Translation operator and rotation operator for the 3D system

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Translation operators are linear and unitary. They are closely related to the momentum operator; for example, a translation operator that moves by an infinitesimal amount in the x direction has a simple relationship to the x-component of the momentum operator. Because of this, conservation of momentum holds when the translation operators commute with the Hamiltonian, i.e. when laws of physics are translation-invariant.

Here we discuss the properties of the translation operator for the 3D system.

1. Translation operator

The state vector:

$$\left|\psi\right\rangle = \int d\boldsymbol{r}\left|\boldsymbol{r}\right\rangle\left\langle\boldsymbol{r}\middle|\psi\right\rangle,$$

where

$$\left\langle\boldsymbol{r}\middle|\boldsymbol{r}'\right\rangle = \delta(\boldsymbol{r}-\boldsymbol{r}'),$$

$$\hat{T}(\boldsymbol{a})\left|\boldsymbol{r}\right\rangle = \left|\boldsymbol{r}+\boldsymbol{a}\right\rangle.$$

Translation operator:

$$\left|\psi\right\rangle \rightarrow \left|\psi'\right\rangle = \hat{T}(\boldsymbol{a})\left|\psi\right\rangle,$$

with

$$\hat{T}^{+}(\boldsymbol{a})\hat{T}(\boldsymbol{a}) = \hat{1}. \qquad \text{Unitary operator}$$

(a)

It is expected from the analogy of classical mechanics that

$$\left\langle\psi'\right|\hat{\boldsymbol{r}}\left|\psi'\right\rangle = \left\langle\psi\right|\hat{\boldsymbol{r}}\left|\psi\right\rangle + \left\langle\psi\right|\boldsymbol{a}\left|\psi\right\rangle.$$

or

$$\langle\psi|\hat{T}^{+}(\boldsymbol{a})\hat{\boldsymbol{r}}\hat{T}(\boldsymbol{a})|\psi\rangle = \langle\psi|\hat{\boldsymbol{r}}+\boldsymbol{a}|\psi\rangle,$$

leading to the relation

$$\hat{T}^{+}(\boldsymbol{a})\hat{\boldsymbol{r}}\hat{T}(\boldsymbol{a}) = \hat{\boldsymbol{r}}+\boldsymbol{a}\hat{1},$$

or

$$\hat{\boldsymbol{r}}\hat{T}(\boldsymbol{a}) = \hat{T}(\boldsymbol{a})\hat{\boldsymbol{r}}+\boldsymbol{a}\hat{T}(\boldsymbol{a}),$$

or

$$[\hat{\boldsymbol{r}},\hat{T}(\boldsymbol{a})] = \boldsymbol{a}\hat{T}(\boldsymbol{a}).$$

or

$$[\hat{\mathbf{r}},\hat{T}(\mathbf{a})] = i\hbar\frac{\partial}{\partial\hat{\mathbf{p}}}\hat{T}(\mathbf{a}) = \mathbf{a}\hat{T}(\mathbf{a})$$

From this differential equation, we get the form of $\hat{T}(\mathbf{a})$ as

$$\hat{T}(\mathbf{a}) = \exp(-\frac{i}{\hbar}\hat{\mathbf{p}}\cdot\mathbf{a})$$

Using the commutation relation, we get

$$\begin{aligned}\hat{\boldsymbol{r}}\hat{T}(\boldsymbol{a})|\boldsymbol{r}\rangle &= \hat{T}(\boldsymbol{a})\hat{\boldsymbol{r}}|\boldsymbol{r}\rangle+\boldsymbol{a}\hat{T}(\boldsymbol{a}))|\boldsymbol{r}\rangle \\ &= (\hat{T}(\boldsymbol{a})\boldsymbol{r}|\boldsymbol{r}\rangle+\boldsymbol{a}\hat{T}(\boldsymbol{a}))|\boldsymbol{r}\rangle, \\ &= (\boldsymbol{r}+\boldsymbol{a})\hat{T}(\boldsymbol{a})|\boldsymbol{r}\rangle\end{aligned}$$

This implies that $\hat{T}(\boldsymbol{a})|\boldsymbol{r}\rangle$ is the eigenket of $\hat{\boldsymbol{r}}$ with the eigenvalue $(\boldsymbol{r}+\boldsymbol{a})$,

$$\hat{T}(\boldsymbol{a})|\boldsymbol{r}\rangle = |\boldsymbol{r}+\boldsymbol{a}\rangle,$$

or

$$\langle \boldsymbol{r}+\boldsymbol{a}| = \langle \boldsymbol{r}|\hat{T}^{+}(\boldsymbol{a}).$$

Note that

$$|\boldsymbol{r}\rangle = \hat{T}^{+}(\boldsymbol{a})|\boldsymbol{r}+\boldsymbol{a}\rangle .$$

When $\boldsymbol{r}\rightarrow\boldsymbol{r}-\boldsymbol{a}$, we get

$$|\boldsymbol{r}-\boldsymbol{a}\rangle = \hat{T}^{+}(\boldsymbol{a})|\boldsymbol{r}\rangle ,$$

or

$$\langle \boldsymbol{r}|\hat{T}(\boldsymbol{a}) = \langle \boldsymbol{r}-\boldsymbol{a}|$$

(b)

It is also expected from the analogy of classical mechanics that

$$\langle \psi'|\hat{\boldsymbol{p}}|\psi'\rangle = \langle \psi|\hat{\boldsymbol{p}}|\psi\rangle .$$

or

$$\langle \psi|\hat{T}^{+}(\boldsymbol{a})\hat{\boldsymbol{p}}\hat{T}(\boldsymbol{a})|\psi\rangle = \langle \psi|\hat{\boldsymbol{p}}|\psi\rangle ,$$

leading to the commutation relation

$$\hat{T}^{+}(\boldsymbol{a})\hat{\boldsymbol{p}}\hat{T}(\boldsymbol{a}) = \hat{\boldsymbol{p}} , \qquad \text{or} \qquad [\hat{\boldsymbol{p}},\hat{T}(\boldsymbol{a})] = 0 .$$

Note that

$$[\hat{\mathbf{p}},\hat{T}(\mathbf{a})] = \frac{\hbar}{i}\frac{\partial}{\partial\hat{\mathbf{r}}}\hat{T}(\mathbf{a}) = 0$$

So that $\hat{T}(\mathbf{a})$ is independent of $\hat{\mathbf{r}}$.

We assume that the infinitesimal translation operator is given by

$$\hat{T}(d\boldsymbol{r}) = \hat{1} - \frac{i}{\hbar}\hat{\boldsymbol{G}}\cdot d\boldsymbol{r} ,$$

where

$$\hat{T}^+(d\boldsymbol{r})\hat{T}(d\boldsymbol{r}) = \hat{1},$$

$$\hat{T}^+(d\boldsymbol{r})\hat{\boldsymbol{r}}\hat{T}(d\boldsymbol{r}) = \hat{\boldsymbol{r}} + d\boldsymbol{r}\hat{1},$$

$$\hat{T}^+(d\boldsymbol{r})\hat{\boldsymbol{p}}\hat{T}(d\boldsymbol{r}) = \hat{\boldsymbol{p}}.$$

(a) $\hat{\boldsymbol{G}}$ is a Hermitian operator.

$$\begin{aligned}\hat{T}^+(d\boldsymbol{r})\hat{T}(d\boldsymbol{r}) &= (\hat{1} + \frac{i}{\hbar}\hat{\boldsymbol{G}}^+ \cdot d\boldsymbol{r})(\hat{1} - \frac{i}{\hbar}\hat{\boldsymbol{G}} \cdot d\boldsymbol{r}) \\ &= \hat{1} + \frac{i}{\hbar}(\hat{\boldsymbol{G}}^+ - \hat{\boldsymbol{G}}) \cdot d\boldsymbol{r} \\ &= \hat{1}\end{aligned}$$

Then we have

$$\hat{\boldsymbol{G}}^+ = \hat{\boldsymbol{G}} \qquad \text{(Hermitian operator)}$$

(b) The commutation relation (I)

$$[\hat{\boldsymbol{p}}, \hat{1} - \frac{i}{\hbar}\hat{\boldsymbol{G}} \cdot d\boldsymbol{r}] = 0,$$

or

$$[\hat{p}_\alpha, \hat{G}_\beta] = 0 \qquad \text{with } \alpha, \beta = x, y, z.$$

