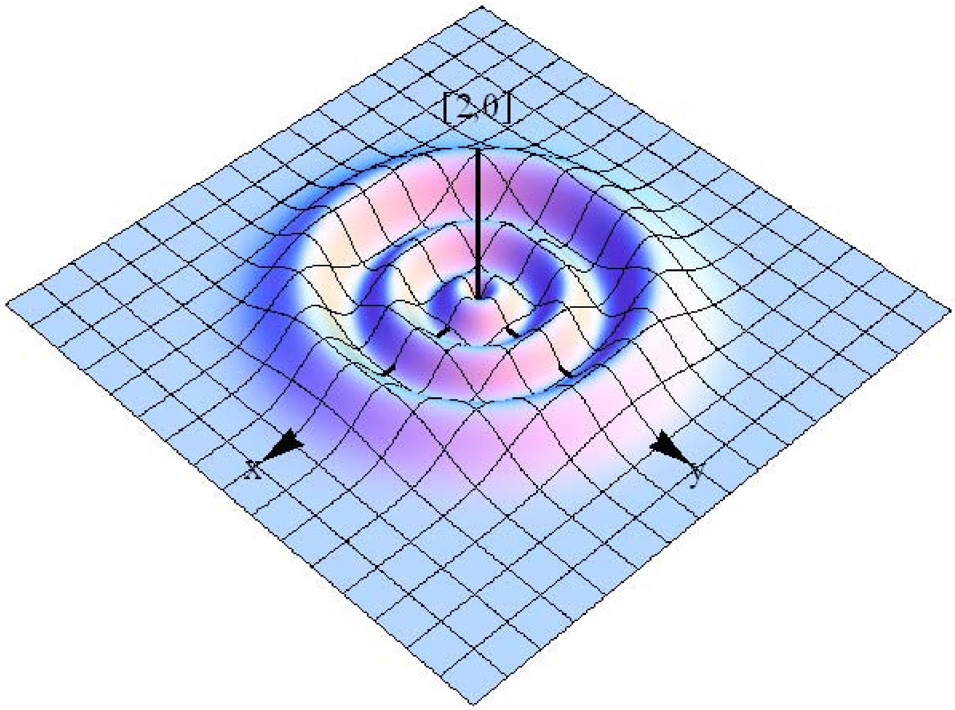

[1, −1]
x
y

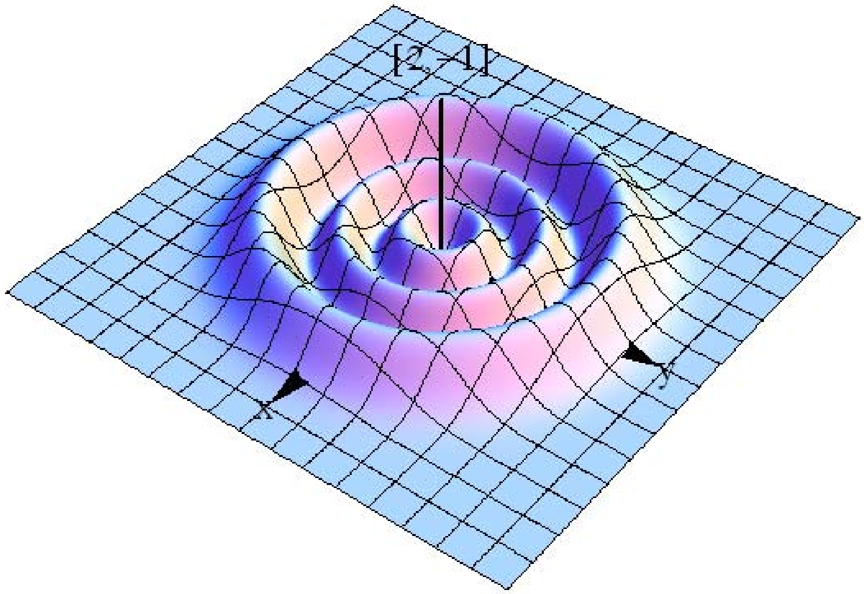

[1, −2]
x
y

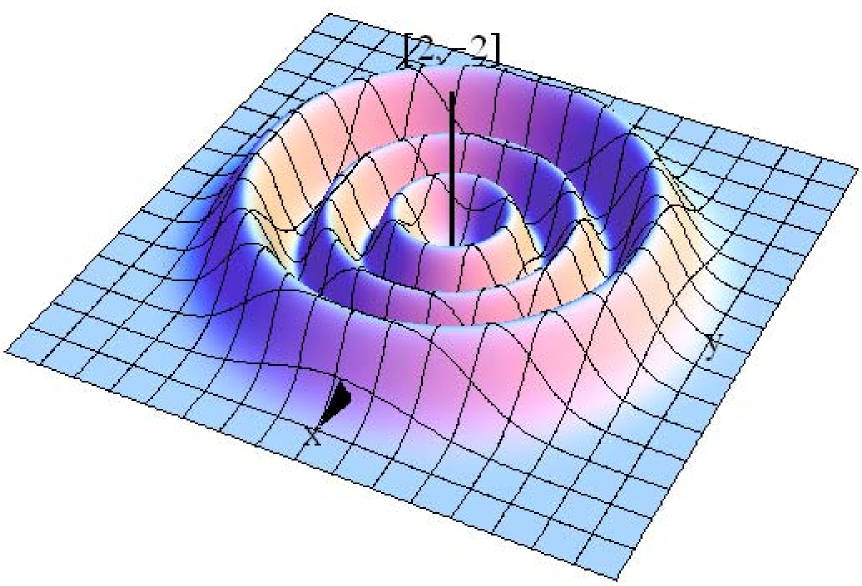
(iii) $n = 2 \quad m = 2, 1, 0, -1, -2$



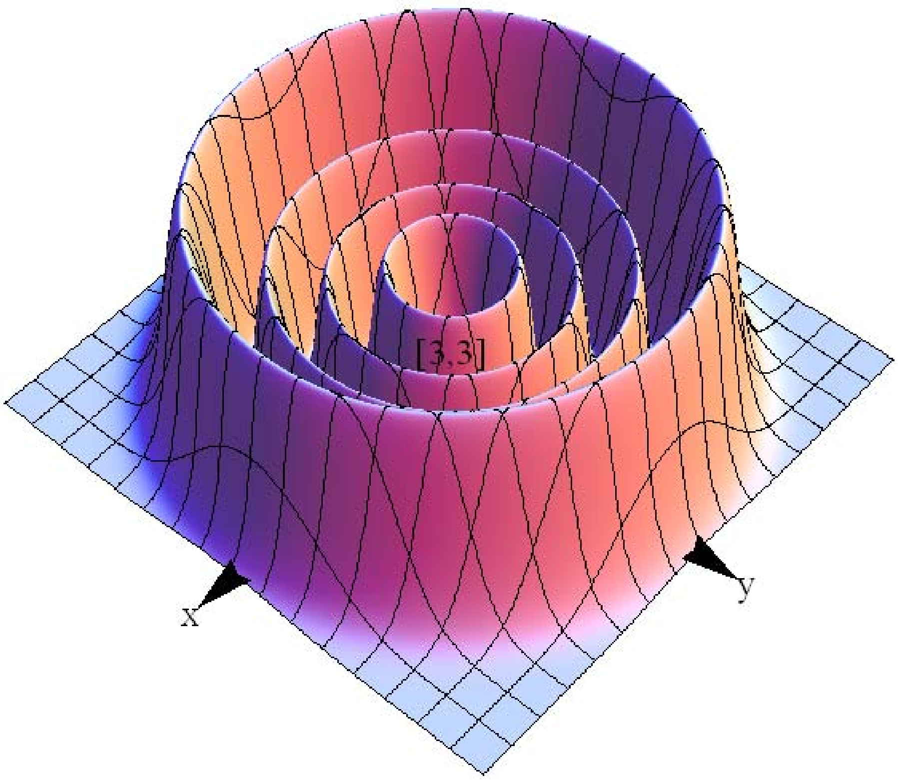


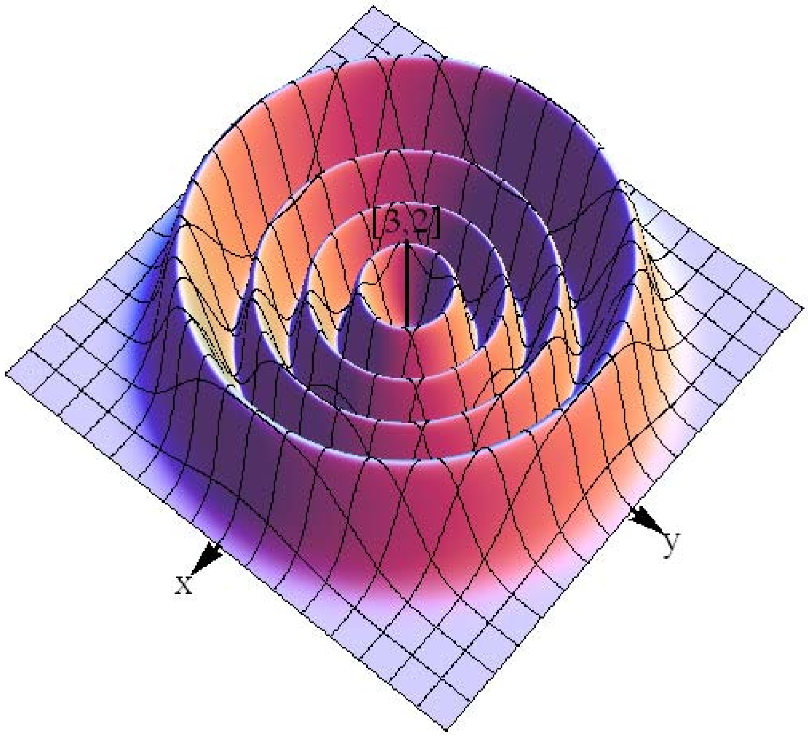

[2,0]
x
y

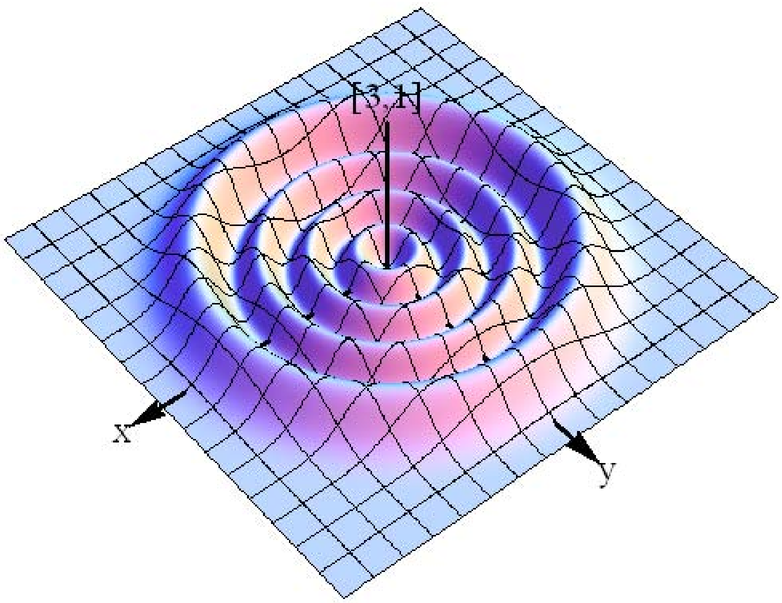

[2,-1]
x
y

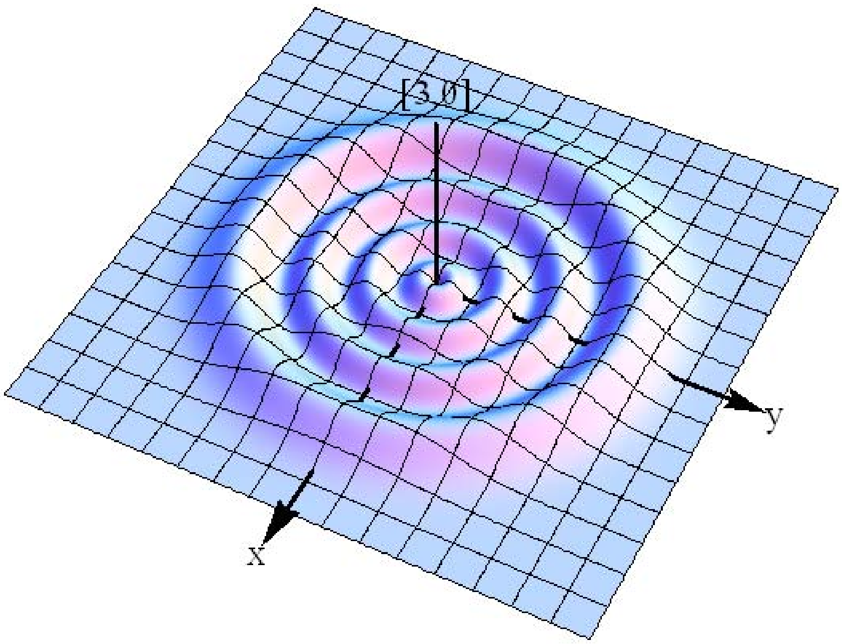

[2,-2]
x
y

(iv) $n = 3$ $m = 0, 1, 2, 3$




[3,2]
x
y


[3,1]
x
y



20. Mathematica: solving the differential equation by series expansion

```
Clear["Global`*"];
```

$$eq1 = x\, f''[x] + f'[x] + \left(\frac{-x}{4} + \eta - \frac{a^2}{4x}\right) f[x];$$

$$rule1 = \{f \to (Exp[-\#/2]\, \#^{a/2}\, u[\#]\ \&)\};$$

a = | m |;

eq11 = eq1 /. rule1 // Simplify

$$\frac{1}{2} e^{-x/2} x^{a/2} (-(1+a-2\eta)\, u[x] + 2\,((1+a-x)\, u'[x] + x\, u''[x]))$$

eq12 =

$$(-(1+a-2\eta)\, u[x] + 2\,((1+a-x)\, u'[x] + x\, u''[x]))/2\ //$$

Expand // Simplify

$$-\frac{1}{2}(1+a-2\eta)\, u[x] + (1+a-x)\, u'[x] + x\, u''[x]$$

$$rule2 = \left\{u \to \left(\left(\#^p \sum_{k=0}^{10} C[k]\, \#^k\right)\ \&\right)\right\};$$

eq21 = eq12 /. rule2 // Expand;

$$eq22 = eq21\, x^{1-p}\ //\ Simplify;$$

```
list1 = Table[{n, Coefficient[eq22, x, n]}, {n, 0, 4}] //
  Simplify;
list1 // TableForm
```

