Two dimensional isotropic simple harmonics model
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We now consider the solution of the two dimensional isotropic simple harmonic oscillator.

$$\hat{H} = \frac{1}{2m}(\hat{p}_x{}^2 + \hat{p}_y{}^2) + \frac{1}{2}m\omega_0{}^2(\hat{x}^2 + \hat{y}^2).$$

((Cohen Tannoudji et al))

The quantum mechanical problem is exactly soluble and does not involve complicated calculations. Furthermore, this subject provides an opportunity to study a simple application of the properties of the orbital angular momentum $\hat{\boldsymbol{L}}$, since the stationary states of such an oscillator can be classified with respect to the possible values of the observable $\hat{L}_z$.

1. Definition

The operators $\hat{x}$ and $\hat{p}_x$ are given by

$$\hat{x} = \frac{1}{\sqrt{2}\beta}\left(\hat{a}_x + \hat{a}_x{}^+\right),$$

$$\hat{p}_x = \frac{1}{\sqrt{2}\beta}\frac{m\omega_0}{i}\left(\hat{a}_x - \hat{a}_x{}^+\right),$$

where

$$[\hat{x}, \hat{p}_x] = \frac{1}{\left(\sqrt{2}\beta\right)^2}\frac{m\omega_0}{i}\left[\hat{a}_x + \hat{a}_x{}^+, \hat{a}_x - \hat{a}_x{}^+\right] = -\frac{\hbar}{i}\left[\hat{a}_x, \hat{a}_x{}^+\right] = -\frac{\hbar}{i}\hat{1}.$$

Similarly, the operators $\hat{y}$ and $\hat{p}_y$ are defined by

$$\hat{y} = \frac{1}{\sqrt{2}\beta}\left(\hat{a}_y + \hat{a}_y{}^+\right),$$

$$\hat{p}_y = \frac{1}{\sqrt{2}\beta}\frac{m\omega_0}{i}\left(\hat{a}_y - \hat{a}_y{}^+\right),$$

where

$$\left[\hat{y}, \hat{p}_y\right] = \frac{1}{\left(\sqrt{2}\beta\right)^2} \frac{m\omega_0}{i} \left[\hat{a}_y + \hat{a}_y{}^+, \hat{a}_y - \hat{a}_y{}^+\right] = -\frac{\hbar}{i}\left[\hat{a}_y, \hat{a}_y{}^+\right] = -\frac{\hbar}{i}\hat{1},$$

with

$$\beta = \sqrt{\frac{m\omega_0}{\hbar}} .$$

The angular momentum along the *z* axis is defined by

$$\begin{aligned}
\hat{L}_z &= \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \\
&= \frac{1}{\sqrt{2}\beta}\left(\hat{a}_x + \hat{a}_x{}^+\right)\frac{1}{\sqrt{2}\beta}\frac{m\omega_0}{i}\left(\hat{a}_y - \hat{a}_y{}^+\right) \\
&\quad - \frac{1}{\sqrt{2}\beta}\left(\hat{a}_y + \hat{a}_y{}^+\right)\frac{1}{\sqrt{2}\beta}\frac{m\omega_0}{i}\left(\hat{a}_x - \hat{a}_x{}^+\right) \\
&= \frac{m\omega_0}{2\beta^2 i}[\left(\hat{a}_x + \hat{a}_x{}^+\right)\left(\hat{a}_y - \hat{a}_y{}^+\right) - \left(\hat{a}_y + \hat{a}_y{}^+\right)\left(\hat{a}_x - \hat{a}_x{}^+\right)] \\
&= i\hbar(\hat{a}_x\hat{a}_y{}^+ - \hat{a}_x{}^+\hat{a}_y)
\end{aligned}$$

where

$$[\hat{a}_x, \hat{a}_x^+] = \hat{1}, \qquad [\hat{a}_y, \hat{a}_y^+] = \hat{1},$$

$$[\hat{a}_x, \hat{a}_y^+] = \hat{0}, \qquad [\hat{a}_y, \hat{a}_x^+] = \hat{0}.$$

The Hamiltonian is invariant under the rotations along the *z* axis.

$$\langle\psi'|\hat{H}|\psi'\rangle = \langle\psi|\hat{R}_z(\delta\theta)^+\hat{H}\hat{R}_z(\delta\theta)|\psi\rangle = \langle\psi|\hat{H}|\psi\rangle$$

where

$$\hat{R}_z(\delta\theta) = \exp(-\frac{i}{\hbar}\hat{J}_z\delta\theta) \approx \hat{1} - -\frac{i}{\hbar}\hat{J}_z\delta\theta$$

Then we have

$$\hat{H}\hat{R}_z(\delta\theta) = \hat{R}_z(\delta\theta)\hat{H}$$

or

$$\hat{H}(\hat{1} - \frac{i}{\hbar}\hat{J}_z\delta\theta) = [\hat{1} - \frac{i}{\hbar}\hat{J}_z\delta\theta)]\hat{H} .$$

So it is concluded that

$$[\hat{H}, \hat{J}_z] = \hat{0} .$$

2. Number operators

The Hamiltonian is given by

$$\hat{H} = \hbar\omega_0(\hat{a}_x{}^+\hat{a}_x + \hat{a}_y{}^+\hat{a}_y + \hat{1}) = \hbar\omega_0(\hat{N}_x + \hat{N}_y + \hat{1}) ,$$

where the number operator is defined as

$$\hat{N}_x = \hat{a}_x^+\hat{a}_x , \qquad \hat{N}_y = \hat{a}_y\hat{a}_y .$$

We assume that the usual commutation relations among { $\hat{a}_x^+, \hat{a}_x, \hat{a}_y^+, \hat{a}_y, \hat{N}_x, \hat{N}_y$ } hold for oscillations of the same direction.