(c) The commutation relation (II)

$$(\hat{1} + \frac{i}{\hbar}\hat{\boldsymbol{G}} \cdot d\boldsymbol{r})\hat{\boldsymbol{r}}(\hat{1} - \frac{i}{\hbar}\hat{\boldsymbol{G}} \cdot d\boldsymbol{r}) = \hat{\boldsymbol{r}} + d\boldsymbol{r}\hat{1},$$

or

$$(\hat{\boldsymbol{G}} \cdot d\boldsymbol{r})\hat{\boldsymbol{r}} - \hat{\boldsymbol{r}}(\hat{\boldsymbol{G}} \cdot d\boldsymbol{r}) = \frac{\hbar}{i} d\boldsymbol{r}\hat{1},$$

or

$$\sum_{\alpha} [\hat{G}_{\alpha}, \hat{x}_{\beta}] dx_{\alpha} = \frac{\hbar}{i} dx_{\beta}\hat{1} = \frac{\hbar}{i}\hat{1}\sum_{\alpha} \delta_{\alpha\beta} dx_{\alpha},$$

So we get the commutation relation

$$[\hat{G}_{\alpha}, \hat{x}_{\beta}] = \frac{\hbar}{i}\hat{1}.$$

From these results, it can be concluded that

$$\hat{\boldsymbol{G}} = \hat{\boldsymbol{p}}.$$

3. Infinitesimal translation operator

$$\begin{aligned}
\hat{T}(d\boldsymbol{r})|\psi\rangle &= \hat{T}(d\boldsymbol{r})\int d\boldsymbol{r}'|\boldsymbol{r}'\rangle\langle\boldsymbol{r}'|\psi\rangle \\
&= \int d\boldsymbol{r}'\hat{T}(d\boldsymbol{r})|\boldsymbol{r}'\rangle\langle\boldsymbol{r}'|\psi\rangle \\
&= \int d\boldsymbol{r}'|\boldsymbol{r}'+d\boldsymbol{r}\rangle\langle\boldsymbol{r}'|\psi\rangle \\
&= \int d\boldsymbol{r}'|\boldsymbol{r}'\rangle\langle\boldsymbol{r}'-d\boldsymbol{r}|\psi\rangle
\end{aligned}$$

Using the Taylor expansion

$$\langle\boldsymbol{r}'-d\boldsymbol{r}|\psi\rangle = \psi(\boldsymbol{r}'-d\boldsymbol{r}) = \psi(\boldsymbol{r}') - \frac{\partial\psi(\boldsymbol{r}')}{\partial\boldsymbol{r}'}d\boldsymbol{r},$$

we have

$$\begin{aligned}
\hat{T}(d\boldsymbol{r})|\psi\rangle &= \int d\boldsymbol{r}'|\boldsymbol{r}'\rangle[\psi(\boldsymbol{r}') - \frac{\partial\psi(\boldsymbol{r}')}{\partial\boldsymbol{r}'}d\boldsymbol{r}] \\
&= \int d\boldsymbol{r}'|\boldsymbol{r}'\rangle[\langle\boldsymbol{r}'|\psi\rangle - \frac{i}{\hbar}\langle\boldsymbol{r}'|\hat{\boldsymbol{p}}|\psi\rangle \cdot d\boldsymbol{r}] \\
&= (\hat{1} - \frac{i}{\hbar}\hat{\boldsymbol{p}} \cdot d\boldsymbol{r})|\psi\rangle
\end{aligned}$$

since

$$\langle \boldsymbol{r}|\hat{\boldsymbol{p}}|\psi\rangle = \frac{\hbar}{i}\nabla_r\langle \boldsymbol{r}|\psi\rangle .$$

Then we have

$$\hat{T}(d\boldsymbol{r}) = \hat{1} - \frac{i}{\hbar}\hat{\boldsymbol{p}}\cdot d\boldsymbol{r} .$$

4. Finite translation operator

The finite translation operator is given by

$$\hat{T}(\boldsymbol{a}) = \lim_{N\to\infty}(\hat{1} - \frac{i}{\hbar}\hat{\boldsymbol{p}}\cdot\frac{\boldsymbol{a}}{N})^N = \exp(-\frac{i}{\hbar}\hat{\boldsymbol{p}}\cdot\boldsymbol{a}),$$

where we use the definition of e^{-x} as

$$e^{-x} = \lim_{N\to\infty}(1 - \frac{x}{N})^N .$$

5. Transformation function

Using the relation

$$\langle \boldsymbol{r}|\hat{\boldsymbol{p}}|\psi\rangle = \frac{\hbar}{i}\nabla\langle \boldsymbol{r}|\psi\rangle = \frac{\hbar}{i}\frac{\partial}{\partial\boldsymbol{r}}\langle \boldsymbol{r}|\psi\rangle .$$

We assume that $|\psi\rangle = |\mathbf{p}\rangle$

$$\langle \mathbf{r}|\hat{\mathbf{p}}|\mathbf{p}\rangle = \frac{\hbar}{i}\frac{\partial}{\partial\mathbf{r}}\langle \mathbf{r}|\mathbf{p}\rangle$$

or

$$\frac{\partial}{\partial\mathbf{r}}\langle \mathbf{r}|\mathbf{p}\rangle = \frac{i}{\hbar}\mathbf{p}\langle \mathbf{r}|\mathbf{p}\rangle$$

or

$$\langle \mathbf{r}|\mathbf{p}\rangle = A\exp(\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}) \qquad \text{(A; constant)}$$

We note that

$$\langle \boldsymbol{p}|\boldsymbol{p}'\rangle = \delta(\boldsymbol{p}-\boldsymbol{p}')$$

Then we have

$$\begin{aligned}\langle \mathbf{p}|\mathbf{p}'\rangle &= \int dr\langle \mathbf{p}|\mathbf{r}\rangle\langle \mathbf{r}|\mathbf{p}'\rangle \\ &= |A|^2\int d\mathbf{r}\exp[\frac{i}{\hbar}(\mathbf{p}'-\mathbf{p})\cdot\mathbf{r}] \\ &= |A|^2(2\pi)^3\delta[\frac{1}{\hbar}(\mathbf{p}'-\mathbf{p})] \\ &= |A|^2(2\pi\hbar)^3\delta(\mathbf{p}'-\mathbf{p})\end{aligned}$$

