Physics of simple pendulum

Masatsugu Suzuki and Itsuko S. Suzuki
Department of Physics, State University of New York at Binghamton
(July 09, 2008)

Abstract

One of the authors (M. S.) has been teaching the Introductory Physics course to freshmen since Fall 2007. When the motion of a simple pendulum is discussed, a small angle approximation is always used. This approximation is the condition necessary for the simple harmonics. For the large angle, the differential equation of the simple pendulum becomes nonlinear. Because the difficulty in mathematics, the physics has not been discussed in classes. In this region, the time dependence of the angle is expressed in terms of Jacobi elliptic functions. This note was originally written as a part of the lecture note (as a challenging topics) for students taking the Introductory Physics (Phys.131). These lecture notes are presented in the Blackboard of SUNY-Binghamton (Binghamton University). The present lecture note is a revised form of the original lecture notes. Here we discuss the time dependence of the angle when the initial angular velocity is changed as a parameter. The differential equations are also solved numerically using the Mathematica (NDSolve). The Fast Fourier Transform (FFT) is used to analyze the higher harmonics of the oscillating waves. We find that the nature of the nonlinearity is strongly enhanced as the angle approaches 180°. A part of our discussion is based on an excellent book written by Prof. Morikazu Toda (Theory of Oscillation, Baifukan, 7-th edition 1977) [in Japanese]. We think that it is very useful for students to study the essential nature of the nonlinear behaviors using a single pendulum. The numerical method is not so powerful in the simple pendulum. When the perturbation effect such as a resistive pendulum is discussed, this method is so powerful and useful in our understanding of physics. The analogy between the pendulum and the Josephson junction in the superconductivity will be discussed.

Contents

1. Theory of simple pendulum

1.1 Formulation

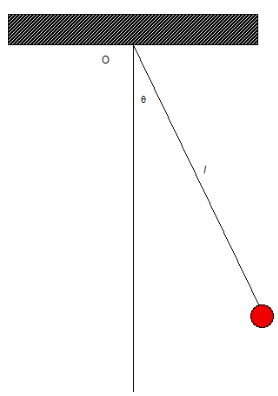


Fig.1 Simple pendulum

We consider a simple pendulum consisting of a particle mass *m* (called the bob of the pendulum) suspended from one end of a stretchable, massless string of length *l*. The string is fixed at the other end, the pivot point on a ceiling. We set up an equation of motion,

$$I\ddot{\theta} = -mgl\sin\theta \tag{1}$$

where g is the acceleration due to gravity and I is the moment of inertia and is given by

$$I = ml^2 \tag{2}$$

Equation (1) can be rewritten as

$$\ddot{\theta} + \varepsilon\sin\theta = 0 \tag{3}$$

where

$$\varepsilon = \frac{g}{l} = \omega_0^{\ 2}. \qquad \text{or} \qquad \omega_0 = \sqrt{\varepsilon}\,.$$

For small θ,

$$\ddot{\theta} + \omega_0^{\ 2}(\theta - \frac{\theta^3}{6} + \frac{\theta^3}{120} + \ldots) = 0\,. \tag{4}$$

The solution of this equation is given by the nonlinear terms

$$\theta = \alpha_1\cos(\omega t + \delta) + \alpha_3\cos(3\omega t + \delta) + \alpha_5\cos(5\omega t + \delta) + \ldots \tag{5}$$

where $|\alpha_1| > |\alpha_2| > |\alpha_3|$ and δ is a phase factor;

$$\omega^2 = \omega_0^{\ 2}(1 - \frac{1}{8}\alpha_1^{\ 2})$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_0}(1 + \frac{\alpha^2}{16}) = T_0(1 + \frac{\alpha^2}{16})$$

$$\alpha_3 \approx \frac{1}{-9\omega^2 + \omega_0^{\ 2}}\frac{1}{24}\omega_0^{\ 2}\alpha_1^{\ 3} \approx -\frac{\alpha_1^{\ 3}}{192} \tag{6}$$

with

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}}\,. \tag{7}$$

In the limit of $\theta \to 0$,

$$\ddot{\theta} + \omega_0^{\ 2}\theta = 0 \quad (8)$$

corresponding to a simple harmonics,

$$\theta = \alpha_1 \cos(\omega_0 t + \delta) \quad (9)$$

1.2. Energy conservation and potential energy

The multiplication of $\dot{\theta}$ on both sides and integration with respect to t leads to

$$\ddot{\theta}\dot{\theta} + \varepsilon \sin\theta\dot{\theta} = 0$$

or

$$\frac{d}{dt}[\frac{1}{2}\dot{\theta}^2 + \varepsilon(1-\cos\theta)] = 0$$

or

$$E = \frac{1}{2}\dot{\theta}^2 + \varepsilon(1-\cos\theta)$$
$$= \frac{1}{2}v^2 + \varepsilon(1-\cos\theta)$$

where the first term is the kinetic energy and the second term is a potential energy defined by

$$U = \varepsilon(1-\cos\theta)$$

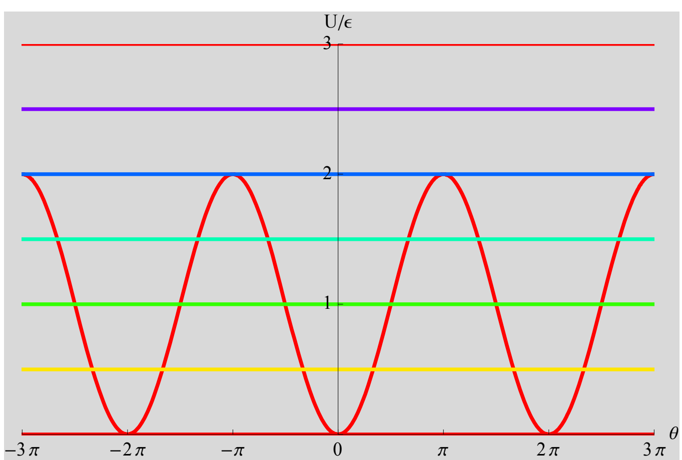


Fig.2 Potential energy U normalized by ε as a function of θ. The horizontal lines corresponds constant energy lines with $E/\varepsilon = 0.5 - 3.0$.

For small θ, U can be approximated by

$$U = \varepsilon(\frac{\theta^2}{2} - \frac{\theta^4}{24} + \frac{\theta^6}{720} - \frac{\theta^8}{40320} + ...)$$

Here we use an initial condition

$$\theta(t=0) = \theta_0$$
$$\dot{\theta}(t=0) = v_0$$

Then the total energy E is expressed by

$$E = \frac{1}{2}v_0{}^2 + \varepsilon(1 - \cos\theta_0)$$

In other words, E increases as v_0 increases and it also increases as θ_0. increases.

1.3 Phase space of θ vs $v = d\theta/dt$

From the two differential equations

$$v = \dot{\theta} \qquad \dot{v} = -\varepsilon\sin\theta$$

we have

$$\frac{dv}{d\theta} = \frac{\dot{v}}{\dot{\theta}} = -\frac{\varepsilon}{v}\sin\theta$$

This is a differential equation (separation variable type). So we can solve easily as

$$\int vdv = -\varepsilon\int\sin\theta d\theta$$
$$\frac{1}{2}v^2 - \varepsilon\cos\theta = C$$

From the energy conservation law,

$$E = \frac{1}{2}v^2 + \varepsilon(1 - \cos\theta) = C + \varepsilon$$

or

$$C = E - \varepsilon$$

(a) Phase-plane trajectory I

We consider the trajectory in the phase space (θ vs v) for

$$\frac{1}{2}v^2 + \varepsilon(1 - \cos\theta) = E,$$

where $E = 2\varepsilon$ and the parameter ε is varied as a parameter. The character of the trajectories inside and outside the curve are rather different. Such a family of curve is called sepratrices.

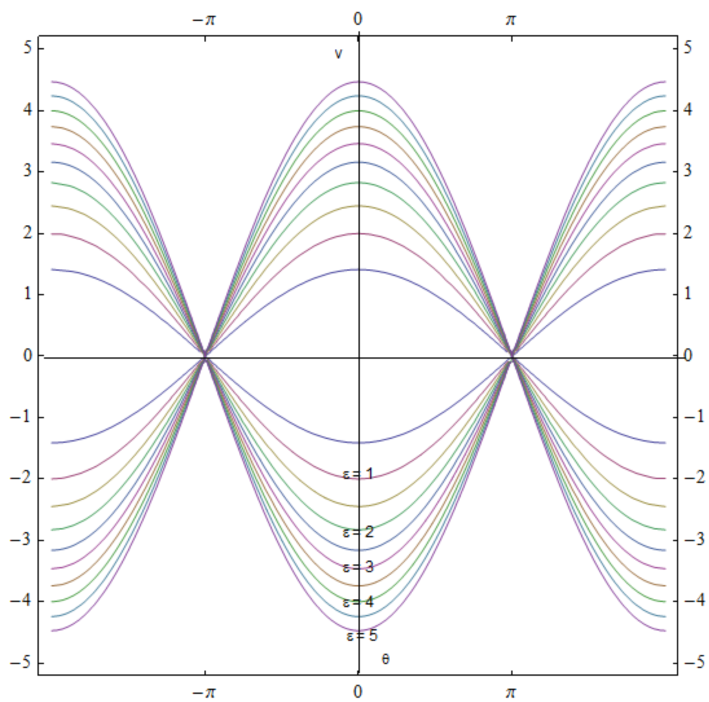


Fig.3 Phase space of v vs θ with $E = 2\varepsilon$, where ε is changes as a parameter.

(b) Phase-plane trajectory II

Next we consider the trajectory in the phase space (θ vs v) for

$$\frac{1}{2}v^2 + \varepsilon(1 - \cos\theta) = E,$$

where $\varepsilon = 1$ and the parameter E is changed as a parameter.

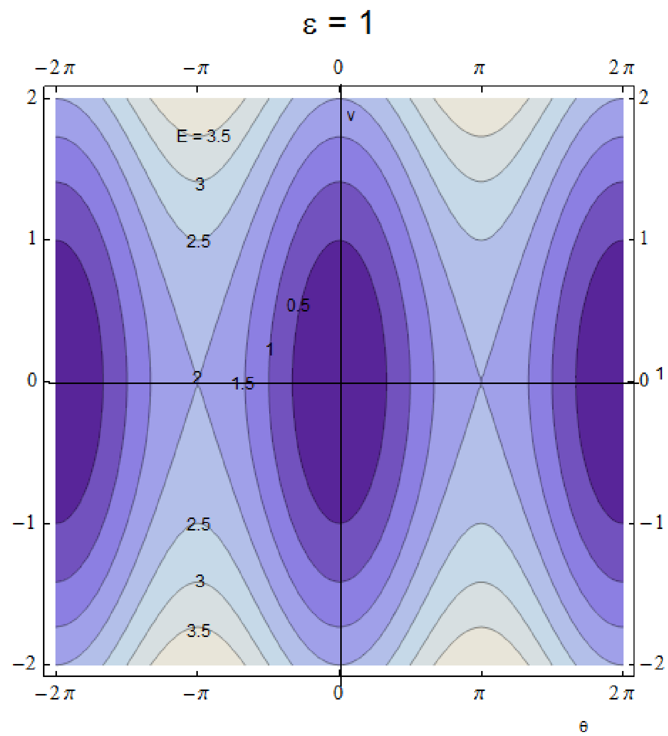


Fig.4 Phase space (v vs θ) for $\varepsilon = 1$. $E = 0.5$, 1.0, 1.5, 2.0, 2.5, 3.0, and 3.5. $E = 2\varepsilon = 2$ is the highest energy of the potential.

