Series expansion method
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Using the series expansion method, we solve the differential equation for the radial wave function for the hydrogen system. We start with

$$u''(r) - [\frac{l(l+1)}{r^2} + \frac{2\mu(\varepsilon_1 - Ze^2/r)}{\hbar^2}]u(r) = 0,$$

with

$$r = \frac{\hbar}{\sqrt{8\mu\varepsilon_1}}\rho = \frac{\rho}{2\kappa},$$

with

$$\varepsilon_1 = \frac{\hbar^2\kappa^2}{2\mu}, \qquad \kappa = \frac{\sqrt{2\mu\varepsilon_1}}{\hbar}.$$

Solution of radial part of the hydrogen atom

$$[\frac{d^2}{d\rho^2} - \frac{l(l+1)}{\rho^2} + \frac{\lambda}{\rho} - \frac{1}{4}]u(\rho) = 0, \tag{1}$$

with

$$\lambda = \frac{Ze^2}{\hbar}\sqrt{\frac{\mu}{2\varepsilon_1}},$$

which corresponds to the eigenvalue. Note that according to the Bohr model, $\lambda = n =$ integer since

$$\varepsilon_1 = \frac{\mu Z^2 e^4}{2n^2\hbar^2}.$$

We solve the differential equation to determine the eigenvalue and eigenfunction.

(a) In the limit of $\rho \to 0$.

We assume that it behaves at the origin like

$$u \approx \rho^s,$$

or

$$[s(s-1)-l(l+1)]\rho^{s-2}+\lambda\rho^{s-1}-\frac{1}{4}\rho^{s}=0\,.$$

Note that the ρ^{s-2} term dominates for small ρ.

$$s(s-1)-l(l+1)=0\,,$$

or

$$(s-\ell-1)(s+l)=0\,,$$

or

$$s=l+1 \text{ or } s=-l.$$

We must discard those solutions that behave as ρ^{-l}. So we get the form around $\rho = 0$:

$$u[\rho]\approx\rho^{l+1}.$$

(b) In the limit of $\rho\to\infty$,

$$(\frac{d^2}{d\rho^2}-\frac{1}{4})u(\rho)=0\,.$$

The solution for this equation is

$$u(\rho)\approx Ae^{-\rho/2}+Be^{\rho/2}.$$

The constant B should be equal to zero ($\rho\to\infty$):

$$u(\rho)\approx e^{-\rho/2}\,.$$

Thus we can attempt to find a solution of the form

$$u(\rho)=\rho^{l+1}e^{-\rho/2}F(\rho)\,.$$

With this substitution, the differential equation (1) becomes

$$\frac{d^2F(\rho)}{d\rho^2}+(\frac{2l+2}{\rho}-1)\frac{dF(\rho)}{d\rho}+(\frac{\lambda}{\rho}-\frac{l+1}{\rho})F(\rho)=0\,.$$

We assume that

$$F(\rho) = \sum_{k=0}^{\infty} C_k \rho^k ,$$

with $C_0 \neq 0$.

$$\begin{aligned}&\sum_{k=2}^{\infty} k(k-1)C_k\rho^{k-2} + \sum_{k=1}^{\infty}(2l+2)kC_k\rho^{k-2} - \sum_{k=1}^{\infty}kC_k\rho^{k-1} \\ &+ \sum_{k=0}^{\infty}[\lambda-(l+1)]C_k\rho^{k-1} = 0\end{aligned} ,$$

