**2D isotropic simple harmonics: operator method**

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Here we discuss the eigenstates of 2D isotropic simple harmonics using the creation and annihilation operators. The eigenstates are the simultaneous ones of the Hamiltonian and angular momentum.

## 1. Hamiltonian and angular momentum in terms of creation operator and annihilation operator and angular momentum

We consider the Hamiltonian for the 2D motion of a particle of mass $\mu$ in a isotropic harmonic oscillator potential

$$\begin{aligned}\hat{H} &= \\ &= \frac{1}{2\mu}(\hat{p}_x{}^2 + \hat{p}_y{}^2) + \frac{1}{2}\mu\omega_0{}^2(\hat{x}^2 + \hat{y}^2) \\ &= (\frac{1}{2\mu}\hat{p}_x{}^2 + \frac{1}{2}\mu\omega_0{}^2\hat{x}^2) + (\frac{1}{2\mu}\hat{p}_y{}^2 + \frac{1}{2}\mu\omega_0{}^2\hat{y}^2) \\ &= \hat{H}_x + \hat{H}_y\end{aligned}$$

We introduce the creation and annihilation operators such that

$$\hat{x} = \frac{1}{\sqrt{2}\beta}\left(\hat{a}_1 + \hat{a}_1{}^+\right) = \sqrt{\frac{\hbar}{2\mu\omega_0}}\left(\hat{a}_1 + \hat{a}_1{}^+\right),$$

$$\hat{y} = \frac{1}{\sqrt{2}\beta}\left(\hat{a}_2 + \hat{a}_2{}^+\right) = \sqrt{\frac{\hbar}{2\mu\omega_0}}\left(\hat{a}_2 + \hat{a}_2{}^+\right),$$

$$\hat{p}_x = \frac{1}{\sqrt{2}\beta}\frac{\mu\omega_0}{i}\left(\hat{a}_1 - \hat{a}_1{}^+\right) = \frac{1}{i}\sqrt{\frac{\mu\hbar\omega_0}{2}}\left(\hat{a}_1 - \hat{a}_1{}^+\right),$$

$$\hat{p}_y = \frac{1}{\sqrt{2}\beta}\frac{\mu\omega_0}{i}\left(\hat{a}_2 - \hat{a}_2{}^+\right) = \frac{1}{i}\sqrt{\frac{\mu\hbar\omega_0}{2}}\left(\hat{a}_2 - \hat{a}_2{}^+\right),$$

where

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \qquad \hat{a}^+|n\rangle = \sqrt{n+1}|n+1\rangle,$$

$$\hat{a}|0\rangle = 0, \qquad \hat{a}|1\rangle = |0\rangle, \qquad \hat{a}|2\rangle = \sqrt{2}|1\rangle, \qquad \hat{a}|3\rangle = \sqrt{3}|2\rangle,$$

$$\hat{a}^+|0\rangle = |1\rangle, \qquad \hat{a}^+|1\rangle = \sqrt{2}|2\rangle, \; \hat{a}^+|2\rangle = \sqrt{3}|3\rangle, \qquad \hat{a}^+|3\rangle = 2|4\rangle.$$

Then the Hamiltonian van be written as

$$\begin{aligned}
\hat{H} &= \frac{1}{2\mu}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2}\mu\omega_0^2(\hat{x}^2 + \hat{y}^2) \\
&= -\frac{\hbar\omega_0}{4}[\left(\hat{a}_1 - \hat{a}_1{}^+\right)\left(\hat{a}_1 - \hat{a}_1{}^+\right) + \left(\hat{a}_2 - \hat{a}_2{}^+\right)\left(\hat{a}_2 - \hat{a}_2{}^+\right)] \\
&\quad + \frac{\hbar\omega_0}{4}[\left(\hat{a}_1 + \hat{a}_1{}^+\right)\left(\hat{a}_1 + \hat{a}_1{}^+\right) + \left(\hat{a}_2 + \hat{a}_2{}^+\right)\left(\hat{a}_2 + \hat{a}_2{}^+\right)] \\
&= \hbar\omega_0(\hat{a}_1{}^+\hat{a}_1 + \hat{a}_2{}^+\hat{a}_2 + \hat{1}) \\
&= \hbar\omega_0(\hat{N}_1 + \hat{N}_2 + \hat{1})
\end{aligned}$$

where

$$[\hat{a}_1, \hat{a}_1{}^+] = \hat{1}, \qquad [\hat{a}_2, \hat{a}_2{}^+] = \hat{1}.$$