2. Exact solution of simple pendulum using Jacobi elliptic function (*k*<1)

2.1 Differential equation for the motion

We consider the energy conservation given by

$$E = \frac{1}{2}v^2 + \varepsilon(1-\cos\theta) = \frac{1}{2}{v_0}^2$$

We assume that

$$v = 0$$

when $\theta = \theta_{max}$. So we have

$$E = \frac{1}{2}v^2 + \varepsilon(1-\cos\theta) = \frac{1}{2}{v_0}^2 = \varepsilon(1-\cos\theta_{max}) = 2\varepsilon\sin^2\frac{\theta_{max}}{2}.$$

From this, we get

$$\frac{1}{2}v^2 = -\varepsilon(1-\cos\theta) + \varepsilon(1-\cos\theta_{max})$$

$$v^2 = \dot{\theta}^2 = \varepsilon(\cos\theta - \cos\theta_{max})$$

Using a formula,

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

we have

$$\dot{\theta} = \frac{d\theta}{dt} = 2\sqrt{\varepsilon}\sqrt{\sin^2\frac{\theta_{\max}}{2} - \sin^2\frac{\theta}{2}} \qquad (1)$$

or

$$\sqrt{\varepsilon}\int_0^t dt' = \sqrt{\varepsilon}t = \frac{1}{2}\int_0^\theta \frac{d\varphi}{\sqrt{\sin^2\frac{\theta_{\max}}{2} - \sin^2\frac{\varphi}{2}}} = \frac{1}{2k}\int_0^\theta \frac{d\varphi}{\sqrt{1 - \frac{1}{k^2}\sin^2\frac{\varphi}{2}}} \qquad (2)$$

where

$$u = \sqrt{\varepsilon}t \qquad \text{and} \qquad k = \sin(\frac{\theta_{\max}}{2})$$

In Eq.(2), we put

$$\xi = \frac{1}{k}\sin\frac{\varphi}{2},$$

leading to

$$d\xi = \frac{1}{2k}\cos\frac{\varphi}{2}d\varphi = \frac{1}{2k}\sqrt{1 - k^2\xi^2}d\varphi$$

or

$$d\varphi = \frac{2kd\xi}{\sqrt{1 - k^2\xi^2}}$$

Noting that the upper limit of the integral is

$$z = \sin\phi = \frac{1}{k}\sin\frac{\theta}{2}, \qquad (3)$$

Eq.(2) can be rewritten as

$$u = \int_0^z \frac{d\xi}{\sqrt{1-\xi^2}\sqrt{1-k^2\xi^2}} \tag{4}$$

or in the form of differential equation, as

$$\frac{du}{dz} = \frac{1}{\sqrt{1-z^2}\sqrt{1-k^2z^2}}$$

Using the relation,

$$z = \sin\phi$$
$$dz = \cos\phi d\phi$$

du can be rewritten as

$$du = \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \tag{5}$$

or in the differential form, as

$$\frac{du}{d\phi} = \frac{1}{\sqrt{1-k^2\sin^2\phi}} \tag{6}$$

2.2 Jacobi elliptic function

Here we introduce the Jacobi elliptic functions. These functions are defined as follows (see the Appendix for the detail).

$$z = \sin\phi = sn(u)$$

$$cn(u) = \cos\phi = \sqrt{1-\sin^2\phi} = \sqrt{1-sn^2(u)}$$

and

$$\sin\frac{\theta}{2} = k \cdot sn(u)$$
$$\cos\frac{\theta}{2} = \sqrt{1-\sin^2\frac{\theta}{2}} = \sqrt{1-k^2sn^2(u)} = dn(u)$$

where

$$sn^2(u) + cn^2(u) = 1 .$$

Using the relation

$$\frac{d\phi}{du} = \sqrt{1-k^2\sin^2\phi} = \sqrt{1-k^2sn^2(u)} = dn(u)$$

we have

$$\frac{dsn(u)}{du} = \frac{d\sin(\phi)}{du} = cn(u)dn(u)$$

$$\frac{dcn(u)}{du} = \frac{d\cos(\phi)}{du} = -sn(u)dn(u)$$

$$\frac{ddn(u)}{du} = \frac{d\sqrt{1-k^2sn^2(u)}}{du} = -k^2sn(u)cn(u)$$

2.3. Expression for the angle as a function of time

For k<1, we get an exact expression for the angle of the single pendulum,

$$z = \sin\phi = \frac{1}{k}\sin\frac{\theta}{2} = sn(u = \sqrt{\varepsilon}t, k),$$

or

$$\theta = 2Arc\sin[k \cdot sn(u = \sqrt{\varepsilon}t, k)].$$

The angle θ is a single-valued function of t. From this, $d\theta/dt$ can be calculated as

$$\cos\frac{\theta}{2}\cdot\frac{1}{2}\dot{\theta} = k\cdot\frac{d}{du}sn(u,k)\cdot\frac{du}{dt} = k\sqrt{\varepsilon}\cdot cn(u,k)dn(u.k),$$

or

$$v = \dot{\theta} = 2k\sqrt{\varepsilon}\cdot cn(u = \sqrt{\varepsilon}t, k) = v_0 cn(u = \sqrt{\varepsilon}t, k),$$

where

$$E = \frac{1}{2}{v_0}^2 = 2\varepsilon\sin^2\frac{\theta_{max}}{2} = 2\varepsilon k^2.$$

or

$$k = \frac{1}{2\sqrt{\varepsilon}} v_0 \ (<1).$$

We note that from the relation, $sn^2(u) + cn^2(u) = 1$, we get the energy conservation law for the pendulum,

$$\frac{1}{2}v^2 + \varepsilon(1 - \cos\theta) = \frac{1}{2}{v_0}^2 .$$

For $k = 1$, we have

$$\sin\frac{\theta}{2} = sn(u = \sqrt{\varepsilon}t, 1) = \tanh(u = \sqrt{\varepsilon}t)$$

or

$$\theta = 2 Arc\sin[\tanh(u = \sqrt{\varepsilon}t)]$$

When $t \to \infty$, θ become π. This means that it takes infinity-time from $\theta = 0$ to $\theta = \pi$.

((Example)) Calculation of θ vs t. $\varepsilon = 1$. v_0 is changed as a parameter.

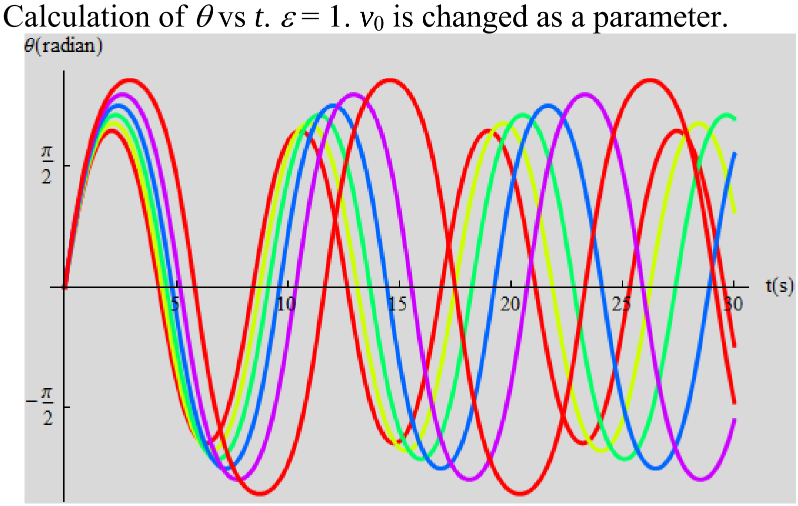


Fig.5(a) Plot of θ vs t. ε = 1. v0 = 1.7 (red), 1.75 (yellow), 1.80 (green), 1.85 (blue), 1.90 (purple), and 1.95.

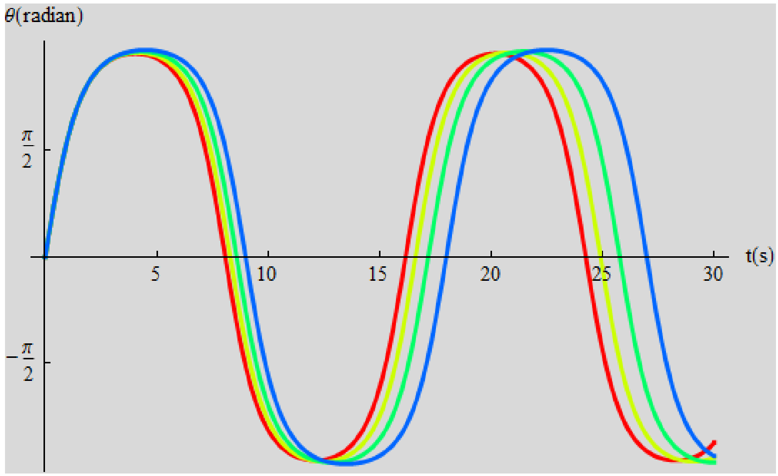


Fig.5(b) Plot of θ vs t. $\varepsilon = 1$. v_0 = 1.995 (red), 1.996 (yellow), 1.997 (green), and 1.998 (blue).

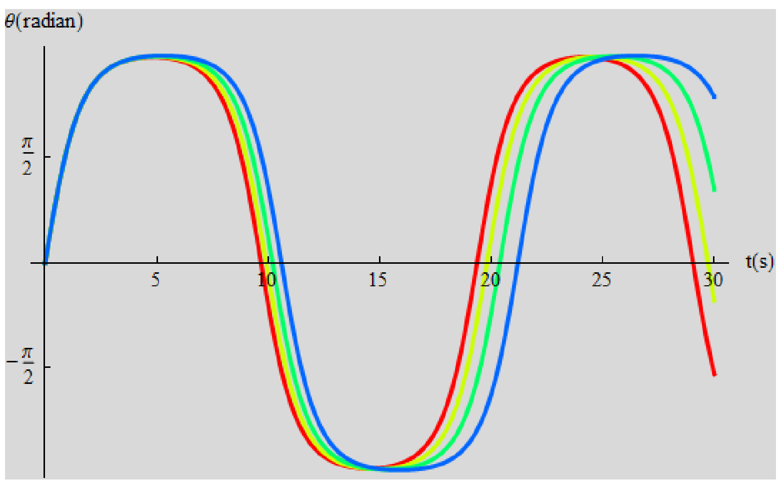


Fig.5(c) Plot of θ vs t. $\varepsilon = 1$. v_0 = 1.9990 (red), 1.9992 (yellow), 1.9994 (green), and 1.9996 (blue).

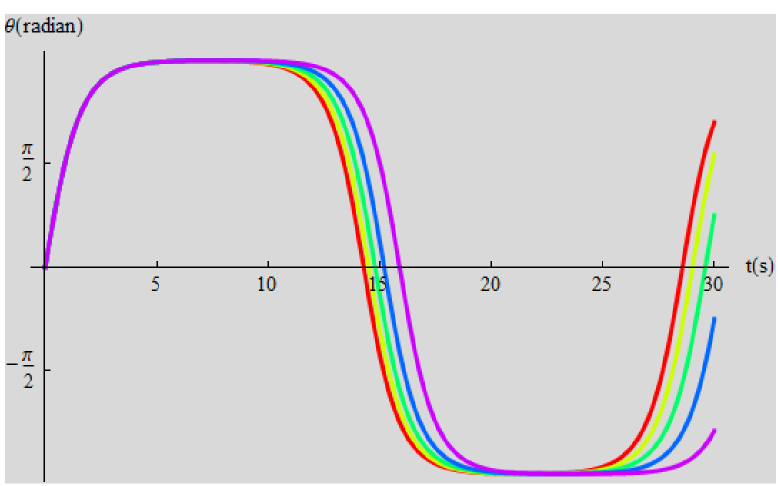


Fig.5(d) Plot of θ vs t. $\varepsilon = 1$. v_0 = 1.999990 (red), 1.999992 (yellow), 1.999994 (green), 1.999996 (blue), and 1.999998 (blue).