0	$p\,(a+p)\, C[0]$
1	$-\frac{1}{2}(1+a+2p-2\eta)\, C[0] + (1+p)\,(1+a+p)\, C[1]$
2	$-\frac{1}{2}(3+a+2p-2\eta)\, C[1] + (2+p)\,(2+a+p)\, C[2]$
3	$-\frac{1}{2}(5+a+2p-2\eta)\, C[2] + (3+p)\,(3+a+p)\, C[3]$
4	$-\frac{1}{2}(7+a+2p-2\eta)\, C[3] + (4+p)\,(4+a+p)\, C[4]$

```
list2 = list1 /. {p → 0}; list2 // TableForm
```

0	0
1	$-\frac{1}{2}(1+a-2\eta)\, C[0] + (1+a)\, C[1]$
2	$-\frac{1}{2}(3+a-2\eta)\, C[1] + 2\,(2+a)\, C[2]$
3	$-\frac{1}{2}(5+a-2\eta)\, C[2] + 3\,(3+a)\, C[3]$
4	$-\frac{1}{2}(7+a-2\eta)\, C[3] + 4\,(4+a)\, C[4]$

Determination of recursion formula

```
rule3 = {u → (Sum[C[n] #^n, {n, k-3, k+3}] &)};

eq3 = eq12 /. rule3 // Expand;

eq31 = eq3 x^(4-k) // Simplify;

list3 = Table[{n, Coefficient[eq31, x, n]}, {n, 3, 5}] //
  Simplify;
list3 // TableForm
```

3	$-\frac{1}{2}(-1+a+2k-2\eta)\,C[-1+k]+k(a+k)\,C[k]$
4	$-\frac{1}{2}(1+a+2k-2\eta)\,C[k]+(1+k)(1+a+k)\,C[1+k]$
5	$-\frac{1}{2}(3+a+2k-2\eta)\,C[1+k]+(2+k)(2+a+k)\,C[2+k]$

```
eq4 = list3[[2, 2]]
```

$$-\frac{1}{2}(1+a+2k-2\eta)\,C[k]+(1+k)(1+a+k)\,C[1+k]$$

```
eq5 = Solve[eq4 == 0, C[k + 1]]
```

$$\left\{\left\{C[1+k]\to\frac{(1+a+2k-2\eta)\,C[k]}{2(1+k)(1+a+k)}\right\}\right\}$$

```
DSolve[eq12 == 0, u[x], x] // Simplify
```

$$\left\{\left\{u[x]\to C[1]\,\text{HypergeometricU}\left[\frac{1}{2}(1+a-2\eta),\,1+a,\,x\right]+C[2]\,\text{LaguerreL}\left[-\frac{1}{2}-\frac{a}{2}+\eta,\,a,\,x\right]\right\}\right\}$$

Solving differential equation:

```
seq1 = x F''[x] + (1 + Abs[m] - x) F'[x] + (nr ) F[x] == 0;

DSolve[seq1, F[x], x]
```

```
{{F[x] → C[1] HypergeometricU[-nr, 1 + Abs[m], x] +
    C[2] LaguerreL[nr, Abs[m], x]}}
```

21. Wavefunction for the Coulomb gauge

Here we use the Coulomb gauge such that $\nabla\cdot\boldsymbol{A}=0$. Then we get

$$\hat{H}=\frac{1}{2m}(\hat{\boldsymbol{p}}+\frac{e}{c}\boldsymbol{A})^2=\frac{1}{2m}[\hat{\boldsymbol{p}}^2+\frac{e^2}{c^2}\boldsymbol{A}^2+\frac{e}{c}(\hat{\boldsymbol{p}}\cdot\boldsymbol{A}+\boldsymbol{A}\cdot\hat{\boldsymbol{p}})]$$

where

$$
\begin{aligned}
\hat{\boldsymbol{p}}\cdot\boldsymbol{A}+\boldsymbol{A}\cdot\hat{\boldsymbol{p}} &= \hat{p}_x A_x + \hat{p}_y A_y + \hat{p}_z A_z + A_x \hat{p}_x + A_y \hat{p}_y + A_z \hat{p}_z \\
&= [\hat{p}_x, A_x] + [\hat{p}_y, A_y] + [\hat{p}_z, A_z] + 2\boldsymbol{A}\cdot\hat{\boldsymbol{p}} \\
&= \frac{\hbar}{i}\nabla\cdot\boldsymbol{A} + 2\boldsymbol{A}\cdot\hat{\boldsymbol{p}}
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\hat{H} &= \frac{1}{2m}[\hat{\boldsymbol{p}}^2 + \frac{e^2}{c^2}\boldsymbol{A}^2 + \frac{e}{c}(\frac{\hbar}{i}\nabla\cdot\boldsymbol{A} + 2\boldsymbol{A}\cdot\hat{\boldsymbol{p}})] \\
&= \frac{1}{2m}(\hat{\boldsymbol{p}}^2 + \frac{e^2}{c^2}\boldsymbol{A}^2 + \frac{e\hbar}{ic}\nabla\cdot\boldsymbol{A} + \frac{2e}{c}\boldsymbol{A}\cdot\hat{\boldsymbol{p}})
\end{aligned}
$$

Since $\nabla\cdot\boldsymbol{A} = 0$, we have

$$
\hat{H} = \frac{1}{2m}(\hat{\mathbf{p}}^2 + \frac{e^2}{c^2}\mathbf{A}^2 + \frac{2e}{c}\mathbf{A}\cdot\hat{\mathbf{p}}) = \frac{1}{2m}\hat{\mathbf{p}}^2 + \frac{e^2B^2}{2mc^2}\hat{x}^2 + \frac{eB}{mc}\hat{x}\hat{p}_y
$$

where

$$
l^2 = \frac{c\hbar}{eB}, \qquad \hbar\omega_c = \frac{\hbar^2}{ml^2} = \frac{\hbar eB}{mc}, \qquad m\omega_c{}^2 = \frac{e^2B^2}{mc^2}
$$

$$
\hat{H} = \frac{1}{2m}\hat{\boldsymbol{p}}^2 + \frac{e^2B^2}{2mc^2}\hat{x}^2 + \omega_c\hat{x}\hat{p}_y = \frac{1}{2m}\hat{\boldsymbol{p}}^2 + \frac{m\omega_c{}^2}{2}\hat{x}^2 + \omega_c\hat{x}\hat{p}_y
$$

The first and second terms of this Hamiltonian are that of the simple harmonics along the *x* axis.

We note that this Hamiltonian $\hat{H}$ commutes with $\hat{p}_y$ and $\hat{p}_z$.

$$
[\hat{H}, \hat{p}_y] = 0 \quad \text{and} \quad [\hat{H}, \hat{p}_z] = 0
$$

Then $\left|n, k_y, k_z\right\rangle$ is a simultaneous eigenket of $\hat{H}$, $\hat{p}_y$ and $\hat{p}_z$.

$$
\hat{H}\left|n, k_y, k_z\right\rangle = E_n\left|n, k_y, k_z\right\rangle
$$

and

$$\hat{p}_y \left| n, k_y, k_z \right\rangle = \hbar k_y \left| n, k_y, k_z \right\rangle,$$

and

$$\hat{p}_z \left| n, k_y, k_z \right\rangle = \hbar k_z \left| n, k_y, k_z \right\rangle$$

Thus the wave function is described by the form,

$$\psi(x, y, z) = \phi_n(x) e^{ik_y y + ik_z z}$$

REFERENCES

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S.F. Ezawa, Quantum Hall Effect Field Theoretical Approach and Related Topics, second edition (World Scientific, 2008).

Schaum's Quantum Mechanics

APPENDIX

A.1 Hamiltonian

$$\hat{H} = \frac{1}{2m}(\hat{\boldsymbol{p}} + \frac{e}{c}\hat{\boldsymbol{A}})^2 - e\phi,$$

where $e>0$. The canonical momentum is defined by

$$\hat{\boldsymbol{\pi}} = m\hat{\boldsymbol{v}} = \hat{\boldsymbol{p}} + \frac{e}{c}\hat{\boldsymbol{A}}.$$

A.2 Guiding center and relative co-ordinate

(a) General case

$$[\hat{x}, \hat{\pi}_x] = i\hbar\hat{1},$$

$$[\hat{y}, \hat{\pi}_y] = i\hbar\hat{1},$$

$$[\hat{x}, \hat{\pi}_y] = 0,$$

$$[\hat{y}, \hat{\pi}_x] = 0,$$

$$[\hat{\pi}_x, \hat{\pi}_y] = -i\frac{\hbar^2}{l^2}, \qquad [\hat{\pi}_y, \hat{\pi}_z] = 0, \quad [\hat{\pi}_z, \hat{\pi}_x] = 0$$

where $\boldsymbol{B}$ is the magnetic field. B is directed along the z axis.

$$l^2 = \frac{c\hbar}{eB}. \qquad \omega_c = \frac{eB}{m_e c}$$

(b) Landau gauge

$$\hat{x} = \hat{X} + \frac{l^2}{\hbar}\hat{\pi}_y, \qquad \hat{y} = \hat{Y} - \frac{l^2}{\hbar}\hat{\pi}_x$$

$$\hat{\xi} = \hat{x} - \hat{X} = \frac{l^2}{\hbar}\hat{\pi}_y, \qquad \hat{\eta} = \hat{y} - \hat{Y} = -\frac{l^2}{\hbar}\hat{\pi}_x$$

$$[\hat{\xi}, \hat{\eta}] = -il^2$$

$$[\hat{X}, \hat{Y}] = il^2$$

$$[\hat{\xi}, \hat{X}] = [\frac{l^2}{\hbar}\hat{\pi}_y, \hat{x} - \frac{l^2}{\hbar}\hat{\pi}_y] = 0$$

$$[\hat{\eta}, \hat{X}] = [-\frac{l^2}{\hbar}\hat{\pi}_x, \hat{x} - \frac{l^2}{\hbar}\hat{\pi}_y] = 0,$$

$$[\hat{\xi}, \hat{Y}] = [\frac{l^2}{\hbar}\hat{\pi}_y, \hat{y} + \frac{l^2}{\hbar}\hat{\pi}_x] = 0$$

$$[\hat{\eta}, \hat{Y}] = [-\frac{l^2}{\hbar}\hat{\pi}_x, \hat{y} + \frac{l^2}{\hbar}\hat{\pi}_x] = 0,$$

The Hamiltonian

$$\hat{H} = \frac{\hbar^2}{2ml^4}(\hat{\xi}^2 + \hat{\eta}^2)$$

$$[\hat{H}, \hat{X}] = \frac{\hbar^2}{2ml^4}[\hat{\xi}^2 + \hat{\eta}^2, \hat{X}] = 0$$

$$[\hat{H}, \hat{Y}] = \frac{\hbar^2}{2ml^4}[\hat{\xi}^2 + \hat{\eta}^2, \hat{Y}] = 0$$

A2: Quantum mechanical operators (Landau gauge)

$$\hat{\boldsymbol{A}} = (0, B\hat{x}, 0)$$

$$\hat{\pi}_x = \hat{p}_x + \frac{e}{c}\hat{A}_x = \hat{p}_x, \qquad \hat{\pi}_y = \hat{p}_y + \frac{e}{c}\hat{A}_y = \hat{p}_y + \frac{eB}{c}\hat{x} = \hat{p}_y + \frac{\hbar}{l^2}\hat{x}$$

$$\hat{X} = \hat{x} - \frac{l^2}{\hbar}\hat{\pi}_y = \hat{x} - \frac{l^2}{\hbar}(\hat{p}_y + \frac{\hbar}{l^2}\hat{x}) = -\frac{l^2}{\hbar}\hat{p}_y$$

$$\hat{Y} = \hat{y} + \frac{l^2}{\hbar}\hat{\pi}_x = \hat{y} + \frac{l^2}{\hbar}\hat{p}_x$$

$$\hat{a} = \frac{\ell}{\sqrt{2}\hbar}(\hat{\pi}_x - i\hat{\pi}_y) = \frac{\ell}{\sqrt{2}\hbar}[\hat{p}_x - i\frac{\hbar}{l^2}(\hat{x} - \hat{X})]$$

$$\hat{a}^+ = \frac{\ell}{\sqrt{2}\hbar}(\hat{\pi}_x + i\hat{\pi}_y) = \frac{\ell}{\sqrt{2}\hbar}[\hat{p}_x + i\frac{\hbar}{l^2}(\hat{x} - \hat{X})]$$

A3: Quantum mechanical operators (Symmetric gauge)

$$\hat{\boldsymbol{A}} = (-\frac{1}{2}B\hat{y}, \frac{1}{2}B\hat{x}, 0)$$

$$\hat{\pi}_x = \hat{p}_x + \frac{e}{c}\hat{A}_x = \hat{p}_x - \frac{\hbar}{2l^2}\hat{y},$$

$$\hat{\pi}_y = \hat{p}_y + \frac{e}{c}\hat{A}_y = \hat{p}_y + \frac{\hbar}{2l^2}\hat{x}$$

$$\hat{X} = \frac{l^2}{\hbar}\hat{\pi}_y = \frac{l^2}{\hbar}(\hat{p}_y + \frac{\hbar}{2l^2}\hat{x}) = \frac{1}{2}\hat{x} + \frac{l^2}{\hbar}\hat{p}_y .$$

$$\hat{Y} = -\frac{l^2}{\hbar}\hat{\pi}_x = -\frac{l^2}{\hbar}(\hat{p}_x - \frac{\hbar}{2l^2}\hat{y}) = \frac{1}{2}\hat{y} - \frac{l^2}{\hbar}\hat{p}_x .$$