The energy eigenstate is defined by

$$\begin{aligned}
\hat{H}|n_1, n_2\rangle &= \hbar\omega_0(\hat{N}_1 + \hat{N}_2 + \hat{1})|n_1, n_2\rangle \\
&= \hbar\omega_0(n+1)|n_1, n_2\rangle
\end{aligned}$$

where

$$E(n) = \hbar\omega_0(n+1),$$

with integer $n$ (=0, 1, 2, …) such that

$$n_1 + n_2 = n.$$

The angular momentum is also written as

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$
$$= \frac{\hbar}{2i}\left(\hat{a}_1 + \hat{a}_1^{+}\right)\left(\hat{a}_2 - \hat{a}_2^{+}\right) - \frac{\hbar}{2i}\left(\hat{a}_2 + \hat{a}_2^{+}\right)\left(\hat{a}_1 - \hat{a}_1^{+}\right)$$
$$= i\hbar(\hat{a}_1\hat{a}_2^{+} - \hat{a}_1^{+}\hat{a}_2)$$

From a symmetry argument, we find that $[\hat{H}, \hat{L}_z] = 0$. Then we have a simultaneous eigenket of $\hat{H}$ and $\hat{L}_z$ such that

$$\hat{H}\left|\psi\right\rangle = E\left|\psi\right\rangle \quad \text{and} \quad \hat{L}_z\left|\psi\right\rangle = m\hbar\left|\psi\right\rangle .$$

## 2. The eigenstates of $E = 2\hbar\omega$ with the combination of states $\left|10\right\rangle, \left|01\right\rangle$)

When $E = 2\hbar\omega_0$, there are two states $\left|n_1 = 1, n_2 = 0\right\rangle$ and $\left|n_1 = 0, n_2 = 1\right\rangle$. These two states are degenerate states with the same energy. The combination of these two states leads to the eigenstate of $\hat{H}$.

$$\hat{L}_z\left|10\right\rangle = i\hbar(\hat{a}_1\hat{a}_2^{+} - \hat{a}_1^{+}\hat{a}_2)\left|10\right\rangle = i\hbar\hat{a}_1\hat{a}_2^{+}\left|10\right\rangle = i\hbar\left|01\right\rangle$$

$$\hat{L}_z\left|01\right\rangle = i\hbar(\hat{a}_1\hat{a}_2^{+} - \hat{a}_1^{+}\hat{a}_2)\left|01\right\rangle = (-i\hbar)\hat{a}_1^{+}\hat{a}_2)\left|01\right\rangle = -i\hbar\left|10\right\rangle$$

$$\hat{L}_z = \hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

under the basis of $\left|10\right\rangle$ and $\left|01\right\rangle$. The eigenstate and eigenvalue for $\hat{L}_z$ is evalued with the use of Mathematica.

For $m = 1$

$$\left|\psi_1\right\rangle = \frac{1}{\sqrt{2}}[\left|10\right\rangle + i\left|01\right\rangle\} .$$

For $m = -1$

$$\left|\psi_{-1}\right\rangle = \frac{1}{\sqrt{2}}[\left|10\right\rangle - i\left|01\right\rangle\} .$$

((**Note**))

We assume that

$$|\psi\rangle = c_1|1,0\rangle + c_2|0,1\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Since $[\hat{H},\hat{L}_z] = 0$, $|\psi\rangle$ is also the eigenstate of $\hat{L}_z$;

$$\hat{L}_z|\psi\rangle = \lambda|\psi\rangle.$$

Since

$$\hat{L}_z|1,0\rangle = i\hbar|0,1\rangle, \qquad \hat{L}_z|0,1\rangle = -i\hbar|1,0\rangle,$$

we have the matrix representation of $\hat{L}_z$ under the basis of $|1,0\rangle$ and $|0,1\rangle$,

$$\hat{L}_z = \hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \hbar\hat{\sigma}_y.$$

Therefore the eigenvalue and eigemstate of $\hat{L}_z$ is

$$|+\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}(|1,0\rangle + i|0,1\rangle), \quad |-\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}}(|1,0\rangle - i|0,1\rangle).$$

((**Mathematica**))

```
Clear["Global`*"]; A = ⅈ ℏ ( 0  -1
                            1   0 );
eq1 = Eigensystem[A]
```

$\{\{-\hbar, \hbar\}, \{\{i, 1\}, \{-i, 1\}\}\}$

```
ψ1 = ⅈ Normalize[eq1[[2, 2]]]
```

$$\left\{\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right\}$$

```
ψ1 = -ⅈ Normalize[eq1[[2, 1]]]
```

