

Orbital magnetic moment
Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
(Date: February 23, 2015)

The Current density for the electron with a charge $(-e)$ and a mass μ , is given by

$$\mathbf{J} = \frac{-e}{\mu} \text{Re}[\psi_{nlm}(\mathbf{r})^* \frac{\hbar}{i} \nabla \psi_{nlm}(\mathbf{r})],$$

with

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}. \quad (\text{the spherical co-ordinate})$$

In an electron for the hydrogen-like atom,

$$R_{nl}(r) = \sqrt{\frac{4Z^3(n-l-1)!}{a^3 n^4(n+l)!}} e^{-\frac{Zr}{na}} \left(\frac{2Zr}{na}\right)^l L_{n-l-1}^{2l+1}\left(\frac{2Zr}{na}\right),$$

$$\psi_{nlm}(\mathbf{r}) = R_{nl}(r) Y_l^m(\theta, \phi) = R_{nl}(r) \Theta_l^m(\theta) \Phi_m(\phi),$$

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}.$$

and the spherical harmonics,

$$Y_l^m(\theta, \phi) = \frac{(-1)^l}{2^l l!} \sqrt{\frac{(2l+1)(l+m)!}{4\pi(l-m)!}} e^{im\phi} \frac{1}{\sin^m \theta} \frac{d^{l-m}}{d(\cos \theta)^{l-m}} (\sin \theta)^{2l},$$

Note that $R_{nl}(r)$ and $\Theta_l^m(\theta)$ are real functions.

We define $Y_l^{-m}(\theta, \phi)$ ($m \geq 0$) by

$$Y_l^{-m}(\theta, \phi) = (-1)^m [Y_l^m(\theta, \phi)]^*, \quad \text{for } m \geq 0,$$

or

$$[Y_l^m(\theta, \phi)]^* = (-1)^m Y_l^{-m}(\theta, \phi).$$

Then we have the current density as

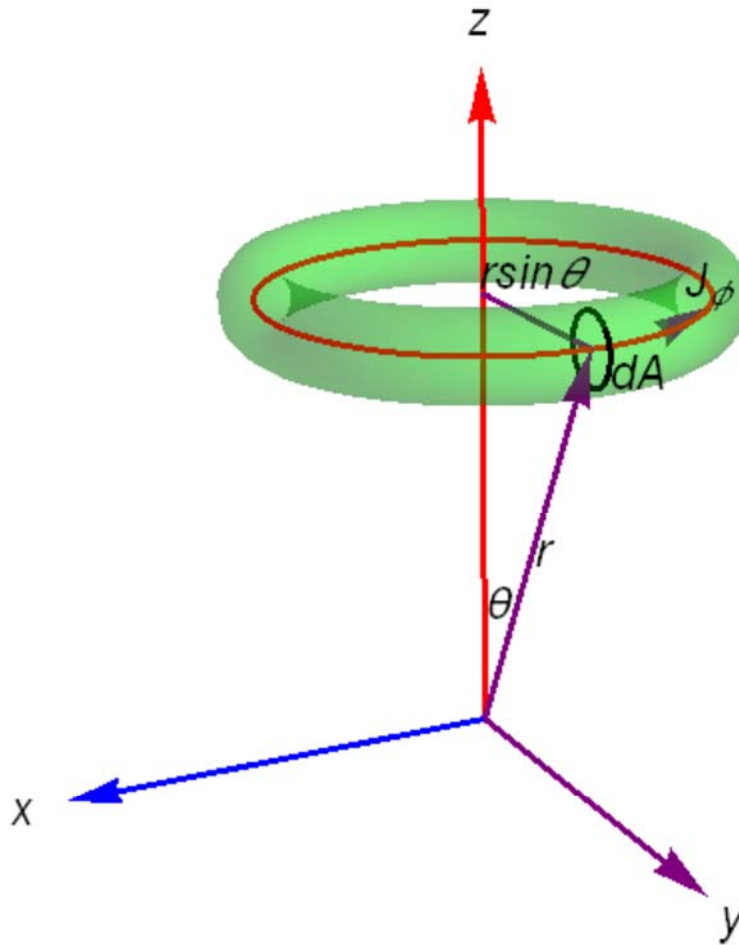
$$\mathbf{J} = -\frac{e\hbar}{\mu} \text{Re}[R_{nl}(r)\Theta_l^m(\theta)\Phi_m^*(\phi)\frac{1}{i}\nabla R_{nl}(r)\Theta_l^m(\theta)\Phi_m^*(\phi)].$$

Noting that

$$\begin{aligned} & \text{Re}[R_{nl}(r)\Theta_l^m(\theta)\Phi_m^*(\phi)\frac{1}{i}\{\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\}R_{nl}(r)\Theta_l^m(\theta)\Phi_m(\phi)] \\ &= \text{Re}\{\frac{1}{i}\mathbf{e}_r R_{nl}(r)\frac{\partial R_{nl}(r)}{\partial r}[\Theta_l^m(\theta)]^2 + \frac{1}{ir}\mathbf{e}_\theta[R_{nl}(r)]^2[\Theta_l^m(\theta)]\frac{\partial\Theta_l^m(\theta)}{\partial\theta} \\ &+ \frac{1}{ir\sin\theta}\mathbf{e}_\phi[R_{nl}(r)]^2[\Theta_l^m(\theta)]^2\Phi_m^*(\phi)\frac{\partial\Phi_m(\phi)}{\partial\phi}\} \\ &= \frac{m}{r\sin\theta}\mathbf{e}_\phi[R_{nl}(r)]^2[\Theta_l^m(\theta)]^2|\Phi_m(\phi)|^2 \\ &= \frac{m}{r\sin\theta}\mathbf{e}_\phi\psi_{nlm}^*(\mathbf{r})\psi_{nlm}(\mathbf{r}) \end{aligned}$$

we obtain the expression for the current density

$$J_\phi = \frac{-e}{\mu} m\hbar \frac{|\psi_{nlm}(\mathbf{r})|^2}{r\sin\theta}.$$



The magnetic moment along the z axis is

$$\begin{aligned}
 d\mu_z &= \frac{\pi(r \sin \theta)^2}{c} J_\phi dA \\
 &= \left(\frac{-e}{\mu} \right) \frac{\pi(r \sin \theta)^2}{c} \left[m\hbar \frac{\psi_{nlm}^*(\mathbf{r})\psi_{nlm}(\mathbf{r})}{r \sin \theta} \right] dA \\
 &= \left(\frac{-e}{\mu} \right) \frac{\pi r \sin \theta}{c} m\hbar |\psi_{nlm}(\mathbf{r})|^2 dA \\
 &= \left(\frac{-e\hbar}{2\mu c} \right) m |\psi_{nlm}(\mathbf{r})|^2 2\pi r \sin \theta dA
 \end{aligned}$$

Since $dV = 2\pi r \sin \theta dA$

$$\mu_z = \left(\frac{-e\hbar}{2\mu c}\right)m \int |\psi_{nlm}(\mathbf{r})|^2 dV = \left(\frac{-e}{2\mu c}\right)m\hbar = \left(\frac{-e\hbar}{2\mu c}\right)\frac{m\hbar}{\hbar}.$$

or

$$\hat{\mu}_L = -\frac{\mu_B}{\hbar} \hat{\mathbf{L}}, \quad (\text{orbital magnetic moment}).$$

where

$$\int |\psi_{nlm}(\mathbf{r})|^2 dV = 1$$

$$\mu_B = \frac{e\hbar}{2\mu c}. \quad (\text{Bohr magneton}).$$