B.1. Landau tubes with the quantum number *n*

Here we draw a series of Landau tubes with different quantum number n for each fixed magnetic field *B*. We start with the relation given by

$$E = \frac{\hbar^2}{2m}k_z{}^2 + \hbar\omega_c(n + \frac{1}{2})$$

where

$$\omega_c = \frac{eB}{mc}.$$

For simplicity, we assume that $\hbar = 1$, $m = 1$. Then we get

$$E = \frac{1}{2}k_z^{\ 2} + \omega_c(n + \frac{1}{2})$$

For a fixed n (0 or positive integer) and a fixed E,

$$k_z = \pm\sqrt{2E - \omega_c(2n+1)}$$

The Landau tube consists of the cylinder with a radius

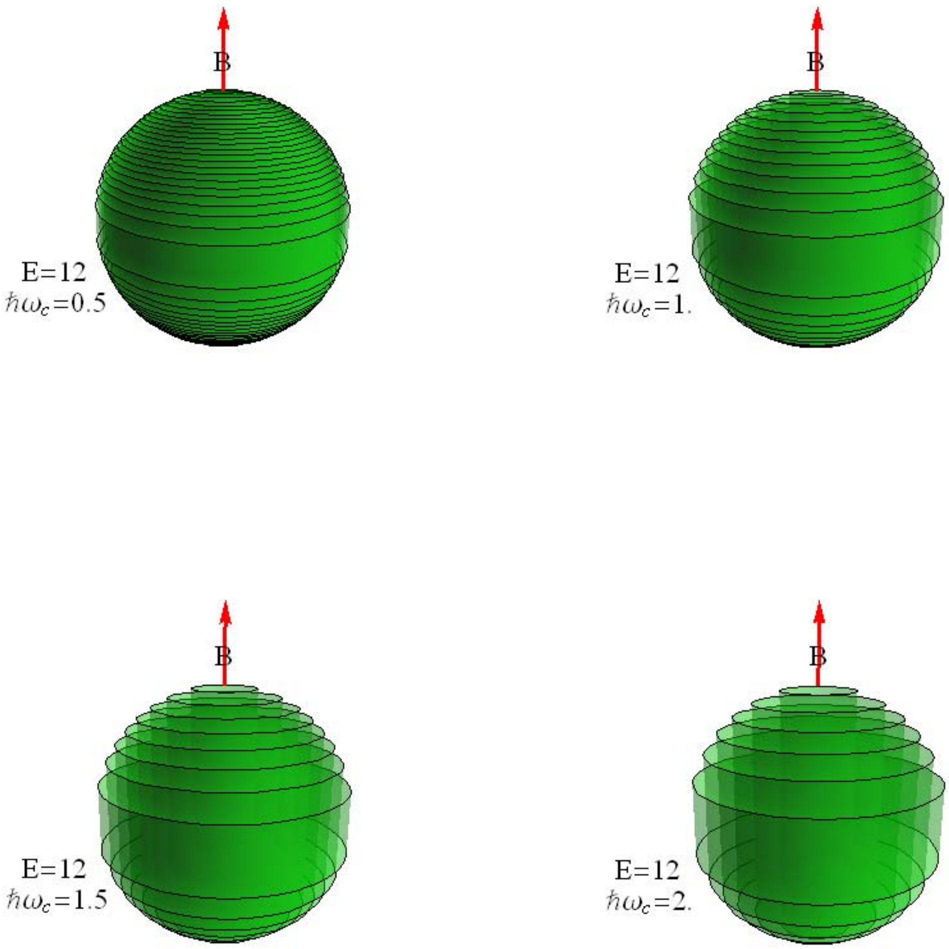
$$k_\perp = k_n = \sqrt{(2n+1)\omega_c}$$

and the length of cylinder

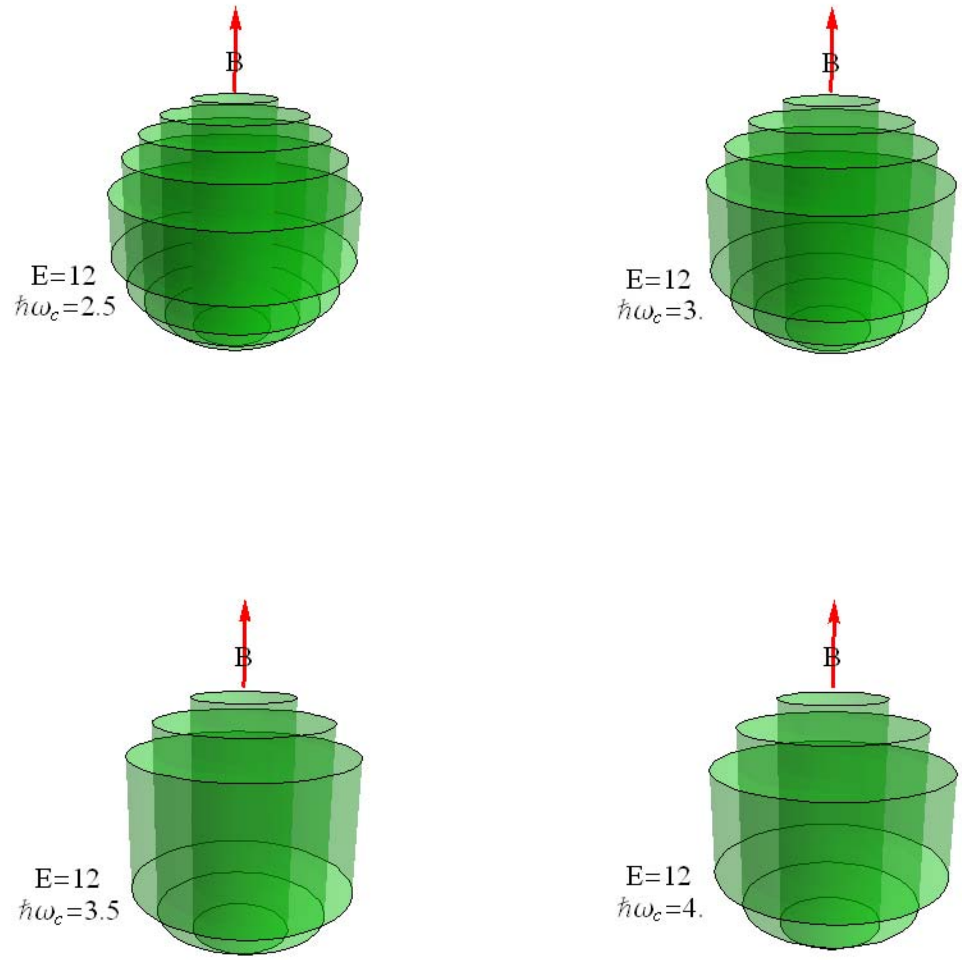
$$\left|k_z\right| \leq \sqrt{2E - \omega_c(2n+1)}$$

We choose; $E = 12$. ω_c is changed as a parameter, where $\hbar = 1$ and $m = 1$.. In the quantum limit, there is only one state with n = 0 inside the Fermi surface. The Mathematica program is shown in the APPENDIX. We show the Landau tubes of quantized magnetic levels (n) at fixed values of $\hbar\omega_c$.

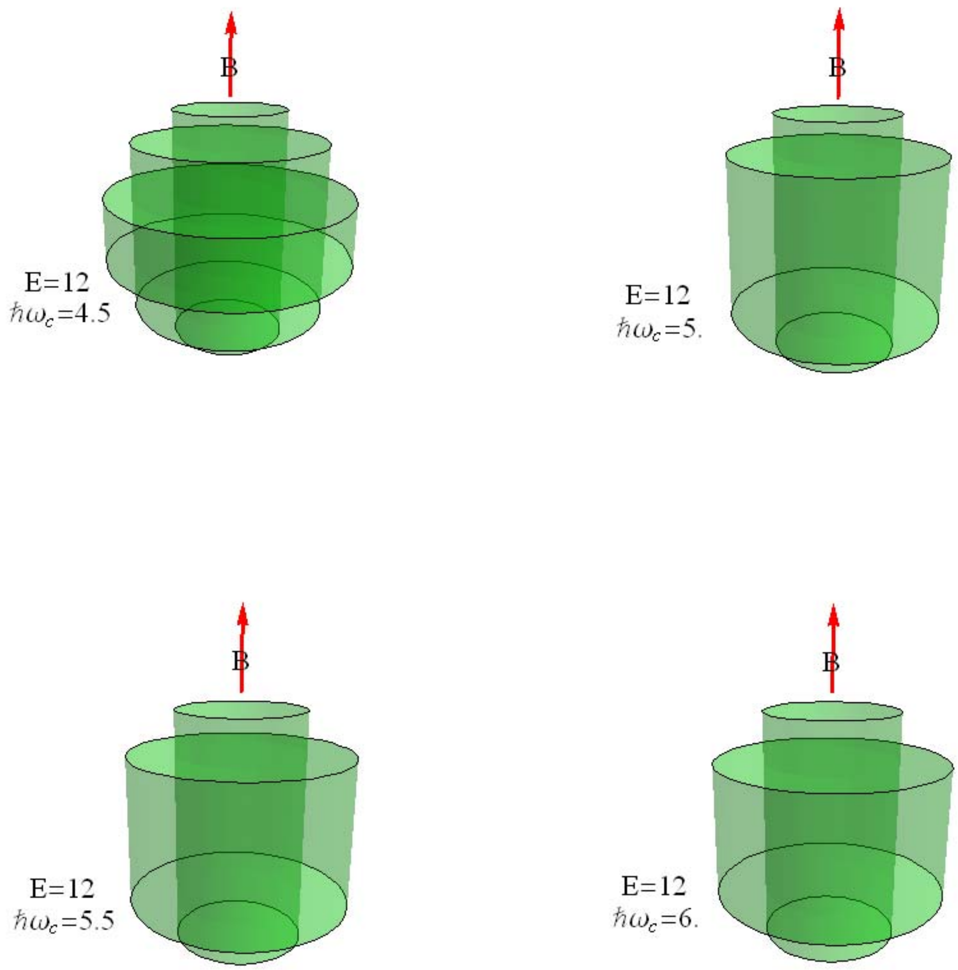
(a) $\hbar\omega_c$ = 0.5, 1, 1.5, and 2.0



(b) $\hbar\omega_c$ = 2.5, 3, 3.5, and 4.0



(c) $\hbar\omega_c$ = 4.5, 5, 5.5, and 6.0



(d) $\hbar\omega_c$ = 6.5, 7, 7.5, 8.0

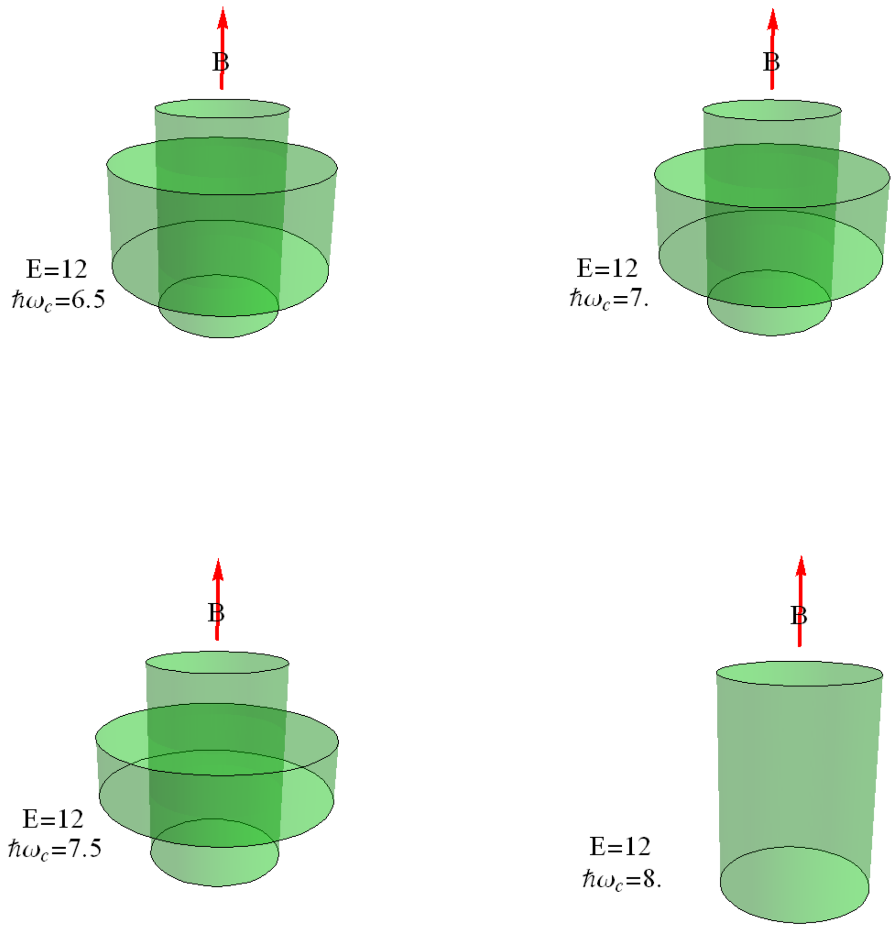


Fig.

The last figure shows the quantum limit where only the Landau tube with $n = 0$ exists inside the Fermi surface.

B.2. Fundamentals of the Landau levels

We now consider the case when $k_z = 0$.

