Relative orbital angular momentum and parity operator for two identical particles

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
(Date: March 26, 2015)

The two spin-1/2 fermions are in a state of orbital angular momentum *l* in **their center-of-mass frame**. A system of two spin-1/2 fermions is in a state of even orbital angular momentum *l* if its spin state is a singlet, and in a state of odd orbital angular momentum *l* if its spin state is a triplet. It is usual to denote the total spin as *S*, the total orbital angular momentum as *L*, the total angular momentum as *J*, and $^{2S+1}L_J$ the state of the two fermions. For example, a $^3P_2$ state corresponds to $S = 1$, $L = 1$, $J = 2$ and a $^1D_2$ state to $S = 0$, $L = 2$, $J = 2$. The case of two spin-zero bosons is even simpler: only states of even orbital angular momentum are allowed.

1. Hamilton Central force problem

We now consider the quantum mechanics of the central force problem, where there is a central force between two identical particles ($m = m_1 = m_2$).

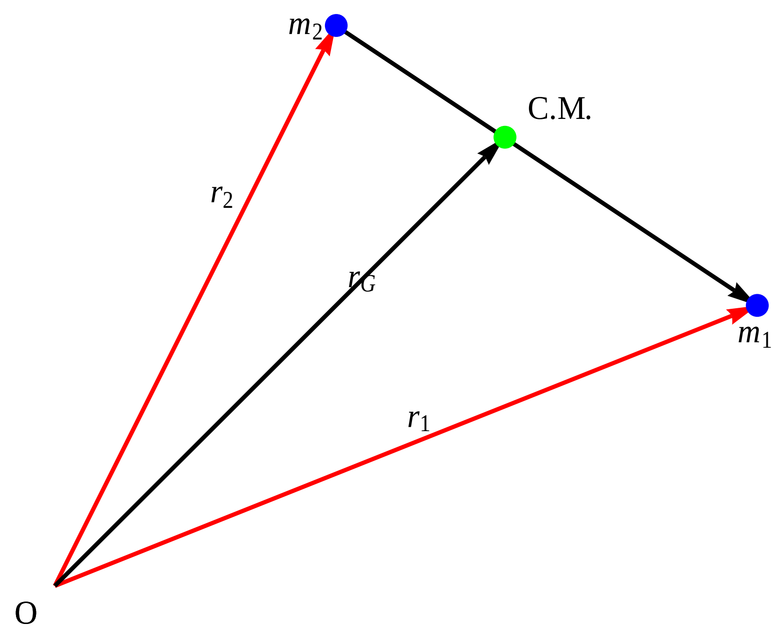

Fig. The central force problem with the identical masses with $m = m_1 = m_2$ There is a central force between two identical masses. The center of mass is at the midpoint between the positions of mass m_1 and mass m_2.

We consider a system of two particles, of masses m_1 and m_2, at positions $\boldsymbol{r}_1$ and r_2, whose interaction is given by a potential $V(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)$. In quantum mechanics, the Hamiltonian is given by

$$\hat{H} = \frac{\hat{\boldsymbol{p}}_1^{\,2}}{2m_1} + \frac{\hat{\boldsymbol{p}}_2^{\,2}}{2m_2} + V(|\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2|)$$

(i) The relative co-ordinate operator:

$$\hat{\boldsymbol{r}} = \hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2 ,$$

(ii) The relative momentum operator:

$$\hat{\boldsymbol{p}} = \frac{m_2 \hat{\boldsymbol{p}}_1 - m_1 \hat{\boldsymbol{p}}_2}{m_1 + m_2} = \frac{1}{2}(\hat{\boldsymbol{p}}_1 - \hat{\boldsymbol{p}}_2) .$$

(iii) The co-ordinate operator for the center of mass:

$$\hat{\boldsymbol{r}}_G = \frac{m_1 \hat{\boldsymbol{r}}_1 + m_2 \hat{\boldsymbol{r}}_2}{m_1 + m_2} = \frac{1}{2}(\hat{\boldsymbol{r}}_1 + \hat{\boldsymbol{r}}_2) .$$

(iv) The momentum operator for the center of mass:

$$\hat{\boldsymbol{p}}_G = \hat{\boldsymbol{p}}_1 + \hat{\boldsymbol{p}}_2 .$$

(v) The total angular momentum operator for the system:

$$\hat{\boldsymbol{L}}_T = \hat{\boldsymbol{L}}_G + \hat{\boldsymbol{L}}$$

with

$$\hat{\boldsymbol{L}}_G = \hat{\boldsymbol{r}}_G \times \hat{\boldsymbol{p}}_G . \qquad \hat{\boldsymbol{L}} = \hat{\boldsymbol{r}} \times \hat{\boldsymbol{p}} \text{ (relative angular momentum)}$$