leading to

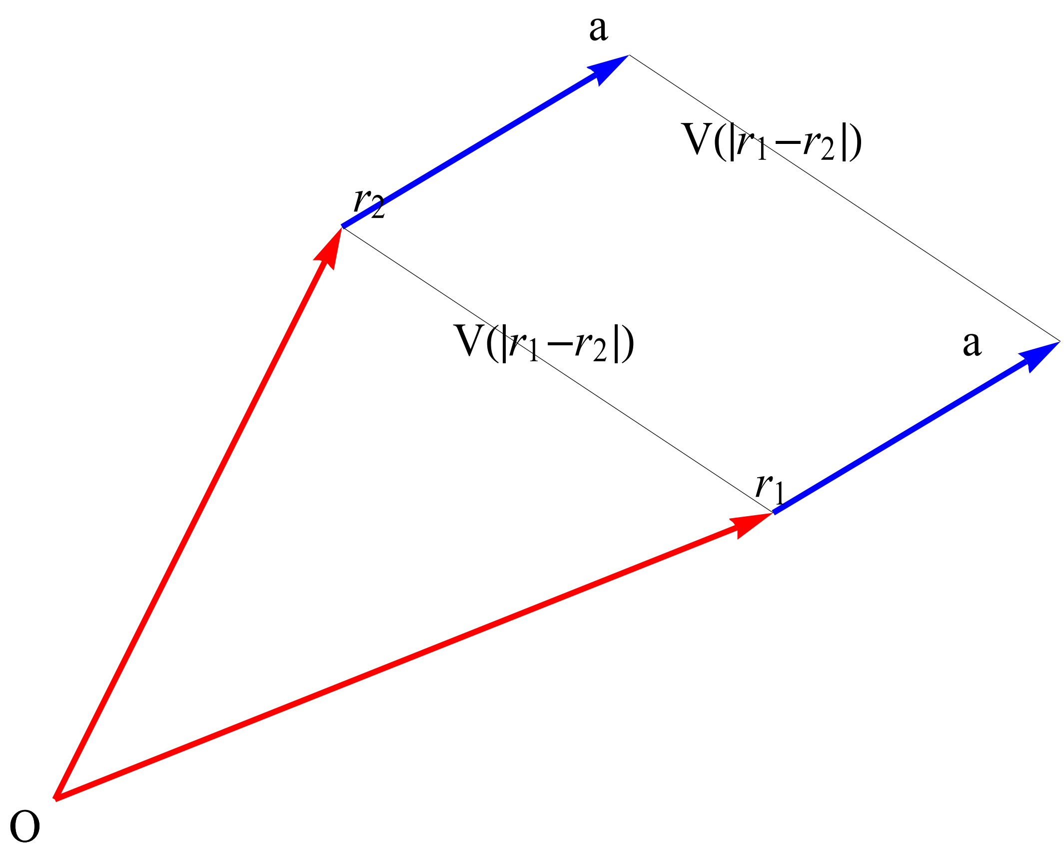
$$|A| = (2\pi\hbar)^{-3/2}$$

In conclusion, we have the transformation function as

$$\langle \boldsymbol{r}|\boldsymbol{p}\rangle = \frac{1}{(2\pi\hbar)^{3/2}}\exp(\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{r}).$$

6. Translation operator for two-body problem

We consider a Hamiltonian of two particles at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are the momentum of particles 1 and 2, respectively.



The Hamiltonian is given by

$$\hat{H} = \frac{1}{2m_1}\hat{\boldsymbol{p}}_1^{\ 2} + \frac{1}{2m_2}\hat{\boldsymbol{p}}_2^{\ 2} + V(|\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2|),$$

where $V(|\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2|)$ is the interaction between two particles with mass m_1 and m_2. This is so-called the central field problem.

((**Definition of Central-force Problem**))

In classical mechanics, the **central-force problem** is to determine the motion of a particle under the influence of a single central force. A central force is a force that points from the particle directly towards (or directly away from) a fixed point in space, the center, and whose magnitude only depends on the distance of the object to the center.

We consider the two particles (denoted by particle 1 and particle 2) located at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, respectively. The position ket vector for these two particles is expressed by

$$|\boldsymbol{r}_1, \boldsymbol{r}_2\rangle = |\boldsymbol{r}_1\rangle_1 \otimes |\boldsymbol{r}_2\rangle_2,$$

using the Kronecker product $\otimes$. Note that we have the commutation relations,

$$[\hat{x}_{1i}, \hat{p}_{1j}] = i\hbar\delta_{ij}, \qquad [\hat{x}_{2i}, \hat{p}_{2j}] = i\hbar\delta_{ij},$$

$$[\hat{x}_{1i}, \hat{p}_{2j}] = 0, \quad [\hat{x}_{2i}, \hat{p}_{1j}] = 0.$$

which means that the operators for particle 1 and particle 2 are completely independent each other.

The total momentum is defined by

$$\hat{\boldsymbol{P}} = \hat{\boldsymbol{p}}_1 + \hat{\boldsymbol{p}}_2.$$

Using the closure relation, we can define the wave function as

$$|\psi\rangle = \int d\boldsymbol{r}_1 d\boldsymbol{r}_2 |\boldsymbol{r}_1, \boldsymbol{r}_2\rangle\langle\boldsymbol{r}_1, \boldsymbol{r}_2|\psi\rangle.$$

We now show that

$$[V(|\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2|), \hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a})] = 0$$

((**Proof**))

$$\begin{aligned}\hat{T}_1(\mathbf{a})\hat{T}_2(\mathbf{a})V(|\hat{\mathbf{r}}_1 - \hat{\mathbf{r}}_2|)|\mathbf{r}_1, \mathbf{r}_2\rangle &= V(|\mathbf{r}_1 - \mathbf{r}_2|)\hat{T}_1(\mathbf{a})\hat{T}_2(\mathbf{a})|\mathbf{r}_1, \mathbf{r}_2\rangle \\ &= V(|\mathbf{r}_1 - \mathbf{r}_2|)|\mathbf{r}_1 + \mathbf{a}, \mathbf{r}_2 + \mathbf{a}\rangle\end{aligned}$$

$$\begin{aligned}V(|\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2|)\hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a})|\boldsymbol{r}_1, \boldsymbol{r}_2\rangle &= V(|\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2|)|\boldsymbol{r}_1 + \boldsymbol{a}, \boldsymbol{r}_2 + \boldsymbol{a}\rangle \\ &= V(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)|\boldsymbol{r}_1 + \boldsymbol{a}, \boldsymbol{r}_2 + \boldsymbol{a}\rangle\end{aligned}.$$

Then we have

$$\hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a})V(|\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2|)|\boldsymbol{r}_1, \boldsymbol{r}_2\rangle = V(|\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2|)\hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a})|\boldsymbol{r}_1, \boldsymbol{r}_2\rangle$$

or

$$[\hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a}), V(|\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2|)] = 0.$$

Since

$$[\hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a}), \frac{1}{2m_1}\hat{\boldsymbol{p}}_1{}^2 + \frac{1}{2m_1}\hat{\boldsymbol{p}}_1{}^2] = 0,$$

we get the commutation relation

$$[\hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a}), \hat{H}] = 0.$$

This means that there is a simultaneous eigenket $|\psi\rangle$ of both $\hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a})$ and $\hat{H}$, such that

$$H|\psi\rangle = E|\psi\rangle, \qquad \hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a})|\psi\rangle = \lambda|\psi\rangle.$$

We also note that

$$\hat{T}_1(\delta\boldsymbol{a})\hat{T}_2(\delta\boldsymbol{a}) = \exp(-\frac{i}{\hbar}\hat{\boldsymbol{P}}\cdot\delta\boldsymbol{a}) = \hat{1} - \frac{i}{\hbar}\hat{\boldsymbol{P}}\cdot\delta\boldsymbol{a}.$$

For any $\delta\boldsymbol{a}$, we have

$$[\hat{H}, \hat{\boldsymbol{P}}] = 0,$$

leading to

$$\frac{d}{dt}\langle\psi(t)|\hat{\boldsymbol{P}}|\psi(t)\rangle = \frac{i}{\hbar}\langle\psi(t)|[\hat{H}, \hat{\boldsymbol{P}}]|\psi(t)\rangle = 0. \qquad \text{(Ehrenfest theorem)}$$

This implies the conservation of the total momentum. In other words, the total momentum is a constant of motion.

((**In summary**))

What is the physical meaning of the above result?

From $H|\psi\rangle = E|\psi\rangle$, the wave function of the Schrödinger equation is given by

$$\psi(\boldsymbol{r}_1, \boldsymbol{r}_2) = \langle \boldsymbol{r}_1, \boldsymbol{r}_2|\psi\rangle.$$

From $\hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a})|\psi\rangle = \lambda|\psi\rangle$, we have

$$\langle \boldsymbol{r}_1, \boldsymbol{r}_2 | \hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a})|\psi\rangle = \langle \boldsymbol{r}_1 - \boldsymbol{a}, \boldsymbol{r}_2 - \boldsymbol{a}|\psi\rangle = \psi(\boldsymbol{r}_1 - \boldsymbol{a}, \boldsymbol{r}_2 - \boldsymbol{a}) = \lambda\psi(\boldsymbol{r}_1, \boldsymbol{r}_2).$$

Suppose that $\boldsymbol{r}_2 - \boldsymbol{a} = 0$ ($\boldsymbol{a}$ can be chosen arbitrarily). Then we get

$$\psi(\boldsymbol{r}_1 - \boldsymbol{r}_2, 0) = \lambda\psi(\boldsymbol{r}_1, \boldsymbol{r}_2).$$

In other words, the wave function $\psi(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is only dependent on the relative coordinate

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2, \qquad \psi(\boldsymbol{r}_1, \boldsymbol{r}_2) = \psi(\boldsymbol{r}).$$

The total momentum is a constant of motion. $\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2$

((**Note**))
The use of Kronecker product for the translation operators. (see the **APPENDIX**).