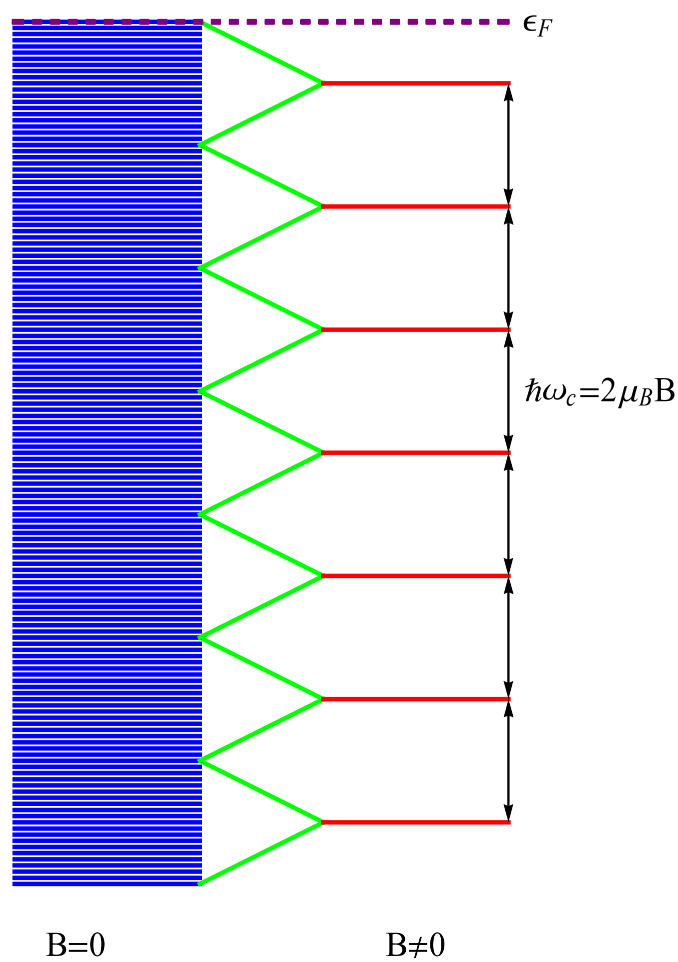


Fig.

This regular periodic motion introduces a new quantization of the energy levels (Landau levels) in the (k_x, k_y) plane, corresponding to those of a harmonic oscillator with frequency ω_c and energy

$$\varepsilon_n = \hbar\omega_c(n + \frac{1}{2}) = \frac{\hbar^2}{2m}k_\perp{}^2, \tag{12}$$

where $k_\perp$ is the magnitude of the in-plane wave vector and the quantum number n takes integer values 0, 1, 2, 3,…... ω_c is the cyclotron angular frequency and is defined by

$$\omega_c = \frac{eB}{mc}.$$

Each Landau ring is associated with an area of $\boldsymbol{k}$ space. The area S_n is the area of the orbit n with the radius $k_\perp = k_n$

$$S_n(\boldsymbol{k}) = \pi k_n^{\ 2} = \frac{2\pi eB}{\hbar c}(n+\frac{1}{2}) = \frac{2\pi}{\ell^2}(n+\frac{1}{2}). \tag{13}$$

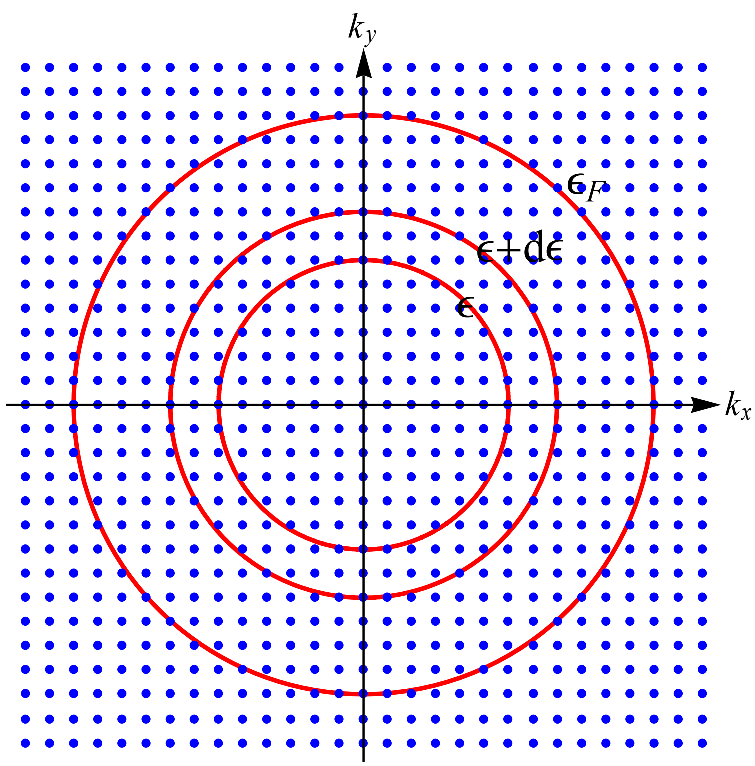
Thus in a magnetic field the area of the orbit in ***k*** space is quantized. Note that

$$l^2 = \frac{\hbar c}{eB}.$$

l is given by

$$l = \frac{256.556}{\sqrt{B[T]}} \ [\mu\text{m}]$$

in the SI units.



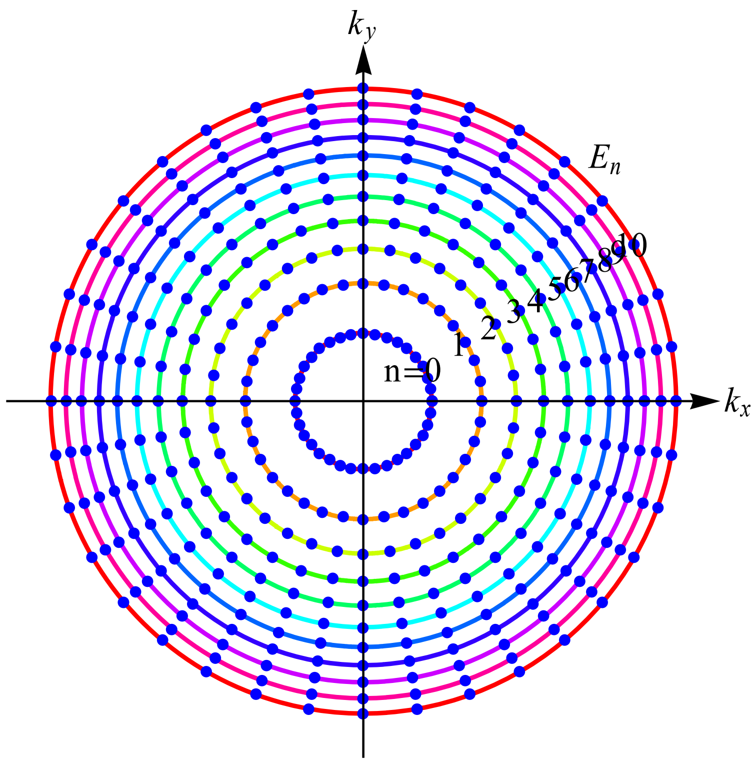


Fig. Quantization scheme for free electrons. Electron states are denoted by points in the ***k*** space in the absence and presence of external magnetic field ***B***. The states on each circle are degenerate. (a) When ***B*** = 0, there is one state per area $(2\pi/L)^2$. L^2 is the area of the system. (b) When ***B*** ≠ 0, the electron energy is quantized into Landau levels. Each circle represents a Landau level with energy $E_n = \hbar\omega_c(n+1/2)$.

B.3. Density of states using the magnetic length

We consider the density of states for the 2D system

$$D_o(\varepsilon)d\varepsilon = \frac{L^2}{(2\pi)^2}2\pi k dk$$

where the electron spin is neglected. The energy dispersion relation is given by

$$\varepsilon_\perp = \frac{\hbar^2}{2m}k_\perp{}^2,$$

with

$$d\varepsilon_\perp = \frac{\hbar^2}{m}k_\perp dk_\perp$$

Then we get

$$D_o(\varepsilon_\wp)d\varepsilon_\perp = \frac{L^2}{(2\pi)^2}2\pi\frac{m}{\hbar^2}d\varepsilon_\perp,$$

or

$$D_o(\varepsilon) = \frac{L^2}{(2\pi)^2}2\pi\frac{m}{\hbar^2} = \frac{mL^2}{2\pi\hbar^2}$$

which is constant. Note that we neglect the electron spin, since only the orbital motion is concerned.

Now we calculate the number of states between the adjacent energy levels ε_n and ε_{n+1}. Since

$$\Delta\varepsilon_\perp = \varepsilon_{n+1} - \varepsilon_n = \hbar\omega \quad \text{(which is independent of the quantum number } n\text{)}$$

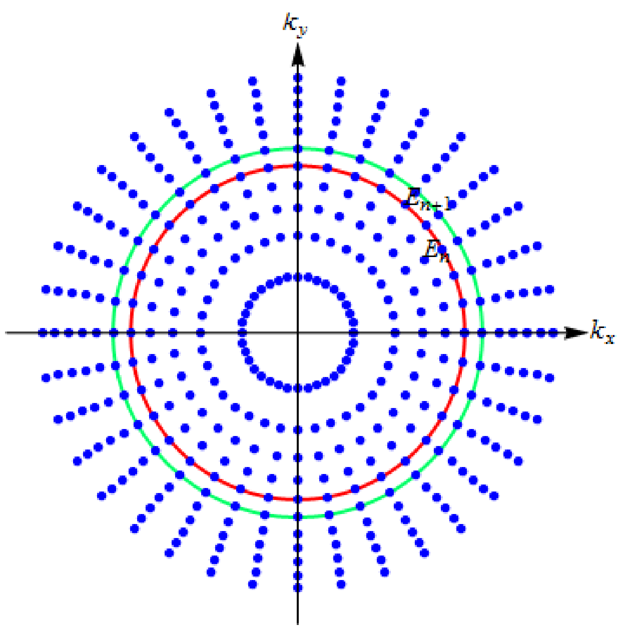


Fig. In a magnetic field the points in the (k_x, k_y) plane may be viewed as restricted circles

Then the number of states is calculated as

$$D = D(\varepsilon_\perp)\Delta\varepsilon_\perp = \frac{mL^2}{2\pi\hbar^2}\Delta\varepsilon_\perp = \frac{mL^2}{2\pi\hbar^2}\hbar\omega_c = \frac{mL^2}{2\pi\hbar^2}\frac{\hbar eB}{mc} = \frac{eBL^2}{2\pi\hbar c} = \rho B.$$

where

$$\rho = \frac{eL^2}{2\pi\hbar c}.$$

and

$$l^2 = \frac{c\hbar}{eB}.$$

APPENDIX

((**General gauge**))

The guiding center is given by

$$\hat{x} = \hat{X} + \frac{l^2}{\hbar}\hat{\pi}_y, \qquad \hat{y} = \hat{Y} - \frac{l^2}{\hbar}\hat{\pi}_x$$

The creation and annihilation operators

$$\hat{a} = \frac{l}{\sqrt{2}\hbar}(\hat{\pi}_x - i\hat{\pi}_y), \qquad \hat{a}^+ = \frac{l}{\sqrt{2}\hbar}(\hat{\pi}_x + i\hat{\pi}_y)$$

$$\hat{b} = \frac{1}{\sqrt{2}l}(\hat{X} + i\hat{Y}), \qquad \hat{b}^+ = \frac{1}{\sqrt{2}l}(\hat{X} - i\hat{Y})$$

$$[\hat{a}, \hat{a}^+] = \hat{1}, \qquad [\hat{b}, \hat{b}^+] = 1, \qquad [\hat{a}, \hat{b}] = 0, \qquad [\hat{a}^+, \hat{b}] = 0$$

(i)

$$\frac{l^2}{2\hbar}(\hat{\pi}_x{}^2 + \hat{\pi}_y{}^2) = \hbar(\hat{a}^+\hat{a} + \frac{1}{2}\hat{1})$$

(ii)

$$\hat{b}^{+}\hat{b} = \frac{1}{\sqrt{2}l}(\hat{X} - i\hat{Y})\frac{1}{\sqrt{2}l}(\hat{X} + i\hat{Y})$$
$$= \frac{1}{2l^2}\{\hat{X}^2 + \hat{Y}^2 + i[\hat{X}, \hat{Y}]\}$$
$$= \frac{1}{2l^2}(\hat{X}^2 + \hat{Y}^2) - \frac{1}{2}\hat{1}$$

or

$$\frac{\hbar}{2l^2}(\hat{X}^2 + \hat{Y}^2) = \hbar(\hat{b}^{+}\hat{b} + \frac{1}{2}\hat{1})$$

where

$$[\hat{X}, \hat{Y}] = il^2\hat{1}$$

since

$$[\hat{\pi}_x, \hat{\pi}_y] = -i\frac{\hbar^2}{l^2}$$

We also have

$$\hat{a} - i\hat{b}^{+} = \frac{l}{\sqrt{2}\hbar}[(\hat{\pi}_x - \frac{\hbar}{l^2}\hat{Y}) - i(\hat{\pi}_y + \frac{\hbar}{l^2}\hat{X})$$
$$= \frac{l}{\sqrt{2}\hbar}\{[\hat{\pi}_x - \frac{\hbar}{l^2}(\hat{y} + \frac{l^2}{\hbar}\hat{\pi}_x)] - i[\hat{\pi}_y + \frac{\hbar}{l^2}(\hat{x} - \frac{l^2}{\hbar}\hat{\pi}_y)]\}$$
$$= -\frac{i}{\sqrt{2}l}(\hat{x} - i\hat{y})$$

and

$$\hat{a}^{+} + i\hat{b} = \frac{i}{\sqrt{2}l}(\hat{x} + i\hat{y})$$

((**Symmetric gauge**))

$$\hat{\pi}_x = \hat{p}_x + \frac{e}{c} A_x = \hat{p}_x - \frac{\hbar}{2l^2} \hat{y}, \qquad \text{(symmetric gauge)}$$

$$\hat{\pi}_y = \hat{p}_y + \frac{e}{c} A_y = \hat{p}_y + \frac{\hbar}{2l^2} \hat{x} \qquad \text{(symmetric gauge)}$$

$$\hat{x} = \hat{X} + \frac{l^2}{\hbar} \hat{\pi}_y, \qquad \hat{y} = \hat{Y} - \frac{l^2}{\hbar} \hat{\pi}_x \qquad \text{(general gauge)}$$

The angular momentum (symmetric gauge)

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

$$\hat{p}_x = \hat{\pi}_x + \frac{\hbar}{2l^2} \hat{y} = \hat{\pi}_x + \frac{\hbar}{2l^2} (\hat{Y} - \frac{l^2}{\hbar} \hat{\pi}_x) = \frac{1}{2} \hat{\pi}_x + \frac{\hbar}{2l^2} \hat{Y} \qquad \text{(symmetric gauge)}$$

$$\hat{p}_y = \hat{\pi}_y - \frac{\hbar}{2l^2} \hat{x} = \hat{\pi}_y - \frac{\hbar}{2l^2} (\hat{X} + \frac{l^2}{\hbar} \hat{\pi}_y) = \frac{1}{2} \hat{\pi}_y - \frac{\hbar}{2l^2} \hat{X} \qquad \text{(symmetric gauge)}$$