$$[\hat{N}_x, \hat{a}_x] = -\hat{a}_x , \qquad [\hat{N}_y, \hat{a}_y] = -\hat{a}_y$$
$$[\hat{N}_x, \hat{a}_x^+] = \hat{a}_x^+ , \qquad [\hat{N}_y, \hat{a}_y^+] = \hat{a}_y^+$$

Since $[\hat{N}_x, \hat{N}_y] = 0$, we can build up simultaneous eigenkets of $\hat{N}_x$ and $\hat{N}_y$ with eigenvalues n_x and n_y, respectively.

$$\hat{N}_x\left|n_x, n_y\right\rangle = n_x\left|n_x, n_y\right\rangle ,$$

$$\hat{N}_y\left|n_x, n_y\right\rangle = n_y\left|n_x, n_y\right\rangle ,$$

$$\hat{a}_x^+\left|n_x, n_y\right\rangle = \sqrt{n_x+1}\left|n_x, n_y\right\rangle,$$

$$\hat{a}_x\left|n_x, n_y\right\rangle = \sqrt{n_x}\left|n_x-1, n_y\right\rangle,$$

$$\hat{a}_y^+\left|n_x, n_y\right\rangle = \sqrt{n_y+1}\left|n_x, n_y+1\right\rangle,$$

$$\hat{a}_y\left|n_x, n_y\right\rangle = \sqrt{n_y}\left|n_x, n_y-1\right\rangle,$$

$$\hat{a}_x\left|0,0\right\rangle = 0 \qquad \hat{a}_y\left|0,0\right\rangle = 0 .$$

$\left|0,0\right\rangle$: vacuum ket

$$\left|n_x, n_y\right\rangle = \frac{(\hat{a}_x^+)^{n_x}(\hat{a}_x^+)^{n_y}}{\sqrt{n_x!}\sqrt{n_y!}}\left|0,0\right\rangle$$

---

3. Simultaneous eigenket

There are simultaneous eigenkets of $\hat{H}$ and $\hat{L}_z$. In other words, if $\left|\psi\right\rangle$ is the eigenket of $\hat{H}$, then it is also the eigen-ket of $\hat{L}_z$. Note that

$$\begin{aligned}\hat{L}_z\left|n_x, n_y\right\rangle &= i\hbar(\hat{a}_x\hat{a}_y{}^+ - \hat{a}_x{}^+\hat{a}_y)\left|n_x, n_y\right\rangle \\ &= i\hbar(\sqrt{n_x}\sqrt{n_y+1}\left|n_x-1, n_y+1\right\rangle - \sqrt{n_x+1}\sqrt{n_y}\left|n_x+1, n_y-1\right\rangle)\end{aligned}$$

$$\hat{H}\left|n_x, n_y\right\rangle = \hbar\omega_0(n_x+n_y+1)\left|n_x, n_y\right\rangle$$

with the energy eigenvalue

$$E(n_x, n_y) = \hbar\omega_0(n_x+n_y+1)$$

Then $\left|n_x, n_y\right\rangle$ is not the simultaneous eigenket of $\hat{H}$ and $\hat{L}_z$, since the matrix of $\hat{L}_z$ under the basis of $\{\left|n_x, n_y\right\rangle\}$ is not a diagonal matrix.

Energy ket vectors

$2\hbar\omega_0$	linear combination of $\|1,0\rangle$ and $\|0,1\rangle$
$3\hbar\omega_0$	linear combination of $\|2,0\rangle$, $\|1,1\rangle$ and $\|0,2\rangle$
$4\hbar\omega_0$	linear combination of $\|3,0\rangle$, $\|2,1\rangle$, $\|1,2\rangle$, and $\|0,3\rangle$.
$5\hbar\omega_0$	linear combination of $\|4,0\rangle$, $\|3,1\rangle$, $\|2,2\rangle$, $\|1,3\rangle$, and $\|0,4\rangle$

(a) States with energy $2\hbar\omega$: $|10\rangle$, $|01\rangle$

$$\hat{L}_z|1,0\rangle = i\hbar|0,1\rangle,$$

$$\hat{L}_z|0,1\rangle = -i\hbar|1,0\rangle).$$

Under the basis of {$|1,0\rangle$, $|0,1\rangle$ }, $\hat{L}_z$ corresponds to $\hbar\hat{\sigma}_y$, where $\hat{\sigma}_y$ is the Pauli spin operator

$$\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Then we have the eigenkets and eigenvalues for $\hat{L}_z$

Eigenvalue $\hbar$ $\quad \frac{1}{\sqrt{2}}[|10\rangle + i|01\rangle] = \frac{1}{\sqrt{2}}[|x\rangle + i|y\rangle] = |R\rangle$.

Eigenvalue $-\hbar$ $\quad \frac{1}{\sqrt{2}}[|10\rangle - i|01\rangle] = \frac{1}{\sqrt{2}}[|x\rangle - i|y\rangle] = |L\rangle$.

Energy eigenvalues:

$$\hat{H}|R\rangle = 2\hbar\omega_0|R\rangle, \qquad \hat{H}|L\rangle = 2\hbar\omega_0|L\rangle.$$

The unitary operator:

$$|R\rangle = \hat{U}|10\rangle, \qquad |L\rangle = \hat{U}|01\rangle$$

with

$$\hat{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

(b) States with energy $3\hbar\omega_0$: $|20\rangle$, $|11\rangle$, $|02\rangle$

$$\hat{L}_z|2,0\rangle = i\hbar(\sqrt{2}|1,1\rangle)$$

$$\hat{L}_z|1,1\rangle = i\hbar(\sqrt{2}|0,2\rangle - \sqrt{2}|2,0\rangle)$$

$$\hat{L}_z|0,2\rangle = -i\hbar\sqrt{2}|1,1\rangle)$$

The matrix $\hat{L}_z$ under the basis of $|20\rangle$, $|11\rangle$, $|02\rangle$

$$\hat{L}_z = \hbar\sqrt{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

Eigenvalue $2\hbar$ $\quad |\chi_1\rangle = \frac{1}{\sqrt{2}}(|2,0\rangle + i|1,1\rangle - |0,2\rangle) = \hat{U}|2,0\rangle$.