Instead of the use of $\hat{T}_1(\mathbf{a})\hat{T}_2(\mathbf{a})$, we can it using the Kronecker product

$$\hat{T}_1(\mathbf{a}) \otimes \hat{T}_2(\mathbf{a}) = \exp[-\frac{i}{\hbar}(\hat{\mathbf{p}}_1 \otimes \hat{1} + \hat{1} \otimes \hat{\mathbf{p}}_2)\cdot\mathbf{a}],$$

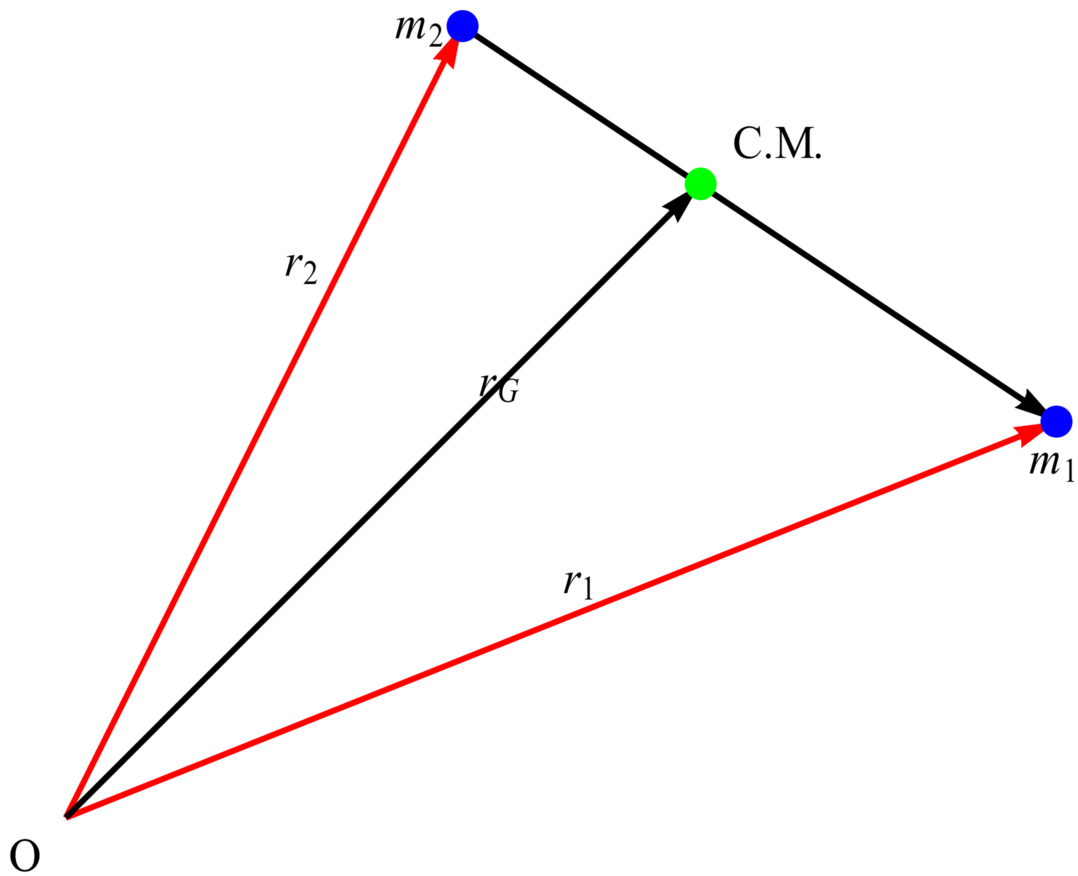
where we define

$$\hat{\mathbf{P}} = \hat{\mathbf{p}}_1 \otimes \hat{1} + \hat{1} \otimes \hat{\mathbf{p}}_2,$$

instead of using the expression of

$$\hat{\mathbf{P}} = \hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2.$$

6. Two-body problems: Classical mechanics



Lagrangian:

$$L = \frac{1}{2} m_1 \left(\frac{d\boldsymbol{r}_1}{dt}\right)^2 + \frac{1}{2} m_2 \left(\frac{d\boldsymbol{r}_2}{dt}\right)^2 - V(|\boldsymbol{r}_1 - \boldsymbol{r}_2|),$$

$$\boldsymbol{p}_1 = m_1 \frac{d\boldsymbol{r}_1}{dt}, \qquad \boldsymbol{p}_2 = m_2 \frac{d\boldsymbol{r}_2}{dt}.$$

Center of mass:

$$\boldsymbol{r}_G = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2}.$$

Relative coordinate:

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2,$$

$$\boldsymbol{r}_1 = \boldsymbol{r}_G + \frac{m_2 \boldsymbol{r}}{m_1 + m_2}, \qquad \boldsymbol{r}_2 = \boldsymbol{r}_G - \frac{m_1 \boldsymbol{r}}{m_1 + m_2}.$$

The Lagrangian L can be written in terms of $\boldsymbol{r}_G$ and $\boldsymbol{r}$

$$L(\boldsymbol{r}, \dot{\boldsymbol{r}}, \boldsymbol{r}_G) = \frac{1}{2} m_1 (\dot{\boldsymbol{r}}_G + \frac{m_2}{m_1 + m_2} \dot{\boldsymbol{r}})^2 + \frac{1}{2} m_2 (\dot{\boldsymbol{r}}_G - \frac{m_1}{m_1 + m_2} \dot{\boldsymbol{r}})^2 - V(|\boldsymbol{r}|),$$

or

$$L(\boldsymbol{r}, \dot{\boldsymbol{r}}, \boldsymbol{r}_G) = \frac{1}{2} M \dot{\boldsymbol{r}}_G{}^2 + \frac{1}{2} \mu \dot{\boldsymbol{r}}^2 - V(|\boldsymbol{r}|)$$

where the total mass is defined by

$$M = m_1 + m_2,$$

and the reduced mass is defined by

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \qquad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}.$$

Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}}\right) = \frac{\partial L}{\partial \boldsymbol{r}}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}_G}\right) = \frac{\partial L}{\partial \boldsymbol{r}_G} = 0.$$

Since $L(\boldsymbol{r}, \dot{\boldsymbol{r}}, \boldsymbol{r}_G)$ is independent of $\boldsymbol{r}_G$, we find that the conjugate momentum

$$\boldsymbol{p}_G = \frac{\partial L}{\partial \dot{\boldsymbol{r}}_G} = M \dot{\boldsymbol{r}}_G = m_1 \dot{\boldsymbol{r}}_1 + m_2 \dot{\boldsymbol{r}}_2 = \boldsymbol{p}_1 + \boldsymbol{p}_2,$$

is a cyclic (time-independent) (which means the momentum conservation because of no external force). The conjugate momentum is given by

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{r}}} = \mu \dot{\boldsymbol{r}} = \mu(\dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{r}}_2).$$

or

$$\boldsymbol{p} = \mu(\frac{\boldsymbol{p}_1}{m_1} - \frac{\boldsymbol{p}_2}{m_2})$$
$$= \frac{m_1 m_2}{m_1 + m_2}(\frac{m_2\boldsymbol{p}_1 - m_1\boldsymbol{p}_2}{m_1 m_2})$$
$$= \frac{m_2\boldsymbol{p}_1 - m_1\boldsymbol{p}_2}{m_1 + m_2}$$

where

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{r}}_2 = \frac{1}{m_1}\boldsymbol{p}_1 - \frac{1}{m_2}\boldsymbol{p}_2 = \frac{m_2\boldsymbol{p}_1 - m_1\boldsymbol{p}_2}{m_1 m_2},$$

Since the momentum of the center of mass is given by

$$\boldsymbol{p}_G = \boldsymbol{p}_1 + \boldsymbol{p}_2,$$

we get

$$\boldsymbol{p}_1 = \boldsymbol{p} + \frac{m_1}{m_1 + m_2}\boldsymbol{p}_G, \qquad \boldsymbol{p}_2 = -\boldsymbol{p} + \frac{m_2}{m_1 + m_2}\boldsymbol{p}_G.$$

