Deuteron

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Deuteron (nuclei of deuterium) consisting of one neutron and one proton

Spin angular momentum $S = 1$ (in the units of $\hbar$)
Orbital angular momentum $L = 0$
The total angular momentum: $J = 1$,
The intrinsic parity of the deuteron is +1.
Isospin $T = 0$: the state is given by $|0,0\rangle$
Magnetic moment: $\mu_{\text{deuteron}} = 0.857436\ \mu_{\text{N}}$
The nuclear binding energy: $B_d = 2.224$ MeV.

1. Deuteron and deuterium

Deuteron:

Nucleus of deuterium (heavy hydrogen) that consists of one proton and one neutron (the notation: *d* or *D*). Deuterons are formed chiefly by ionizing deuterium (stripping the single electron away from the atom) and are used as projectiles to produce nuclear reactions after accumulating high energies in particle accelerators. A deuteron also results from the capture of a slow neutron by a proton, accompanied by the emission of a gamma photon.

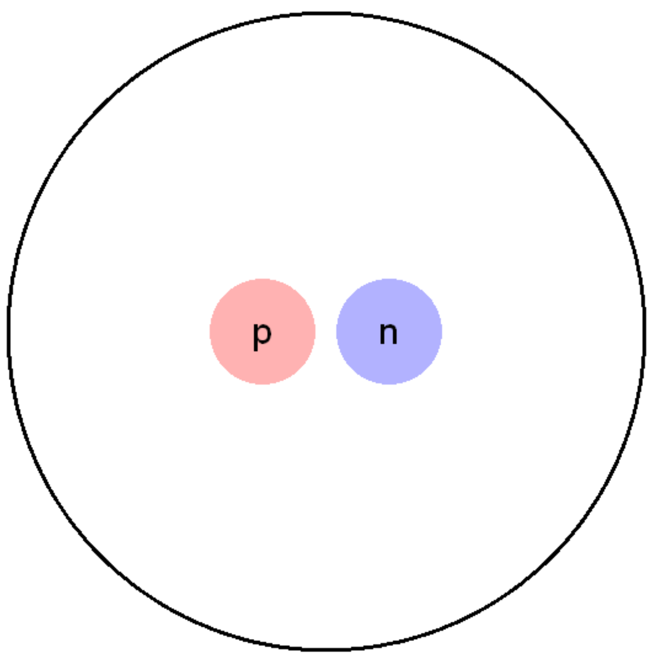


Fig. Deuteron consisting of one proton and one neutron.

Deuterium (heavy hydrogen, ^{2}H): A heavy isotope of hydrogen that consists of one proton, one neutron, and one electron; The nuclei of deuterium, called *deuterons* (d), consist of one proton (p) and one neutron (n).

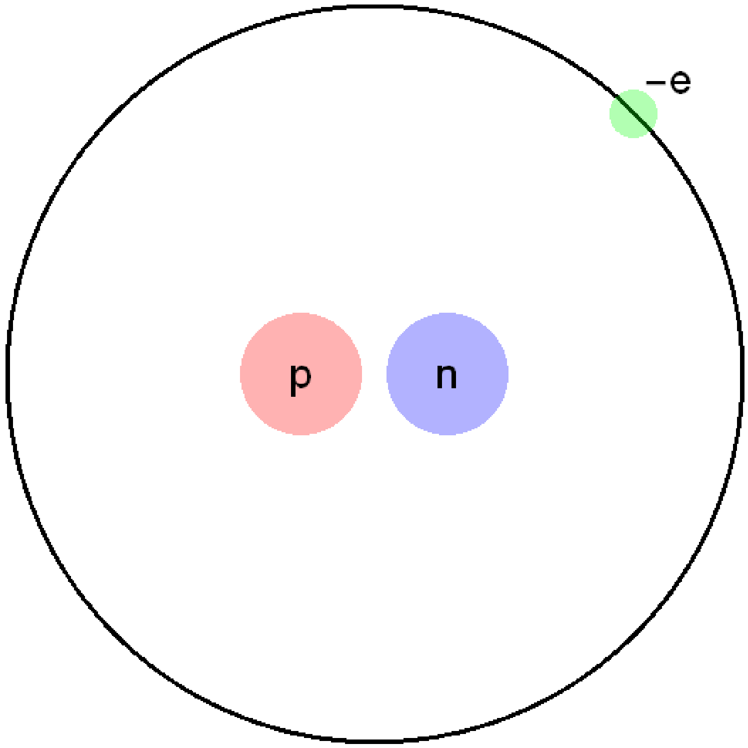


Fig. Deuterium (heavy hydrogen), consisting of one proton, one neutron, and one electron.

((**Note**)) **Deuteron and Big Bang model**

The stability of the deuteron is an important part of the story of the universe. In the Big Bang model it is presumed that in early stages there were equal numbers of neutrons and protons since the available energies were much higher than the 0.78 MeV required to convert a proton and electron to a neutron. When the temperature dropped to the point where neutrons could no longer be produced from protons, the decay of free neutrons began to diminish their population. Those which combined with protons to form deuterons were protected from further decay. This is fortunate for us because if all the neutrons had decayed, there would be no universe as we know it, and we wouldn't be here!

2. Isotopic spin

The charge independence of nuclear forces allows us to consider the proton and the neutron as different states of the same particles, or nucleon. Thus, in addition to position and spin variables, the state of nucleon requires for its specification a "charge" variable which can take two values. Therefore in the study of nucleon systems a formalism similar to the one developed for the eigenstates of spin 1/2 can be used. It is called the isotopic spin formalism.

The neutron and proton are very similar apart from a small mass difference (0.1%) and of course the difference in electric charge. Both play an equally important role in determining the properties of nuclei. Heisenberg (1932) postulates that *n*, *p* are two sub-states of a nucleon with iso-spin 1/2 by analogy with ordinary spin *s* =1/2.

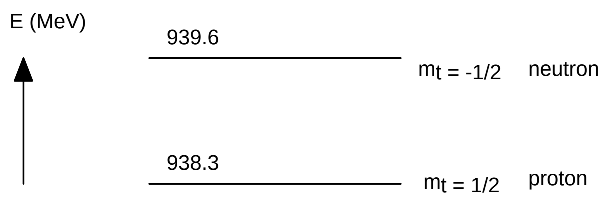
Briefly, the base vectors of a 2D isotopic spin space can be defined to be

$$|p\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and the operator representing isotopic spin and nucleon charge are then

$$\hat{\boldsymbol{T}} = \frac{1}{2}\hat{\boldsymbol{\tau}}, \qquad \hat{Q} = \frac{1}{2}(1+\hat{\tau}_3)$$

where the operator of $\hat{\boldsymbol{\tau}} = (\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3)$ on the base vector *s* can be represented by Pauli matrices.



Both neutrons and protons have spin *s* = 1/2. *s* and *T* are independent quantum number. *T* has no classical analog. It is a quantum mechanical vector, literally "like spin" (iso=like). So it follows the same addition as *s*, *l*, *j*, etc.