Eigenvalue $0\hbar$ $\quad |\chi_2\rangle = \frac{1}{\sqrt{2}}(|2,0\rangle + |0,2\rangle) = \hat{U}|1,1\rangle$.

Eigenvalue $-2\hbar$ $\quad |\chi_3\rangle = \frac{1}{\sqrt{2}}(|2,0\rangle - i|1,1\rangle - |0,2\rangle) = \hat{U}|0,2\rangle$.

Energy eigenvalues:

$$\hat{H}|\chi_1\rangle = 3\hbar\omega_0|\chi_1\rangle,$$

$$\hat{H}|\chi_2\rangle = 3\hbar\omega_0|\chi_2\rangle,$$

$$\hat{H}|\chi_3\rangle = 3\hbar\omega_0|\chi_3\rangle.$$

The unitary operator $\hat{U}$ is given by

$$\hat{U} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ i\sqrt{2} & 0 & -i\sqrt{2} \\ -1 & \sqrt{2} & -1 \end{pmatrix}$$

((**Mathematica**))

```
Clear["Global`*"];
exp_* :=
 exp /. {Complex[re_, im_] :→ Complex[re, -im]};
Lz = ℏ √2 ( 0  -ⅈ  0
            ⅈ   0  -ⅈ
            0   ⅈ   0 );
eq1 = Eigensystem[Lz]

{{-2 ℏ, 2 ℏ, 0},
 {{-1, ⅈ √2, 1}, {-1, -ⅈ √2, 1}, {1, 0, 1}}}

χ1 = -Normalize[eq1[[2, 2]]]

{1/2, ⅈ/√2, -1/2}

χ2 = Normalize[eq1[[2, 3]]]

{1/√2, 0, 1/√2}
```

```
χ3 = -Normalize[eq1[[2, 1]]]
```

$\left\{\frac{1}{2}, -\frac{i}{\sqrt{2}}, -\frac{1}{2}\right\}$

```
{χ1*.χ2, χ2*.χ3, χ3*.χ1}
```

{0, 0, 0}

```
UT = {χ1, χ2, χ3};
U = Transpose[UT]; UH = UT*;
```

$\left\{\left\{\frac{1}{2}, -\frac{i}{\sqrt{2}}, -\frac{1}{2}\right\}, \left\{\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{2}, \frac{i}{\sqrt{2}}, -\frac{1}{2}\right\}\right\}$

```
U // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix}$$

```
UH.U
```

{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}

```
UH.Lz.U
```

$\{\{2\hbar, 0, 0\}, \{0, 0, 0\}, \{0, 0, -2\hbar\}\}$

(c) States with energy $4\hbar\omega_0$: $|30\rangle$, $|21\rangle$, $|12\rangle$, $|03\rangle$

$\hat{L}_z|3,0\rangle = i\hbar(\sqrt{3}|2,1\rangle)$,

$\hat{L}_z|2,1\rangle = i\hbar(2|1,2\rangle - \sqrt{3}|3,0\rangle)$,

$\hat{L}_z|1,2\rangle = i\hbar(\sqrt{3}|0,3\rangle - 2|2,1\rangle)$,

$$\hat{L}_z|0,3\rangle = -i\hbar\sqrt{3}|1,2\rangle).$$

The matrix $\hat{L}_z$ under the basis of $|3,0\rangle$, $|2,1\rangle$, $|1,2\rangle$, and $|0,3\rangle$

$$\hat{L}_z = \hbar\begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix}$$

Eigenvalue $3\hbar$ $\quad |\phi_1\rangle = \frac{1}{2\sqrt{2}}(|3,0\rangle + i\sqrt{3}|2,1\rangle - \sqrt{3}|1,2\rangle - i|0,3\rangle) = \hat{U}|3,0\rangle$

Eigenvalue $\hbar$ $\quad |\phi_2\rangle = \frac{1}{2\sqrt{2}}(\sqrt{3}|3,0\rangle + i|2,1\rangle + |1,2\rangle + \sqrt{3}i|0,3\rangle) = \hat{U}|2,1\rangle$

Eigenvalue $-\hbar$ $\quad |\phi_3\rangle = \frac{1}{2\sqrt{2}}(\sqrt{3}|3,0\rangle - i|2,1\rangle + |1,2\rangle - \sqrt{3}i|0,3\rangle) = \hat{U}|1,2\rangle$

Eigenvalue $-3\hbar$ $\quad |\phi_4\rangle = \frac{1}{2\sqrt{2}}(|3,0\rangle - i\sqrt{3}|2,1\rangle - \sqrt{3}|1,2\rangle + i|0,3\rangle) = \hat{U}|0,3\rangle$

Energy eigenvalues:

$$\hat{H}|\phi_1\rangle = 4\hbar\omega_0|\phi_1\rangle,$$

$$\hat{H}|\phi_2\rangle = 4\hbar\omega_0|\phi_2\rangle,$$

$$\hat{H}|\phi_3\rangle = 4\hbar\omega_0|\phi_3\rangle,$$

$$\hat{H}|\phi_4\rangle = 4\hbar\omega_0|\phi_4\rangle.$$

The unitary operator $\hat{U}$ is given by

$$\hat{U} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & \sqrt{3} & \sqrt{3} & 1 \\ i\sqrt{3} & i & -i & -i\sqrt{3} \\ -\sqrt{3} & 1 & 1 & -\sqrt{3} \\ -i & i\sqrt{3} & -i\sqrt{3} & i \end{pmatrix}.$$

((**Mathematica**))