The Hamiltonian H can be written as

$$H = \boldsymbol{p}_G \cdot \dot{\boldsymbol{r}}_G + \boldsymbol{p} \cdot \dot{\boldsymbol{r}} - L$$
$$= \boldsymbol{p}_G \cdot \dot{\boldsymbol{r}}_G + \boldsymbol{p} \cdot \dot{\boldsymbol{r}} - [\frac{1}{2}M\dot{\boldsymbol{r}}_G{}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 - V(|\boldsymbol{r}|)].$$
$$= \frac{\boldsymbol{p}_G{}^2}{2M} + \frac{\boldsymbol{p}^2}{2\mu} + V(|\boldsymbol{r}|)$$

where

$$\dot{\boldsymbol{r}}_G = \frac{1}{M}\boldsymbol{p}_G, \qquad \dot{\boldsymbol{r}} = \frac{1}{\mu}\boldsymbol{p}$$

The total orbital angular momentum:

$$
\begin{aligned}
\boldsymbol{L}_T &= \boldsymbol{L}_1 + \boldsymbol{L}_2 \\
&= \boldsymbol{r}_1 \times \boldsymbol{p}_1 + \boldsymbol{r}_2 \times \boldsymbol{p}_2 \\
&= (\boldsymbol{r}_G + \frac{m_2 \boldsymbol{r}}{m_1 + m_2}) \times (\boldsymbol{p} + \frac{m_1}{m_1 + m_2} \boldsymbol{p}_G) + (\boldsymbol{r}_G - \frac{m_1 \boldsymbol{r}}{m_1 + m_2}) \times (-\boldsymbol{p} + \frac{m_2}{m_1 + m_2} \boldsymbol{p}_G) \\
&= \boldsymbol{r}_G \times \boldsymbol{p}_G + \boldsymbol{r} \times \boldsymbol{p}
\end{aligned}
$$

or

$$\boldsymbol{L}_T = \boldsymbol{L}_G + \boldsymbol{L}$$

with

$$\boldsymbol{L}_G = \boldsymbol{r}_G \times \boldsymbol{p}_G . \qquad \boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$

7. Quantum Kepler problem

We now consider the quantum mechanics of the central force problem.

(i) The relative co-ordinate operator:

$$\hat{\boldsymbol{r}} = \hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2 ,$$

(ii) The relative momentum operator:

$$\hat{\boldsymbol{p}} = \frac{m_2 \hat{\boldsymbol{p}}_1 - m_1 \hat{\boldsymbol{p}}_2}{m_1 + m_2} .$$

(iii) The co-ordinate operator for the center of mass:

$$\hat{\boldsymbol{r}}_G = \frac{m_1 \hat{\boldsymbol{r}}_1 + m_2 \hat{\boldsymbol{r}}_2}{m_1 + m_2} .$$

(iv) The momentum operator for the center of mass:

$$\hat{\boldsymbol{p}}_G = \hat{\boldsymbol{p}}_1 + \hat{\boldsymbol{p}}_2 .$$

Note that $\hat{\boldsymbol{p}}_G = \hat{\boldsymbol{p}}_1 + \hat{\boldsymbol{p}}_2 = \hat{\boldsymbol{P}}$ (total momentum)

(v) The total angular momentum operator for the system:

$$\hat{\boldsymbol{L}}_T = \hat{\boldsymbol{L}}_G + \hat{\boldsymbol{L}} ,$$

with

$$\hat{\boldsymbol{L}}_G = \hat{\boldsymbol{r}}_G \times \hat{\boldsymbol{p}}_G .$$

$$\hat{\boldsymbol{L}} = \hat{\boldsymbol{r}} \times \hat{\boldsymbol{p}} . \qquad \text{(internal angular momentum)}$$

The reduced mass is defined as

$$\mu = \frac{m_1 m_2}{m_1 + m_2} .$$

8. The commutation relation:

We assume that

$$[\hat{x}_{1i}, \hat{x}_{1j}] = 0 , \qquad [\hat{x}_{2i}, \hat{x}_{2j}] = 0 ,$$

$$[\hat{p}_{1i}, \hat{p}_{1j}] = 0 , \qquad [\hat{p}_{2i}, \hat{p}_{2j}] = 0 ,$$

We note that

$$[\hat{x}_{1i}, \hat{p}_{1j}] = i\hbar\delta_{ij} , \qquad [\hat{x}_{2i}, \hat{p}_{2j}] = i\hbar\delta_{ij} ,$$

for the same particle, and

$$[\hat{x}_{1i}, \hat{p}_{2j}] = 0 , \qquad [\hat{x}_{2i}, \hat{p}_{1j}] = 0 ,$$

$$[\hat{x}_{1i}, \hat{x}_{2j}] = 0 , \qquad [\hat{p}_{1i}, \hat{p}_{2j}] = 0 ,$$

for the different particles, where $i = x, y, z$, and $j = x, y, z$.

Based on the above relations, we discuss the commutation relations between $\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}}, \hat{\boldsymbol{r}}_G, \hat{\boldsymbol{p}}_G$, as follows.

$$
\begin{aligned}
[\hat{x}_i, \hat{p}_j] &= [\hat{x}_{1i} - \hat{x}_{2i}, \frac{m_2 \hat{p}_{1j} - m_1 \hat{p}_{2j}}{m_1 + m_2}] \\
&= \frac{m_2}{m_1 + m_2}[\hat{x}_{1i}, \hat{p}_{1j}] + \frac{m_1}{m_1 + m_2}[\hat{x}_{2i}, \hat{p}_{2j}] \\
&= i\hbar \delta_{ij} \hat{1}
\end{aligned}
$$

$$
\begin{aligned}
[\hat{x}_i, \hat{p}_{Gj}] &= [\hat{x}_{1i} - \hat{x}_{2i}, \hat{p}_{1j} + \hat{p}_{2j}] \\
&= [\hat{x}_{1i}, \hat{p}_{1j}] - [\hat{x}_{2i}, \hat{p}_{2j}] \\
&= i\hbar \delta_{ij} \hat{1} - i\hbar \delta_{ij} \hat{1} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
[\hat{x}_{Gi}, \hat{p}_{Gj}] &= [\frac{m_1 \hat{x}_{1i} + m_2 \hat{x}_{2i}}{m_1 + m_2}, \hat{p}_{1j} + \hat{p}_{2j}] \\
&= \frac{m_1}{m_1 + m_2}[\hat{x}_{1i}, \hat{p}_{1j}] + \frac{m_2}{m_1 + m_2}[\hat{x}_{2i}, \hat{p}_{2j}] \\
&= i\hbar \delta_{ij} \hat{1}
\end{aligned}
$$

$$
\begin{aligned}
[\hat{x}_{Gi}, \hat{p}_j] &= [\frac{m_1 \hat{x}_{1i} + m_2 \hat{x}_{2i}}{m_1 + m_2}, \frac{m_2 \hat{p}_{1j} - m_1 \hat{p}_{2j}}{m_1 + m_2}] \\
&= \frac{m_1 m_2}{m_1 + m_2}[\hat{x}_{1i}, \hat{p}_{1j}] - \frac{m_1 m_2}{m_1 + m_2}[\hat{x}_{2i}, \hat{p}_{2j}] \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
[\hat{p}_{Gi}, \hat{p}_j] &= [\hat{p}_{1i} + \hat{p}_{2i}, \frac{m_2 \hat{p}_{1j} - m_1 \hat{p}_{2j}}{m_1 + m_2}] \\
&= \frac{m_2}{m_1 + m_2}[\hat{p}_{1i}, \hat{p}_{1j}] - \frac{m_{12}}{m_1 + m_2}[\hat{p}_{2i}, \hat{p}_{2j}] \\
&= 0
\end{aligned}
$$

We note that the original Hamiltonian

$$
\hat{H} = \frac{1}{2m_1}\hat{\boldsymbol{p}}_1^{\,2} + \frac{1}{2m_2}\hat{\boldsymbol{p}}_2^{\,2} + V(|\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2|),
$$

can be rewritten as

$$\hat{H} = \hat{H}_G + \hat{H}_{rel} = \frac{\hat{\boldsymbol{p}}_G{}^2}{2M} + \frac{\hat{\boldsymbol{p}}^2}{2\mu} + V(|\hat{\boldsymbol{r}}|).$$

with

$$\hat{H}_{rel} = \frac{\hat{\boldsymbol{p}}^2}{2\mu} + V(|\hat{\boldsymbol{r}}|).$$