3. $^3S_1$ state and $^3D_1$ state in deuteron

Deuteron consists of one proton and one neutron where proton (fermion) has spin 1/2 and neutron (fermion) has spin 1/2. The intrinsic parity of deuteron is +1, since the intrinsic parity of proton and neutron is +1. The intrinsic parity of the deuteron is given by $(+1)(+1)(-1)^l = +1$. Thus we have

$$l = 0, 2, 4, 6, ...$$

where *l* is the relative orbital angular momentum.

The total angular momentum of deuteron is given by

$$\boldsymbol{j} = \boldsymbol{l} + \boldsymbol{s}$$

where $\boldsymbol{l}$ is the relative angular momentum of the neutron and proton. *s* is the spin angular momentum. For the resultant spin,

$D_{1/2} \times D_{1/2} = D_1 + D_0$ $\qquad$ $s = 1$ (symm) or $s = 0$ (antisymm)

Of the possible quantum numbers, $l = 0$ has the lowest energy. This value of orbital angular momentum $l = 0$ combines with these spin values.

$s = 1$ (symm),	$l = 0$ (symm):	$j = 1$	$^3S_1$
$s = 0$ (antisymm),	$l = 0$ (symm):	$j = 0$	$^1S_0$

as well as

$s = 1$ (symm),	$l = 2$ (symm) :	$j = 3, 2, 1$	$^3D_{1,2,3}$
$s = 0$ (antisymm),	$l = 2$ (symm) :	$j = 2$	$^1D_2$

Note that isospin of the deuteron is $T = 0$ (antisymmetric) or $T = 1$ (antisymmetric).

$D_{1/2} \times D_{1/2} = D_1 + D_0.$

The total wave function for two identical fermions has to be antisymmetric with respect to particle exchange

$$|\psi\rangle = |\psi_{space}\rangle |\psi_{spin}\rangle |\chi_{isospin}\rangle$$

$s = 1$ (symm),	**$l = 0$ (symm):**	**$T = 0$ (antisymm)**	**$^3S_1$**
$s = 1$ (symm),	**$l = 2$ (symm):**	**$T = 1$ (symm)**	**$^3D_1$**

Thus the total wavefunction is antisymmetic under the exchange of neutron and proton. The isospin of deuteron is given by $|0,0\rangle$.

4. Magnetic moment of deuteron

In fact the ground state is a mixture of $\sqrt{0.96}$ $^3S_1$ and $\sqrt{0.04}$ $^3D_1$.

or

$$|\psi_G\rangle = a_S |^3S_1\rangle + a_D |^3D_1\rangle$$

with $a_S = \sqrt{0.96} = 0.98$ and $a_D = \sqrt{0.04} = 0.2$.

As $a_D << a_S$, for the purpose of calculating the magnetic moment of the deuteron, it suffices to take $|\psi_G\rangle$ to be entirely an S state (l = 0).

((**Note**))

μ_{proton} = 2.79 μ_N (I. Rabi)

$\mu_{deuteron}$ = 0.8573 μ_N

$\mu_{deuteron} = \mu_{neutron} + \mu_{proton}$

or

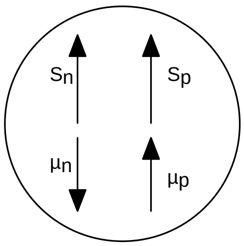
0.8573 μ_N = 2.79 μ_N + $\mu_{neutron}$

Then we have

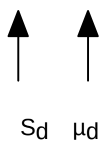
$\mu_{neutron}$ = -1.93 μ_N

The negative sign shows that the direction of the magnetic moment is opposite to that of spin.

Neutron and proton



Deuteron



5. Binding energy of deuteron

^1H Hydrogen
^2H Deuterium
d: Deuteron

p: proton
n: neutron

The binding energy of deuteron, B_d

$$M(^2H) = m_n + M(^1H) - \frac{B_d}{c^2}$$

where

$$m_n = 1.008665u \quad \text{neutron mass}$$
$$M(^1H) = 1.007825u \quad \text{atomic hydrogen mass}$$
$$M(^2H) = 2.014102u \quad \text{atomic deuterium mass}$$

The nuclear binding energy is obtained as

$$B_d = 2.224\ \text{MeV}.$$

We find that the atomic electron binding energy of 13.6 eV is much smaller than the nuclear binding energy of 2.2 MeV.

This binding energy can be determined from the photodisintegration or a photonuclear reaction. We scatter gamma rays (photons) from deuterium gas and look for the breakup of a deuteron into a neutron and a proton;

$$\gamma + d \rightarrow n + p .$$

The minimum energy required for the photodisintegration is given by

$$hf_{\min} = B_d[1 + \frac{B_d}{2M(^2H)c^2}] \approx B_d .$$

Experiment shows that a photon of energy less than 2.22 MeV cannot dissociate a deuteron. This energy is necessary to separate the two nucleons completely. No excited states of the deuteron have been found, and it is surmised that the deuteron has only one bound state.

6. Symmetric states of the nuclear motion of D_2

D has spin 1 (boson)

s = 2 and *s* = 0 (symmetric)

$\boxed{}\boxed{}$: $\boxed{1}\boxed{1}$, $\boxed{1}\boxed{2}$, $\boxed{1}\boxed{3}$, $\boxed{2}\boxed{2}$, $\boxed{2}\boxed{3}$, $\boxed{3}\boxed{3}$,

6 states (j = 2 and 0)

m=2 m=1 m=0 m=0 m=-1 m=-2

s = 1 (anti-symmetric

$\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$: $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$, $\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}$, $\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array}$, 3 states ($j$=1)

m=1 m=0 m=-1

The relative angular momentum

$$\hat{\pi}|l,m\rangle = \hat{P}|l,m\rangle = (-1)^l |l,m\rangle$$

$|l,m\rangle$: symmetric for even l and anti-symmetric for odd l.

D is boson: the wave function should be symmetric.

$|l_{even}, m_l\rangle |s=2, m_s\rangle$: 5 states

or

$|l_{even}, m_l\rangle |s=0, m_s=0\rangle$: 1 state (ortho state)

$|l_{odd}, m_l\rangle |s=1, m_s\rangle$: 3 states (para state)

7. The Schrödinger equation for Dueteron

A deuteron (^{2}H nucleus) consists of neutron and a proton. It is the simplest bound state of nucleons and therefore gives us an ideal system of studying the nucleon-nucleon interaction. The binding energy of the deuteron is

2.23 ± 0.03 MeV.