```
Clear["Global`*"];
exp_* :=
 exp /. {Complex[re_, im_] :> Complex[re, -im]};
```

$$\text{Lz} = \hbar \begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix};$$

```
eq1 = Eigensystem[Lz]
```

$$\left\{\{-3\hbar, 3\hbar, -\hbar, \hbar\}, \left\{\left\{-i, -\sqrt{3}, i\sqrt{3}, 1\right\}, \left\{i, -\sqrt{3}, -i\sqrt{3}, 1\right\}, \left\{i, \frac{1}{\sqrt{3}}, \frac{i}{\sqrt{3}}, 1\right\}, \left\{-i, \frac{1}{\sqrt{3}}, -\frac{i}{\sqrt{3}}, 1\right\}\right\}\right\}$$

```
χ1 = -i Normalize[eq1[[2, 2]]] // Simplify
```

$$\left\{\frac{1}{2\sqrt{2}}, \frac{1}{2} i \sqrt{\frac{3}{2}}, -\frac{\sqrt{\frac{3}{2}}}{2}, -\frac{i}{2\sqrt{2}}\right\}$$

```
χ2 = ⅈ Normalize[eq1[[2, 4]]] // Simplify
```

$$\left\{\frac{\sqrt{\frac{3}{2}}}{2}, \frac{i}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2} i \sqrt{\frac{3}{2}}\right\}$$

```
χ3 = - ⅈ Normalize[eq1[[2, 3]]]
```

$$\left\{\frac{\sqrt{\frac{3}{2}}}{2}, -\frac{i}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, -\frac{1}{2} i \sqrt{\frac{3}{2}}\right\}$$

```
χ4 = ⅈ Normalize[eq1[[2, 1]]]
```

$$\left\{\frac{1}{2\sqrt{2}}, -\frac{1}{2} i \sqrt{\frac{3}{2}}, -\frac{\sqrt{\frac{3}{2}}}{2}, \frac{i}{2\sqrt{2}}\right\}$$

```
{χ1*.χ2, χ2*.χ3, χ3*.χ4, χ1*.χ3, χ1*.χ4,
 χ2*.χ4}
```

{0, 0, 0, 0, 0, 0}

```
UT = {χ1, χ2, χ3, χ4};
U = Transpose[UT]; UH = UT*;
U // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2\sqrt{2}} & \frac{\sqrt{\frac{3}{2}}}{2} & \frac{\sqrt{\frac{3}{2}}}{2} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2}\, i\, \sqrt{\frac{3}{2}} & \frac{i}{2\sqrt{2}} & -\frac{i}{2\sqrt{2}} & -\frac{1}{2}\, i\, \sqrt{\frac{3}{2}} \\ -\frac{\sqrt{\frac{3}{2}}}{2} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{i}{2\sqrt{2}} & \frac{1}{2}\, i\, \sqrt{\frac{3}{2}} & -\frac{1}{2}\, i\, \sqrt{\frac{3}{2}} & \frac{i}{2\sqrt{2}} \end{pmatrix}$$

```
UH.U
```

{{1, 0, 0, 0}, {0, 1, 0, 0},
 {0, 0, 1, 0}, {0, 0, 0, 1}}

```
UH.Lz.U
```

$\{\{3\hbar, 0, 0, 0\}, \{0, \hbar, 0, 0\},$
$\{0, 0, -\hbar, 0\}, \{0, 0, 0, -3\hbar\}\}$

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4. Circularly polarized light

$$\hat{x} = \frac{1}{\sqrt{2}\beta}\left(\hat{a}_x + \hat{a}_x^{\ +}\right),$$

$$\hat{p}_x = \frac{1}{\sqrt{2}\beta}\frac{m\omega_0}{i}\left(\hat{a}_x - \hat{a}_x^{\ +}\right),$$

$$\hat{y} = \frac{1}{\sqrt{2}\beta}\left(\hat{a}_y + \hat{a}_y{}^+\right),$$

$$\hat{p}_y = \frac{1}{\sqrt{2}\beta}\frac{m\omega_0}{i}\left(\hat{a}_y - \hat{a}_y{}^+\right),$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = i\hbar(\hat{a}_x\hat{a}_y{}^+ - \hat{a}_x{}^+\hat{a}_y)\,.$$

Here we introduce new operators

$$\hat{a}_R^+ = -\frac{1}{\sqrt{2}}(\hat{a}_x^+ + i\hat{a}_y^+)\,, \qquad \hat{a}_L^+ = \frac{1}{\sqrt{2}}(\hat{a}_x^+ - i\hat{a}_y^+)\,,$$

or

$$\hat{a}_R = -\frac{1}{\sqrt{2}}(\hat{a}_x - i\hat{a}_y)\,, \qquad \hat{a}_L = \frac{1}{\sqrt{2}}(\hat{a}_x + i\hat{a}_y)\,.$$

The commutation relations:

$$[\hat{a}_R, \hat{a}_R^+] = \frac{1}{2}[\hat{a}_x - i\hat{a}_y, \hat{a}_x^+ + i\hat{a}_y^+] = \hat{1}\,,$$

$$[\hat{a}_L, \hat{a}_L^+] = \frac{1}{2}[\hat{a}_x + i\hat{a}_y, \hat{a}_x^+ - i\hat{a}_y^+] = \hat{1}\,,$$

$$[\hat{a}_R, \hat{a}_L] = -\frac{1}{2}[\hat{a}_x - i\hat{a}_y, \hat{a}_x + i\hat{a}_y] = 0\,,$$

$$[\hat{a}_R, \hat{a}_L{}^+] = -\frac{1}{2}[\hat{a}_x - i\hat{a}_y, \hat{a}_x^+ - i\hat{a}_y^+] = 0\,,$$

$$[\hat{a}_L, \hat{a}_R{}^+] = -\frac{1}{2}[\hat{a}_x + i\hat{a}_y, \hat{a}_x^+ + i\hat{a}_y^+] = 0\,.$$