((**Mathematica**))
Using the commutation relations, we can directly show that

$$\frac{1}{2m_1}\hat{\boldsymbol{p}}_1{}^2 + \frac{1}{2m_2}\hat{\boldsymbol{p}}_2{}^2 = \frac{\hat{\boldsymbol{p}}_G{}^2}{2M} + \frac{\hat{\boldsymbol{p}}^2}{2\mu}.$$

```
Clear["Global`*"]; p1 = {p1x, p1y, p1z};
p2 = {p2x, p2y, p2z}; μ = (m1 m2)/(m1 + m2); M1 = m1 + m2;
p = (m2 p1 - m1 p2)/(m1 + m2);
pG = p1 + p2;

K1 = pG.pG/(2 M1) + p.p/(2 μ) // FullSimplify;
K2 = p1.p1/(2 m1) + p2.p2/(2 m2) // Simplify;

K1 - K2 // Simplify

0
```

9. Reduction of the two-body problem

We note that

$$[\hat{\boldsymbol{p}}_G, \hat{H}_{rel}] = 0,$$

and

$$[\hat{H}, \hat{\boldsymbol{p}}_G] = [\hat{H}_G + \hat{H}_{rel}, \hat{\boldsymbol{p}}_G] = [\hat{H}_{rel}, \hat{\boldsymbol{p}}_G] = 0.$$

Then $\hat{H}_{rel}$ and $\hat{\boldsymbol{p}}_G$ can all be simultaneously diagonalized. In other words, there exists a simultaneous eigenstate $|\boldsymbol{p}_G, E_r\rangle$.

$$\hat{H}_G\,|\boldsymbol{p}_G, E_r\rangle = E_G\,|\boldsymbol{p}_G, E_r\rangle, \qquad \hat{H}_{rel}|\boldsymbol{p}_G, E_r\rangle = E_r|\boldsymbol{p}_G, E_r\rangle,$$

and

$$\hat{H}|\boldsymbol{p}_G, E_r\rangle = (\hat{H}_G + \hat{H}_{rel})|\boldsymbol{p}_G, E_r\rangle = (E_G + E_r)|\boldsymbol{p}_G, E_r\rangle.$$

We note that

$$\hat{H}_G|\boldsymbol{p}_G\rangle = \frac{\boldsymbol{p}_G{}^2}{2M}|\boldsymbol{p}_G\rangle = E_G|\boldsymbol{p}_G\rangle,$$

where

$$E_G = \frac{\boldsymbol{p}_G{}^2}{2M}.$$

The wave function can be described by

$$|\psi\rangle = |\boldsymbol{p}_G\rangle \otimes |E_r\rangle = |\boldsymbol{p}_G\rangle|\psi_r\rangle, \qquad \text{or} \qquad |\psi\rangle = |\boldsymbol{p}_G, E_r\rangle$$

where

$$|E_r\rangle = |\psi_r\rangle.$$

10. The representation of $|\boldsymbol{r}_G, \boldsymbol{r}\rangle = |\boldsymbol{r}_G\rangle \otimes |\boldsymbol{r}\rangle$

Based on the commutation relations,

$$[\hat{x}_{Gi}, \hat{p}_{Gj}] = i\hbar\delta_{ij}\hat{1}, \qquad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}\hat{1},$$

we can use the basis

$$|\boldsymbol{r}_G, \boldsymbol{r}\rangle = |\boldsymbol{r}_G\rangle \otimes |\boldsymbol{r}\rangle,$$

for both the center-of mass co-ordinate and relative co-ordinate, corresponding to the basis for the momentum basis

$$|\boldsymbol{p}_G,\boldsymbol{p}\rangle = |\boldsymbol{p}_G\rangle \otimes |\boldsymbol{p}\rangle .$$

The transformation functions are defined by

$$\langle \boldsymbol{r}_G | \boldsymbol{p}_G \rangle = \frac{1}{(2\pi\hbar)^{3/2}} \exp(\frac{i}{\hbar}\boldsymbol{p}_G \cdot \boldsymbol{r}_G) ,$$

and

$$\langle \boldsymbol{r} | \boldsymbol{p} \rangle = \frac{1}{(2\pi\hbar)^{3/2}} \exp(\frac{i}{\hbar}\boldsymbol{p} \cdot \boldsymbol{r}) .$$

The wave function in the position representation can be described by

$$|\psi\rangle = |\boldsymbol{p}_G\rangle |E_r\rangle = |\boldsymbol{p}_G\rangle |\psi_r\rangle .$$

The representation of the wave function in the positional representation

$$\langle \boldsymbol{r}_G, \boldsymbol{r} | \psi \rangle = \langle \boldsymbol{r}_G | \boldsymbol{p}_G \rangle \langle r | \psi_r \rangle = \frac{1}{(2\pi\hbar)^{3/2}} \exp(\frac{i}{\hbar}\boldsymbol{p}_G \cdot \boldsymbol{r}_G) \langle \boldsymbol{r} | \psi_r \rangle .$$

11. Ehrenfest theorem for $\langle \hat{\boldsymbol{p}}_G \rangle$

We note that

$$[\hat{H}, \hat{\boldsymbol{p}}_G] = 0 .$$

From the Ehrenfest theorem, we have

$$\frac{d}{dt}\langle \hat{\boldsymbol{p}}_G \rangle = \frac{1}{i\hbar}\langle [\hat{\boldsymbol{p}}_G, \hat{H}] \rangle = 0 ,$$

leading to $\langle \hat{\boldsymbol{p}}_G \rangle$=constant of motion. For simplicity, we assume that

$$\hat{\boldsymbol{p}}_G = 0 .$$

The we have the final form of the Hamiltonian as

$$\hat{H} = \hat{H}_{rel} = \frac{\hat{\boldsymbol{p}}^2}{2\mu} + V(\hat{\boldsymbol{r}}).$$

The Schrodinger equation is given by

$$[\frac{\hat{\boldsymbol{p}}^2}{2\mu} + V(\hat{\boldsymbol{r}})]|\psi_r\rangle = E_r|\psi_r\rangle$$

or

$$[-\frac{\hbar^2}{2\mu}\nabla_r{}^2 + V(\boldsymbol{r})]\langle\boldsymbol{r}|\psi_r\rangle = E_r\langle\boldsymbol{r}|\psi_r\rangle.$$

12. Rotation operator in Quantum mechanics

After the geometrical rotation;

$$\boldsymbol{r} \to \Re\boldsymbol{r} = \boldsymbol{r}' , \qquad \text{(geometrical rotation)}$$

we assume that the state vector changes from the old state $|\psi\rangle$ to the new state $|\psi'\rangle$.

$$|\psi'\rangle = \hat{R}|\psi\rangle ,$$

or

$$\langle\psi'| = \langle\psi|\hat{R}^+ ,$$

where $\hat{R}$ is a rotation operator in the quantum mechanics. It is natural to assume that

$$\langle\psi'|\hat{\boldsymbol{r}}|\psi'\rangle = \langle\psi|\hat{\boldsymbol{r}}'|\psi\rangle = \langle\psi|\Re\hat{\boldsymbol{r}}|\psi\rangle ,$$

or

$$\langle\psi|\hat{R}^+\hat{\boldsymbol{r}}\hat{R}|\psi\rangle = \langle\psi|\Re\hat{\boldsymbol{r}}|\psi\rangle ,$$

or

$$\hat{R}^{+}\hat{\boldsymbol{r}}\hat{R} = \Re\hat{\boldsymbol{r}}. \qquad (1)$$

The rotation operator is a unitary operator.

$$\langle\psi'|\psi'\rangle = \langle\psi|\psi\rangle,$$

or

$$\hat{R}^{+}\hat{R} = \hat{R}\hat{R}^{+} = \hat{1} \text{ (Unitary operator)}$$

From Eq. (1),

$$\hat{\boldsymbol{r}}\hat{R} = \hat{R}\Re\hat{\boldsymbol{r}}.$$

Here we calculate

$$\hat{\boldsymbol{r}}\hat{R}|\boldsymbol{r}\rangle = \hat{R}\Re\hat{\boldsymbol{r}}|\boldsymbol{r}\rangle = \hat{R}\Re\boldsymbol{r}|\boldsymbol{r}\rangle = \Re\boldsymbol{r}\hat{R}|\boldsymbol{r}\rangle.$$