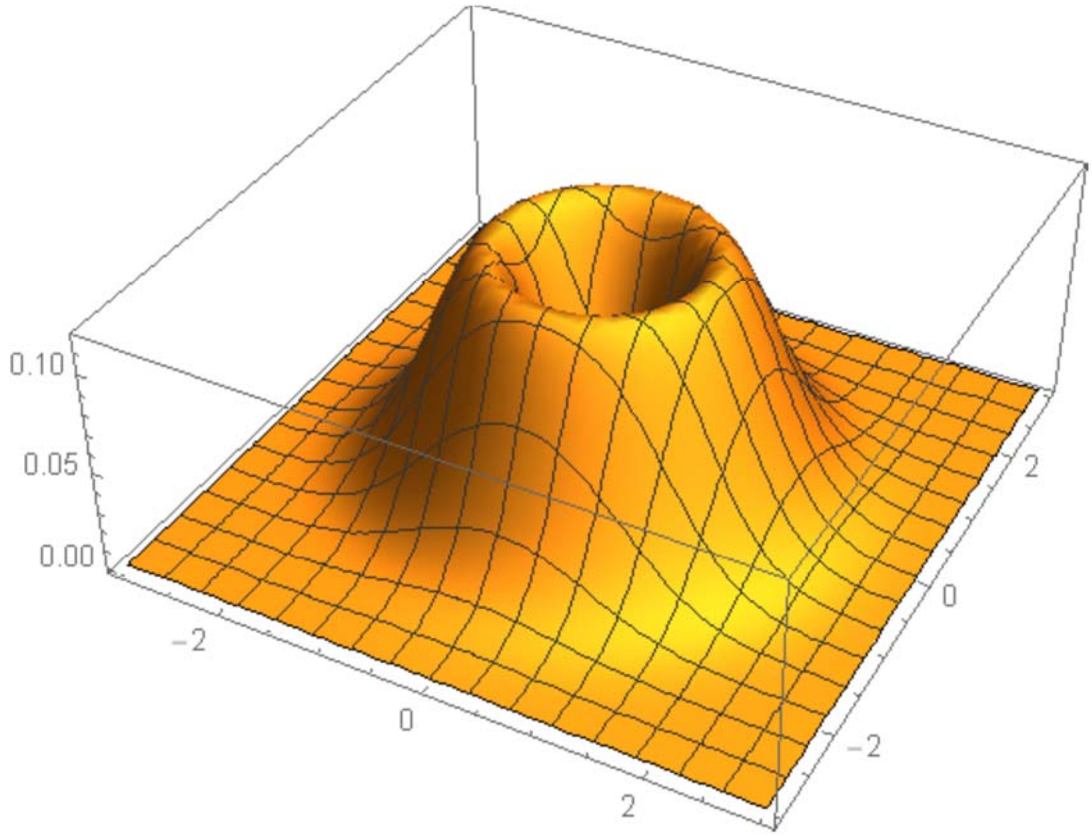
$$\left\{\frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}\right\}$$

((**Plot3D**))
We make a plot3D of the probability density:

$$\left|\langle x, y|\psi_1\rangle\right|^2 \qquad (E = 2\hbar\omega, \text{ and } m = 1)$$

as a function of *x* and *y*, where

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}[|10\rangle + i|01\rangle\}$$




where

$$\langle x, y | n_1, n_2 \rangle = 2^{-n} \pi^{-1/2} (n_1!)^{-1/2} (n_2!)^{-1/2} \exp(-\frac{x^2 + y^2}{2}) H[n_1, x] H[n_2, y]$$

## 3. The eigenstates of $E = 3\hbar\omega$ with the combination of states $|20\rangle$, $|11\rangle$, and $|02\rangle$)

When $E = 3\hbar\omega_0$, there are three states $|n_1 = 2, n_2 = 0\rangle$, $|n_1 = 1, n_2 = 1\rangle$, and $|n_1 = 0, n_2 = 2\rangle$. These three states are degenerate states with the same energy. The combination of these three states leads to the eigenstate of $\hat{H}$.