The angular momentum is expressed by

$$\hat{a}_R^+\hat{a}_R = \frac{1}{2}(\hat{a}_x^+ + i\hat{a}_y^+)(\hat{a}_x - i\hat{a}_y) = \frac{1}{2}(\hat{a}_x^+\hat{a}_x + \hat{a}_y^+\hat{a}_y - i\hat{a}_x^+\hat{a}_y + i\hat{a}_y^+\hat{a}_x)\,,$$

$$\hat{a}_L^+\hat{a}_L = \frac{1}{2}(\hat{a}_x^+ - i\hat{a}_y^+)(\hat{a}_x + i\hat{a}_y) = \frac{1}{2}(\hat{a}_x^+\hat{a}_x + \hat{a}_y^+\hat{a}_y + i\hat{a}_x^+\hat{a}_y - i\hat{a}_y^+\hat{a}_x),$$

$$\hat{a}_R^+\hat{a}_R - \hat{a}_L^+\hat{a}_L = -i(\hat{a}_x^+\hat{a}_y - \hat{a}_y^+\hat{a}_x),$$

$$\hat{L}_z = i\hbar(\hat{a}_x\hat{a}_y{}^+ - \hat{a}_x{}^+\hat{a}_y) = \hbar(\hat{a}_R^+\hat{a}_R - \hat{a}_L^+\hat{a}_L) = \hbar(\hat{N}_R - \hat{N}_L).$$

The Hamiltonian is expressed by

$$\begin{aligned}\hat{H} &= \hbar\omega_0(\hat{a}_x^+\hat{a}_x + \hat{a}_y^+\hat{a}_y + \hat{1}) \\ &= \hbar\omega_0(\hat{a}_R^+\hat{a}_R + \hat{a}_L^+\hat{a}_L + \hat{1}) \\ &= \hbar\omega_0(\hat{N}_R + \hat{N}_L + \hat{1})\end{aligned}$$

The simultaneous eigenket of $\hat{H}$ and $\hat{L}$ is given by $|N_R, N_L\rangle$;

$$\hat{L}_z|N_R, N_L\rangle = \hbar(\hat{N}_R - \hat{N}_L)|N_R, N_L\rangle = \hbar(N_R - N_L)|N_R, N_L\rangle,$$

$$\hat{H}_z|N_R, N_L\rangle = \hbar\omega_0(\hat{N}_R + \hat{N}_L + \hat{1})|N_R, N_L\rangle = \hbar(N_R + N_L + 1)|N_R, N_L\rangle,$$

where

$$|N_R, N_L\rangle = \frac{(\hat{a}_R)^{N_R}(\hat{a}_L)^{N_L}}{\sqrt{N_R!N_L!}}|0,0\rangle.$$

$\|N_R, N_L\rangle$	E	L_z	
$\|0,0\rangle$	$\hbar\omega_0$	0	
$\|1,0\rangle$	$2\hbar\omega_0$	$\hbar$	(right-hand circulary ket)
$\|0,1\rangle$	$2\hbar\omega_0$	$-\hbar$	(left-hand circularly ket)
$\|2,0\rangle$	$3\hbar\omega_0$	$2\hbar$	

$\|1,1\rangle$	$3\hbar\omega_0$	0
$\|0,2\rangle$	$3\hbar\omega_0$	$-2\hbar$
$\|3,0\rangle$	$4\hbar\omega_0$	$3\hbar$
$\|2,1\rangle$	$4\hbar\omega_0$	$\hbar$
$\|1,2\rangle$	$4\hbar\omega_0$	$-\hbar$
$\|0,3\rangle$	$4\hbar\omega_0$	$-3\hbar$

5. ContourPlot of the probability in the x-y plane

Using the Mathematica, we make a ContourPlot of

$$\left|\langle \xi,\eta | n_x, n_y\rangle\right|^2 = \left|\langle \xi | n_x\rangle\right|^2 \left|\langle \xi | n_y\rangle\right|^2,$$

as a function of ξ and η, where the wavefunction of the simple harmonics is given by

$$\langle \xi | n_x\rangle = 2^{-n_x/2}\pi^{-1/4}(n_x!)^{-1/2}e^{-\xi^2/2}H_{n_x}(\xi),$$

and

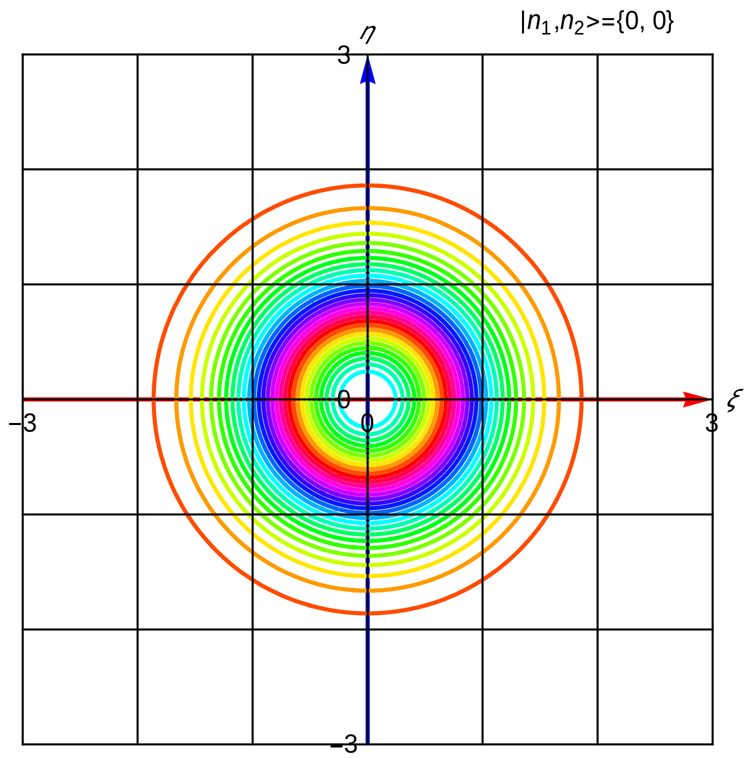
$$\langle \eta | n_y\rangle = 2^{-n_y/2}\pi^{-1/4}(n_y!)^{-1/2}e^{-\eta^2/2}H_{n_y}(\eta).$$