2.4 Wave analysis by Fast Fourier transform

The sine wave is found at small vo. The form of the sine wave is deformed as vo increases. In the vicinity of $v_0 \approx 2$, the form changes from the sine wave to a square wave, indicating the enhancement of higher order harmonics. The frequency of the first harmonic is reduced to zero at $v_0 = 2$. Using the fast Fourier transform, we analyze how the higer order terms due to the nonlinearity of the system is enhanced when v_0 approaches 2.

(a) $v_0 = 0.2$

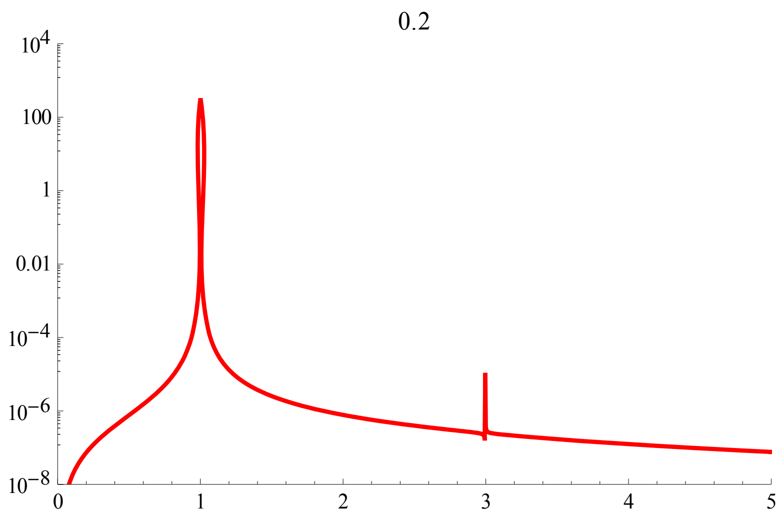


Fig.6(a) The square of amplitude in FFT Fourier spectrum vs ω (rad/s). $\varepsilon = 1$. v_0 = 0.2

(b) $v_0 = 1.60$

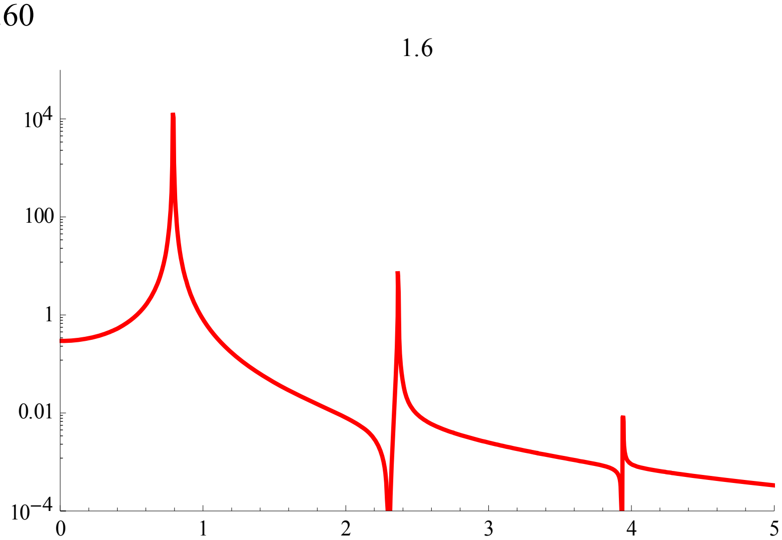


Fig.6(b) The square of amplitude in FFT Fourier spectrum vs ω (rad/s). $\varepsilon = 1$. $v_0 = 1.6$.

(c) $v_0 = 1.92$

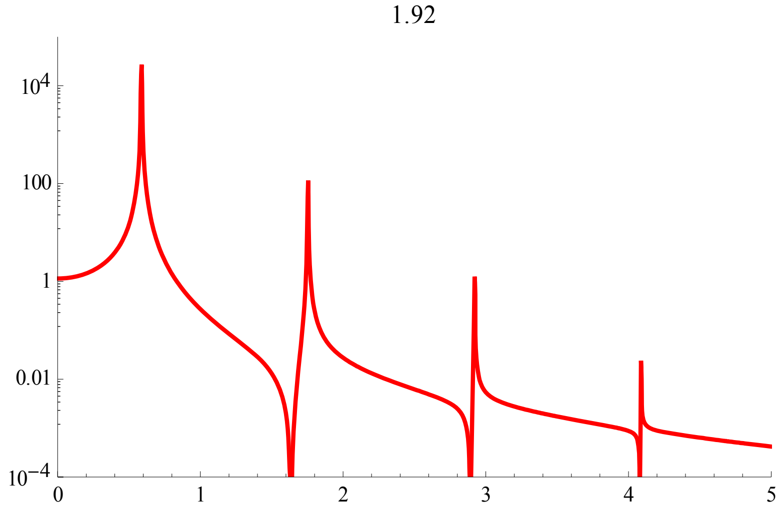


Fig.6(c) Fourier The square of amplitude in FFT Fourier spectrum vs ω (rad/s). $\varepsilon = 1$. $v_0 = 1.92$.

(d) $v_0 = 1.98$

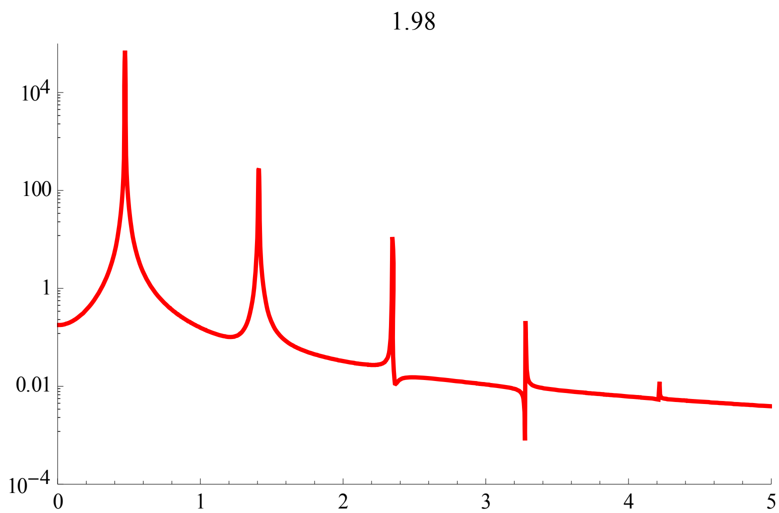


Fig.6(d) The square of amplitude in FFT Fourier spectrum vs ω (rad/s). $\varepsilon = 1$. $v_0 = 1.98$.

(e) $v_0 = 1.998$

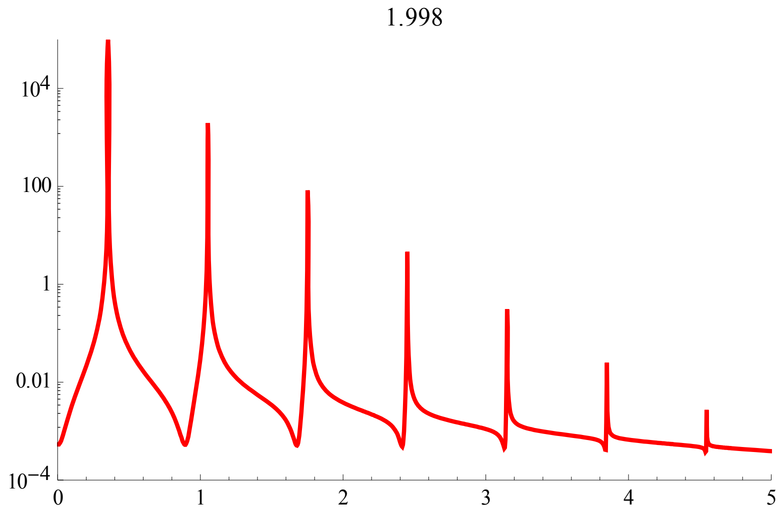


Fig.6(e) Fourier spectrum of the square of amplitude vs ω (rad/s). ε = 1. v_0 = 1.998

2.5 Trajectories in the phase space

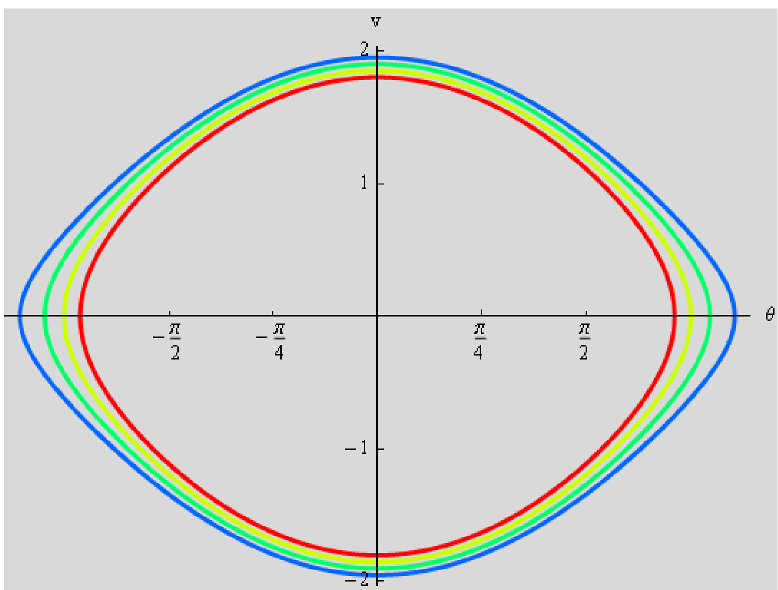


Fig.7(a) ParametricPlot of θ vs v. ε = 1. v_0 is changes as a parameter. v_0 = 1.80 (red), 1.85 (yellow), 1.90 (Green), and 1.95 (blue) .

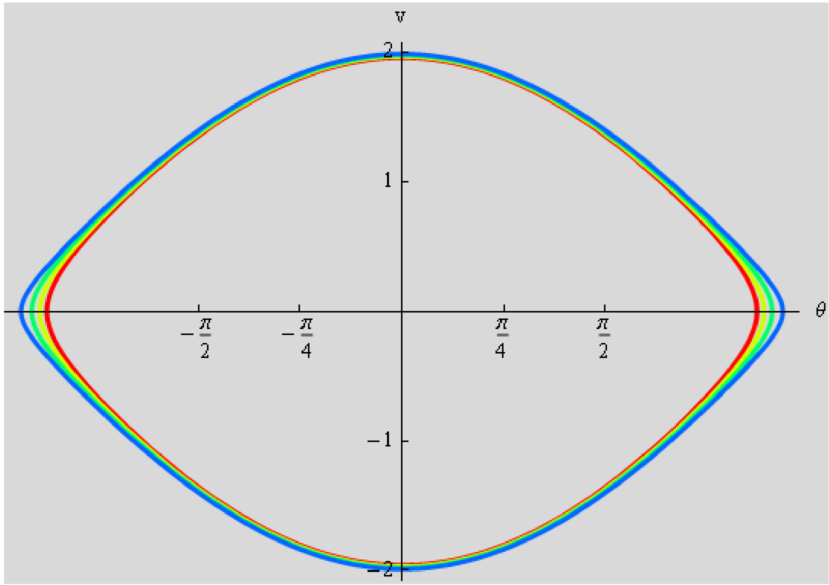


Fig.7(b) ParametricPlot of θ vs v. $\varepsilon = 1$. v_0 is changes as a parameter. $v_0 = 1.96$ (red), 1.97 (yellow), 1.98 (Green), and 1.99 (blue) .