$$\hat{L}_z |20\rangle = i\hbar(\hat{a}_1\hat{a}_2^+ - \hat{a}_1{}^+\hat{a}_2)|20\rangle = i\hbar\hat{a}_1\hat{a}_2^+|20\rangle = i\hbar\sqrt{2}|11\rangle,$$

$$\hat{L}_z |11\rangle = i\hbar(\hat{a}_1\hat{a}_2^+ - \hat{a}_1{}^+\hat{a}_2)|11\rangle = i\hbar\sqrt{2}(|02\rangle - |20\rangle),$$

$$\hat{L}_z |02\rangle = i\hbar(\hat{a}_1\hat{a}_2^+ - \hat{a}_1{}^+\hat{a}_2)|02\rangle = -i\hbar\hat{a}_1{}^+\hat{a}_2|02\rangle = -\hbar\sqrt{2}|11\rangle,$$

$$\hat{L}_z = i\hbar \begin{pmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

under the basis of $|20\rangle$, $|11\rangle$, and $|02\rangle$. We use the Mathematica to get the eigenstates and eigenvalues.

For $m = 2$

$$|\psi_2\rangle = \frac{1}{2}|20\rangle + \frac{i}{\sqrt{2}}|11\rangle - \frac{1}{2}|02\rangle$$

For $m = 0$

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|20\rangle + \frac{1}{\sqrt{2}}|02\rangle$$

For $m = -2$

$$|\psi_{-2}\rangle = \frac{1}{2}|20\rangle - \frac{i}{\sqrt{2}}|11\rangle - \frac{1}{2}|02\rangle$$

((**Mathematica**))

```
B = ⅈ ℏ √2 ( 0  -1   0
              1   0  -1
              0   1   0 ); eq2 = Eigensystem[B]
```

$$\left\{\{-2\hbar, 2\hbar, 0\}, \left\{\left\{-1, i\sqrt{2}, 1\right\}, \left\{-1, -i\sqrt{2}, 1\right\}, \{1, 0, 1\}\right\}\right\}$$

```
ψ1 = - Normalize[eq2[[2, 2]]]
```

$$\left\{\frac{1}{2}, \frac{i}{\sqrt{2}}, -\frac{1}{2}\right\}$$

```
ψ2 = Normalize[eq2[[2, 3]]]
```

$$\left\{\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}$$

```
ψ3 = - Normalize[eq2[[2, 1]]]
```

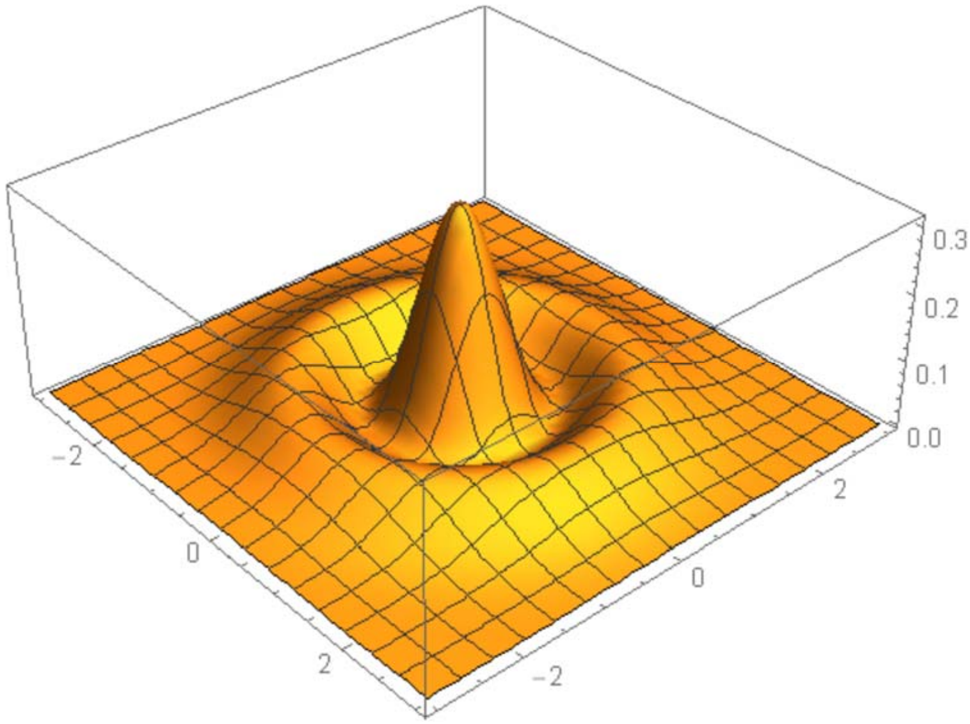
$$\left\{\frac{1}{2}, -\frac{i}{\sqrt{2}}, -\frac{1}{2}\right\}$$