What is the nuclear force holding neutron and proton together? This force cannot be electrical as the neutron is uncharged. So we must accept the nuclear force as a new type of force and try to find out more about it. We assume a central force. The interaction potential of neutron and proton is given by a function $V(r)$, where r is the distance between the particles. Second, the nuclear force has a short range.

m is the reduced mass of proton and neutron.

$$m = \frac{M_n + M_p}{M_n + M_p} = \frac{M}{2}$$

The Schrödinger equation for the wave function $\psi = RY_l^m$:

$$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial}{\partial r}r\frac{1}{r}\frac{\partial}{\partial r}(rR) + \frac{\hbar^2 l(l+1)}{2mr^2}R + V(r)R = ER$$

Suppose that

$$u = rR$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 u}{\partial r^2} + [\frac{\hbar^2 l(l+1)}{2mr^2} + V(r)]u = Eu$$

When l = 0 (S-wave), we have

$$-\frac{\hbar^2}{2m}\frac{d^2 u}{dr^2} + V(r)u = Eu$$

with

$$E = -\frac{\hbar^2\alpha^2}{2m} \quad \text{(bound state)}$$

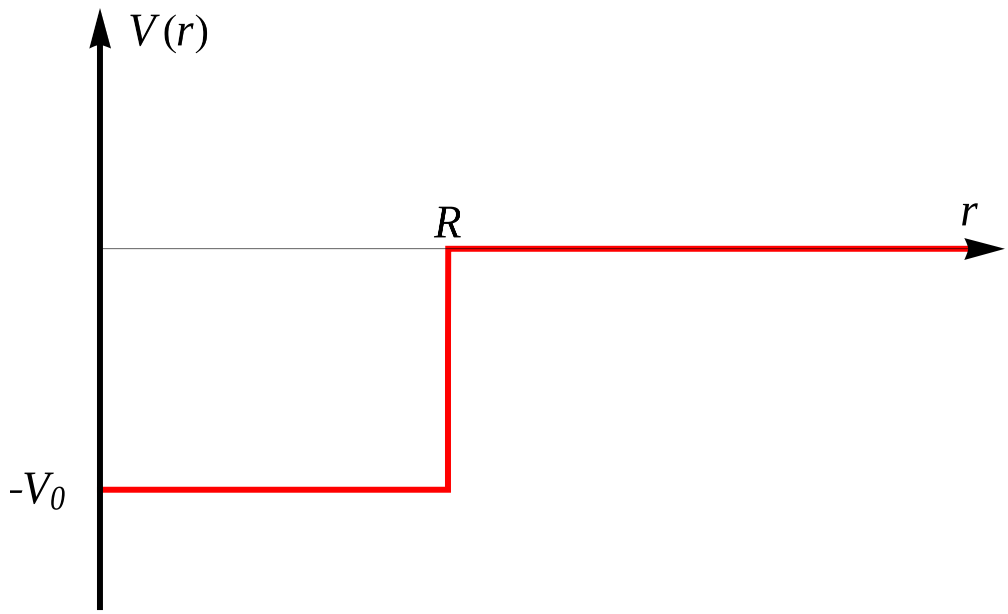


Fig. Attractive square-well potential well for deuteron. V_0 = 35 MeV. The binding energy is -2.2 MeV.

We assume a square-well potential as $V(r)$. There are two parameters, width R and the depth of the well ($-V_0$).

$$V(r) = \begin{cases} -V_0 & r < R \\ 0 & r > R \end{cases}$$

(i) Schrödinger equation for $r<R$ ($l = 0$)

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} - V_0 u = Eu = -\frac{\hbar^2\alpha^2}{2m}u$$

or

$$\frac{d^2u}{dr^2} + \frac{2mV_0}{\hbar^2}u = \alpha^2 u$$

or

$$u'' + (-\alpha^2 + k_0^{\ 2})u = 0$$

or

$$u'' + \kappa^2 u = 0$$

where

$$\frac{2mV_0}{\hbar^2} = U_0 = k_0{}^2,$$

$$\kappa^2 = -\alpha^2 + k_0{}^2$$

Then we get the solution as

$$u = B\sin(\kappa r)$$

with the boundary condition, $u(r) = 0$.

This means that *u* has the first peak at

$$r = r_{\max} = \frac{\pi}{2\kappa}.$$

The condition for the existence of the bound state is that

$$r_{\max} < R$$

or

$$\kappa R > \frac{\pi}{2}.$$

(ii) Schrödinger equation for *r*>*R*

$$u'' - \alpha^2 u = 0,$$

or

$$u = Ae^{-\alpha r}$$

The boundary condition at *r* = *R*, leads to

$$B\sin(\kappa R) = Ae^{-\alpha R}$$

$$B\kappa\cos(\kappa R) = -A\alpha e^{-\alpha R}$$

Then we have

$$\kappa R\cot(\kappa R) = -\alpha R$$

with

$$\kappa^2 + \alpha^2 = k_0{}^2$$

We determine the constants *A* and *B*.

$$A = \frac{e^{\alpha R}\sqrt{\kappa R\alpha R}\sin(\kappa R)}{\sqrt{\pi R}\sqrt{2\kappa R\alpha R + 2\kappa R\sin^2(\kappa R) - 2\alpha R\sin(2\kappa R)}}$$

$$B = \frac{\sqrt{\kappa R\alpha R}}{\sqrt{\pi R}\sqrt{2\kappa R\alpha R + 2\kappa R\sin^2(\kappa R) - 2\alpha R\sin(2\kappa R)}}$$

from the normalization of the wave function such that

$$4\pi\int R^2 r^2 dr = 4\pi\int u^2 dr = 1.$$

In the limit of $\alpha R \to 0$,

$$A = \sqrt{\frac{\alpha}{2\pi}}, \qquad B = \frac{\sqrt{\dfrac{\alpha}{2\pi}}}{\left|\sin(\kappa R)\right|}.$$

8. Numerical solution for the Schrodinger equation (*l* = 0)

We solve the problem using the Mathematica.

$$x = \kappa R, \qquad y = \alpha R$$

In this case,

$$y = -x\cot x, \qquad x^2 + y^2 = k_0{}^2 R^2 = a_0{}^2 \quad \text{(circle)}$$

where

$$a_0 = k_0 R, \qquad U_0 = \left(\frac{\pi}{2R}\right)^2,$$

$$\frac{2mV_0}{\hbar^2} = U_0 = k_0{}^2$$

We make a plot of these two curves, where a_0 is varied as a parameter. The intersection of two curves is the solution of the values of x and y for the fixed value of a_0. The results are as follows.

(i) $\frac{\pi}{2} < a_0 < \frac{3\pi}{2}$ one bound state.

(ii) $\frac{3\pi}{2} < a_0 < \frac{5\pi}{2}$ two bound states.

(iii) $\frac{5\pi}{2} < a_0 < \frac{7\pi}{2}$ three bound states.