2.6 Period *T*

Noting that $u = \sqrt{\varepsilon}t$, the period T is given by

$$T = \frac{2\pi}{\omega} = \frac{4}{\sqrt{\varepsilon}} K(k)$$

where $K(k)$ is the complete elliptic integral of the first kind and is defined by

$$K(k) = \int_0^1 \frac{dz}{\sqrt{1-\xi^2}\sqrt{1-k^2\xi^2}} \qquad \text{(= EllipticK}[m\ (= k^2)]).$$

We note that $K(k)$ can be also defined as

$$K(k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

The complete elliptic integral of the first kind is implemented in Mathematica as EllipticK[m (= k^2)]. Note the use of the parameter $m = k^2$ instead of the modulus k. The parameter k is related to v_o and ε as

$$k = \frac{1}{2\sqrt{\varepsilon}} v_0 ,$$

because of the energy conservation given by

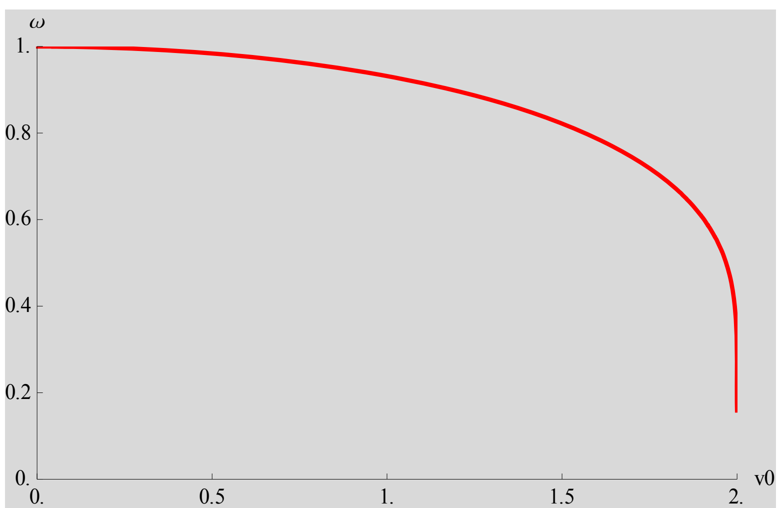
$$\frac{1}{2}v_0^{\ 2} = \varepsilon(1-\cos\theta_{\max}) = 2\varepsilon\sin^2\frac{\theta_{\max}}{2} = 2\varepsilon k^2$$

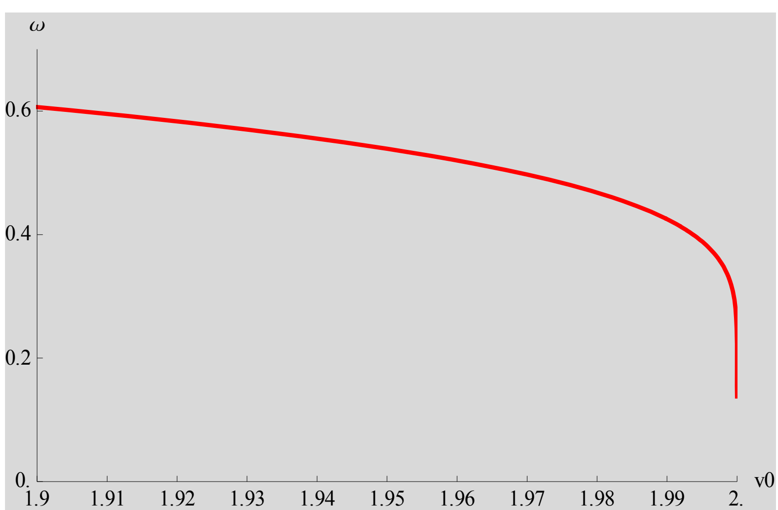
In summary

$$\omega = \frac{\pi\sqrt{\varepsilon}}{2}\frac{1}{K(k = \frac{v_0}{2\sqrt{\varepsilon}})}$$

Here we show the result of calculation of ω as a function of v_0. For simplicity, we assume that $\varepsilon = 1$. We make a plot of ω vs v_0 for $\varepsilon = 1$. Note that

$$\lim_{v0\to 2-0} \omega = 0$$





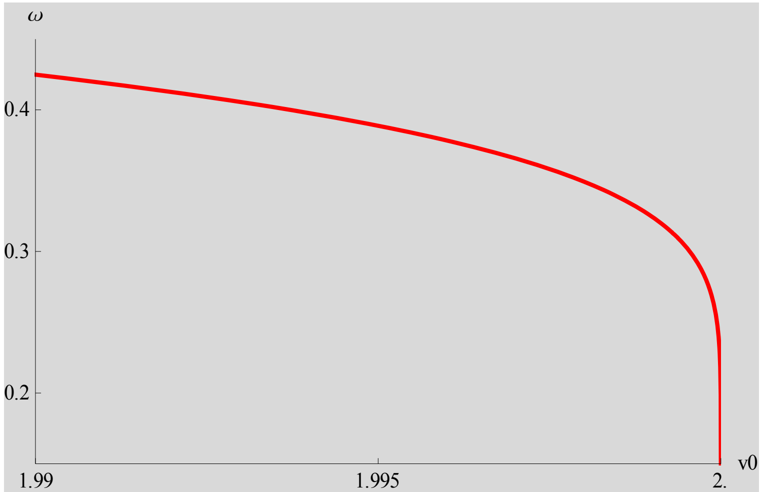


Fig.8 (a), (b) and (c) Plot of ω vs v_0 for $\varepsilon = 1$. The angular frequency reduces to zero at $v_0 = 2$.

2.7 The behavior of ω in the vicinity of $v_0 = 2$ for $\varepsilon = 1$.

We consider the value of ω in the vicinity of the critical value $v_0 = 2$. For convenience we introduce the parameter t which is defined by

$$v_0 = 2 - t$$

Then the critical behavior of ω can be expressed by

$$\omega = \frac{\pi}{2}\frac{1}{K(k = \frac{v_0}{2\sqrt{\varepsilon}})} = \frac{\pi}{2}\frac{1}{K(k = 1 - \frac{t}{2})}$$

around $t = 0$, where $\varepsilon = 1$. Using the Mathematica, it can be easily demonstrated that

$$EllipticK[1-t] \approx Log[4t^{-1/2}]$$

in the limit of $t \to 0$. This approximation in the Mathematica can be rewritten as

$$K(k = \sqrt{1-t}) \approx K(k = 1 - \frac{t}{2}) \approx \ln(4t^{-1/2})$$

in the standard form. Using this formula, we find that ω $(=\omega_l)$ tends to zero as

$$\omega = \omega_l \approx \frac{\pi}{2}\frac{1}{\ln(4t^{-1/2})}$$

in the limit of $t \to 0$.

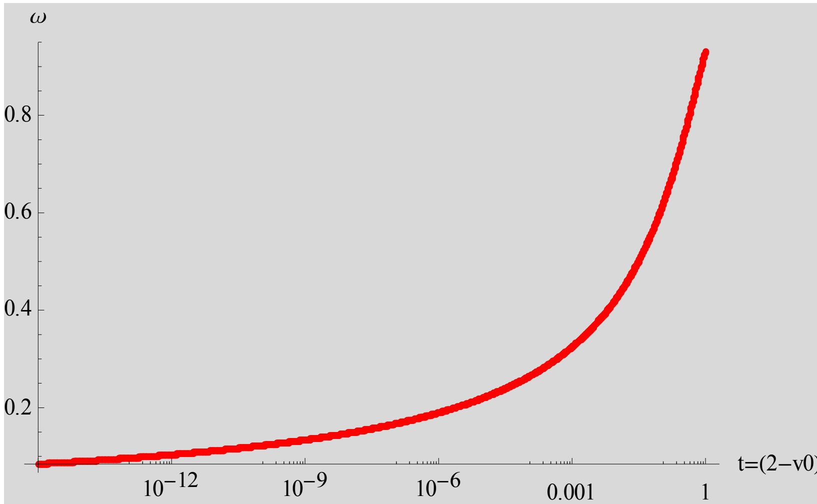


Fig.9 ω (rad/s) vs t ($t = = 2 - v_0$) in the vicinity of $v_0 = 2$. $\varepsilon = 1$.

2.8 θ_{max} vs v_0

The maximum angle θ_{max} is related to v_0 as

$$\theta_{max} = 2\arcsin(k) = 2\arcsin(\frac{v_0}{2\sqrt{\varepsilon}})$$

We make a plot of θ_{max} vs v_0 for $\varepsilon = 1$ (denoted by red curve). For comparison we also show a plot of the straight line (denoted by a blue line); $\theta_{max} = 2v_0/\sqrt{\varepsilon}$. The red curve almost t coincides with the blue line for $v_0<0.8$. This difference becomes large when v_0 tends to 2 from the smaller side of v_0.

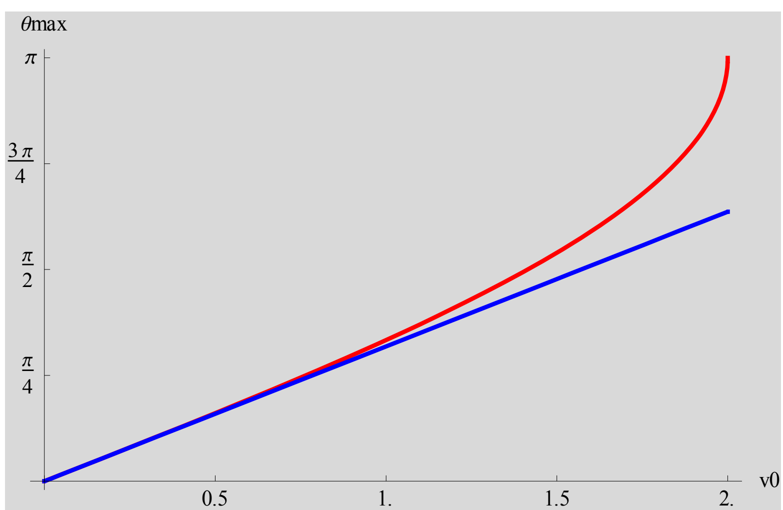


Fig.10 Plot of θ_{max} vs v_0 (red) for $\varepsilon = 1$. The blue line is a prediction of q_{max} vs v_0 in the small angle limit approximation.

3. Exact solution for *k*>1

3.1 Time dependence of the rotation angle

We now discuss the case when $E>2\varepsilon$,

$$E=\frac{1}{2}v^2+\varepsilon(1-\cos\theta)=\frac{1}{2}v^2+2\varepsilon\sin^2\frac{\theta}{2}=\frac{1}{2}{v_0}^2>2\varepsilon$$

In this case, the mass continues to rotate. From this equation, we get a differential equation for θ,

$$v=\dot{\theta}=\sqrt{{v_0}^2-4\varepsilon\sin^2\frac{\theta}{2}}$$

Here we introduce a new parameter κ defined by

$$\kappa=\frac{2\sqrt{\varepsilon}}{v_0}=\frac{1}{k}<1$$

Then we get

$$\dot{\theta}=\frac{d\theta}{dt}=\frac{2\sqrt{\varepsilon}}{\kappa}\sqrt{1-\kappa^2\sin^2\frac{\theta}{2}}$$

This equation can be rewritten as

$$\frac{2\sqrt{\varepsilon}}{\kappa}t=\int_0^{\theta}\frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\frac{\varphi}{2}}}$$

We put

$$\xi=\sin\frac{\varphi}{2}$$

$$d\xi=\frac{1}{2}\cos\frac{\varphi}{2}d\varphi=\frac{1}{2}\sqrt{1-\xi^2}d\varphi$$

Then we get

$$\frac{\sqrt{\varepsilon}}{\kappa}t=u=\int_0^{\sin(\theta/2)}\frac{d\varphi}{\sqrt{1-\kappa^2\xi^2}\sqrt{1-\kappa^2\xi^2}}$$

The solution of this equation is given by

$$\sin\frac{\theta}{2} = sn(u,\kappa) = sn(\frac{\sqrt{\varepsilon}}{\kappa}t,\kappa)$$

((Note))

$$cn(u,\kappa) = \sqrt{1 - sn^2(u,\kappa)}$$
$$dn(u,\kappa) = \sqrt{1 - \kappa^2 sn^2(u,\kappa)}$$

The angle θ is a multi-valued function of t. We make a plot of θ vs t using a ContourPlot (Mathematica). We assume that $\varepsilon = 1$.