((**Plot3D**))
We make a Plot3D of the probability density:

$$\left|\langle x, y|\psi_0\rangle\right|^2 \qquad (E = 3\hbar\omega\text{, and } m = 0)$$

in the *x* and *y* plane where

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|20\rangle + \frac{1}{\sqrt{2}}|02\rangle$$




## 4. Eigenstates of $E = 4\hbar\omega$ with the combination of states $|30\rangle$, $|21\rangle$, $|12\rangle$, and $|03\rangle$ )

When $E = 4\hbar\omega_0$, there are four states $|n_1 = 3, n_2 = 0\rangle$, $|n_1 = 2, n_2 = 1\rangle$, $|n_1 = 1, n_2 = 2\rangle$, and, $|n_1 = 0, n_2 = 3\rangle$. These four states are degenerate states with the same energy. The combination of these four states leads to the eigenstate of $\hat{H}$.

$$\hat{L}_z|30\rangle = i\hbar(\hat{a}_1\hat{a}_2^+ - \hat{a}_1{}^+\hat{a}_2)|30\rangle = i\hbar\hat{a}_1\hat{a}_2^+|30\rangle = i\hbar\sqrt{3}|21\rangle,$$

$$\hat{L}_z|21\rangle = i\hbar(\hat{a}_1\hat{a}_2^+ - \hat{a}_1{}^+\hat{a}_2)|21\rangle = i\hbar(2|12\rangle - \sqrt{3}|3,0\rangle),$$

$$\hat{L}_z|12\rangle = i\hbar(\hat{a}_1\hat{a}_2^+ - \hat{a}_1{}^+\hat{a}_2)|12\rangle = i\hbar(\sqrt{3}|03\rangle - 2|21\rangle),$$

$$\hat{L}_z|03\rangle = i\hbar(\hat{a}_1\hat{a}_2^+ - \hat{a}_1{}^+\hat{a}_2)|03\rangle = -i\hbar\hat{a}_1{}^+\hat{a}_2|30\rangle = -i\hbar\sqrt{3}|12\rangle,$$

$$\hat{L}_z = i\hbar\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}.$$

under the basis of $|30\rangle$, $|21\rangle$, $|12\rangle$, and $|03\rangle$. We use the Mathematica to get the eigenstates and eigenvalues.

For $m$ = 3

$$|\psi_3\rangle = \frac{1}{2\sqrt{2}}[|30\rangle + i\sqrt{3}|21\rangle - \sqrt{3}|12\rangle - i|03\rangle].$$

For $m$ = 1

$$|\psi_1\rangle = \frac{1}{2\sqrt{2}}[\sqrt{3}|30\rangle + i|21\rangle + |12\rangle + i\sqrt{3}|03\rangle].$$

For $m$ = -1

$$|\psi_{-1}\rangle = \frac{1}{2\sqrt{2}}[\sqrt{3}|30\rangle - i|21\rangle + |12\rangle - i\sqrt{3}|03\rangle].$$

For $m$ = -3

$$|\psi_{-3}\rangle = \frac{1}{2\sqrt{2}}[|30\rangle - i\sqrt{3}|21\rangle - \sqrt{3}|12\rangle + i|03\rangle].$$

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((**Mathematica**))

$$\texttt{C1} = i\hbar\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix};$$

```
eq3 = Eigensystem[C1]
```

$$\left\{\{-3\hbar, 3\hbar, -\hbar, \hbar\}, \left\{\left\{-i, -\sqrt{3}, i\sqrt{3}, 1\right\}, \left\{i, -\sqrt{3}, -i\sqrt{3}, 1\right\}, \left\{i, \frac{1}{\sqrt{3}}, \frac{i}{\sqrt{3}}, 1\right\}, \left\{-i, \frac{1}{\sqrt{3}}, -\frac{i}{\sqrt{3}}, 1\right\}\right\}\right\}$$

```
ψ1 = -i Normalize[eq3[[2, 2]]] // Simplify
```

$$\left\{\frac{1}{2\sqrt{2}}, \frac{1}{2} i \sqrt{\frac{3}{2}}, -\frac{\sqrt{\frac{3}{2}}}{2}, -\frac{i}{2\sqrt{2}}\right\}$$

```
ψ2 = i Normalize[eq3[[2, 4]]] // Simplify
```