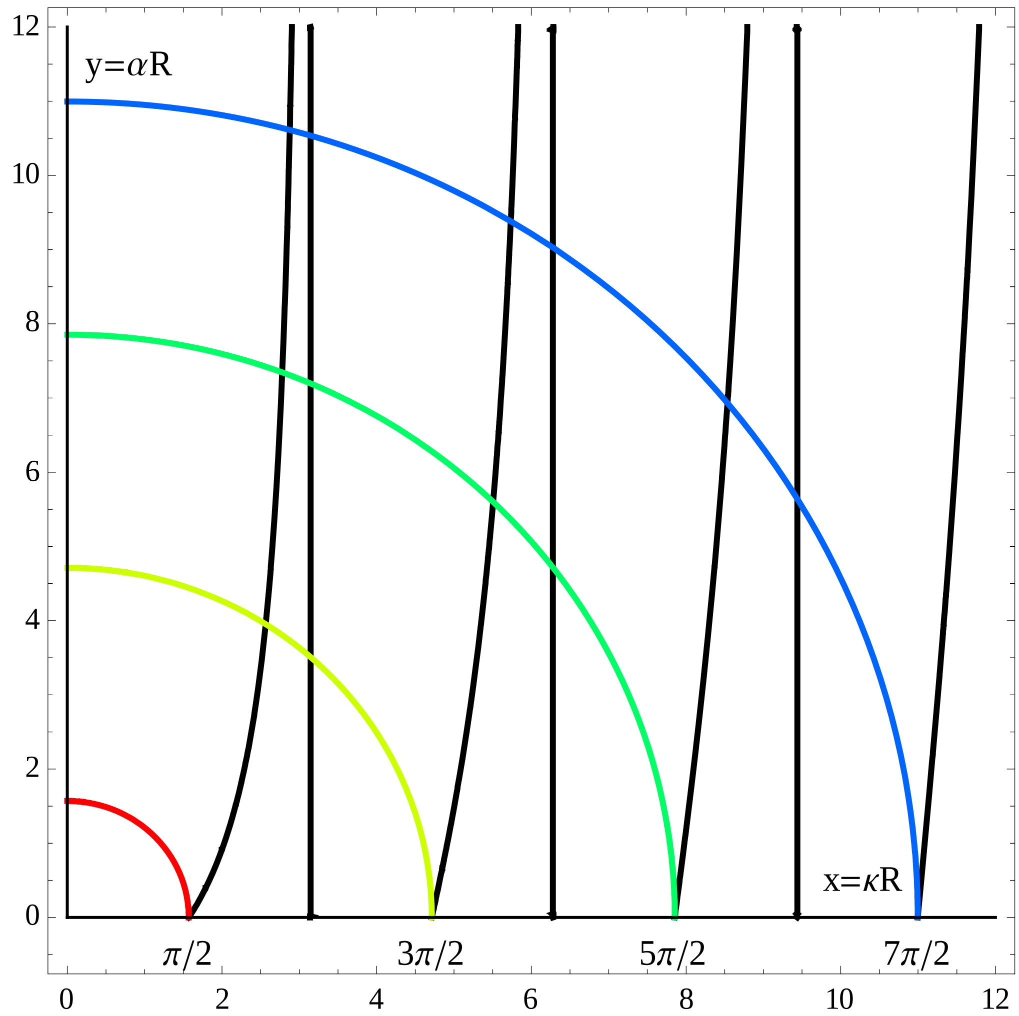


Fig. Plot of $y = -x \cot x$ (red line) and $x^2 + y^2 = a_0{}^2 = (k_0 R)^2$. a_0 = $\pi/2$, a_0 = $3\pi/2$ (light green line), a_0 = $5\pi/2$ (green line), and a_0 = $7\pi/2$ (blue line), $x = \kappa R$. $y = \alpha R$. One bound state for a_0 = $\pi/2$. Two bound states for a_0 = $3\pi/2$. Three bound states for $a_0 = 5\pi/2$) As a_0 increases, the potential well becomes deep.

((**Note**))
The simplest way to determine the condition for the existence for the bound state is as follows.

We start with

$$y = -x\cot x\ .$$

When $y = 0$, we have

$$x\cot x = 0$$

or

$$x = \pi/2,\ 3\pi/2,\ 5\pi/2, ...$$

9. The form of wave function for $r>R$ ($l = 0$)

For $r>R$, we have the wave function as

$$u = Ae^{-\alpha r}$$

with

$$A = \frac{e^{\alpha R}\sqrt{\kappa R\alpha R}\sin(\kappa R)}{\sqrt{\pi R}\sqrt{2\kappa R\alpha R + 2\kappa R\sin^2(\kappa R) - 2\alpha R\sin(2\kappa R)}}$$

This function is a typical exponential decay. The quantity $1/\alpha$ can be taken as a measure of the size of the deuteron. It was shown above that the radius of the deuteron is considerably larger than the range of nuclear forces, i.e.,

$$\frac{1}{\alpha} >> R$$

Thus most of the area under $u(r)$ occurs for $r>R$.

10. Schrödinger equation for higher $l = 1$.

Here we show that the deuteron has no bound excited states for states of higher l.

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + [\frac{\hbar^2 l(l+1)}{2mr^2} + V(r)]u = Eu$$

where

$$E = -\frac{\hbar^2}{2m}\alpha^2, \qquad U_0 = \frac{2m}{\hbar^2}V_0 = k_0{}^2$$

The effective potential is given by

$$V_{eff} = \frac{\hbar^2 l(l+1)}{2mr^2} + V(r)$$

For $r<R$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + [\frac{\hbar^2 l(l+1)}{2mr^2} - V_0]u = -\frac{\hbar^2}{2m}\alpha^2 u$$

or

$$\frac{\partial^2 u}{\partial r^2} + [-\frac{l(l+1)}{r^2} + U_0]u = \alpha^2 u$$

11. The case for *l* = 1

We consider the case for l = 1 (P=wave). Using the Mathematica, the solution for u is obtained as

$$u = B[\frac{\sin(\kappa r)}{\kappa r} - \cos(\kappa r)]$$

with $u(0) = 0$ (boundary condition)

$$\kappa^2 = U_0 - \alpha^2 = k_0{}^2 - \alpha^2$$

For $r>R$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \frac{\hbar^2 l(l+1)}{2mr^2}u = -\frac{\hbar^2}{2m}\alpha^2 u$$

or

$$\frac{d^2u}{dr^2} - \frac{l(l+1)}{r^2}u = \alpha^2 u$$

The solution of u is

$$u = Ae^{-\alpha r}\left(\frac{1}{\alpha r} + 1\right)$$

The function u and its derivative should be continuous at $r = R$

$$\frac{\kappa^2 R^2 \sin(\kappa R)}{\sin(\kappa R) - \kappa R \cos(\kappa R)} = -\frac{\alpha^2 R^2}{1 + R\alpha}$$