We note that

$$\sum_{k=2}^{\infty} k(k-1)C_{k1}\rho^{k-2} = \sum_{k=1}^{\infty} k(k+1)C_{k+1}\rho^{k-1} = \sum_{k=0}^{\infty} k(k+1)C_{k+1}\rho^{k-1} ,$$

$$\sum_{k=1}^{\infty}(2l+2)kC_k\rho^{k-2} = \sum_{k=0}^{\infty}(2l+2)(k+1)C_{k+1}\rho^{k-1} ,$$

$$\sum_{k=1}^{\infty} kC_k\rho^{k-1} = \sum_{k=0} kC_k\rho^{k-1} .$$

Then we get

$$\sum_{k=0}^{\infty} k(k+1)C_{k+1}\rho^{k-1} + \sum_{k=0}^{\infty}(2l+2)(k+1)C_{k+1}\rho^{k-1} + \sum_{k=0}^{\infty}[-k+\lambda-(l+1)]C_k\rho^{k-1} = 0$$

or

$$\sum_{k=0}^{\infty}\{(k+1)(k+2l+2)]C_{k+1} - (k+l+1-\lambda)C_k\}\rho^{k-1} = 0 ,$$

Since the coefficient of ρ^{k-1} should be zero, we get the recursion relation as

$$\frac{C_{k+1}}{C_k} = \frac{k+l+1-\lambda}{(k+1)(k+2l+2)} .$$

Note that

$$\frac{C_{k+1}}{C_k} \to \frac{1}{k},$$

which is the same asymptotic behavior as e^{ρ}. Thus, unless the series terminate, $u(\rho)$ will grow exponentially like $e^{\rho/2}$.

To avoid this, we must have

$$n_r + l + 1 - \lambda = 0,$$

or

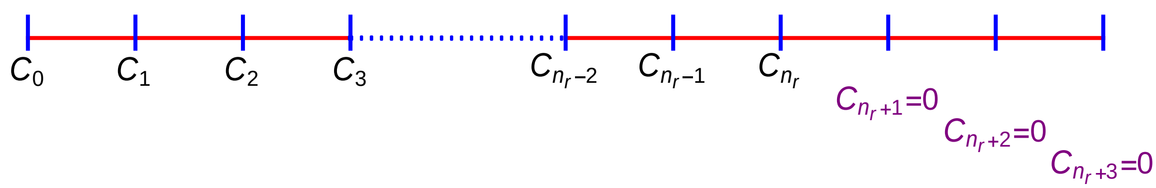
$$\lambda = n_r + l + 1,$$

for $k_{\max} = n_r$. Then we have

$C_0, C_1, C_2, C_3, \ldots, C_{n_r-1}, C_{n_r},$ (finite terms)

and

$$C_{n_r+1} = C_{n_r+2} = \ldots = C_{\infty} = 0.$$



The function F will thus be a polynomial of degree of n_r, known as an associated Laguerre polynimial.

$$\lambda = \frac{Ze^2}{\hbar}\sqrt{\frac{\mu}{2\varepsilon_1}} = l + 1 + n_r,$$

or

$$E = -\varepsilon_1 = -\frac{\mu Z^2 e^4}{2\hbar^2 (l + 1 + n_r)^2}.$$

Since $l = 0, 1, 2, 3, \ldots$, $n_r = 0, 1, 2, \ldots$, we introduced a principal quantum number n,

$$n = l + 1 + n_r,$$

with $n = 1, 2, 3, \ldots$.

Thus, in terms of n, the energy eigenvalue can be rewritten as

$$E_n = -\frac{\mu Z^2 e^4}{2\hbar^2 n^2}.$$

When $n_r = 0$, $l = l_{\max}$. Thus we have

$$l_{\max} = n - 1.$$

((**Note**))

n:	principal quantum number
l:	azimuthal quantum number
m:	Magnetic quantum number

((**Note**))