$$\left\{\frac{\sqrt{\frac{3}{2}}}{2}, \frac{i}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2} i \sqrt{\frac{3}{2}}\right\}$$

```
ψ3 = -i Normalize[eq3[[2, 3]]] // Simplify
```

$$\left\{\frac{\sqrt{\frac{3}{2}}}{2}, -\frac{i}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, -\frac{1}{2} i \sqrt{\frac{3}{2}}\right\}$$

```
ψ4 = i Normalize[eq3[[2, 1]]] // Simplify
```

$$\left\{\frac{1}{2\sqrt{2}}, -\frac{1}{2} i \sqrt{\frac{3}{2}}, -\frac{\sqrt{\frac{3}{2}}}{2}, \frac{i}{2\sqrt{2}}\right\}$$

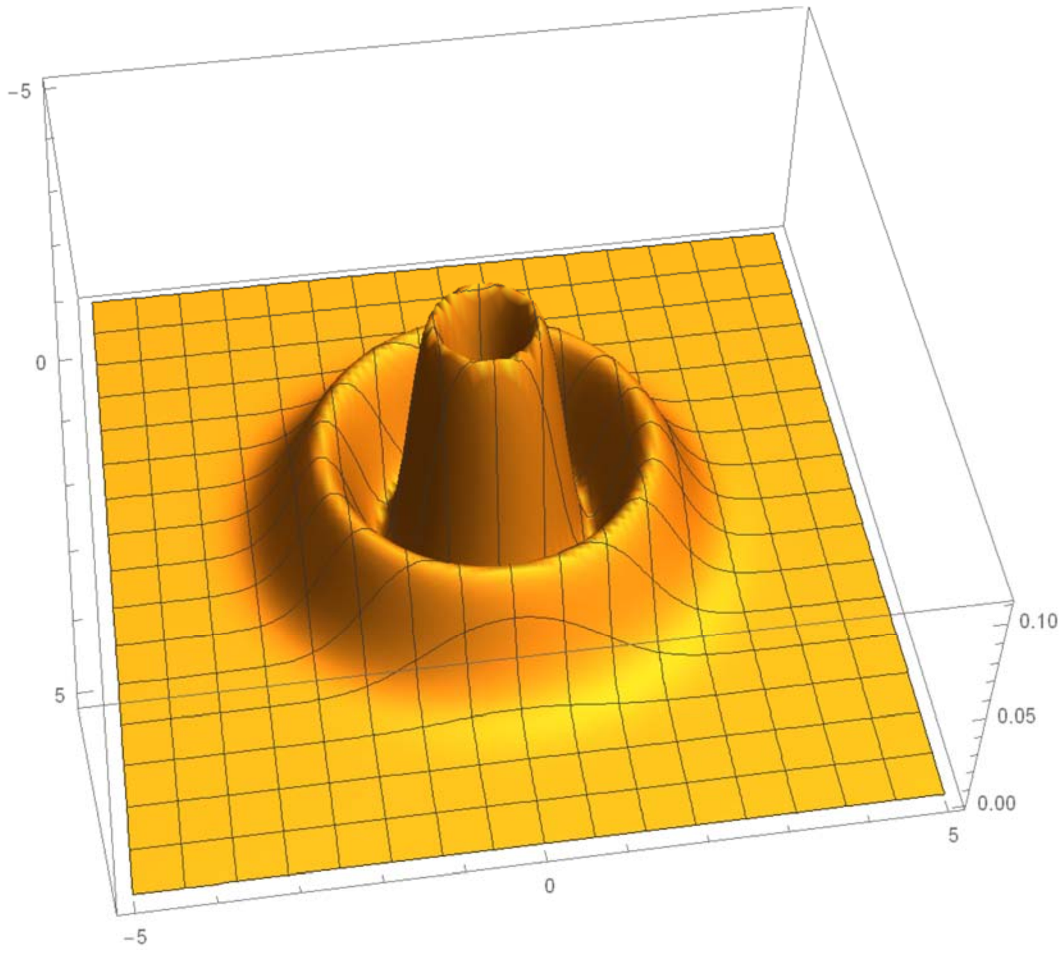
((**Plot3D**))

We make a Plot3D of the probability density,

$$\left|\langle x, y|\psi_1\rangle\right|^2 \qquad (E = 4\hbar\omega\text{, and } m = 1)$$

where

$$|\psi_1\rangle = \frac{1}{2\sqrt{2}}[\sqrt{3}|30\rangle + i|21\rangle + |12\rangle + i\sqrt{3}|03\rangle]$$





## REFERENCES

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