Then we have

$$\begin{aligned}
\hat{L}_z &= \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \\
&= (\hat{X} + \frac{l^2}{\hbar} \hat{\pi}_y)(\frac{1}{2} \hat{\pi}_y - \frac{\hbar}{2l^2} \hat{X}) - (\hat{Y} - \frac{l^2}{\hbar} \hat{\pi}_x)(\frac{1}{2} \hat{\pi}_x + \frac{\hbar}{2l^2} \hat{Y}) \\
&= -\frac{\hbar^2}{2l^2} (\hat{X}^2 + \hat{Y}^2) + \frac{l^2}{2\hbar} (\hat{\pi}_x{}^2 + \hat{\pi}_y{}^2)
\end{aligned}$$

where we use the commutation relation;

$$[\hat{X}, \hat{\pi}_y] = [\hat{x} - \frac{l^2}{\hbar} \hat{\pi}_y, \hat{\pi}_y] = [\hat{x}, \hat{\pi}_y] = 0 . \qquad \text{(general gauge)}$$

$$[\hat{Y}, \hat{\pi}_x] = [\hat{y} + \frac{l^2}{\hbar} \hat{\pi}_x, \hat{\pi}_x] = [\hat{y}, \hat{\pi}_x] = 0 \qquad \text{(general gauge)}$$

The angular momentum is expressed by

$$\hat{L}_z = -\frac{\hbar^2}{2l^2}(\hat{X}^2+\hat{Y}^2)+\frac{l^2}{2\hbar}(\hat{\pi}_x{}^2+\hat{\pi}_y{}^2)$$
$$=\hbar(\hat{a}^+\hat{a}+\frac{1}{2}\hat{1})-\hbar(\hat{b}^+\hat{b}+\frac{1}{2}\hat{1})$$
$$=\hbar(\hat{a}^+\hat{a}-\hat{b}^+\hat{b})$$

Here we use the notation of the operators which are used in Yoshioka's book (personally I know him very well since the days of Ph.D (in physics). students in University of Tokyo)

Symmetric gauge

$$\hat{\pi}_x = \hat{p}_x - \frac{\hbar}{2l^2}\hat{y} \qquad \text{(symmetric gauge)}$$

$$\hat{\pi}_y = \hat{p}_y + \frac{\hbar}{2l^2}\hat{x} \qquad \text{(symmetric gauge)}$$

$$\hat{X} = \hat{x} - \frac{l^2}{\hbar}\hat{\pi}_y = \hat{x} - \frac{l^2}{\hbar}(\hat{p}_y + \frac{\hbar}{2l^2}\hat{x}) = \frac{1}{2}\hat{x} - \frac{l^2}{\hbar}\hat{p}_y \qquad \text{(symmetric gauge)}$$

$$\hat{Y} = \hat{y} + \frac{l^2}{\hbar}\hat{\pi}_x = \hat{y} + \frac{l^2}{\hbar}(\hat{p}_x - \frac{\hbar}{2l^2}\hat{y}) = \frac{1}{2}\hat{y} + \frac{l^2}{\hbar}\hat{p}_x \qquad \text{(symmetric gauge)}$$

(i) Operators $\hat{a}$ and $\hat{a}^+$

$$\hat{a} = \frac{l}{\sqrt{2}\hbar}(\hat{\pi}_x - i\hat{\pi}_y)$$
$$= \frac{l}{\sqrt{2}\hbar}[(\hat{p}_x - \frac{\hbar}{2l^2}\hat{y}) - i(\hat{p}_y + \frac{\hbar}{2l^2}\hat{x})]$$
$$= \frac{l}{\sqrt{2}\hbar}[(\hat{p}_x - i\hat{p}_y) - \frac{\hbar}{2l^2}i(\hat{x} - i\hat{y})]$$
$$\rightarrow \frac{1}{\sqrt{2}l}[-\frac{1}{2}i(\hat{x}-i\hat{y}) - il^2(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})]$$

$$
\begin{aligned}
\hat{a}^{+} &= \frac{l}{\sqrt{2}\hbar}(\hat{\pi}_x + i\hat{\pi}_y) \\
&= \frac{l}{\sqrt{2}\hbar}[(\hat{p}_x - \frac{\hbar}{2l^2}\hat{y}) + i(\hat{p}_y + \frac{\hbar}{2l^2}\hat{x})] \\
&= \frac{l}{\sqrt{2}\hbar}[(\hat{p}_x + i\hat{p}_y) + \frac{\hbar}{2l^2}i(\hat{x} + i\hat{y})] \\
&\to \frac{1}{\sqrt{2}l}[\frac{1}{2}i(\hat{x} + i\hat{y}) - il^2(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})]
\end{aligned}
$$

(ii) Operators $\hat{b}$ and $\hat{b}^{+}$

$$
\begin{aligned}
\hat{b} &= \frac{1}{\sqrt{2}l}(\hat{X} + i\hat{Y}) \\
&= \frac{1}{\sqrt{2}l}[(\frac{1}{2}\hat{x} - \frac{l^2}{\hbar}\hat{p}_y) + i(\frac{1}{2}\hat{y} + \frac{l^2}{\hbar}\hat{p}_x)] \\
&= \frac{1}{\sqrt{2}l}[\frac{1}{2}(\hat{x} + i\hat{y}) + i\frac{l^2}{\hbar}(\hat{p}_x + i\hat{p}_y)] \\
&\to \frac{1}{\sqrt{2}l}[\frac{1}{2}(x + iy) + l^2(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})]
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{b}^{+} &= \frac{1}{\sqrt{2}l}(\hat{X} - i\hat{Y}) \\
&= \frac{1}{\sqrt{2}l}[(\frac{1}{2}\hat{x} - \frac{l^2}{\hbar}\hat{p}_y) - i(\frac{1}{2}\hat{y} + \frac{l^2}{\hbar}\hat{p}_x)] \\
&= \frac{1}{\sqrt{2}l}[\frac{1}{2}(\hat{x} - i\hat{y}) - i\frac{l^2}{\hbar}(\hat{p}_x - i\hat{p}_y)] \\
&\to \frac{1}{\sqrt{2}l}[\frac{1}{2}(x - iy) - l^2(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})]
\end{aligned}
$$

We introduce complex mu,ber and its complex conjugate (x and y are real)

$$
z = \frac{1}{l}(x - iy), \qquad z^{*} = \frac{1}{l}(x + iy)
$$

We note that

$$\frac{\partial}{\partial z}=\frac{l}{2}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}), \qquad \frac{\partial}{\partial z^*}=\frac{l}{2}(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}),$$

where the factor 1/2 arises from the condition that $\frac{\partial z}{\partial z}=1$, and $\frac{\partial z^*}{\partial z^*}=1$. Then we get

$$\hat{a}\rightarrow\frac{-i}{2\sqrt{2}}(z+4\frac{\partial}{\partial z^*})$$
$$=-i\sqrt{2}\exp(-\frac{|z|^2}{4})\frac{\partial}{\partial z^*}\exp(\frac{|z|^2}{4})$$

$$\hat{a}^+\rightarrow\frac{i}{2\sqrt{2}}(z^*-4\frac{\partial}{\partial z})$$
$$=\frac{i}{\sqrt{2}}\exp(-\frac{|z|^2}{4})(z^*-2\frac{\partial}{\partial z})\exp(\frac{|z|^2}{4})$$

$$\hat{b}\rightarrow\frac{1}{2\sqrt{2}}(z^*+4\frac{\partial}{\partial z})$$
$$=\sqrt{2}\exp(-\frac{|z|^2}{4})\frac{\partial}{\partial z}\exp(\frac{|z|^2}{4})$$

$$\hat{b}^+\rightarrow\frac{1}{2\sqrt{2}}(z-4\frac{\partial}{\partial z^*})$$
$$=\frac{1}{\sqrt{2}}\exp(-\frac{|z|^2}{4})(z-2\frac{\partial}{\partial z^*})\exp(\frac{|z|^2}{4})$$

((**Mathematica**))
The commutation relations.

```
Clear["Global`*"];
A1 := -ⅈ √2 Exp[-z1 z2 / 4] D[Exp[z1 z2 / 4] #, z2] &;
A2 :=
  ⅈ/√2 Exp[-z1 z2 / 4]
   (z2 Exp[z1 z2 / 4] # - 2 D[Exp[z1 z2 / 4] #, z1]) &;
B1 := √2 Exp[-z1 z2 / 4] D[Exp[z1 z2 / 4] #, z1] &;
B2 :=
  1/√2 Exp[-z1 z2 / 4]
   (z1 Exp[z1 z2 / 4] # - 2 D[Exp[z1 z2 / 4] #, z2]) &;
```

```
A1[f[z1, z2]] // Simplify
```

$$-\frac{i\left(z1\, f[z1,\, z2] + 4\, f^{(0,1)}[z1,\, z2]\right)}{2\sqrt{2}}$$

```
A2[f[z1, z2]] // Simplify
```

$$\frac{i\left(z2\, f[z1,\, z2] - 4\, f^{(1,0)}[z1,\, z2]\right)}{2\sqrt{2}}$$

```
B1[f[z1, z2]] // Simplify
```

$$\frac{z2\, f[z1,\, z2] + 4\, f^{(1,0)}[z1,\, z2]}{2\sqrt{2}}$$

```
B2[f[z1, z2]] // Simplify
```

$$\frac{z1\, f[z1,\, z2] - 4\, f^{(0,1)}[z1,\, z2]}{2\sqrt{2}}$$

```
A1[A2[f[z1, z2]]] - A2[A1[f[z1, z2]]] // Simplify
f[z1, z2]
```

```
B1[B2[f[z1, z2]]] - B2[B1[f[z1, z2]]] // Simplify
f[z1, z2]
```

Quantum mechanics

The ket $|n,m\rangle$ is the eigenstate of the angular momentum with the eigenvalue $\hbar m$ and is also the eigenstate of $\hat{a}^{+}a$ with the eigenvalue n.

$$\hat{L}_z|n,m\rangle = \hbar(\hat{a}^{+}a - \hat{b}^{+}\hat{b})|n,m\rangle = \hbar m|n,m\rangle$$

$$\hat{a}^{+}a\left|n,m\right\rangle = n\left|n,m\right\rangle$$

Then we have

$$\hat{b}^{+}\hat{b}\left|n,m\right\rangle = (n-m)\left|n,m\right\rangle$$

where $n \geq m$.

Here we introduce a new eigenket such that

$$\hat{a}^{+}\hat{a}\left|\Phi(n,n_b)\right\rangle = n\left|\Phi(n,n_b)\right\rangle, \quad \hat{b}^{+}\hat{b}\left|\Phi(n,n_b)\right\rangle = n_b\left|\Phi(n,n_b)\right\rangle$$

where

$$n_b = n - m$$

and

$$\left|n, m = n - n_b\right\rangle = \left|\Phi(n,n_b)\right\rangle$$

The eigenket $\left|\Phi(n,n_b)\right\rangle$ can be expressed by

$$\left|\Phi(n,n_b)\right\rangle = \frac{1}{\sqrt{n!}}\frac{1}{\sqrt{n_b!}}(\hat{a}^{+})^{n}(\hat{b}^{+})^{n_b}\left|0,0\right\rangle$$

The ground state is defined by

$$\left\langle \boldsymbol{r}\right|\hat{a}\left|0,0\right\rangle = \left\langle \boldsymbol{r}\right|\hat{b}\left|0,0\right\rangle = 0$$

or

$$\frac{\partial}{\partial z^{*}}\exp\left(\frac{|z|^{2}}{4}\right)\psi_{0,0}(\boldsymbol{r}) = 0, \qquad \frac{\partial}{\partial z}\exp\left(\frac{|z|^{2}}{4}\right)\psi_{0,0}(\boldsymbol{r}) = 0$$

where $\left\langle \boldsymbol{r}|0,0\right\rangle = \psi_{0,0}(\boldsymbol{r})$. We have the wavefunction of the ground stae as

$$\psi_{0,0}(\boldsymbol{r}) = A\exp[-\frac{|z|^2}{4}] = A\exp(-\frac{\rho^2}{4l^2})$$

The normalization of the wavefunction:

$$\int_0^\infty 2\pi\rho d\rho |A|^2 \exp(-\frac{\rho^2}{2l^2}) = 2\pi|A|^2 \int_0^\infty \rho d\rho \exp(-\frac{\rho^2}{2l^2}) = 1$$

and

$$|A|^2 = \frac{1}{2\pi l^2},$$

since

$$\int_0^\infty \rho d\rho \exp(-\frac{\rho^2}{2l^2}) = l^2 .$$

Then the normalized wavefunction of the ground state obtained as

$$\psi_{0,0}(\boldsymbol{\rho}) = \frac{1}{\sqrt{2\pi} l}\exp(-\frac{\rho^2}{4l^2})$$

The wavefunction of $\langle \boldsymbol{r} | \Phi(n, n_b) \rangle$ is given by

$$\begin{aligned}
\langle \boldsymbol{r} | \Phi(n, n_b) \rangle &= \frac{1}{\sqrt{n!}}\frac{1}{\sqrt{n_b!}}\langle \boldsymbol{r} | (\hat{a}^+)^n (\hat{b}^+)^{n_b} | 0,0 \rangle \\
&= \frac{1}{\sqrt{n!}}\frac{1}{\sqrt{n_b!}}\left(\frac{i}{2\sqrt{2}}\right)^n \left(\frac{1}{2\sqrt{2}}\right)^{n_b} (z^* - 4\frac{\partial}{\partial z})^n (z - 4\frac{\partial}{\partial z^*})^{n_b} \\
&\quad \frac{1}{\sqrt{2\pi} l}\exp[-\frac{zz^*}{4}] \\
&= \frac{1}{\sqrt{2\pi} l}\frac{i^n}{\sqrt{n! n_b!}}\left(\frac{1}{2\sqrt{2}}\right)^{n+n_b} (z^* - 4\frac{\partial}{\partial z})^n (z - 4\frac{\partial}{\partial z^*})^{n_b} \exp(-\frac{\rho^2}{4l^2})
\end{aligned}$$

where

$$\hat{a}^{+} = \frac{i}{2\sqrt{2}}(z^{*} - 4\frac{\partial}{\partial z}), \qquad \hat{b}^{+} = \frac{1}{2\sqrt{2}}(z - 4\frac{\partial}{\partial z^{*}})$$