Coefficient of ρ^0:

$$C_1 = \frac{(l+1-\lambda)}{1(2l+2)} C_0.$$

Coefficient of ρ^1:

$$C_2 = \frac{(l+2-\lambda)}{2(2l+3)} C_1.$$

Coefficient of ρ^2:

$$C_3 = \frac{(l+3-\lambda)}{3(2l+4)} C_2.$$

Coefficient of ρ^3:

$$C_4 = \frac{(l+4-\lambda)}{4(2l+5)} C_3.$$

Coefficient of ρ^4:

$$C_5 = \frac{(l+5-\lambda)}{5(2l+6)} C_4.$$

Coefficient of ρ^5:

$$C_6 = \frac{(l+6-\lambda)}{6(2l+7)} C_5$$

Radial wave function

(i) For $n_r = 0$, $\lambda = n = l+1+n_r = l+1$

$$F(\rho) = C_0.$$

(ii) For $n_r = 1$, $\lambda = n = l+1+n_r = l+2$

$$F(\rho) = C_0 + C_1\rho = C_0(1 - \frac{1}{2l+2}\rho).$$

with

$$C_1 = \frac{l+1-\lambda}{2(1+l)} C_0 = -\frac{1}{2l+2} C_0.$$

(iii) For $n_r = 2$, $\lambda = n = l+1+n_r = l+3$

$$F(\rho) = C_0 + C_1\rho + C_2\rho^2 = C_0[1 - \frac{1}{l+1}\rho + \frac{1}{(2l+2)(2l+3)}\rho^2]C_0,$$

with

$$C_1 = \frac{l+1-\lambda}{2(1+l)} C_0 = -\frac{1}{(l+1)} C_0.$$

$$C_2 = \frac{l+2-\lambda}{2(2l+3)} C_1 = -\frac{1}{2(2l+3)} C_1 = \frac{1}{(2l+2)(2l+3)} C_0.$$

(iv) For $n_r = 3$, $\lambda = l+1+n_r = l+4$

$$F(\rho) = C_0 + C_1\rho + C_2\rho^2 + C_3\rho^3$$
$$= [1 - \frac{3}{2l+2}\rho + \frac{3}{(2l+2)(2l+3)}\rho^2 - \frac{1}{(2l+2)(2l+3)(2l+4)}\rho^3]\text{'}$$

with

$$C_1 = \frac{l+1-\lambda}{2(1+l)} C_0 = -\frac{3}{2l+2} C_0 ,$$

$$C_2 = \frac{l+2-\lambda}{2(2l+3)} C_1 = \frac{3}{(2l+2)(2l+3)} C_0 ,$$

$$C_3 = \frac{l+3-\lambda}{3(2l+4)} C_2 = -\frac{1}{(2l+2)(2l+3)(2l+4)} C_0 ,.$$

((Physics))

We discuss the radial wave function for a given n (=1, 2, 3, 4,…). Since

$$n = l + 1 + n_r,$$

When n is given, l should be equal to

$l = n-1$, for $n_r = 0$, (C_0)

$l = n-2$, for $n_r = 1$, $(C_0 + C_1\rho)$

$l = n-3$, for $n_r = 2$, $(C_0 + C_1\rho + C_2\rho^2)$

…………………………

$l = 1$, for $n_r = n - 2$, $(C_0 + C_1\rho + C_2\rho^2 + ... + C_{n-2}\rho^{n-2})$

$l = 0$, for $n_r = n - 1$. $(C_0 + C_1\rho + C_2\rho^2 + ... + C_{n-1}\rho^{n-1})$

(i) $n = 1$

$l = 0$ $n_r = 0$ (1s)

$$F(\rho) = C_0 .$$

(i) $n = 2$

$l = 1$ $n_r = 0$ (2p)

$$F(\rho) = C_0 .$$

$l = 0$ $n_r = 1$ (2s)

$$F(\rho) = C_0(1 - \frac{1}{2l+2}\rho) = C_0(1 - \frac{1}{2}\rho) = \frac{C_0}{2}(2-\rho).$$

(ii) $n = 3$

$l = 2 \quad n_r = 0$ (3d)

$$F(\rho) = C_0,$$

$l = 1 \quad n_r = 1$ (3p)

$$F(\rho) = C_0(1 - \frac{1}{2l+2}\rho) = \frac{C_0}{4}(4-\rho).$$

$l = 0 \quad n_r = 2$ (3s)

$$F(\rho) = C_0[1 - \frac{1}{l+1}\rho + \frac{1}{(2l+2)(2l+3)}\rho^2]$$
$$= \frac{C_0}{6}(6 - 6\rho + \rho^2)$$