We put

$$x = \kappa R, \qquad y = \alpha R$$

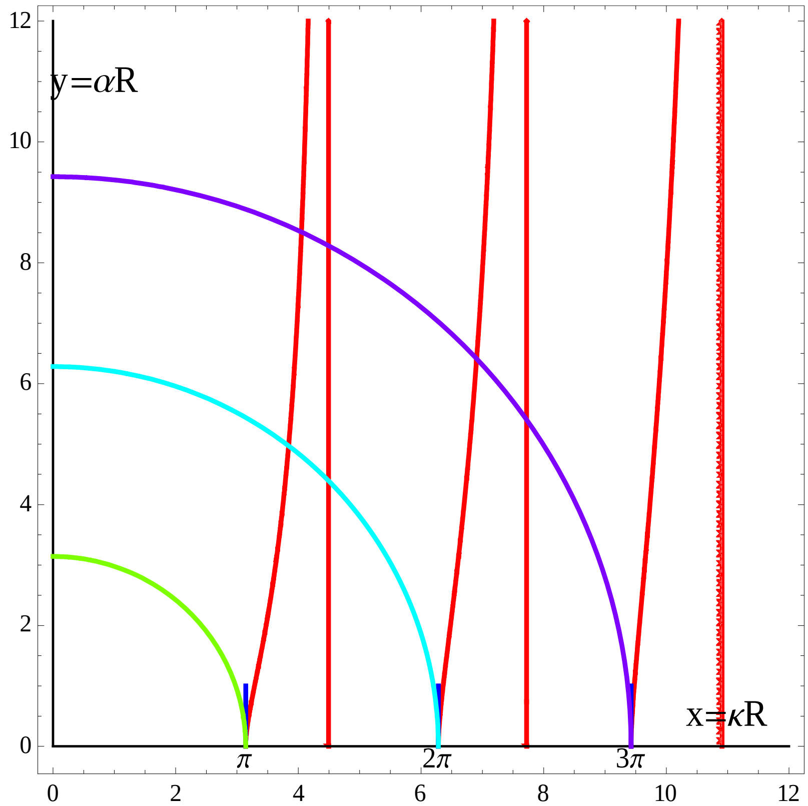
$$\frac{x^2 \sin(x)}{\sin(x) - x\cos(x)} = -\frac{y^2}{1 + y}$$

with

$$x^2 + y^2 = a_0^{\ 2} \qquad \text{(circle)}$$

We make a plot of these two curves, where a_0 is varied as a parameter. The intersection of two curves is the solution of the values of x and y for the fixed value of a_0. The results are as follows.

(i) $\pi < a_0 < 2\pi$ one bound state.
(ii) $2\pi < a_0 < 3\pi$ two bound states.
(iii) $3\pi < a_0 < 4\pi$ three bound states.



From this figure, we have the bound state under the condition that $x > \pi$.

$$\frac{2m}{\hbar^2}V_0R^2 = \frac{M_n}{\hbar^2}V_0R^2 = a_0{}^2 = k_0{}^2R^2 > \pi^2$$

((**Note-1**))

This required well depth V_0 is almost four times as large as the actual well depth in the ground state

$$\frac{2m}{\hbar^2}V_0R^2 = \frac{M_n}{\hbar^2}V_0R^2 = a_0{}^2 = k_0{}^2R^2 \approx \frac{\pi^2}{4}.$$

Note that a_0 is slightly larger than $\pi/2$ but definitely less than π.

((**Note-II**))

The simplest way to determine the condition for the existence for the bound state is as follows.

We start with

$$\frac{x^2\sin(x)}{\sin(x) - x\cos(x)} = -\frac{y^2}{1+y}.$$

When $y = 0$, we have

$$x \sin x = 0$$

or

$$x = \pi, 2\pi, 3\pi, ...$$

12. The bound state of deuteron

There are three parameters to be determined,

(i) V_0 is the depth of the square-well potential.
(ii) R is the width of the potential.
(iii) E is the energy of bound state.

Here we assume that the values of E and R are given. We determine the value of V_0. We show that the depth of the potential as

$$V_0 = 33.73 \text{ MeV}.$$

which is close to the accepted value $V_0 = 35$ MeV.

Suppose that the binding energy is $E = -2.23$ MeV and the width of the potential is $R = 2.1$ fm, where

$$\text{fm} = 10^{-13} \text{ cm (fm: Fermi).}$$

We use the following relations

$$y = -x \cot x, \qquad x^2 + y^2 = k_0{}^2 R^2 = a_0{}^2$$

$$x = \kappa R, \qquad y = \alpha R = \sqrt{\frac{2m|E|}{\hbar^2}} R = 0.231969 \ R[\text{fm}]$$

where

$$E = -\frac{\hbar^2 \alpha^2}{2m} = -2.24 \text{ MeV} \qquad \text{(energy of bound state)}$$

$$a_0{}^2 = k_0{}^2 R^2 = \frac{2m}{\hbar^2} V_0 R^2 = \frac{M_n}{\hbar^2} V_0 R^2$$

and m is the reduced mass, $m = \dfrac{M_n}{2}$ (M_n is the mass of neutron).

((**Mathematica**))

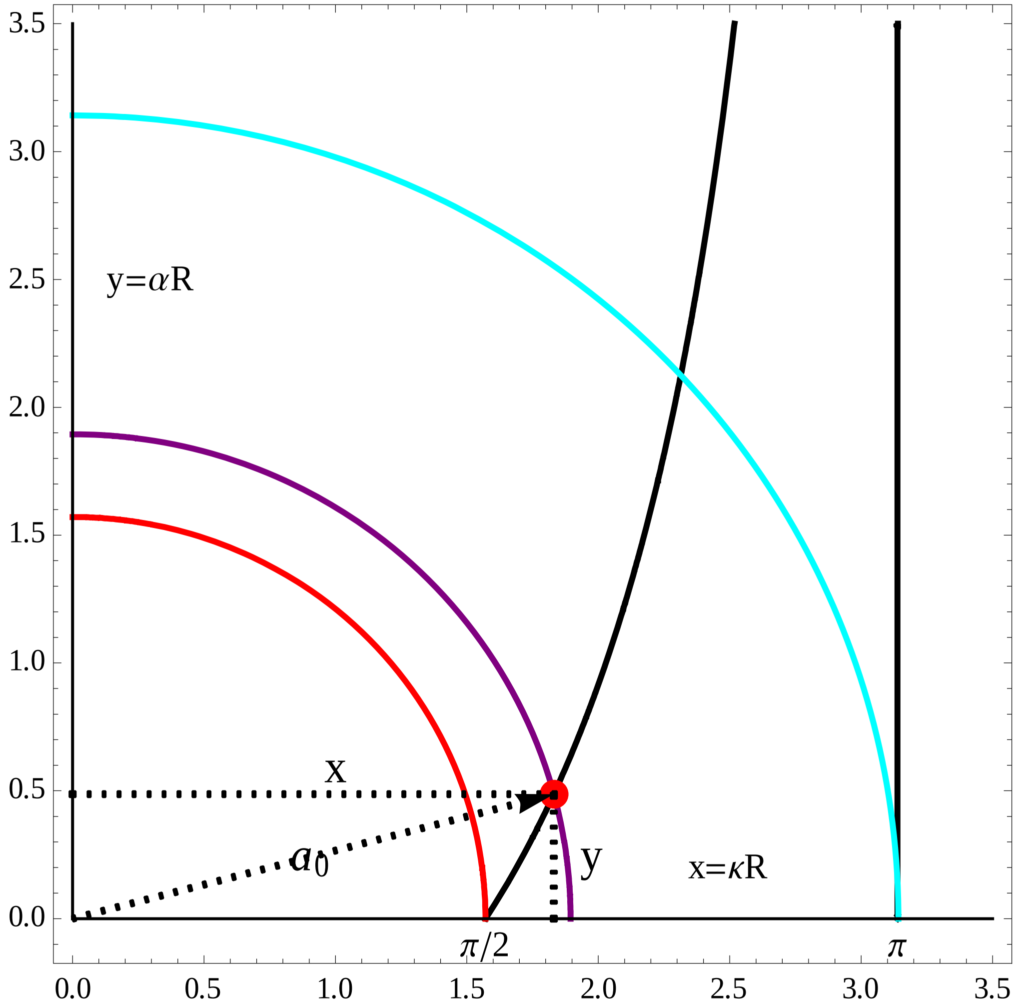


Fig. Graphical solution (x = κR, y = αR) of the bound state of deuteron. $|E| = 2.23$ MeV. $R = 2.1$ fm. The value of V_0 is obtained as 33.73 MeV. The red point denote the solution of x and y. The black thick line is $y = -x\cot x$. The purple circle is denoted by $x^2 + y^2 = a_0^{\,2}$ with a_0 = 1.89454.