((**Mathematica-1**))

```
Clear["Global`*"]; exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
f1[z1_, z2_] := 1/(Sqrt[2 π] L) Exp[-1/4 z1 z2]; A2 := I/(2 Sqrt[2]) (z2 # - 4 D[#, z1]) &;
B2 := 1/(2 Sqrt[2]) (z1 # - 4 D[#, z2]) &; A2N[n_] := Nest[A2, #, n] &;
A2Nb[nb_] := Nest[B2, #, nb] &;
rule1 = {z1 -> (x - I y)/L, z2 -> (x + I y)/L};

F1[n_, nb_] := 1/Sqrt[n! nb!] A2N[n][A2Nb[nb][f1[z1, z2]]];

F1[0, 0] /. rule1 // Simplify
```

$$\frac{e^{-\frac{x^2+y^2}{4 L^2}}}{L\sqrt{2\pi}}$$

```
F1[0, 1] /. rule1 // Simplify
```

$$\frac{e^{-\frac{x^2+y^2}{4 L^2}}(x - i y)}{2 L^2 \sqrt{\pi}}$$

```
F1[0, 2] /. rule1 // Simplify
```

$$\frac{e^{-\frac{x^2+y^2}{4 L^2}} (x - i y)^2}{4 L^3 \sqrt{\pi}}$$

```
F1[0, 3] /. rule1 // Simplify
```

$$\frac{e^{-\frac{x^2+y^2}{4 L^2}} (x - i y)^3}{4 L^4 \sqrt{6 \pi}}$$

```
F1[1, 0] /. rule1 // Simplify
```

$$\frac{e^{-\frac{x^2+y^2}{4 L^2}} (i x - y)}{2 L^2 \sqrt{\pi}}$$

```
F1[1, 1] /. rule1 // Simplify
```

$$-\frac{i e^{-\frac{x^2+y^2}{4 L^2}} \left(2 L^2 - x^2 - y^2\right)}{2 L^3 \sqrt{2 \pi}}$$

```
F1[1, 2] /. rule1 // Simplify
```

$$\frac{e^{-\frac{x^2+y^2}{4 L^2}} (i x + y) \left(-4 L^2 + x^2 + y^2\right)}{4 L^4 \sqrt{2 \pi}}$$

```
F1[1, 3] /. rule1 // Simplify
```

$$\frac{i e^{-\frac{x^2+y^2}{4 L^2}} (x - i y)^2 \left(-6 L^2 + x^2 + y^2\right)}{8 L^5 \sqrt{3 \pi}}$$

```
F1[2, 0] /. rule1 // Simplify
```

$$-\frac{e^{-\frac{x^2+y^2}{4 L^2}} (x + i y)^2}{4 L^3 \sqrt{\pi}}$$

```
F1[2, 1] /. rule1 // Simplify
```

$$-\frac{e^{-\frac{x^2+y^2}{4 L^2}} (x + i y) \left(-4 L^2 + x^2 + y^2\right)}{4 L^4 \sqrt{2 \pi}}$$

```
F1[2, 2] /. rule1 // Simplify
```

$$\frac{e^{-\frac{x^2+y^2}{4 L^2}} \left(-8 L^4 + 8 L^2 \left(x^2 + y^2\right) - \left(x^2 + y^2\right)^2\right)}{8 L^5 \sqrt{2 \pi}}$$

```
F1[3, 0] /. rule1 // Simplify
```

$$\frac{e^{-\frac{x^2+y^2}{4 L^2}} (i x - y)^3}{4 L^4 \sqrt{6 \pi}}$$

```
F1[3, 1] /. rule1 // Simplify
```

$$-\frac{i e^{-\frac{x^2+y^2}{4 L^2}} (x + i y)^2 \left(-6 L^2 + x^2 + y^2\right)}{8 L^5 \sqrt{3 \pi}}$$

```
F1[3, 2] /. rule1 // Simplify
```

$$\frac{e^{-\frac{x^2+y^2}{4 L^2}} (-i x + y) \left(24 L^4 - 12 L^2 \left(x^2 + y^2\right) + \left(x^2 + y^2\right)^2\right)}{16 L^6 \sqrt{3 \pi}}$$

```
F1[3, 3] /. rule1 // Simplify
```

$$\frac{1}{48\,L^7\sqrt{2\,\pi}}\,\mathrm{i}\,e^{-\frac{x^2+y^2}{4\,L^2}}\left(48\,L^6-72\,L^4\left(x^2+y^2\right)+18\,L^2\left(x^2+y^2\right)^2-\left(x^2+y^2\right)^3\right)$$

```
F1[4, 0] /. rule1 // Simplify
```

$$\frac{e^{-\frac{x^2+y^2}{4\,L^2}}\,(x+\mathrm{i}\,y)^4}{16\,L^5\sqrt{3\,\pi}}$$

```
F1[4, 1] /. rule1 // Simplify
```

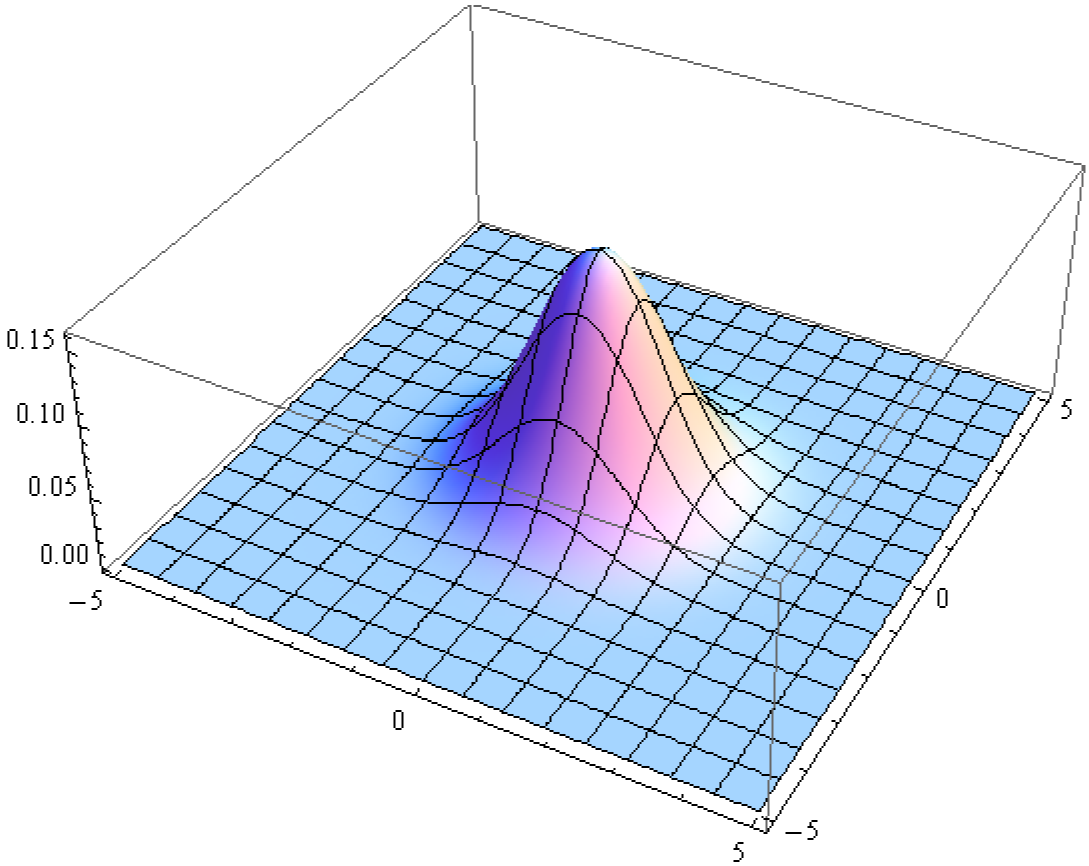
$$\frac{e^{-\frac{x^2+y^2}{4\,L^2}}\,(x+\mathrm{i}\,y)^3\left(-8\,L^2+x^2+y^2\right)}{16\,L^6\sqrt{6\,\pi}}$$

```
F1[4, 2] /. rule1 // Simplify
```

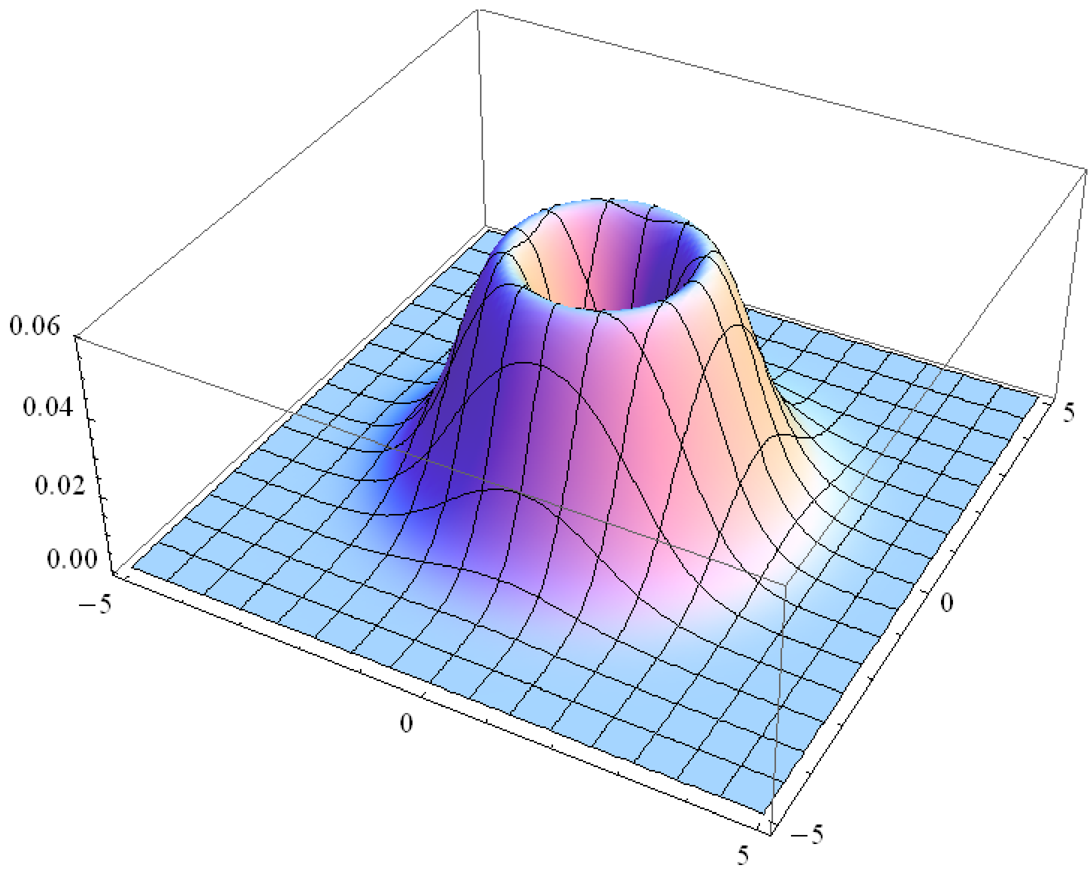
$$\frac{e^{-\frac{x^2+y^2}{4\,L^2}}\,(x+\mathrm{i}\,y)^2\left(48\,L^4-16\,L^2\left(x^2+y^2\right)+\left(x^2+y^2\right)^2\right)}{32\,L^7\sqrt{6\,\pi}}$$

((**Mathematica-2**))