(iii) $n = 4$

$l = 3 \quad n_r = 0$ (4f)

$$F(\rho) = C_0.$$

$l = 2 \quad n_r = 1$ (4d)

$$F(\rho) = C_0(1 - \frac{1}{2l+2}\rho) = \frac{C_0}{6}(6-\rho).$$

$l = 1 \quad n_r = 2$ (4p)

$$F(\rho) = C_0[1 - \frac{1}{l+1}\rho + \frac{1}{(2l+2)(2l+3)}\rho^2]_0$$
$$= \frac{C_0}{20}(20 - 10\rho + \rho^2)$$

$l = 0, \quad n_r = 3$ (4s)

$$F(\rho)=[1-\frac{3}{2l+2}\rho+\frac{3}{(2l+2)(2l+3)}\rho^2-\frac{1}{(2l+2)(2l+3)(2l+4)}\rho^3]C_0$$
$$=\frac{C_0}{24}(24-36\rho+12\rho^2-\rho^3)$$

We note that $F(\rho)$ coincides with the **associated Laguerrel polynomial** $L_{n-l-1}^{2l+1}(x)$.

$n=1$

$l=0$ $\quad 1$

$n=2$

$l=0$ $\quad 2-\rho$

$l=1$ $\quad 1$

$n=3$

$l=0$ $\quad \frac{1}{2}(6-6\rho+\rho^2)$

$l=1$ $\quad 4-\rho$

$l=2$ $\quad 1$

$n=4$

$l=0$ $\quad \frac{1}{6}(24-36\rho+12\rho^2-\rho^3)$

$l=1$ $\quad \frac{1}{2}(20-10\rho+\rho^2)$

$l=2$ $\quad 6-\rho$

$l=3$ $\quad 1$

$n=5$

$l = 0$ $\frac{1}{24}(120 - 240\rho + 120\rho^2 - 20\rho^3 + \rho^4)$

$l = 1$ $\frac{1}{6}(120 - 90\rho + 18\rho^2 - \rho^3)$

$l = 2$ $\frac{1}{2}(42 - 14\rho + \rho^2)$

$l = 3$ $8 - \rho$

$l = 4$ 1

APPENDIX
Method of series expansion
((**Mathematica-1**))

Series expansion: radial part of the hydrogen atom

```
Clear["Global`*"];
```

```
vchange[Eq_, ψ_, x_, z_, f_] :=
  Eq /. {D[ψ[x], {x, n_}] :> Nest[(1/D[f, z] D[#, z] &), ψ[z], n],
    ψ[x] :> ψ[z], x :> f}
```

$$\text{Seq1} = \frac{\left(2\, e^2\, r\, Z\, \mu - 2\, r^2\, \varepsilon 1\, \mu - \ell\, \hbar^2 - \ell^2\, \hbar^2\right) u[r]}{r^2\, \hbar^2} + u''[r] == 0$$

$$\frac{\left(2\, e^2\, r\, Z\, \mu - 2\, r^2\, \varepsilon 1\, \mu - \ell\, \hbar^2 - \ell^2\, \hbar^2\right) u[r]}{r^2\, \hbar^2} + u''[r] == 0$$

$$\text{Seq2} = \text{vchange}\left[\text{Seq1},\, u,\, r,\, \rho,\, \frac{\hbar\, \rho}{\sqrt{8\, \mu\, \varepsilon 1}}\right] \text{ // PowerExpand}$$

$$\frac{8\, \varepsilon 1\, \mu \left(\frac{e^2\, Z\, \sqrt{\mu}\, \rho\, \hbar}{\sqrt{2}\, \sqrt{\varepsilon 1}} - \ell\, \hbar^2 - \ell^2\, \hbar^2 - \frac{\rho^2\, \hbar^2}{4}\right) u[\rho]}{\rho^2\, \hbar^4} + \frac{8\, \varepsilon 1\, \mu\, u''[\rho]}{\hbar^2} == 0$$