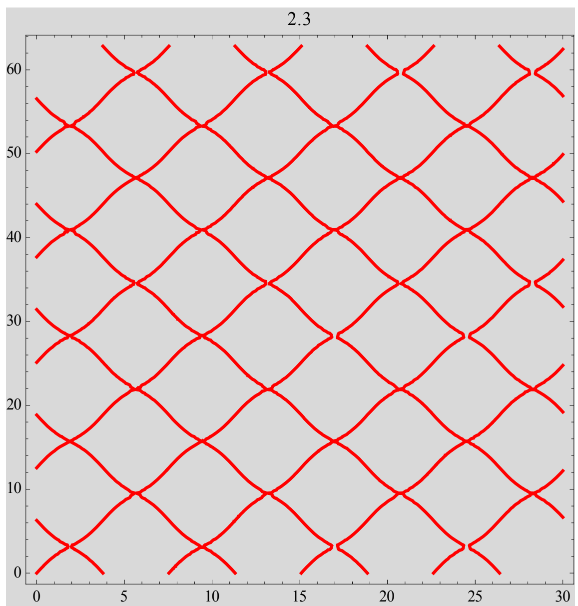


Fig.10(a) Plot of θ (rad) vs t (s) for $v_0 = 2.3$. The appropriate curve passes through the origin among a family of curves.

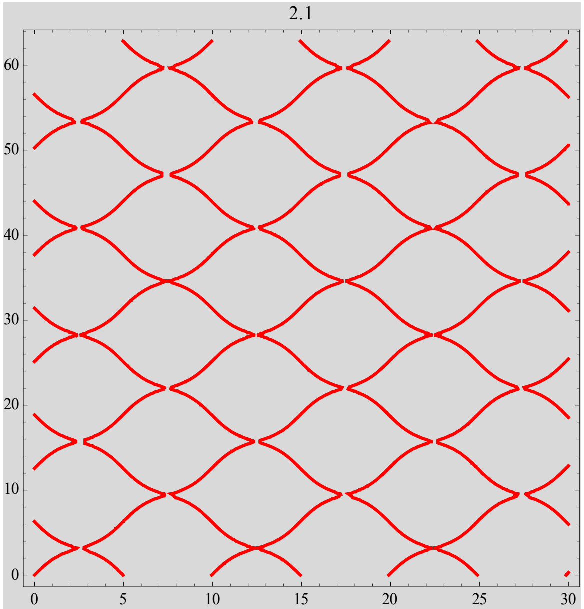


Fig.10(b) Plot of θ (rad) vs t (s) for v_0 = 2. 1. The appropriate curve passes through the origin.

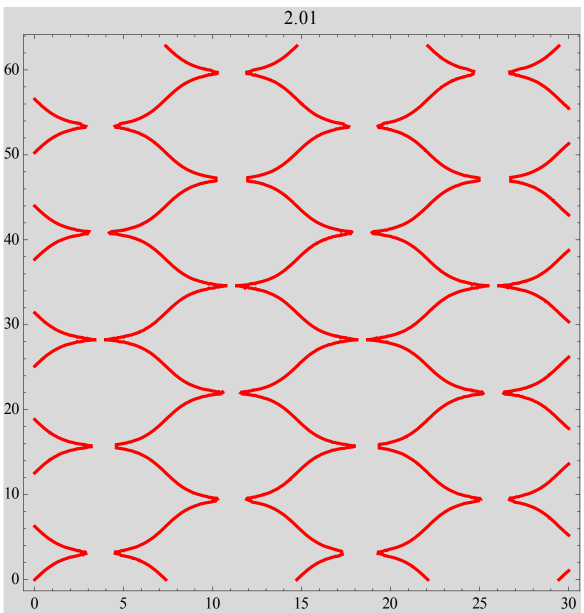


Fig.10(c) Plot of θ (rad) vs t (s) for v_0 = 2.01. The appropriate curve passes through the origin.

3.2 Period *T*

The period *T* is calculated in the following way.

$$\frac{2\sqrt{\varepsilon}}{\kappa}T = 2\int_0^{\pi}\frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\frac{\varphi}{2}}} = 4\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}}$$

or

$$T = \frac{2\kappa}{\sqrt{\varepsilon}}\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}} = \frac{2\kappa}{\sqrt{\varepsilon}}K(\kappa)$$

The angular frequency ω is obtained as

$$\omega = \frac{2\pi}{T} = \frac{\pi\sqrt{\varepsilon}}{\kappa}\frac{1}{K(\kappa)}$$

We make a plot of ω vs v_0 for $\varepsilon = 1$, where $\kappa = \frac{2\sqrt{\varepsilon}}{v_0} = \frac{2}{v_0}$ and $v_0>2$. We note that

$$\lim_{v0\to 2+0}\omega = 0$$

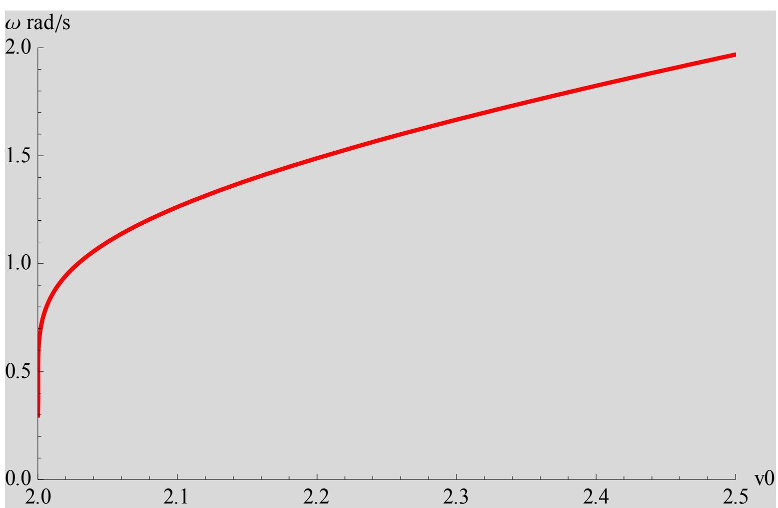


Fig.11 Plot of ω vs v_0 for $v_0>2$. $\varepsilon = 1$. $\kappa = \frac{2\sqrt{\varepsilon}}{v_0} = \frac{2}{v_0}$

3.3 Critical behavior of ω vs v_0 in the vicinity of $v_0 = 2$

We consider the value of ω in the vicinity of the critical value $v_0 = 2$ ($v_0>2$). For convenience we use the parameter t defined by

$$v_0 = 2 + t\,.$$

Then the parameter κ is given by

$$\kappa = \frac{2}{2+t} \approx 1 - \frac{t}{2}$$

for $\varepsilon = 1$. The critical behavior of ω around $t = 0$ is described by

$$\omega = \frac{\pi}{1-\frac{t}{2}} \frac{1}{K(\kappa = 1-\frac{t}{2})} \approx \frac{\pi}{K(\kappa = 1-\frac{t}{2})}.$$

Here we use the approximation given by

$$K(\kappa = \sqrt{1-t}) \approx K(\kappa = 1-\frac{t}{2}) \approx \ln(4t^{-1/2}),$$

in the limit of $t \to 0$. Then ω (= ω_u) tends to zero as

$$\omega = \omega_u \approx \pi \frac{1}{\ln(4t^{-1/2})}$$

in the limit of $t \to 0$. Note that the ratio $\omega_u/\omega_l = 2$ at the same t, indicating the asymmetric behavior of ω with respect to the axis of $v_0 = 2$. We show the plot of ω vs t where $t = 2 - v_0$.

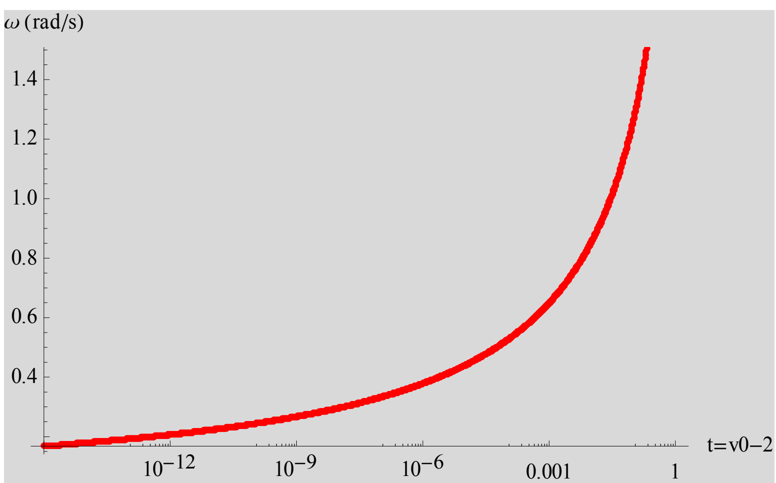


Fig.12 ω (rad/s) vs t ($t = v_0 - 2$) in the vicinity of $t = 0$. $\varepsilon = 1$.

In summary, Figure 13 show the plot of ω vs v_0 in the entire range of v_0. Note that $v_0 = 2$ is the critical value for $\varepsilon = 1$.

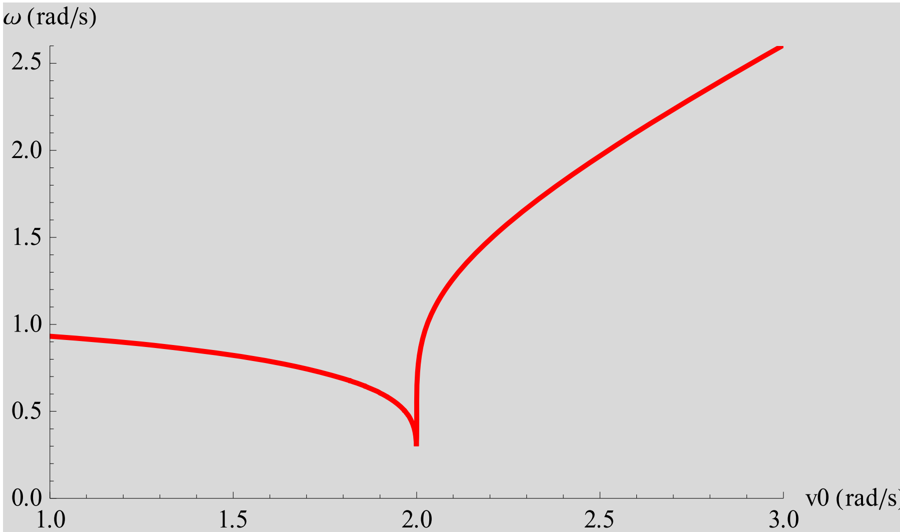


Fig.13 Plot of ω vs v_0 for $v_0<2$ ($k<1$) and $v_0>2$ ($k = 1/\kappa>1$). $\varepsilon = 1$.