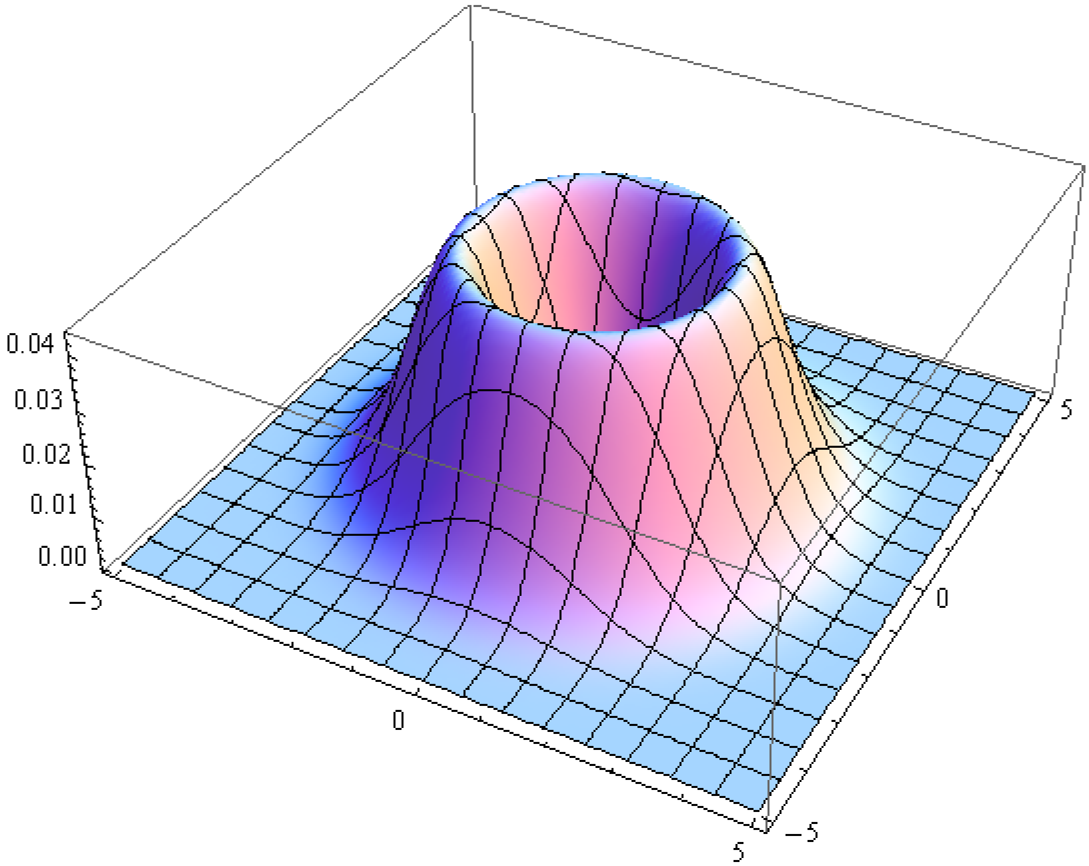
$n = 0$, $n_b = 0$, $m = n - n_b = 0$



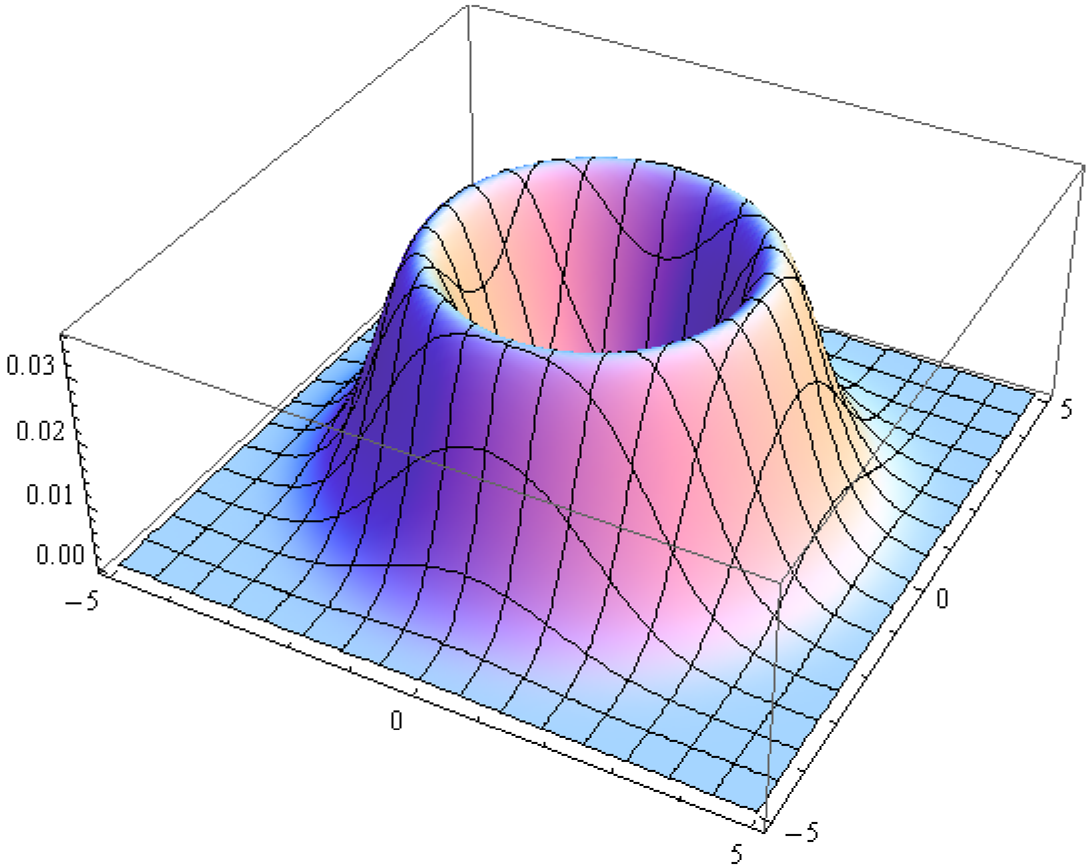
$n = 0, n_{\mathrm{b}} = 1, \qquad m = n - n_b = -1$



$n = 0$, $n_{\mathrm{b}} = 2$, $m = n - n_b = -2$

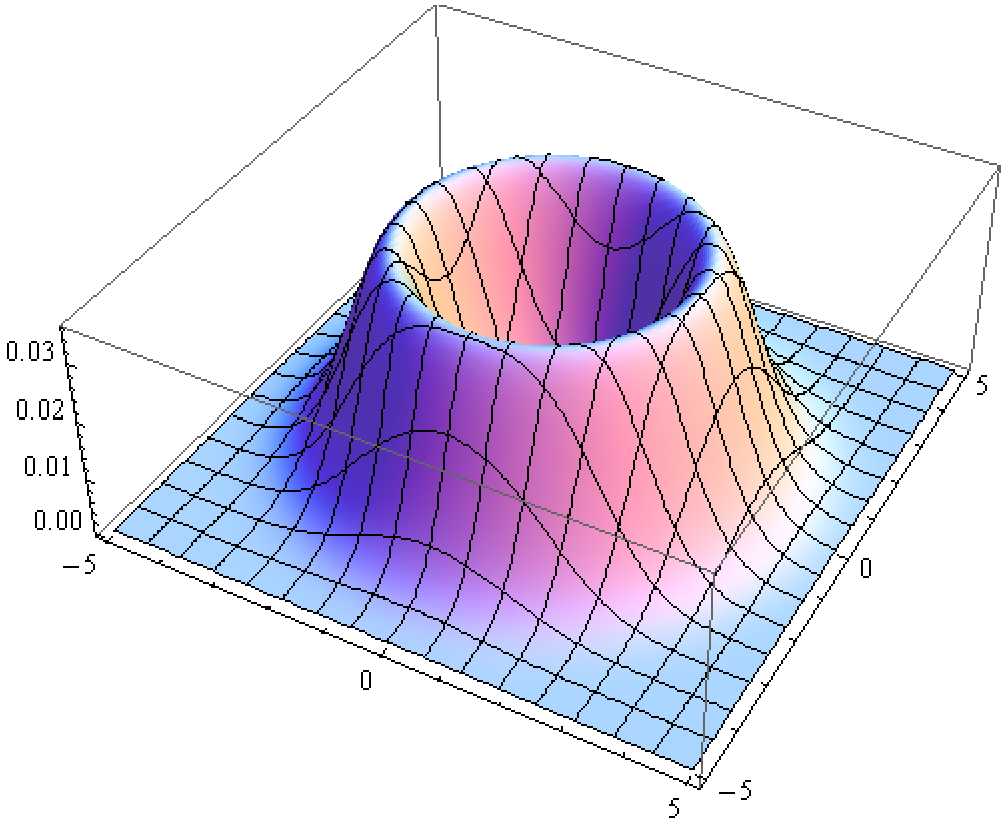


$n = 0$, $n_{\mathrm{b}} = 3$, $m = n - n_b = -3$

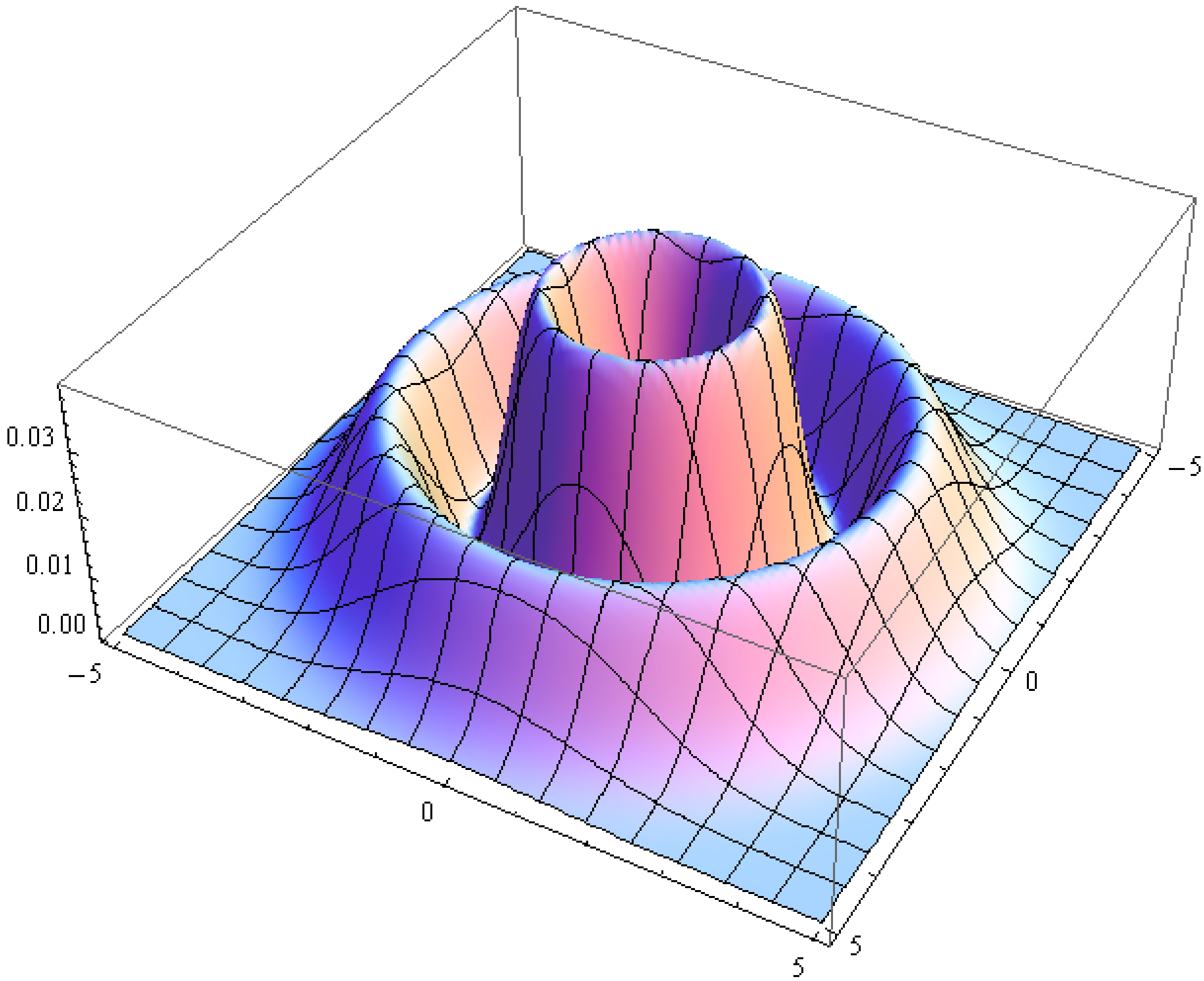


((**Mathematica-2**))