$$\lambda = \frac{Z\, e^2}{\hbar} \sqrt{\frac{\mu}{2\, \varepsilon 1}}$$

$$\text{rule1} = \left\{\varepsilon 1 \to \frac{1}{2}\left(\frac{\sqrt{\mu}\, Z\, e^2}{\hbar}\, \frac{1}{\lambda}\right)^2\right\};$$

```
Seq3 = Seq2 /. rule1 // PowerExpand;
```

```
Seq4 = Solve[Seq3, u''[ρ]]
```

$$\left\{\left\{u''[\rho] \to \frac{\left(4\,\ell + 4\,\ell^2 - 4\,\lambda\,\rho + \rho^2\right)\,u[\rho]}{4\,\rho^2}\right\}\right\}$$

```
F = D[u[ρ], {ρ, 2}] - ℓ (ℓ + 1)/ρ^2 u[ρ] + (λ/ρ - 1/4) u[ρ];
```

```
rule1 = {u → (#^s &)};
```

```
F /. rule1 // Simplify
```

$$-\frac{1}{4}\,\rho^{-2+s}\,\left(4\,s - 4\,s^2 + 4\,\ell + 4\,\ell^2 + \rho\,(-4\,\lambda + \rho)\right)$$

```
rule2 = {u → (#^(1+ℓ) Exp[-#/2] F1[#] &)};
```

```
eq1 = F /. rule2 // Simplify
```

$$-e^{-\rho/2}\,\rho^{\ell}\,\left((1 + \ell - \lambda)\,F1[\rho] + (-2 - 2\,\ell + \rho)\,F1'[\rho] - \rho\,F1''[\rho]\right)$$

```
eq2 = -((1 + ℓ - λ) F1[ρ] + (-2 - 2 ℓ + ρ) F1'[ρ] - ρ F1''[ρ])
```

$$-(1 + \ell - \lambda)\,F1[\rho] - (-2 - 2\,\ell + \rho)\,F1'[\rho] + \rho\,F1''[\rho]$$

```
rule3 = {F1 → (Sum[C[k] #^k, {k, 0, 10}] &)};
```

```
eq3 = eq2 /. rule3 // Expand;
```

```
list1 = Table[{n, Coefficient[eq3, ρ, n]}, {n, 0, 9}] // Simplify;
list1 // TableForm
```

0	$(-1-\ell+\lambda)\,C[0]+2\,(1+\ell)\,C[1]$
1	$(-2-\ell+\lambda)\,C[1]+2\,(3+2\,\ell)\,C[2]$
2	$(-3-\ell+\lambda)\,C[2]+6\,(2+\ell)\,C[3]$
3	$(-4-\ell+\lambda)\,C[3]+4\,(5+2\,\ell)\,C[4]$
4	$(-5-\ell+\lambda)\,C[4]+10\,(3+\ell)\,C[5]$
5	$(-6-\ell+\lambda)\,C[5]+6\,(7+2\,\ell)\,C[6]$
6	$(-7-\ell+\lambda)\,C[6]+14\,(4+\ell)\,C[7]$
7	$(-8-\ell+\lambda)\,C[7]+8\,(9+2\,\ell)\,C[8]$
8	$(-9-\ell+\lambda)\,C[8]+18\,(5+\ell)\,C[9]$
9	$(-10-\ell+\lambda)\,C[9]+10\,(11+2\,\ell)\,C[10]$

Determination of recursion formula

$$\text{rule4}=\left\{F1\to\left(\sum_{n=k-3}^{k+3}C[n]\,\#^{n}\,\&\right)\right\};$$

$$\text{eq4}=\left(\text{eq2}\,/\,\rho^{-4+k}\right)\ /.\ \text{rule4}\ //\ \text{Expand};$$

```
list2 = Table[{n, Coefficient[eq4, ρ, n]}, {n, 0, 7}] // Simplify;
list2 // TableForm
```