For $|E| = 2.23$ MeV. $R = 2.1$ fm.

$$y = \alpha R = \sqrt{\frac{2m|E|}{\hbar^2}}R = 0.231969\,R[\text{fm}] = 0.487134.$$

From the relations, $y = -x\cot x$ and $x^2 + y^2 = a_0^{\,2}$, we get

x = 1.83084. a_0 = 1.89454,

Then we have

$$a_0 = \sqrt{\frac{2m}{\hbar^2}V_0 R^2} = 0.15534\sqrt{V_0(MeV)}R(fm).$$

where $m = M_n/2$. Since R = 2.1 fm, we have

V_0 = 33.73 MeV.

13. Wave function of the bound state for deuteron

We make a plot of the wave function (u/u_{max}) of the bound state for deuteron. Note that u is proportional to $\sin(\kappa r)$ for $r<R$. The function u has a peak at

$$\frac{r}{R} = \frac{\pi/2}{\kappa R} = \frac{\pi/2}{x} = 0.858$$

where x = 1.83084. For $r>R$, the wave function is proportional to $e^{-\alpha r}$. The characteristic distance is

$$\frac{1}{\alpha} = \frac{R}{\alpha R} = \frac{R}{y} = 2.05R = 4.3 \text{ fm.}$$

where y = 0.487134 and R = 2.1 fm.

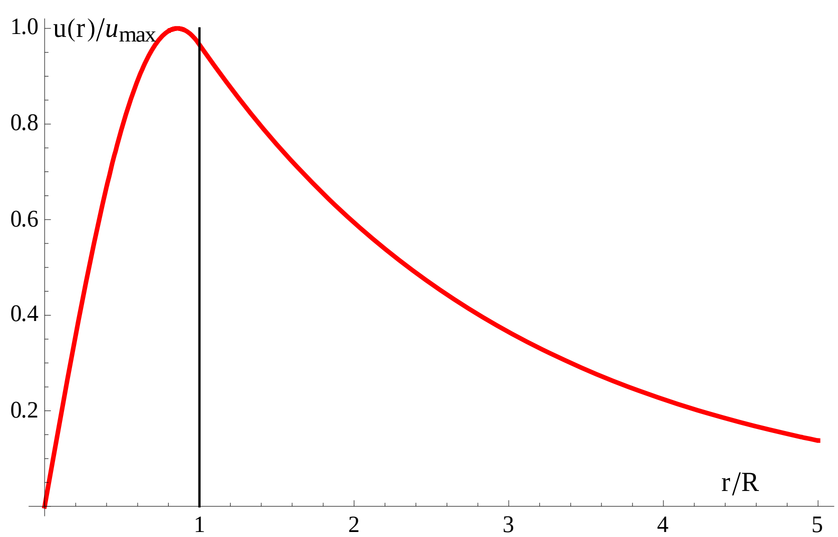


Fig. The plot of $u(r)/u_{max}$ as a function of r/R. R = 2.1 fm. E = -2.23 MeV

14. Relation between a_0 and R

In order to discuss the relation between a_0 and R, we use

$$a_0 = \sqrt{2.41298\times 10^{-2} V_0 R^2} = 1.5534\times 10^{-1}\ \sqrt{V_0[MeV]}R[fm]$$

where V_0 is in the unit of MeV and R is the unit of fm. We assume that V_0 is given as V_0 = 35 MeV. The plot of a_0 vs R is shown below. When R_0 = 2.1 fm, a_0 is beween $\pi/2$ and π. This means that there is no bound state for l = 1, 2,.... There is only one bound state for l = 0.

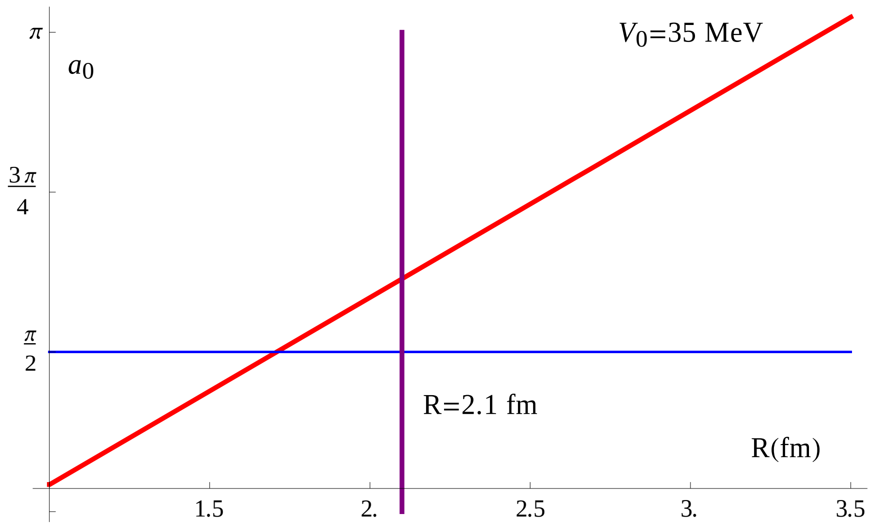


Fig. The relation between a_0 and R [fm]. V_0 = 35 MeV.

14. Pion

14.1. Yukawa's idea ((Eisberg and Resnick))

In 1935, Yukawa proposed that a nucleon frequently emits a particle with an appreciable rest mass, now called a *n meson* or *pion*. This particle hovers near the nucleon in the so-called π *meson field* for a very short time, and then is absorbed by the nucleon. During the process the nucleon maintains its normal rest mass, and so while it is happening there is a violation of the law of mass-energy conservation because there is more rest mass present than there is before the π meson is emitted or after it is absorbed. The energy-time uncertainty principle shows, however, that such a violation is not impossible if it lasts for a sufficiently short time. Of course, the meson cannot permanently escape the nucleon because that would permanently violate the mass-energy conservation law. Such a pion is called a *virtual particle* because it has a very short existence limited by its violation of mass-energy conservation. If two nucleons are close enough for their meson fields to overlap, it is possible for a meson to leave one field and join the other, without permanently changing the total energy of the system of two nucleons. In the interaction, the momentum carried by the π meson is transferred from one field to the other, and therefore from one nucleon to the other. But if momentum is transferred, the effect is the same as if a force is acting between the nucleons. Thus the *exchange* of a virtual pion between two nucleons leads to the nucleon force acting between them, according to Yukawa. Yukawa was guided by two analogies available to him at the time. One is the covalent binding in the H_2^+ molecule. In this process, a force arises from the sharing, or exchange, of an electron between two atoms.