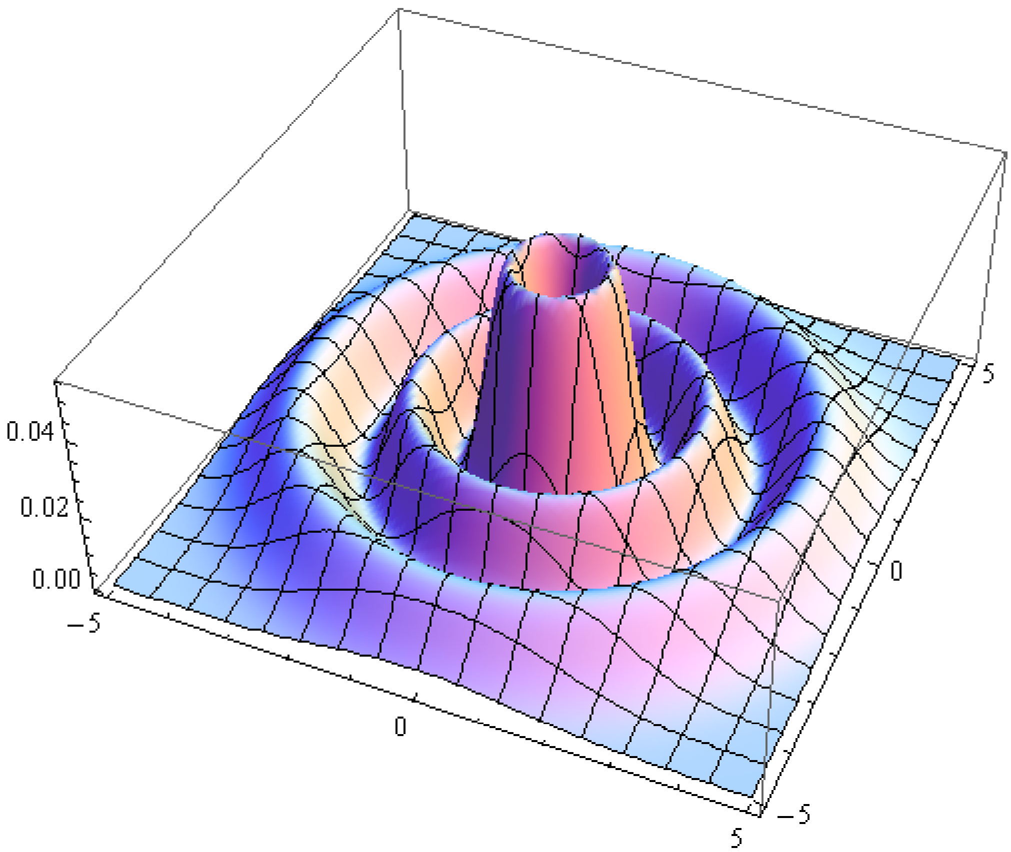
$n = 3$, $n_b = 0$, $m = n - n_b = 3$



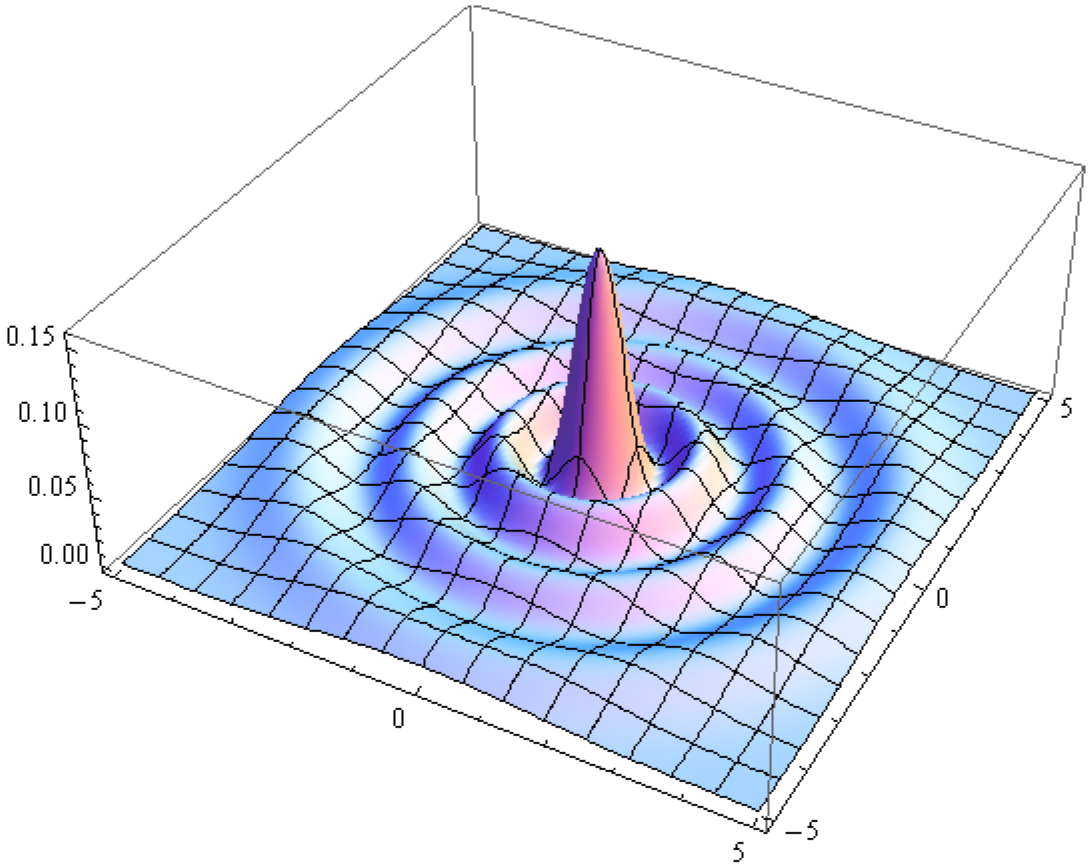
$n = 3$, $n_\mathrm{b} = 1$, $m = n - n_b = 2$



$n = 3$, $n_b = 3$, $m = n - n_b = 0$



$n = 3$, $n_b = 1$, $m = n - n_b = 2$



APPENDIX

1. Definition of the coherent state $|\alpha\rangle$

The coherent state is defined as

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle,$$

where α is a complex number. Suppose that

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

$$\hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} c_n \hat{a}|n\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n}|n-1\rangle = \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}|n\rangle$$

This is equal to

$$\alpha|\alpha\rangle = \sum_{n=0}^{\infty} c_n \alpha|n\rangle$$

Then we get the relation

$$c_{n+1} = c_n \frac{\alpha}{\sqrt{n+1}}$$

which leads to

$$c_n = c_0 \frac{\alpha^n}{\sqrt{n!}}$$

Consequently we have

$$|\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

The bra vector $\langle\alpha|$ is given by

$$\langle\alpha| = c_0^* \sum_{n=0}^{\infty} \frac{(\alpha^*)^n}{\sqrt{n!}} \langle n|$$

The condition of the normalization

$$\begin{aligned}
\langle\alpha|\alpha\rangle &= |c_0|^2 \sum_{m=0}^{\infty} \frac{(\alpha^*)^m}{\sqrt{m!}} \langle m| \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \\
&= |c_0|^2 \sum_{m=0}^{\infty} \frac{(\alpha^*)^m}{\sqrt{m!}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \langle m|n\rangle \\
&= |c_0|^2 \sum_{m=0}^{\infty} \frac{(\alpha^*)^m}{\sqrt{m!}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \delta_{n,m} \\
&= |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{|\alpha|^2} = 1
\end{aligned}$$

yields

$$|c_0| = e^{-\frac{|\alpha|^2}{2}}.$$

Finally we have

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

2. Displacement operator $\hat{D}_\alpha$

The displacement operator $\hat{D}_\alpha$ is defined as

$$\hat{D}_\alpha = \exp(\alpha \hat{a}^+ - \alpha^* \hat{a}).$$

Then the coherent state $|\alpha\rangle$ can be expressed by

$$|\alpha\rangle = D_\alpha |0\rangle .$$

where

$$\hat{D}_\alpha{}^+ \hat{a} \hat{D}_\alpha = \alpha \hat{1} + \hat{a}$$

Note that

$$\hat{D}_\alpha{}^+ = \hat{D}_{-\alpha} = \exp(-\alpha \hat{a}^+ + \alpha^* \hat{a}),$$

$$\hat{D}_\alpha{}^+ \hat{a}^+ \hat{D}_\alpha = \alpha^* \hat{1} + \hat{a}^+,$$

$$\hat{D}_\alpha{}^+ \hat{D}_\alpha = \hat{1}. \quad \text{(Unitary operator).}$$

Since

$$\hat{a} \hat{D}_\alpha = \hat{D}_\alpha \hat{a} + \alpha \hat{D}_\alpha$$

we have

$$\hat{a} \hat{D}_\alpha |0\rangle = \hat{D}_\alpha \hat{a} |0\rangle + \alpha \hat{D}_\alpha |0\rangle$$

or

$$\hat{a} \hat{D}_\alpha |0\rangle = \alpha \hat{D}_\alpha |0\rangle$$

Then $\hat{D}_\alpha |0\rangle$ is the eigenket of $\hat{a}$ with the eigenvalue α,

$$|\alpha\rangle = D_\alpha|0\rangle$$

The form of $|\alpha\rangle$ can be expressed as follows.

$$|\alpha\rangle = \hat{D}_\alpha|0\rangle = \exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha\hat{a}^+)\exp(-\alpha^*\hat{a})|0\rangle$$

Using

$$\exp(-\alpha^*\hat{a})|0\rangle = |0\rangle,$$

$$[\exp(\alpha\hat{a}^+), \hat{a}] = -\exp(\alpha\hat{a}^+)\alpha\ ,$$

or

$$\exp(\alpha\hat{a}^+)(\alpha\hat{1} + \hat{a}) = \hat{a}\exp(\alpha\hat{a}^+),$$

we have

$$\begin{aligned}
\hat{a}|\alpha\rangle &= \hat{a}\hat{D}_\alpha|0\rangle \\
&= \exp(-\frac{1}{2}|\alpha|^2)\hat{a}\exp(\alpha\hat{a}^+)|0\rangle \\
&= \exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha\hat{a}^+)(\alpha\hat{1} + \hat{a})|0\rangle \\
&= \alpha\exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha\hat{a}^+)|0\rangle = \alpha|\alpha\rangle
\end{aligned}$$

or

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

which means that $|\alpha\rangle$ is the eigenket of $\hat{a}$ with the eigenvalue α.
((Note))

$$\langle\beta|\alpha\rangle = \exp(-\frac{|\alpha|^2 + |\beta|^2}{2} + \alpha\beta^*).$$

3. Coherent state

$$\hat{a} \to \frac{-i}{2\sqrt{2}}(z + 4\frac{\partial}{\partial z^*})$$
$$= -i\sqrt{2}\exp(-\frac{|z|^2}{4})\frac{\partial}{\partial z^*}\exp(\frac{|z|^2}{4})$$

$$\hat{b} \to \frac{1}{2\sqrt{2}}(z^* + 4\frac{\partial}{\partial z})$$
$$= \sqrt{2}\exp(-\frac{|z|^2}{4})\frac{\partial}{\partial z}\exp(\frac{|z|^2}{4})$$

(i) $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$,

$$\langle z|\hat{a}|\alpha\rangle = \langle z|\alpha|\alpha\rangle = \alpha\langle z|\alpha\rangle$$

$$-i\sqrt{2}\exp(-\frac{|z|^2}{4})\frac{\partial}{\partial z^*}\exp(\frac{|z|^2}{4})\langle z|\alpha\rangle = \alpha\langle z|\alpha\rangle$$

or

$$\frac{\partial}{\partial z^*}[\exp(\frac{|z|^2}{4})\langle z|\alpha\rangle] = \frac{i\alpha}{\sqrt{2}}[\exp(\frac{|z|^2}{4})\langle z|\alpha\rangle]$$

Then we have

$$\exp(\frac{|z|^2}{4})\langle z|\alpha\rangle = A\exp(\frac{i\alpha z^*}{\sqrt{2}})$$

or

$$\langle z|\alpha\rangle = N\exp(\frac{i\alpha z^*}{\sqrt{2}})\exp(-\frac{|z|^2}{4} - \frac{1}{2}|\alpha|^2)$$
$$= N\exp[-(\frac{|z|^2}{4} - \frac{i\alpha z^*}{\sqrt{2}} + \frac{1}{2}|\alpha|^2)]$$
$$= N\exp(-\frac{1}{2}|\alpha|^2)\exp(\frac{i\alpha z^*}{\sqrt{2}})\exp(-\frac{|z|^2}{4})$$

Note that the coherent state is not a linear combination of just ground-state wavefunctions, but contain excited states.

$$\left|\langle z|\alpha\rangle\right| = N^2 \exp(-|\alpha|^2)\exp(\frac{i\alpha z^*}{\sqrt{2}})\exp(-\frac{|z|^2}{4})$$

(ii)

$$\hat{b}|\beta\rangle = \beta|\beta\rangle$$

$$\langle z|\hat{b}|\beta\rangle = \langle z|\beta|\beta\rangle = \beta\langle z|\alpha\rangle$$

$$\sqrt{2}\exp(-\frac{|z|^2}{4})\frac{\partial}{\partial z}\exp(\frac{|z|^2}{4})\langle z|\beta\rangle = \beta\langle z|\beta\rangle$$

or

$$\frac{\partial}{\partial z}[\exp(\frac{|z|^2}{4})\langle z|\beta\rangle] = \frac{\beta}{\sqrt{2}}[\exp(\frac{|z|^2}{4})\langle z|\beta\rangle]$$

Then we have

$$\exp(\frac{|z|^2}{4})\langle z|\beta\rangle = \frac{1}{\sqrt{2\pi}}\exp(\frac{\beta z}{\sqrt{2}})$$

or

$$\langle z|\beta\rangle = N\exp(-\frac{1}{2}|\beta|^2)\exp(\frac{\beta z}{\sqrt{2}})\exp(-\frac{|z|^2}{4})$$

We can define the coherent state as

$$|\alpha,\beta\rangle = \exp(\alpha\hat{a}^+ - \alpha^*\hat{a})\exp(\beta\hat{b}^+ - \beta^*\hat{b})|0,0\rangle$$

$$\langle z|\alpha,\beta\rangle = N\exp[-\frac{1}{2}(|\alpha|^2+|\beta|^2)\exp(\frac{i\alpha\bar{z}_1+\beta z_2}{\sqrt{2}})\exp(-\frac{|z_1|^2+|z_2|^2}{4})$$

1.

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

$$\langle z|\alpha\rangle = \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{4}z\bar{z}+\frac{i}{\sqrt{2}}\bar{z}\alpha-\frac{1}{2}\alpha\bar{\alpha})$$

$\hat{a}^2|\alpha\rangle = \alpha^2|\alpha\rangle$, $\qquad \hat{a}^n|\alpha\rangle = \alpha^n|\alpha\rangle \qquad (n = 1, 2, 3, ...)$

$$\langle z|\frac{\hat{b}^+}{\sqrt{1!}}|\alpha\rangle = \langle z|\alpha\rangle\frac{1}{\sqrt{1!}}(\frac{z-\sqrt{2}\alpha i}{\sqrt{2}})$$

$$\langle z|\frac{\hat{b}^{+2}}{\sqrt{2!}}|\alpha\rangle = \langle z|\alpha\rangle^2\frac{1}{\sqrt{2!}}(\frac{z-\sqrt{2}\alpha i}{\sqrt{2}})^2$$

$$\langle z|\frac{\hat{b}^{+3}}{\sqrt{3!}}|\alpha\rangle = \langle z|\alpha\rangle\frac{1}{\sqrt{3!}}(\frac{z-\sqrt{2}\alpha i}{\sqrt{2}})^3$$

In general

$$\langle z|\frac{\hat{b}^{+n}}{\sqrt{n!}}|\alpha\rangle = \langle z|\alpha\rangle\frac{1}{\sqrt{n!}}(\frac{z-\sqrt{2}\alpha i}{\sqrt{2}})^n$$

2.

$$\hat{b}|\beta\rangle = \beta|\beta\rangle$$

$$\langle z|\beta\rangle = \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{4}z\bar{z}+\frac{1}{\sqrt{2}}z\beta-\frac{1}{2}\beta\bar{\beta})$$

$\hat{b}^2|\beta\rangle = \beta^2|\beta\rangle$, $\qquad \hat{b}^n|\beta\rangle = \beta^n|\beta\rangle \qquad (n = 1, 2, 3, ...)$

$$\langle z|\frac{\hat{a}^{+}}{\sqrt{1!}}|\beta\rangle=\langle z|\beta\rangle\frac{i}{\sqrt{1!}}(\frac{\bar{z}-\sqrt{2}\beta}{\sqrt{2}})$$

$$\langle z|\frac{\left(\hat{a}^{+}\right)^{2}}{\sqrt{2!}}|\beta\rangle=\langle z|\beta\rangle\frac{i^{2}}{\sqrt{2!}}(\frac{\bar{z}-\sqrt{2}\beta}{\sqrt{2}})^{2}$$

$$\langle z|\frac{\left(\hat{a}^{+}\right)^{3}}{\sqrt{3!}}|\beta\rangle=\langle z|\beta\rangle\frac{i^{3}}{\sqrt{3!}}(\frac{\bar{z}-\sqrt{2}\beta}{\sqrt{2}})^{3}$$

In general,

$$\langle z|\frac{\left(\hat{a}^{+}\right)^{n}}{\sqrt{n!}}|\beta\rangle=\langle z|\beta\rangle\frac{i^{n}}{\sqrt{n!}}(\frac{\bar{z}-\sqrt{2}\beta}{\sqrt{2}})^{n}$$

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