0	$(-3+k)\,(-2+k+2\,\ell)\,C[-3+k]$
1	$-(-2+k+\ell-\lambda)\,C[-3+k]+(-2+k)\,(-1+k+2\,\ell)\,C[-2+k]$
2	$-(-1+k+\ell-\lambda)\,C[-2+k]+(-1+k)\,(k+2\,\ell)\,C[-1+k]$
3	$-(k+\ell-\lambda)\,C[-1+k]+k\,(1+k+2\,\ell)\,C[k]$
4	$-(1+k+\ell-\lambda)\,C[k]+(1+k)\,(2+k+2\,\ell)\,C[1+k]$
5	$-(2+k+\ell-\lambda)\,C[1+k]+(2+k)\,(3+k+2\,\ell)\,C[2+k]$
6	$-(3+k+\ell-\lambda)\,C[2+k]+(3+k)\,(4+k+2\,\ell)\,C[3+k]$
7	$-(4+k+\ell-\lambda)\,C[3+k]$

```
eq5 = list2[[5, 2]] == 0
```

$$-(1+k+\ell-\lambda)\,C[k]+(1+k)\,(2+k+2\,\ell)\,C[1+k]==0$$

```
Solve[eq5, C[k + 1]]
```

$$\left\{\left\{C[1+k]\to\frac{(1+k+\ell-\lambda)\,C[k]}{(1+k)\,(2+k+2\,\ell)}\right\}\right\}$$

((Mathematica-2))

Radial wave function

Hydrogenic atom: Radial wave function

```
Clear["Global`*"];
```

rwave[n_, ℓ_, r_] :=

$$\frac{1}{\sqrt{(n+\ell)\,!}}\left(2^{1+\ell}\, a^{-\ell-\frac{3}{2}}\, e^{-\frac{Z\,r}{a\,n}}\, n^{-\ell-2}\, Z^{\ell+\frac{3}{2}}\, r^{\ell}\, \sqrt{(n-\ell-1)\,!}\;\text{LaguerreL}\left[-1+n-\ell,\ 1+2\,\ell,\ \frac{2\,Z\,r}{a\,n}\right]\right)$$

```
Table[rwave[n, ℓ, r], {n, 1, 3}, {ℓ, 0, n - 1}] // TableForm[#,
    TableHeadings → {{"n=1", "n=2", "n=3"},
      {"ℓ=0", "ℓ=1", "ℓ=2"}}] &
```

	ℓ=0	ℓ=1	ℓ=2
n=1	$\frac{2\, e^{-\frac{r\,Z}{a}}\, Z^{3/2}}{a^{3/2}}$		
n=2	$\frac{e^{-\frac{r\,Z}{2\,a}}\, Z^{3/2}\left(2-\frac{r\,Z}{a}\right)}{2\sqrt{2}\; a^{3/2}}$	$\frac{e^{-\frac{r\,Z}{2\,a}}\, r\, Z^{5/2}}{2\sqrt{6}\; a^{5/2}}$	
n=3	$\frac{2\, e^{-\frac{r\,Z}{3\,a}}\, Z^{3/2}\left(27\,a^2-18\,a\,r\,Z+2\,r^2\,Z^2\right)}{81\sqrt{3}\; a^{7/2}}$	$\frac{\sqrt{\frac{2}{3}}\; e^{-\frac{r\,Z}{3\,a}}\, r\, Z^{5/2}\left(4-\frac{2\,r\,Z}{3\,a}\right)}{27\, a^{5/2}}$	$\frac{2\sqrt{\frac{2}{15}}\; e^{-\frac{r\,Z}{3\,a}}\, r^2\, Z^{7/2}}{81\, a^{7/2}}$