$\hat{R}|\boldsymbol{r}\rangle$ is the eigenket of $\hat{\boldsymbol{r}}$ with the eigenvalue $\Re\boldsymbol{r}$. So that we can write

$$\hat{R}|\boldsymbol{r}\rangle = |\Re\boldsymbol{r}\rangle.$$

When

$$\Re\boldsymbol{r} = \boldsymbol{r}_0,$$

or

$$\boldsymbol{r} = \Re^{-1}\boldsymbol{r}_0,$$

$$\hat{R}|\Re^{-1}\boldsymbol{r}_0\rangle = |\boldsymbol{r}_0\rangle,$$

or

$$\left|\Re^{-1}\boldsymbol{r}_0\right\rangle = \hat{R}^{+}\left|\boldsymbol{r}_0\right\rangle.$$

For any $\boldsymbol{r}$,

$$\left|\Re^{-1}\boldsymbol{r}\right\rangle = \hat{R}^{+}\left|\boldsymbol{r}\right\rangle,$$

$$\hat{R}\hat{R}^{+}\left|\boldsymbol{r}\right\rangle = \hat{R}\left|\Re^{-1}\boldsymbol{r}\right\rangle = \left|\Re\Re^{-1}\boldsymbol{r}\right\rangle = \left|\boldsymbol{r}\right\rangle.$$

In summary, we have

(1) $\hat{R}^{+}\hat{R} = \hat{R}\hat{R}^{+} = \hat{1}$.

(2) $\hat{R}\left|\boldsymbol{r}\right\rangle = \left|\Re\boldsymbol{r}\right\rangle$.

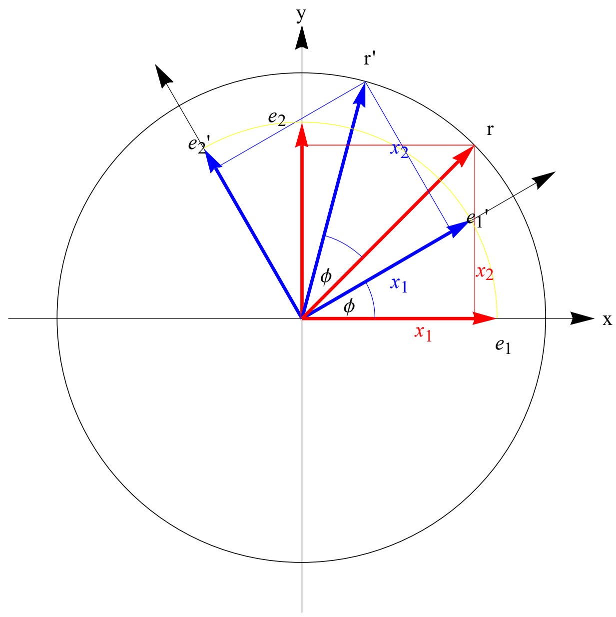
(3) $\left\langle\boldsymbol{r}\right|\hat{R}^{+} = \left\langle\Re\boldsymbol{r}\right|$.

(4) $\hat{R}^{+}\left|\boldsymbol{r}\right\rangle = \left|\Re^{-1}\boldsymbol{r}\right\rangle$.

(5) $\left\langle\boldsymbol{r}\right|\hat{R} = \left\langle\Re^{-1}\boldsymbol{r}\right|$.

13. Rotation matrix

Suppose that the vector $\boldsymbol{r}$ is rotated through θ (counter-clock wise) around the z axis. The position vector $\boldsymbol{r}$ is changed into $\boldsymbol{r}'$ in the same orthogonal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$.



In this Fig, we have

$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_1' = \cos\phi \qquad \boldsymbol{e}_2 \cdot \boldsymbol{e}_1' = \sin\phi$$
$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_2' = -\sin\phi, \qquad \boldsymbol{e}_2 \cdot \boldsymbol{e}_2' = \cos\phi$$

We define ***r*** and ***r'*** as

$$\boldsymbol{r}' = x_1'\boldsymbol{e}_1 + x_2'\boldsymbol{e}_2 = x_1\boldsymbol{e}_1' + x_2\boldsymbol{e}_2',$$

and

$$\boldsymbol{r} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2.$$

Using the relation

$$\boldsymbol{e}_1 \cdot \boldsymbol{r}' = \boldsymbol{e}_1 \cdot (x_1'\boldsymbol{e}_1 + x_2'\boldsymbol{e}_2) = \boldsymbol{e}_1 \cdot (x_1\boldsymbol{e}_1' + x_2\boldsymbol{e}_2')$$
$$\boldsymbol{e}_2 \cdot \boldsymbol{r}' = \boldsymbol{e}_2 \cdot (x_1'\boldsymbol{e}_1 + x_2'\boldsymbol{e}_2) = \boldsymbol{e}_2 \cdot (x_1\boldsymbol{e}_1' + x_2\boldsymbol{e}_2')$$

we have

$$x_1' = \mathbf{e}_1 \cdot (x_1\mathbf{e}_1' + x_2\mathbf{e}_2') = x_1\cos\phi - x_2\sin\phi$$
$$x_2' = \mathbf{e}_2 \cdot (x_1\mathbf{e}_1' + x_2\mathbf{e}_2') = x_1\sin\phi + x_2\cos\phi$$

or including the x_3 axis,

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \Re_z(\phi) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

We note that

$$\Re_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$\Re_z^{-1}(\phi) = \begin{pmatrix} \cos(-\phi) & -\sin(-\phi) & 0 \\ \sin(-\phi) & \cos(-\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

---

14. Infinitesimal rotation matrix around the *z* axis

We assume that $\phi = d\alpha$ (infinitesimally small angle);

$$\boldsymbol{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \Re_z^{-1}(d\alpha)\boldsymbol{r}$$

$$= \begin{pmatrix} \cos(d\alpha) & \sin(d\alpha) & 0 \\ -\sin(d\alpha) & \cos(d\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \approx \begin{pmatrix} 1 & d\alpha & 0 \\ -d\alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} x + y d\alpha \\ -x d\alpha + y \\ z \end{pmatrix}$$

or

$$x' = x + y d\alpha$$
$$y' = y - x d\alpha$$
$$z' = z$$

Then we have

$$\begin{aligned}
\langle \boldsymbol{r} | \psi' \rangle &= \langle \boldsymbol{r} | \hat{R}_z(d\alpha) | \psi \rangle \\
&= \langle \Re_z^{-1}(d\alpha)\boldsymbol{r} | \psi \rangle \\
&= \langle x + y d\alpha, y - x d\alpha, z | \psi \rangle \\
&= \psi(x + y d\alpha, y - x d\alpha, z) \\
&= \psi(x, y, z) - d\alpha(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})\psi(x, y, z) \\
&= \psi + d\alpha(y\frac{\partial \psi}{\partial x} - x\frac{\partial \psi}{\partial y}) \\
&= \langle \boldsymbol{r} | \hat{1} - \frac{i}{\hbar} d\alpha \hat{L}_z | \psi \rangle
\end{aligned}$$

where we use the Taylor expansion and the angular (orbital) momentum is defined by

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x .$$

Then we have the expression of the infinitesimal rotation operator as

$$\hat{R}_z(d\alpha) = \hat{1} - \frac{i}{\hbar} d\alpha \hat{L}_z .$$

((**Note**))

$$\langle \boldsymbol{r} | (\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) | \psi \rangle = \langle r | \hat{L}_z | \psi \rangle = \frac{\hbar}{i}(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) \langle \boldsymbol{r} | \psi \rangle .$$

15. Positional-space representation of *L* in spherical co-ordinates

We also use the ket vector $|\boldsymbol{r}\rangle = |r, \theta, \phi\rangle$, where r, θ, and ϕ are the spherical coordinates.

$$\hat{R}_z(d\alpha) | r, \theta, \phi \rangle = | r, \theta, \phi + d\alpha \rangle ,$$

$$\hat{R}_z^{+}(d\alpha)\left|r,\theta,\phi\right\rangle = \left|r,\theta,\phi - d\alpha\right\rangle.$$

$$\left\langle r,\theta,\phi - d\alpha\right| = \left\langle r,\theta,\phi\right|\hat{R}_z(d\alpha).$$

thus we have

$$\left\langle r,\theta,\phi\right|\hat{R}_z(d\alpha)\left|\psi\right\rangle = \left\langle r,\theta,\phi - d\alpha\middle|\psi\right\rangle = \left\langle r,\theta,\phi\middle|\psi\right\rangle - d\alpha\frac{\partial}{\partial\phi}\left\langle r,\theta,\phi\middle|\psi\right\rangle$$