The Hamiltonian can be rewritten as

$$\hat{H} = \hat{H}_G + \hat{H}_{rel} = \frac{\hat{\boldsymbol{p}}_G{}^2}{2M} + [\frac{\hat{\boldsymbol{p}}^2}{2\mu} + V(|\hat{\boldsymbol{r}}|)]$$

with

$$\hat{H}_G = \frac{\hat{\boldsymbol{p}}_G{}^2}{2M} , \qquad \hat{H}_{rel} = \frac{\hat{\boldsymbol{p}}^2}{2\mu} + V(|\hat{\boldsymbol{r}}|)$$

where M is the total mass and m is the reduced mass,

$$M = m_1 + m_2 = 2m , \qquad \mu = \frac{m_1 m_2}{m1 + m_2} = \frac{m}{2} .$$

Here we have the commutation relations.

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}\hat{1},$$

$$[\hat{x}_i, \hat{p}_{Gj}] = 0$$

$$[\hat{x}_{Gi}, \hat{p}_{Gj}] = i\hbar\delta_{ij}\hat{1}$$

$$[\hat{x}_{Gi}, \hat{p}_j] = 0$$

$$[\hat{p}_{Gi}, \hat{p}_j] = 0$$

In other words, the position and momentum operators of the center of mass and of the relative motion obeys the canonical commutation relations. Any variable associated with the center of mass motion commutes with any variable associated with the relative motion. These commutation relations imply that

$$[\hat{\boldsymbol{p}}_G, \hat{H}_{rel}] = 0, \qquad [\hat{\boldsymbol{p}}_G, \hat{H}] = 0,$$

$$[\hat{H}_G, \hat{H}_{rel}] = 0, \qquad [\hat{H}, \hat{H}_{rel}] = [\hat{H}_G + \hat{H}_{rel}, \hat{H}_{rel}] = 0$$

C0nsequently, we can find the eigenstate of $\hat{H}$, which is a simultaneous eigenstate of $\hat{\boldsymbol{p}}_G$ and $\hat{H}_{rel}$, $|\boldsymbol{p}_G, E_r\rangle$.

$$\hat{H}_G|\boldsymbol{p}_G, E_r\rangle = E_G|\boldsymbol{p}_G, E_r\rangle, \qquad \hat{H}_r|\boldsymbol{p}_G, E_r\rangle = E_r|\boldsymbol{p}_G, E_r\rangle$$

and

$$(\hat{H}_G + \hat{H}_r)|\boldsymbol{p}_G, E_r\rangle = (E_G + E_r)|\boldsymbol{p}_G, E_r\rangle$$

where

$$\hat{H}_G|\boldsymbol{p}_G\rangle = \frac{\boldsymbol{p}_G^{\ 2}}{2M}|\boldsymbol{p}_G\rangle = E_G|\boldsymbol{p}_G\rangle$$

and

$$E_G = \frac{\boldsymbol{p}_G{}^2}{2M}.$$

Since

$$[\hat{H}, \hat{\boldsymbol{p}}_G] = [\hat{H}_G + \hat{H}_r, \hat{\boldsymbol{p}}_G] = [\hat{H}_r, \hat{\boldsymbol{p}}_G] = 0$$

we get

$$\langle \hat{\boldsymbol{p}}_G \rangle = 0$$

For simplicity, we assume that

$$\hat{\boldsymbol{p}}_G = 0$$

Then we have the final form of the Hamiltonian as

$$\hat{H} = \hat{H}_{rel} = \frac{\hat{\boldsymbol{p}}^2}{2\mu} + V(\hat{\boldsymbol{r}}).$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{1}{2} m.$$

2. Parity operator and exchange operator for two identical particles

Here we show that the exchange operator is the same as the parity operator for the two identical particles

(i) Relative co-ordinate operator: $\hat{\boldsymbol{r}} = \hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2$

Using the general relation

$$\hat{P}_{12}\hat{A}_1\hat{P}_{12}{}^{-1} = \hat{A}_2, \qquad \hat{P}_{12}\hat{A}_2\hat{P}_{12}{}^{-1} = \hat{A}_1$$

we get

$$\hat{P}_{12}(\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2)\hat{P}_{12}{}^{-1} = \hat{\boldsymbol{r}}_2 - \hat{\boldsymbol{r}}_1$$

or

$$\hat{P}_{12}\hat{\boldsymbol{r}}\hat{P}_{12}{}^{-1} = -\hat{\boldsymbol{r}}$$

(ii) Relative momentum: $\hat{\boldsymbol{p}} = \dfrac{m_2 \hat{\boldsymbol{p}}_1 - m_1 \hat{\boldsymbol{p}}_2}{m_1 + m_2} = \dfrac{1}{2}(\hat{\boldsymbol{p}}_1 - \hat{\boldsymbol{p}}_2)$