The dimensionless parameters ξ and η are related to the co-ordinates x and y as

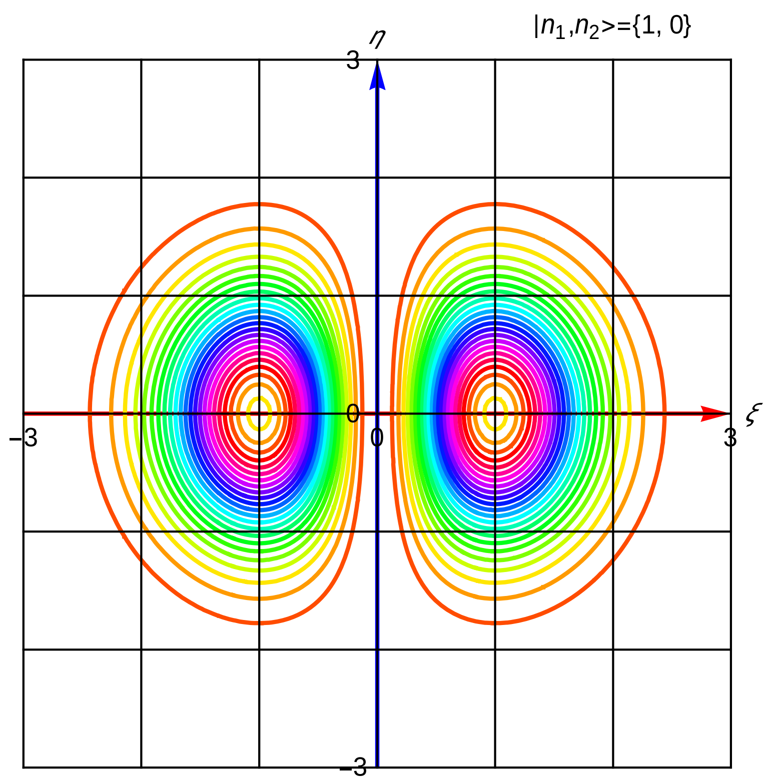
$$\xi = \beta x, \qquad \eta = \beta y$$

with $\beta = \sqrt{\dfrac{m\omega_0}{\hbar}}$.

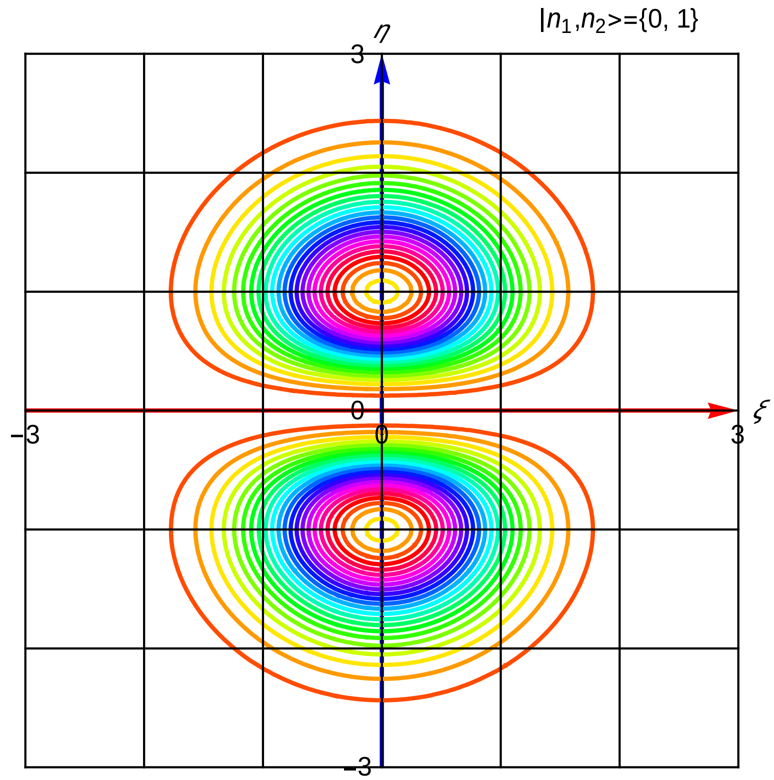
(a) $\left|\langle \xi,\eta | n_x = 0, n_y = 0\rangle\right|^2$