4. Geometrical consideration

4.1 Oscillation with $k<1$

(a) Geometry for ΔABE with a point D between the points A and E

In this figure, the point P oscillates on the arc BAB_1. $\angle AOB$ is the maximum angle. We draw a circle AD. BB' is a tangential line to this circle. The projection of P (PN) intersect with the circle at a point Q. $\angle ADQ = \phi$. $\angle AOP = \theta$

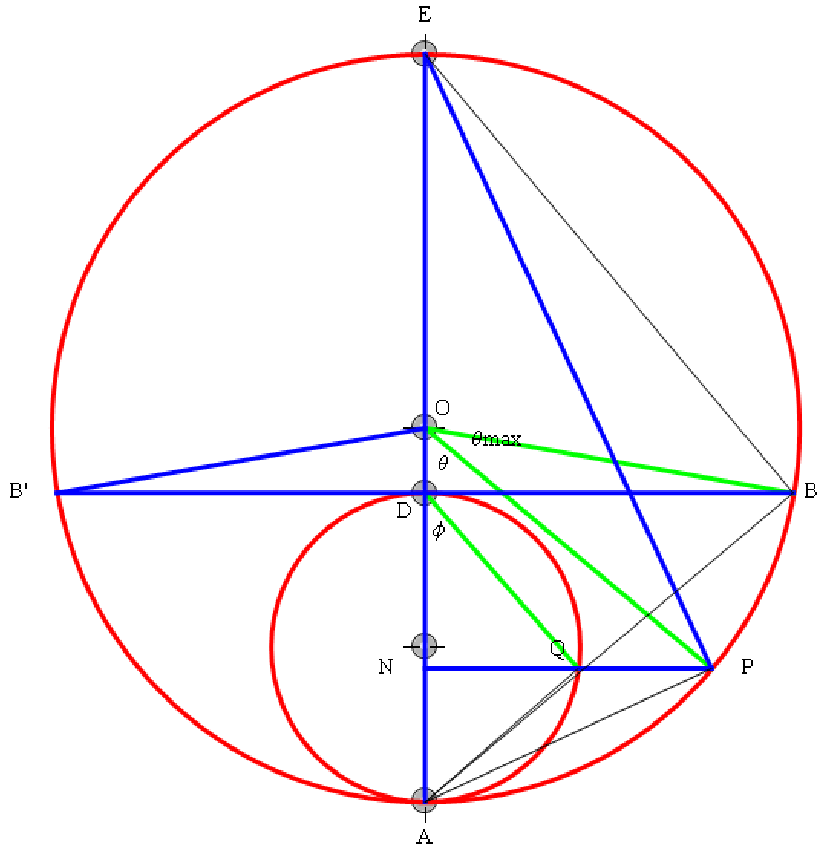


Fig.14 Plot of the geometry using the Graphics of the Mathematica. θ_{max} =80º and θ= 50º

First we consider the geometrical relations for ΔABE with a point D between the points A and E

$$\angle AOB = \theta_{\max}$$
$$\angle AEB = \frac{1}{2}\theta_{\max}$$
$$\angle ABE = \frac{\pi}{2}$$
$$\angle ABD = \frac{1}{2}\theta_{\max}$$
$$AE = 2l$$
$$OB = l$$
$$AB = AE \cdot \sin\frac{\theta_{\max}}{2} = 2lk$$
$$AD = AB \cdot \sin\frac{\theta_{\max}}{2} = 2lk^2$$
$$BE = AE \cdot \cos\frac{\theta_{\max}}{2} = 2l\cos\frac{\theta_{\max}}{2} = 2l\sqrt{1-k^2}$$

where

$$k = \sin\frac{\theta_{\max}}{2} \quad \text{and} \quad \cos\frac{\theta_{\max}}{2} = \sqrt{1-k^2}$$

We note that ΔABE is similar to ΔADB. This leads to the following relation,

$$k = \frac{AD}{AB} = \frac{AB}{AE}, \qquad \text{or} \qquad k^2 = \frac{AD}{AB}\frac{AB}{AE} = \frac{AD}{AE}$$

(b) Geometry for ΔAEP with a point N between the points A and E

We consider the geometrical relations for ΔAEP with a point N between the points A and E

$$\angle AOP = \theta$$

$$\angle AEP = \frac{1}{2}\theta$$

$$\angle APE = \frac{\pi}{2}$$

$$\angle APN = \frac{1}{2}\theta$$

$$EP = AE \cdot \cos\frac{\theta}{2} = 2l \cdot \cos\frac{\theta}{2} = 2l \cdot dn(u)$$

$$AD = AO - DO = l \cdot (1 - \cos\theta_{\max}) = 2l \cdot \sin^2\frac{\theta_{\max}}{2} = 2lk^2$$

$$AP = AE \cdot \sin\frac{\theta}{2} = 2l \cdot \sin\frac{\theta}{2} = 2lk \cdot sn(u)$$

$$AN = AP \cdot \sin\frac{\theta}{2} = 2l \cdot \sin^2\frac{\theta}{2} = 2lk^2 \cdot sn^2(u)$$

$$ND = AD - AN = 2lk^2 \cdot [1 - sn^2(u)] = 2lk^2 \cdot cn^2(u)$$

$$EN = AE - AN = 2l \cdot [1 - k^2 \cdot sn^2(u)] = 2l \cdot dn^2(u)$$

$$NP = AP \cdot \cos\frac{\theta}{2} = 2lk \cdot sn(u) \cdot dn(u)$$

We note that ΔPEN is similar to ΔAPN. This leads to the relation

$$\frac{EN}{NP} = \frac{NP}{AN},$$

or

$$NP^2 = EN \cdot AN = 2l \cdot dn^2(u) \cdot 2lk^2 \cdot sn^2(u) = 4l^2k^2 dn^2(u) \cdot sn^2(u)$$

(c) Geometry for ΔADQ with a point N between the points A and D

We consider the geometrical relation for ΔADQ with a point N between the points A and D

$$\angle ADQ = \phi$$

$$\angle AQD = \frac{\pi}{2}$$

$$\angle AQN = \phi$$

$$AQ = AD \cdot \sin\phi = 2lk^2 \cdot \sin\phi = 2lk^2 \cdot sn(u)$$

$$AN = AQ \cdot \sin\phi = AD \cdot \sin^2\phi = 2lk^2 \cdot \sin^2\phi = 2lk^2 \cdot sn^2(u)$$

$$NQ = AQ \cdot \cos\phi = 2lk^2 \sin\phi\cos\phi = 2lk^2 \cdot sn(u)cn(u)$$

We note that ΔDNQ is similar to ΔQNA. This leads to the relation

$$\frac{DN}{NQ} = \frac{NQ}{AN},$$

or

$$NQ^2 = DN \cdot AN = 2lk^2 \cdot cn^2(u) \cdot 2lk^2 \cdot sn^2(u)$$

or

$$NQ = 2lk^2 \cdot sn(u) \cdot cn(u)$$

(d) Relation between θ and ϕ

From the above consideration, we have the relation, the length AN is described by

$$AN = 2lk^2 \cdot \sin^2 \phi$$

in terms of ϕ. The length AN is also described by

$$AN = 2l \sin^2 \frac{\theta}{2}$$

in terms of θ. From these relations, we obtain

$$\sin \frac{\theta}{2} = k \cdot \sin \phi = z$$

((Example))

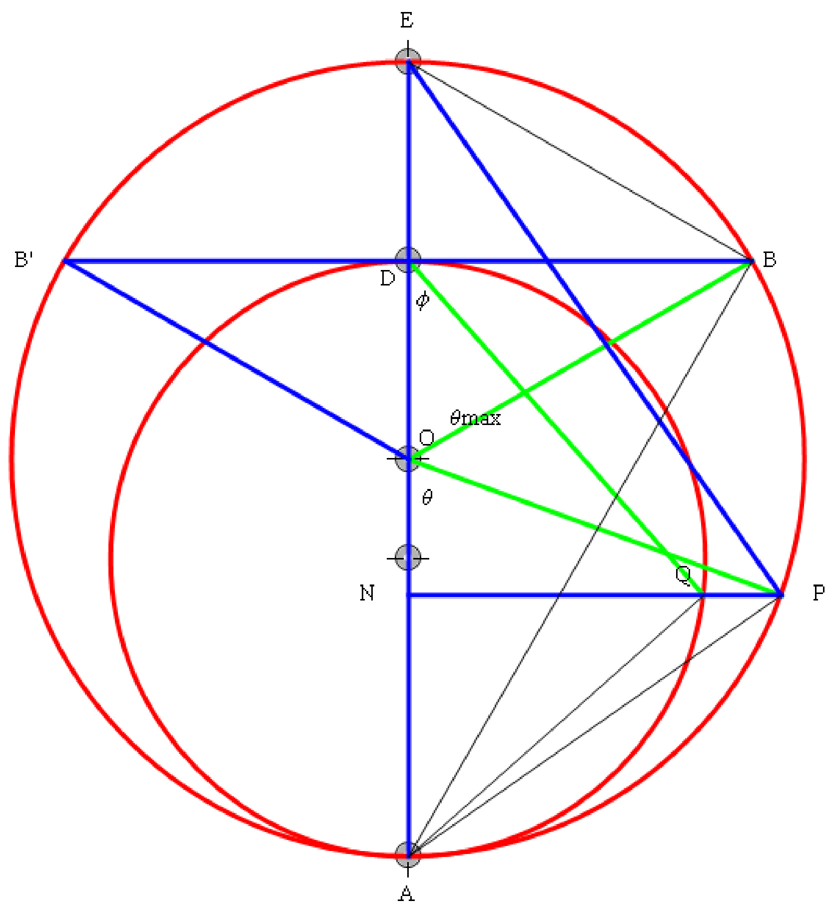


Fig.15 Plot of the geometry using the Graphics of the Mathematica. θ_{max} =120º and θ = 70º

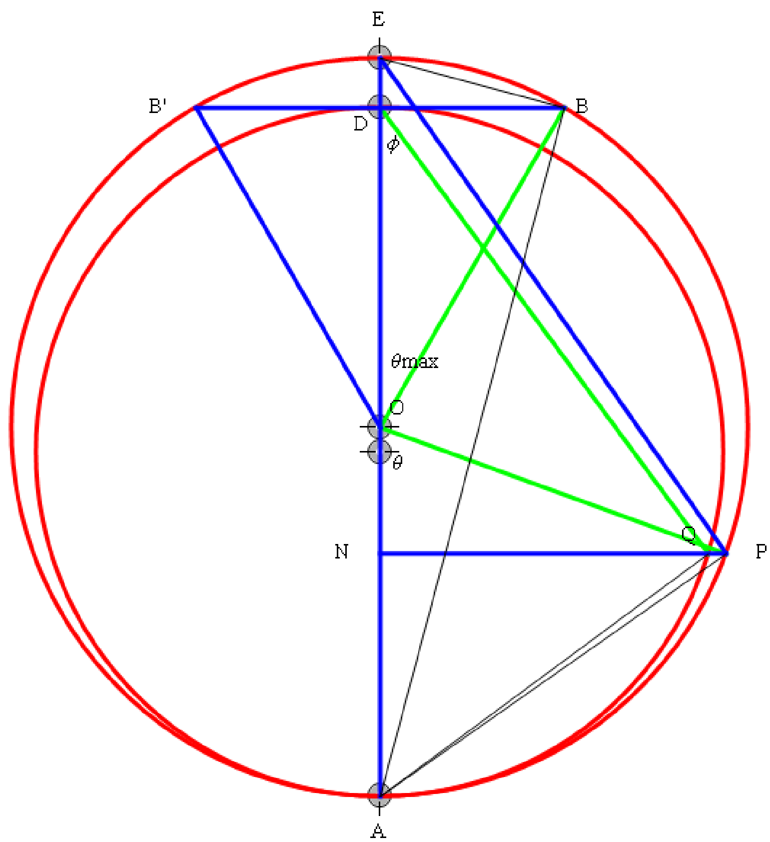


Fig.16 Plot of the geometry using the Graphics of the Mathematica. θ_{max} = 150º, θ = 70º