14.2 Evaluation of mass of π *meson* (Heisenberg's principle of uncertainty)

The range of the nucleon force is of the order of the radius r' of the n-meson field surrounding a nucleon, since two nucleons experience that force only when their meson fields overlap. To estimate the radius of the field, consider a process in which a nucleon emits a meson of rest mass m_π, which travels out to the limits of the field, and then returns to the nucleon where it is absorbed. In this process, the π meson travels a distance of the order of r'. While it is happening there is a violation of the conservation of mass-energy. The reason is that the total energy of the system equals one nucleon rest mass energy before and after the process, and one nucleon rest mass energy plus at least one π-meson rest mass energy during the process. But the energy-time uncertainty principle shows that a violation of energy conservation by an amount

$$\Delta E \approx m_\pi c^2,$$

is not impossible if it does not happen for a time longer than Δt, where

$$\Delta E \Delta t \approx \hbar .$$

The reason is that such a violation could not be detected because the energy cannot be measured in a time Δt more accurately than ΔE. Since the speed of the pion can be no greater than c, the time required for it to travel a distance of the order of r' is at least

$$\Delta t \approx \frac{r'}{c}.$$

These three relations give

$$m_\pi \approx \frac{\hbar}{r'c},$$

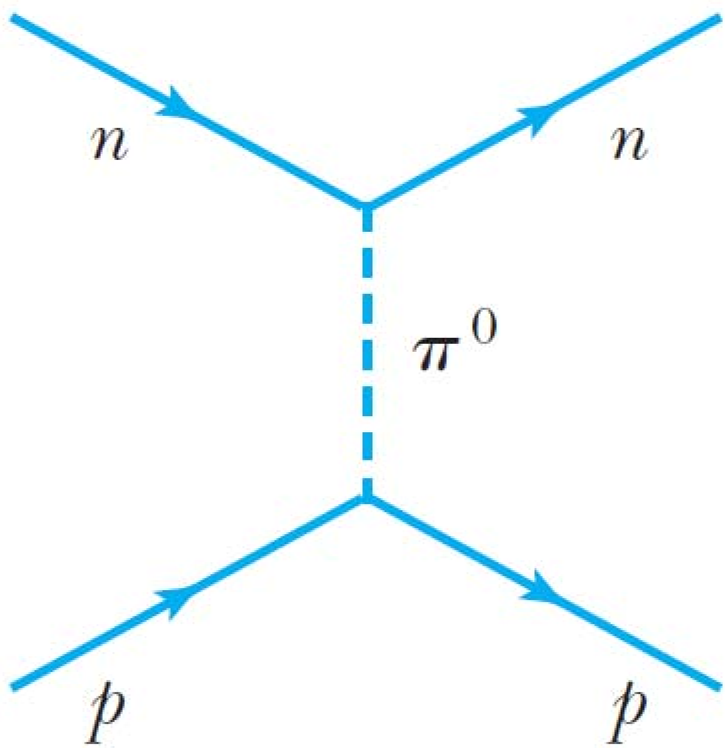
or

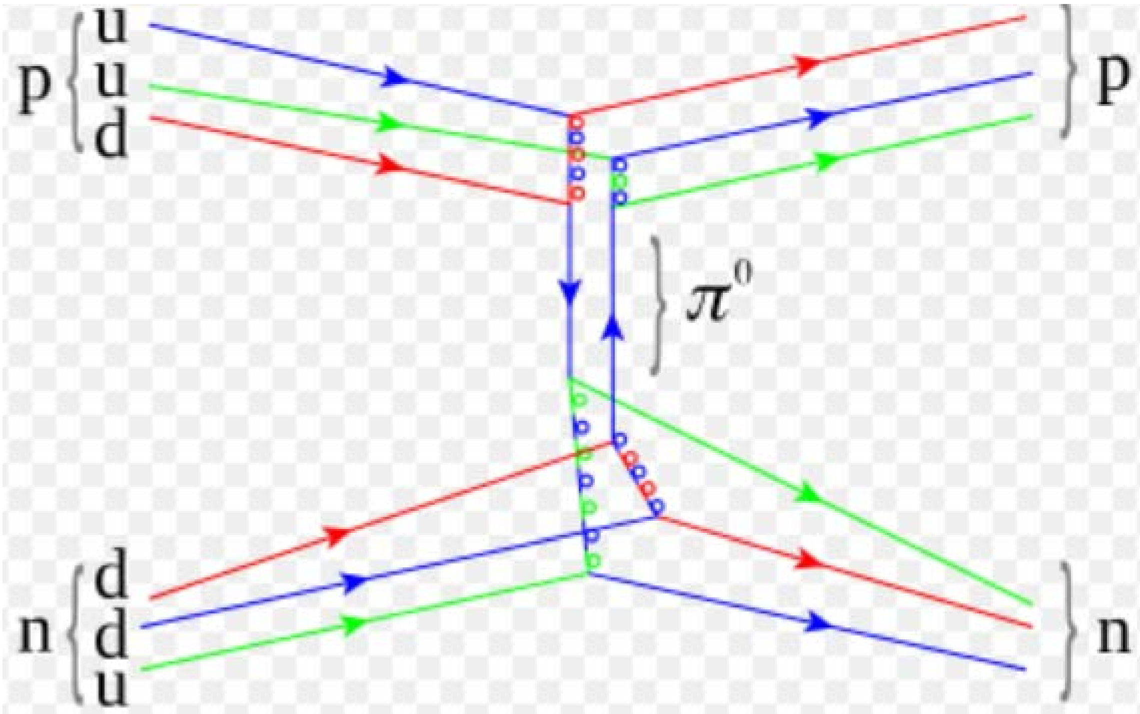
If we take $r' = 2F = 2 \text{ x } 10^{-13}$ cm, the π-meson rest mass m_π can be evaluated as

$m_\pi = 193.1\ m_e.$

or

$m_\pi c^2 = 98.66$ MeV.





15. π^+, π^0, and π^- meson

15.1 π^+

$u\bar{d}$

Rest mass:	139.56995 ± 0.00035 MeV
Mean life:	(2.6033 ± 0.0005) x 10^{-8} s.
Spin	0 (boson)
Isospin:	$\lvert 1,1 \rangle$; $T = 1$, $T_z = 1$.
Electric charge:	$+e$
Typical reactions	

$$p + p \to \pi^+ + d$$
$$\pi^+ \to \mu^+ + \nu_\mu$$
$$\pi^+ \to \pi^0 + e^+ + \nu_e$$

15.2 π^0

$u\bar{u}$, $d\bar{d}$

Rest mass:	134.9766 ±0.0006 MeV
Spin:	0 (boson)
Isospin	$\lvert 1,0 \rangle$; $T = 1$, $T_z = 0$.
Intrinsic parity:	odd
Electric charge:	0

Mean life: $\tau = (8.4 \pm 0.6) \times 10^{-17}$ s

Typical reaction:

$$\pi^0 \to \gamma + \gamma$$

15.3 π^-

$d\bar{u}$

Rest mass:	139.56995 ± 0.00035 MeV
The intrinsic parity:	odd
Spin	0 (boson)
Isospin:	$\lvert 1,-1\rangle$; T = 1, T_z = -1.
Charge:	-e
Typical reactions:	$\pi^- + d \to n + n$
	$\pi^- \to \mu^- + \bar{\nu}_\mu$
	$\pi^- \to \pi^0 + e^- + \bar{\nu}_e$

where e>0.