Similarly we have

$$\hat{P}_{12}(\hat{\boldsymbol{p}}_1 - \hat{\boldsymbol{p}}_2)\hat{P}_{12}^{-1} = \hat{\boldsymbol{p}}_2 - \hat{\boldsymbol{p}}_1$$

or

$$\hat{P}_{12}\hat{\boldsymbol{p}}\hat{P}_{12}^{-1} = -\hat{\boldsymbol{p}}$$

Then we can conclude that the exchange operator is equivalent to the parity operator.

$$\hat{P}_{12} = \hat{\pi}$$

since

$$\hat{\pi}\hat{\boldsymbol{r}}\hat{\pi}^{-1} = -\hat{\boldsymbol{r}}, \quad \hat{\pi}\hat{\boldsymbol{p}}\hat{\pi}^{-1} = -\hat{\boldsymbol{p}}$$

and

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \qquad \text{(commutation relation).}$$

(iii) Hamiltonian

$$\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2\mu} + V(|\hat{\boldsymbol{r}}|)$$

$$\hat{P}_{12} = \hat{\pi} \qquad \text{(the parity operator).}$$

Then we have the commutation relation

$$[\hat{H}, \hat{\pi}] = 0$$

So there is an simultaneous eigenket of $\hat{H}$ and $\hat{\pi}$.

$$\hat{H}|\psi\rangle = E|\psi\rangle, \qquad \hat{\pi}|\psi\rangle = \lambda|\psi\rangle$$

Since $\hat{\pi}^2 = \hat{\pi}$, we have

$$\hat{\pi}|\psi_e\rangle = |\psi_e\rangle, \qquad \hat{\pi}|\psi_o\rangle = -|\psi_o\rangle$$

In other words, $|\psi_e\rangle$ has the even parity, while $|\psi_o\rangle$ has the odd parity. Suppose that

$$|\psi\rangle = |n, l, m\rangle$$

We know that

$$\hat{\pi}|l, m\rangle = (-1)^l |l, m\rangle$$

So

(i) The wave function has odd parity for l = 1, 3, 5, …, (=$|\psi_o\rangle$), and

(ii) The wave function has even parity for l = 0, 2, 4, 6, … ($|\psi_e\rangle$.

Then the orbital state with even integer of l has the even parity, while the orbital state with odd integer of l has the odd parity.

3. Classification of the symmetry for superconductivity

When we take into account of the spin states, the symmetry of the resultant eigenket should be anti-symmetric. In other words,

S = 0 (antisymmetric) with l = even.
S = 1 (symmetric) with l = odd.

(i) S-state (BCS Cooper pair)

l =0, S = 0 ^1S j = 1 (3 states)

(ii) P-state (liquid ^{3}He superfluidity)

l =1, S = 1 ^3P j = 2, 1, 0 (9 states; 5+3+1=9)

(iii) D-state (high T_c superconductor)

l =2, S = 0 ^1D j = 2 (5 states)

The onset of superconductivity occurs with the condensation of electron pairs. These electron pairs, called the Cooper pair, can be in a state of either total spin S=0 (spin singlet) or 1 (spin triplet). Being fermions, electrons anticommute. Therefore the antisymmetric spin-singlet state is accompanied by a symmetric orbital wave function (even parity) and vice versa, in order to preserve the anti-symmetry of the total wave function.

4. Deuteron

The spin angular momentum and the orbital angular momentum of deuteron consisting of one neutron and one proton) can be discussed using the above discussion, since the neutron and proton are regarded as identical particles.

Both the proton (p) and the neutron (n) have s spin 1/2. Then the total spin is S = 0 (singlet, antisymmetric) or S = 1 (triplet, symmetric). The relative orbital angular moment of the deuteron is

L = 0, 2, 4, (even parity, or symmetric) or L = 1, 3, 5, … (odd parity, or antisymmetric).

(i) S = 0 (antisymm)

Since the proton and neutron are fermions, the symmetry of the wave function should be antisymmetric. So we have L = 0, 2, … (symmetric). In this case the total angular momentum is

$D_0 \times D_0 = D_0$ (S = 0, L = 0, J =0). ${}^1S_0$

$D_0 \times D_2 = D_2$ (S = 0, L = 2, J =2) ${}^1D_2$

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(ii) S = 1 (symm)

In this case, we have L = 1, 3, … (antisymmetric). The total angular momentum is

$D_1 \times D_1 = D_2 + D_1 + D_0$ (S = 1, L = 1, J =2, 1, 0). ${}^3P_{2,1,0}$

$D_1 \times D_3 = D_4 + D_3 + D_2$ (S = 1, L = 3, J =4, 3, 2) ${}^3F_{4,3,2}$