On the other hand, we get

$$\left\langle r,\theta,\phi\right|\hat{R}_z(d\alpha)\left|\psi\right\rangle = \left\langle r,\theta,\phi\right|\hat{1} - \frac{i}{\hbar}\hat{L}_z d\alpha\left|\psi\right\rangle = \left\langle r,\theta,\phi\middle|\psi\right\rangle - \frac{i}{\hbar}d\alpha\left\langle r,\theta,\phi\right|\hat{L}_z\left|\psi\right\rangle$$

Then we have

$$\left\langle r,\theta,\phi\right|\hat{L}_z\left|\psi\right\rangle = \frac{\hbar}{i}\frac{\partial}{\partial\phi}\left\langle r,\theta,\phi\middle|\psi\right\rangle$$

or

$$L_z\frac{\partial}{\partial\phi}\psi(\mathbf{r}) = \frac{\hbar}{i}\frac{\partial}{\partial\phi}\psi(\mathbf{r}).$$

16. Finite rotation

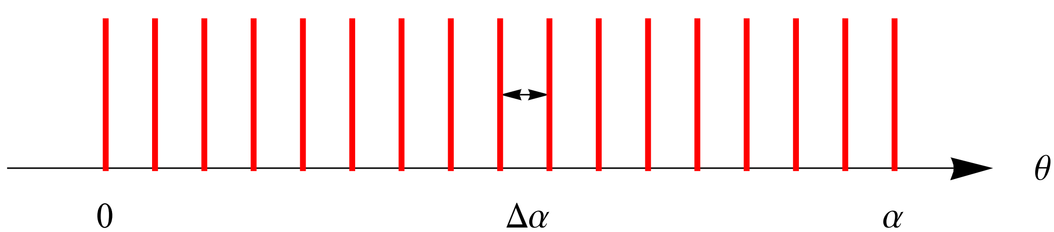


Fig. $\alpha = N\Delta\alpha$.

$$\hat{R}_z(\alpha = 0) = \hat{1},$$

$$\hat{R}_z(\alpha) = \lim_{N\to\infty}[\hat{R}_z(\Delta\alpha)]^N = \lim_{N\to\infty}(\hat{1} - \frac{i}{\hbar}\Delta\alpha\hat{L}_z)^N = \lim_{N\to\infty}(\hat{1} - \frac{i}{\hbar}\frac{\alpha}{N}\hat{L}_z)^N$$
$$= \exp(-\frac{i}{\hbar}\alpha\hat{L}_z)$$
,

((**Note**))

$$\lim_{N\to\infty}(\hat{1} - \frac{i}{\hbar}\frac{\alpha}{N}\hat{L}_z)^N = \lim_{N\to\infty}[(\hat{1} + \frac{\mu}{N})^{\frac{N}{\mu}}]^{\mu} = e^{\mu},$$

where

$$\mu = -\frac{i}{\hbar}\alpha\hat{L}_z.$$

In general, we have the rotation operator

$$\hat{R}_u(\alpha) = \exp(-\frac{i}{\hbar}\alpha\hat{\boldsymbol{L}}\cdot\boldsymbol{u}).$$

In the case of an arbitrary quantum mechanical system, using the general angular momentum $\hat{\boldsymbol{J}}$ instead of $\hat{\boldsymbol{L}}$:

$$\hat{R}_u(\alpha) = \exp(-\frac{i}{\hbar}\alpha\hat{\boldsymbol{J}}\cdot\boldsymbol{u}).$$

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APPENDIX-I

We show the following theorem (see **Steeb**)

Theorem:

$$\exp(\hat{A}\otimes\hat{1}+\hat{1}\otimes\hat{B})=\exp(\hat{A}\otimes\hat{1})\exp(\hat{1}\otimes\hat{B})$$
$$=\exp(\hat{A})\otimes\exp(\hat{B})$$

((**Proof**))
Since

$$[\hat{A}\otimes\hat{1},\hat{1}\otimes\hat{B}]=(\hat{A}\otimes\hat{1})(\hat{1}\otimes\hat{B})-(\hat{1}\otimes\hat{B})(\hat{A}\otimes\hat{1})$$
$$=\hat{A}\otimes\hat{B}-\hat{A}\otimes\hat{B}$$
$$=0$$

we have

$$\exp(\hat{A}\otimes\hat{1}+\hat{1}\otimes\hat{B})=\exp(\hat{A}\otimes\hat{1})\exp(\hat{1}\otimes\hat{B})$$

Now,

$$\exp(\hat{A}\otimes\hat{1})=\sum_n\frac{1}{n!}(\hat{A}\otimes\hat{1})^n \quad , \qquad \exp(\hat{1}\otimes\hat{B})=\sum_n\frac{1}{n!}(\hat{1}\otimes\hat{B})^n$$

So that, we have

$$\exp(\hat{A}\otimes\hat{1})\exp(\hat{1}\otimes\hat{B})=\sum_{n,m}\frac{1}{n!m!}(\hat{A}\otimes\hat{1})^n(\hat{1}\otimes\hat{B})^m$$

Note that

$$(\hat{A}\otimes\hat{1})^n(\hat{1}\otimes\hat{B})^m=\hat{A}^n\otimes\hat{B}^m$$

Thus, we get

$$\exp(\hat{A}\otimes\hat{1})\exp(\hat{1}\otimes\hat{B})=\sum_{n,m}\frac{1}{n!m!}\hat{A}^n\otimes\hat{B}^m$$
$$=\sum_n\frac{1}{n!}\hat{A}^n\otimes\sum_m\frac{1}{m!}\hat{B}^m$$
$$=\exp(\hat{A})\otimes\exp(\hat{B})$$

leading to

$$\exp(\hat{A}\otimes\hat{1}+\hat{1}\otimes\hat{B}) = \exp(\hat{A}\otimes\hat{1})\exp(\hat{1}\otimes\hat{B})$$
$$= \exp(\hat{A})\otimes\exp(\hat{B})$$

From this theorem we also get

$$\exp(\hat{B}\otimes\hat{1}+\hat{1}\otimes\hat{A}) = \exp(\hat{B}\otimes\hat{1})\exp(\hat{1}\otimes\hat{A})$$
$$= \exp(\hat{B})\otimes\exp(\hat{A})$$

APPENDIX-II

Product of the translation operators

$$\hat{T}_1(\mathbf{a})\otimes\hat{T}_2(\mathbf{a}) = \exp[-\frac{i}{\hbar}(\hat{\mathbf{p}}_1\otimes\hat{1}+\hat{1}\otimes\hat{\mathbf{p}}_2)\cdot\mathbf{a}]$$

((**Proof**))

$$\hat{T}_1(\mathbf{a})\otimes\hat{T}_2(\mathbf{a}) = \exp(-\frac{i}{\hbar}\hat{\mathbf{p}}_1\cdot\mathbf{a})\otimes\exp(-\frac{i}{\hbar}\hat{\mathbf{p}}_2\cdot\mathbf{a})$$
$$= \exp\{-\frac{i}{\hbar}[(\hat{\mathbf{p}}_1\cdot\mathbf{a})\otimes\hat{1}+\hat{1}\otimes(\hat{\mathbf{p}}_2\cdot\mathbf{a})]\}$$
$$= \exp[-\frac{i}{\hbar}(\hat{\mathbf{p}}_1\otimes\hat{1}+\hat{1}\otimes\hat{\mathbf{p}}_2)\cdot\mathbf{a}]$$
$$= \exp(-\frac{i}{\hbar}\hat{\mathbf{P}}\cdot\mathbf{a})$$

where

$$\hat{\mathbf{P}} = \hat{\mathbf{p}}_1\otimes\hat{1}+\hat{1}\otimes\hat{\mathbf{p}}_2$$

with

$$[\hat{\boldsymbol{p}}_1,\hat{\boldsymbol{p}}_2] = 0,$$

since $\hat{\mathbf{p}}_1$ and $\hat{\mathbf{p}}_2$ are the momenta of particles 1 and 2, respectively.