4.2 *k* = 1 (critical state)

The circle AD coincides with the circle AE.

$$\frac{\theta}{2} = \phi$$

$$\sin\frac{\theta}{2} = \sin\phi = sn(u, k = 1) = \tanh(u)$$

$$\cos\phi = cn(u, k = 1) = \sec h(u)$$

where

$$sn(u, k = 1) = \tanh u$$

$$\angle AOP = \theta$$

$$\angle OEP = \frac{\phi}{2}$$

$$\angle APN = \phi$$

$$AP = 2l \cdot \sin\phi = 2l \cdot \sin\frac{\theta}{2} = 2l \cdot sn(u, k=1) = 2l \cdot \tanh(u)$$

$$AN = AP \cdot \sin\phi = 2l \cdot \sin^2\phi = 2l \cdot \sin^2\frac{\theta}{2} = 2l \cdot sn^2(u, k=1) = 2l \cdot \tanh^2 u$$

$$NP = AP \cdot \cos\phi = 2l \cdot \sin\phi\cos\phi = 2l \cdot sn(u, k=1) \cdot cn(u, k=1) = 2l \cdot \tanh(u) \cdot \sec h(u)$$

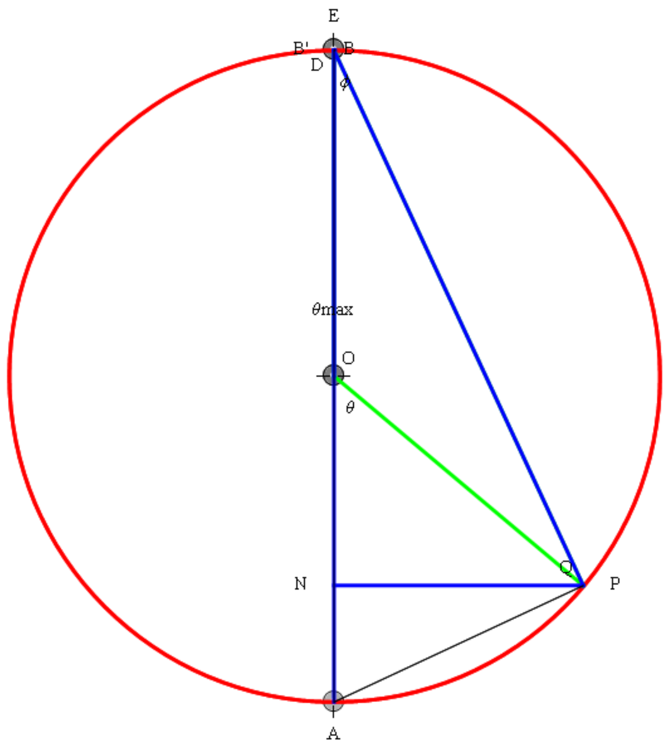


Fig.17 Plot of the geometry using the Graphics of the Mathematica. θ_{max} = 180º, θ= 50º

4.3. *k*>1 (the case of rotating pendulum)

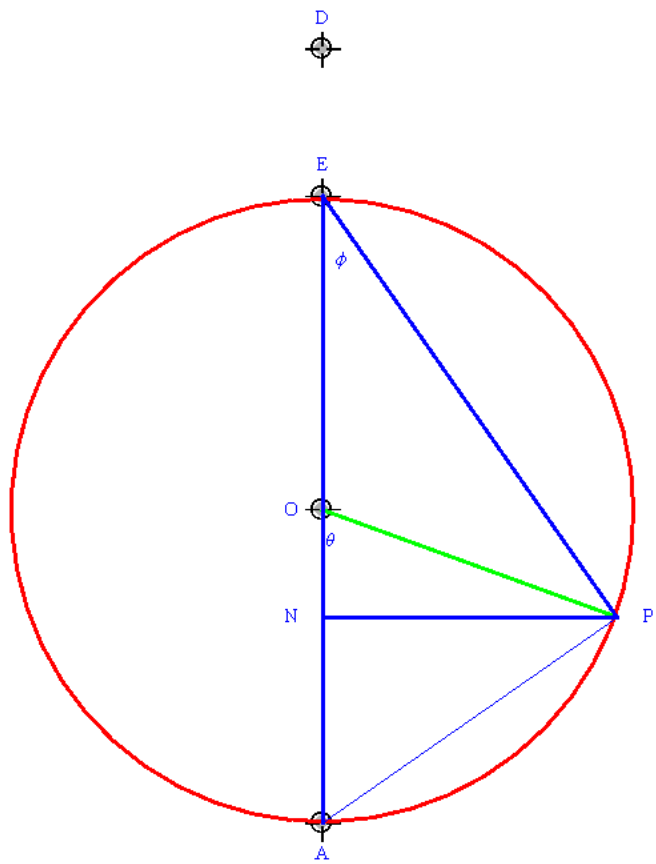


Fig.18 Plot of the geometry using the Graphics of the Mathematica. $\kappa = 0.90$ and $\theta = 70°$.

From the energy conservation law,

$$E = \frac{1}{2}v^2 + \varepsilon(1-\cos\theta) = \frac{1}{2}{v_0}^2$$

the angular velocity at the point E ($\theta = \pi$) can be calculated as

$$v = \dot{\theta} = \sqrt{{v_0}^2 - 4\varepsilon} .$$

We note that the velocity at the point E is

$$V_E = lv = l\sqrt{{v_0}^2 - 4\varepsilon}$$

We choose the point D such that the velocity V_{ff} at the point E after the free fall from the point D is equal to the velocity V_E when it reaches at the top of the circle (the point E).

The distance DE is related to the velocity V_{ff} at the point E is

$$DE = \frac{V_{ff}^{\ 2}}{2g} = \frac{V_{ff}^{\ 2}}{\varepsilon l}$$

where $\varepsilon = g/l$ or $g = \varepsilon l$ and $V_{ff} = V_E$. Then the distance is rewritten as

$$DE = \frac{(v_0^{\ 2} - 4\varepsilon)l^2}{2\varepsilon l} = \frac{(\frac{4\varepsilon}{\kappa^2} - 4\varepsilon)l}{2\varepsilon} = 2l(\frac{1}{\kappa^2} - 1)$$

where

$$v_0 = \frac{2\sqrt{\varepsilon}}{\kappa}$$

Then the distance AD is given by

$$AD = AE + DE = 2l + 2l(\frac{1}{\kappa^2} - 1) = \frac{2l}{\kappa^2}$$

From the above figure, we have the following relations.

$$\frac{\theta}{2} = \phi$$

$$\sin\frac{\theta}{2} = \sin\phi = sn(u,\kappa) = sn(\sqrt{\varepsilon}t,\kappa)$$

$$AP = AE \cdot \sin\phi = 2l \cdot sn(u,\kappa)$$
$$EP = AE \cdot \cos\phi = 2l \cdot cn(u,\kappa)$$
$$AN = AP \cdot \sin\phi = AE \cdot \sin^2\phi = 2l \cdot sn^2(u,\kappa)$$
$$EN = EP \cdot \cos\phi = AE \cdot \cos^2\phi = 2l \cdot cn^2(u,\kappa)$$
$$DN = AD - AN = \frac{2l}{\kappa^2} - 2l \cdot \sin^2\phi = \frac{2l}{\kappa^2}(1 - \kappa^2 sn^2(u,\kappa)) = \frac{2l}{\kappa^2} \cdot dn^2(u,\kappa)$$
$$NP = AP \cdot \cos\phi = AE \cdot \sin\phi\cos\phi = 2l \cdot sn(u,\kappa)cn(u,\kappa)$$
$$AE = 2l$$

5. Numerical calculation using Mathematica for simple pendulum

Here we show the numerical calculation using Mathematica. We use NDSolve to solve the differential equation with the appropriate initial conditions. The advantage of this method is that we do not need any mathematical knowledge on the Jacobi elliptic functions. It is interesting to compare our numerical calculations with the results obtained from the exact solutions above described.

5.1 Formulation

For convenience, the differential equation is separated into two parts

$$v = \dot{\theta} \qquad \dot{v} = -\varepsilon \sin\theta$$

1. First we solve this differential equation using numerical method (*Mathematica*, NDSolve)

2. We analyze the Fourier component by using fast Fourier transform (FFT) (*Mathematica* , Fourier).

   a. We choose the N (= 2^n) data, typically the data between 0 and T (= t_{max}).
   b. The minimum time division is $\Delta = T/N$, which corresponds to the maximum ω_{max} =

   $$\omega_{\max} = \frac{2\pi}{T} N$$

   c. The Fourier spectrum is plotted as a function of the angular frequency defined by channel number multiplied by $(2\pi)/T$.

For simplicity, we consider the case of $\theta_0 = 0$. Then we have

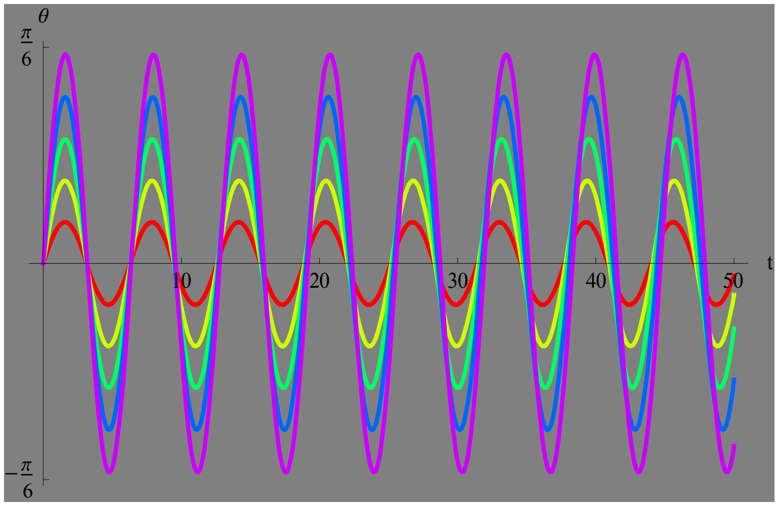
$$E = \frac{1}{2} v_0^{\ 2}$$

When $v_0 < 2\sqrt{\varepsilon}$ (or $E < 2\varepsilon$), the value of θ is limited between $-\pi$ and π. When $v_0 > 2\sqrt{\varepsilon}$ (or $E > 2\varepsilon$), the value of θ is unlimited.