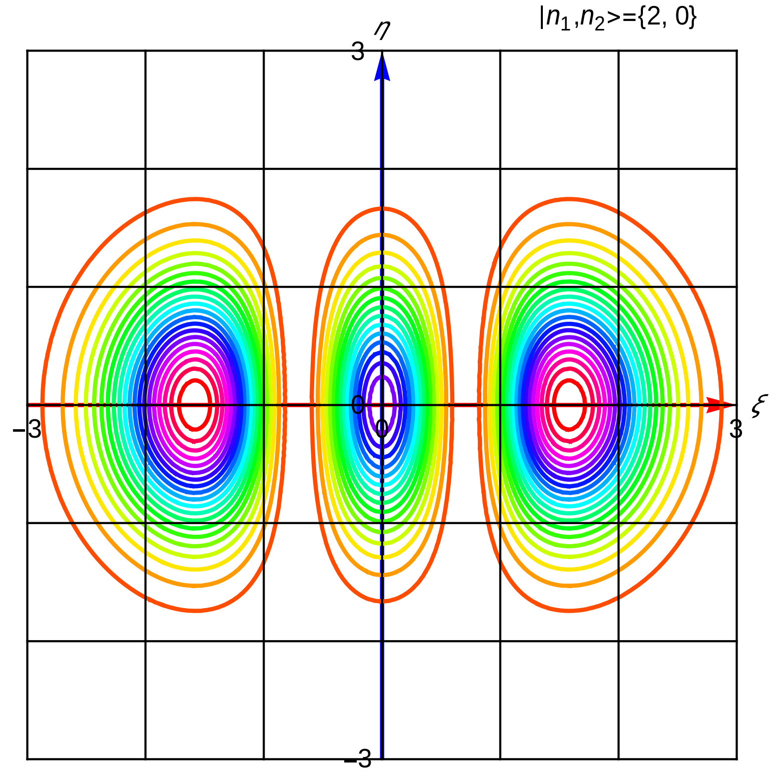
(b) $\left|\left\langle \xi,\eta \middle| n_x = 1, n_y = 0\right\rangle\right|^2$



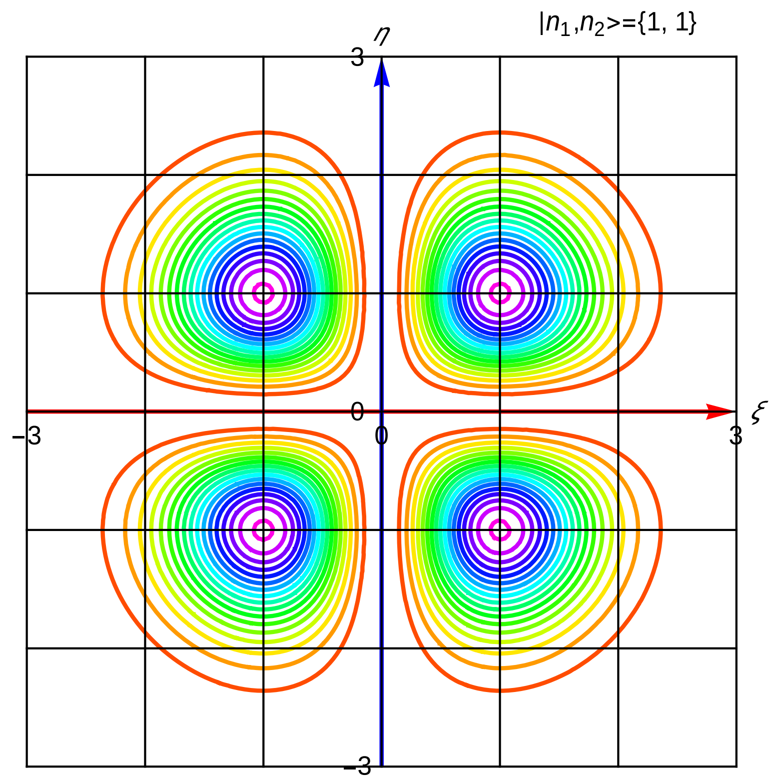
(c) $\left|\left\langle \xi,\eta \middle| n_x = 0, n_y = 1\right\rangle\right|^2$



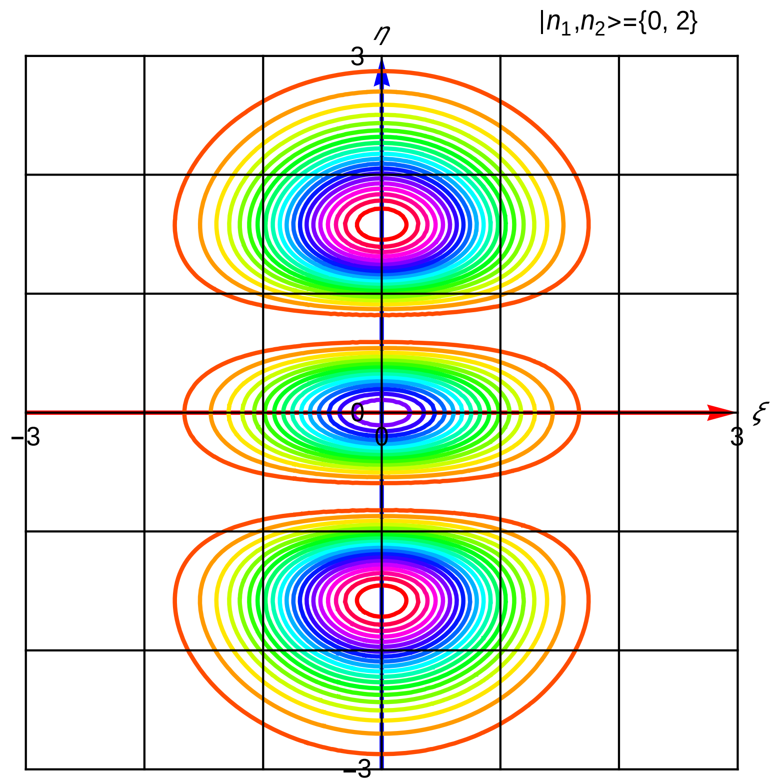
(d) $\left|\left\langle \xi,\eta \middle| n_x = 2, n_y = 0\right\rangle\right|^2$