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APPENDIX
Effective range theory and scattering cross section

(a) The wave function of the bound state

$$\frac{d^2u_b}{dr^2} + (-\alpha^2 + U_0)u_b = 0 \qquad \text{for } 0<r<R$$

$$u_b(r) = A_b \sin(\kappa_b r)$$

$$\frac{d^2u_b}{dr^2} - \alpha^2 u_b = 0\,, \qquad \text{for } r>R$$

$$u_b(r) = B_b e^{-\alpha r}$$

with

$$\kappa_b{}^2 = -\alpha^2 + U_0$$

(b) The wave function of scattering wave;

$$\frac{d^2u_s}{dr^2} + (k^2 + U_0)u_s = 0 \qquad \text{for } 0<r<R$$

$$u_s(r) = A_b \sin(\kappa_b r)$$

$$\frac{d^2u_s}{dr^2} + k^2 u_s = 0 \qquad \text{for } r>R$$

$$u_s(r) = B_b \sin(kr + \delta_0)$$

where

$$\kappa_b{}^2 = k^2 + U_0$$

Multiplying the first by u_s, the second by u_b, and subtracting, gives

$$[u_s \frac{d^2u_b}{dr^2} + (-\alpha^2 + U_0)u_s u_b] - [u_b \frac{d^2u_s}{dr^2} + (k^2 + U_0)u_b u_s = 0$$

or

$$(u_s \frac{d^2u_b}{dr^2} - u_b \frac{d^2u_s}{dr^2}) - (\alpha^2 + k^2)u_s u_b = 0 .$$

Integrating from 0 to R, we get

$$\int_0^R (u_s \frac{d^2u_b}{dr^2} - u_b \frac{d^2u_s}{dr^2})dr - (\alpha^2 + k^2)\int_0^R u_s u_b dr = 0$$

or

$$(u_s \frac{du_b}{dr} - u_b \frac{du_s}{dr})]_0^R = (\alpha^2 + k^2)\int_0^R u_s u_b dr .$$

Since $u_s(0) = u_b(0) = 0$, we have

$$(u_s \frac{du_b}{\partial r} - u_b \frac{du_s}{\partial r})_{r=R} = (\alpha^2 + k^2)\int_0^R u_s u_b dr \,.$$

Dividing by $u_s(R)u_b(R)$, we get

$$(\frac{1}{u_b}\frac{du_b}{dr} - \frac{1}{u_s}\frac{du_s}{dr})_{r=R} = \frac{(\alpha^2 + k^2)}{u_s(R)u_b(R)}\int_0^R u_s u_b dr$$

We note that

$$\frac{1}{u_b}\frac{du_b}{dr}|_{r=R-0} = \kappa_b \cot(\kappa_b R),$$

$$\frac{1}{u_b}\frac{du_b}{dr}|_{r=R+0} = -\alpha$$

leading to the relation

$$\kappa_b \cot(\kappa_b R) = -\alpha$$

We also note that

$$\frac{1}{u_s}\frac{du_s}{dr}|_{r=R-0} = \kappa_s \cot(\kappa_s R)$$

$$\frac{1}{u_s}\frac{du_s}{dr}|_{r=R+0} = k\cot(kR + \delta_0)$$

leading to

$$\kappa_s \cot(\kappa_s R) = k\cot(kR + \delta_0)$$

We need to calculate the integral as

$$\begin{aligned}\frac{1}{u_s(R)u_b(R)}\int_0^R u_s u_b dr &= \frac{-\kappa_b \cot(\kappa_b R) + \kappa_s \cot(\kappa_s R)}{\kappa_b^2 - \kappa_s^2} \\ &= \frac{-\kappa_b \cot(\kappa_b R) + \kappa_s \cot(\kappa_s R)}{-(\alpha^2 + k^2)}\end{aligned}.$$

or

$$\frac{(\alpha^2+k^2)}{u_s(R)u_b(R)}\int_0^R u_s u_b dr = \kappa_b \cot(\kappa_b R) - \kappa_s \cot(\kappa_s R)$$

Then we get

$$\begin{aligned} k\cot(kR+\delta_0)+\alpha &= \kappa_s \cot(\kappa_s R) - \kappa_b \cot(\kappa_b R) \\ &= \alpha + \kappa_s \cot(\kappa_s R) \end{aligned}.$$

We expand $\kappa_s \cot(\kappa_s R)$ around $\kappa_s = \kappa_b$ by using the Taylor expansion.

$$\begin{aligned} \kappa_s \cot(\kappa_s R) &= \kappa_b \cot(\kappa_b R) + [\cot(\kappa_b R) - \kappa_b R \csc^2(\kappa_b R)](\kappa_b - \kappa_b) + ... \\ &\approx -\alpha - (\kappa_s - \kappa_g)[\frac{\alpha}{\kappa_b} + \kappa_b R(1+\cot^2(\kappa_b R)] \\ &= -\alpha - (\kappa_s - \kappa_b)[\frac{\alpha}{\kappa_b} + \kappa_b R(1+\frac{\alpha^2}{\kappa_b{}^2})] \\ &\approx -\alpha - (\kappa_s - \kappa_b)\kappa_b R \end{aligned}$$

We also have

$$\kappa_s - \kappa_b = \frac{\kappa_s{}^2 - \kappa_b{}^2}{\kappa_b + \kappa_b} \approx \frac{\kappa_s{}^2 - \kappa_b{}^2}{2\kappa_b} = \frac{\kappa_s{}^2 - \kappa_b{}^2}{2\kappa_b} = \frac{\alpha^2 + k^2}{2\kappa_b},$$

Finally we have

$$k\cot(kR+\delta_0) = -\alpha - \frac{1}{2}(\alpha^2+k^2)R.$$

Suppose that $kR << 1$, then we have

$$\cot(\delta_0) = -\frac{\alpha}{k}$$

Then the total cross section is given by

$$
\begin{aligned}
\sigma &= \frac{4\pi}{k^2}\sin^2\delta_0 \\
&= \frac{4\pi}{k^2}\frac{1}{1+\cot^2\delta_0} \\
&= \frac{4\pi}{k^2+\alpha^2} \\
&= \frac{2\pi\hbar^2}{m}\frac{1}{E+\dfrac{\hbar^2\alpha^2}{2m}}
\end{aligned}
$$

with $m = M_n / 2$.