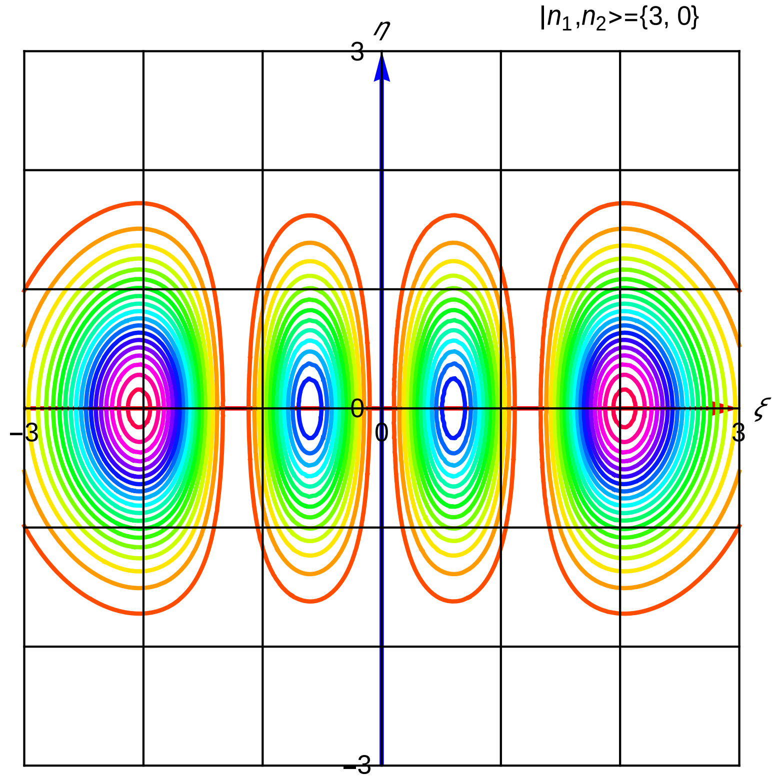
(e) $\left|\left\langle \xi,\eta \middle| n_x = 1, n_y = 1\right\rangle\right|^2$



(f) $\left|\left\langle \xi,\eta \middle| n_x = 0, n_y = 2\right\rangle\right|^2$



(g) $\left|\left\langle \xi,\eta \middle| n_x = 3, n_y = 0\right\rangle\right|^2$



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APPENDIX

Mathematica

```
Clear["Global`*"];
```

Defintion of operators

$$Ax = \frac{\beta}{\sqrt{2}}\left(x\,\# + \frac{\hbar}{m\,\omega} D[\#, x]\right) \&;\ Ay = \frac{\beta}{\sqrt{2}}\left(y\,\# + \frac{\hbar}{m\,\omega} D[\#, y]\right) \&;$$

$$Cx = \frac{\beta}{\sqrt{2}}\left(x\,\# - \frac{\hbar}{m\,\omega} D[\#, x]\right) \&;\ Cy = \frac{\beta}{\sqrt{2}}\left(y\,\# - \frac{\hbar}{m\,\omega} D[\#, y]\right) \&;$$

$$CR := -\frac{1}{\sqrt{2}}\,(Cx[\#] + i\,Cy[\#]) \&;\ AR := -\frac{1}{\sqrt{2}}\,(Ax[\#] - i\,Ay[\#]) \&;$$

$$CL := \frac{1}{\sqrt{2}}\,(Cx[\#] - i\,Cy[\#]) \&;\ AL := \frac{1}{\sqrt{2}}\,(Ax[\#] + i\,Ay[\#]) \&;$$

$$H1 := \hbar\,\omega\,(CR[AR[\#]] + CL[AL[\#]] + \#) \&;\ H0 := \hbar\,\omega\,(Cx[Ax[\#]] + Cy[Ay[\#]] + \#) \&;$$

$$Lz1 := i\,\hbar\,(Ax[Cy[\#]] - Cx[Ay[\#]]) \&;\ Lz2 := \hbar\,(CR[AR[\#]] - CL[AL[\#]]) \&;$$

$$rule1 = \left\{\beta \to \sqrt{\frac{m\,\omega}{\hbar}}\right\};$$

Haniltonian H:

```
H1[f[x, y]] - H0[f[x, y]] /. rule1 // Simplify
```

0

```
H0[f[x, y]] /. rule1 // Simplify
```

$$\frac{m^2\left(x^2+y^2\right)\omega^2 f[x,y] - \hbar^2\left(f^{(0,2)}[x,y] + f^{(2,0)}[x,y]\right)}{2\,m}$$

```
H1[f[x, y]] /. rule1 // Simplify
```

$$\frac{m^2\left(x^2+y^2\right)\omega^2 f[x,y] - \hbar^2\left(f^{(0,2)}[x,y] + f^{(2,0)}[x,y]\right)}{2\,m}$$

Commutation relation

```
AR[CR[f[x, y]]] - CR[AR[f[x, y]]] /. rule1 // Simplify
```

f[x, y]

```
AL[CL[f[x, y]]] - CL[AL[f[x, y]]] /. rule1 // Simplify
```

f[x, y]

Angular momentum Lz:

```
Lz1[f[x, y]] /. rule1 // Simplify
```

$-i\hbar\left(x\, f^{(0,1)}[x, y] - y\, f^{(1,0)}[x, y]\right)$

```
Lz2[f[x, y]] /. rule1 // Simplify
```

$i\hbar\left(-x\, f^{(0,1)}[x, y] + y\, f^{(1,0)}[x, y]\right)$

```
Lz2[f[x, y]] - Lz1[f[x, y]] /. rule1 // Simplify
```

0