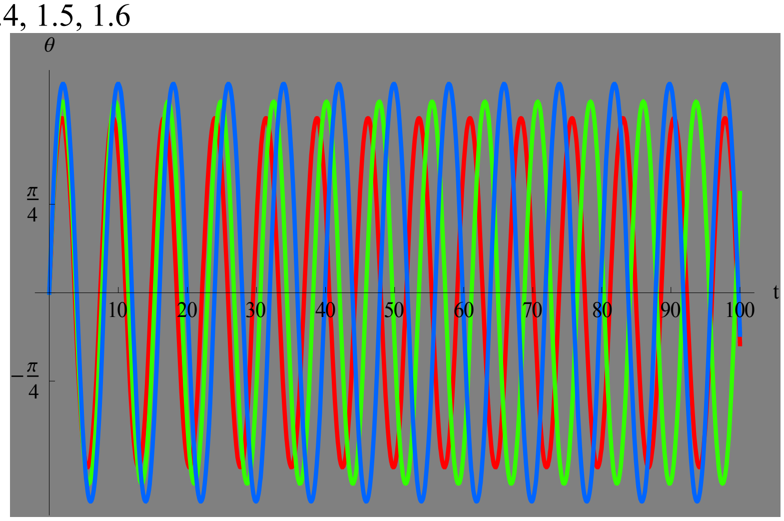
5.2 Time dependence (θ vs t)

We show the time dependence of θ as a function of t with v_0 being changed as a parameter. Here we choose. $\varepsilon = 1$

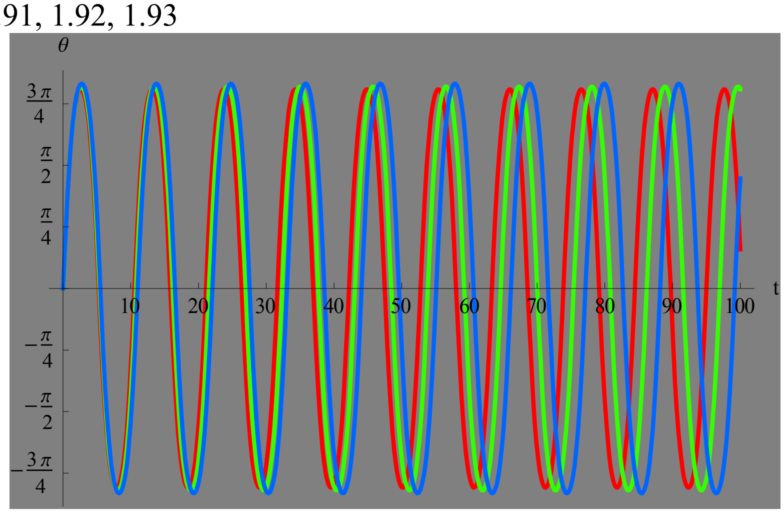
(a) v_0 = 0.1, 0.2, 0.3, 0.4, 0.5



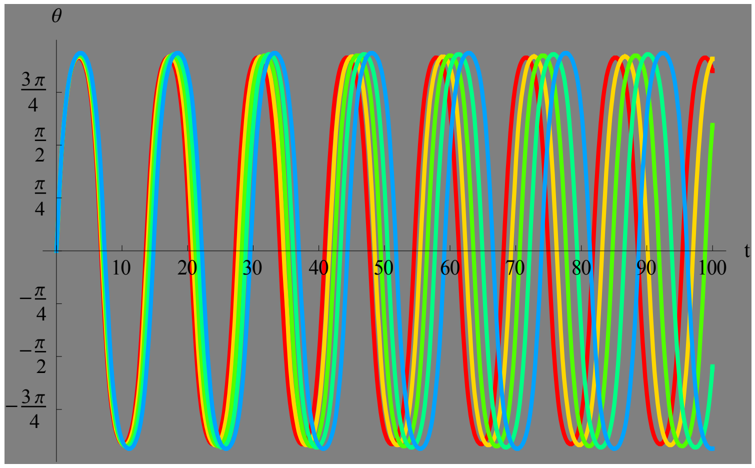
(b) v_0 = 1.4, 1.5, 1.6



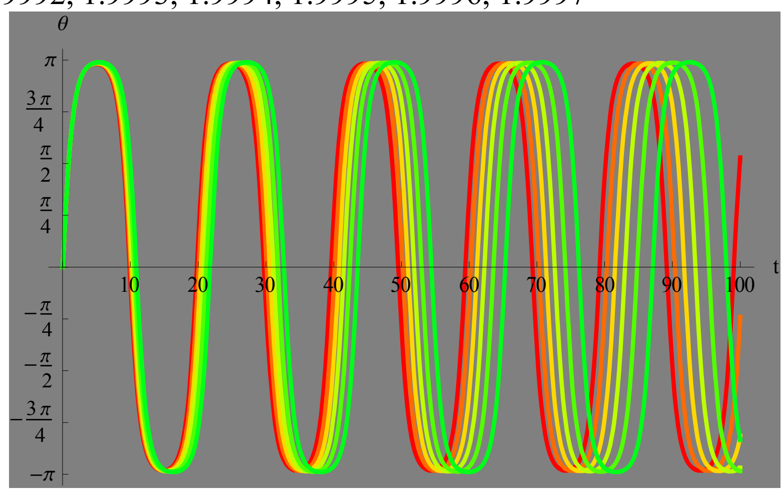
(c) v_0 = 1.91, 1.92, 1.93



(d) v_0 = 1.982, 1.984, 1.986, 1.988, 1.990

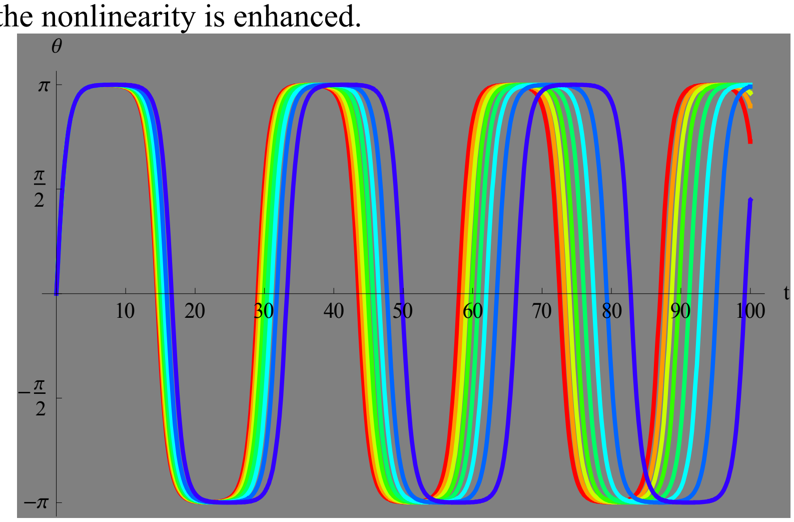


(e) v_0 = 1.9992, 1.9993, 1.9994, 1.9995, 1.9996, 1.9997



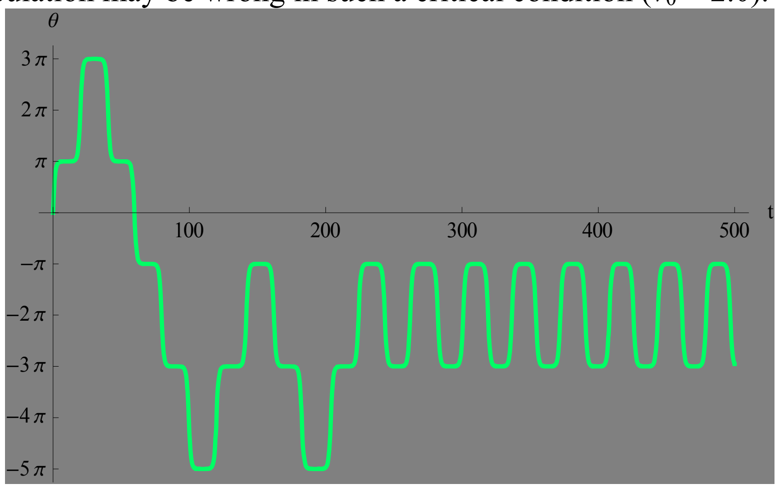
(f) v_0 = 1.999992, 1.999993, 1.999994, 1.999995, 1.999996, 1.999997, 1.999998, 1.999999

As v_0 approaches 2.0, the shape of θ vs t curve becomes square-like. This means that the effect of the nonlinearity is enhanced.



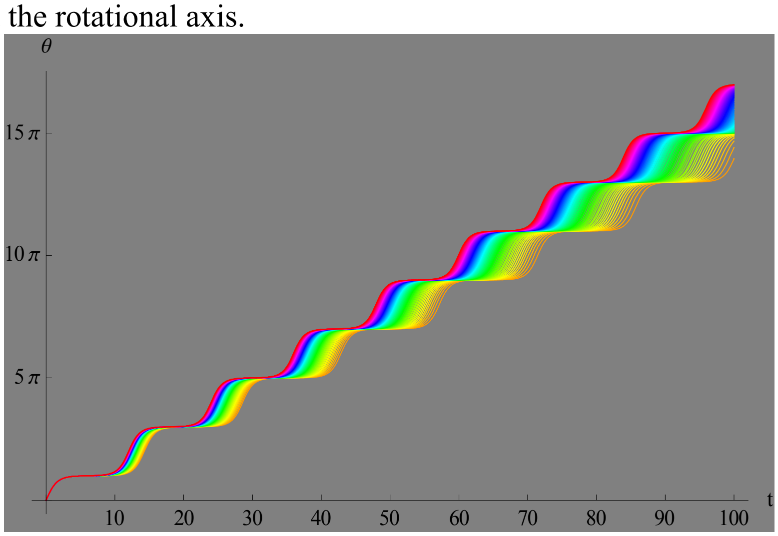
(g) $v_0 = 2$

In this case, the total energy E is equal to the peak height of the potential energy. This is a result form the computer calculation. This result does not agree with the theoretical prediction. Nevertheless we show this result here, indicating that the numerical calculation may be wrong in such a critical condition ($v_0 = 2.0$).

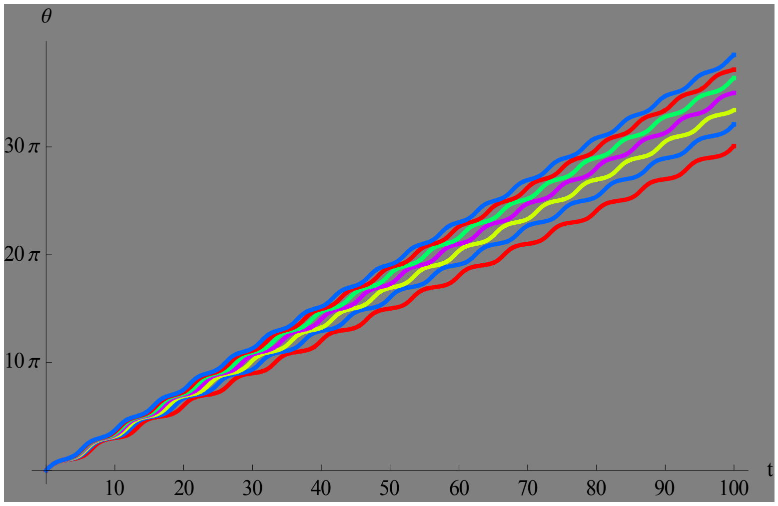


(h) v_0 = 2.00001, 2.00002, 2.00003, 2.00004, 2.0005, 2.0006, 2.0007, 2.00008, 2.00009, 2.0001

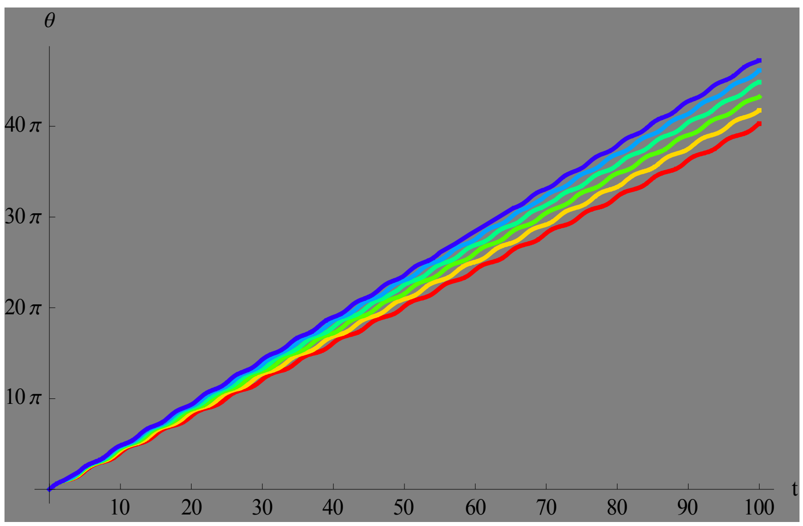
For v_0>2, the value of θ increases with increasing t. In other words, the pendulum rotates around the rotational axis.



(i) v_0 = 2.02, 2.03, 2.04, 2.05, 2.07. 2.08,



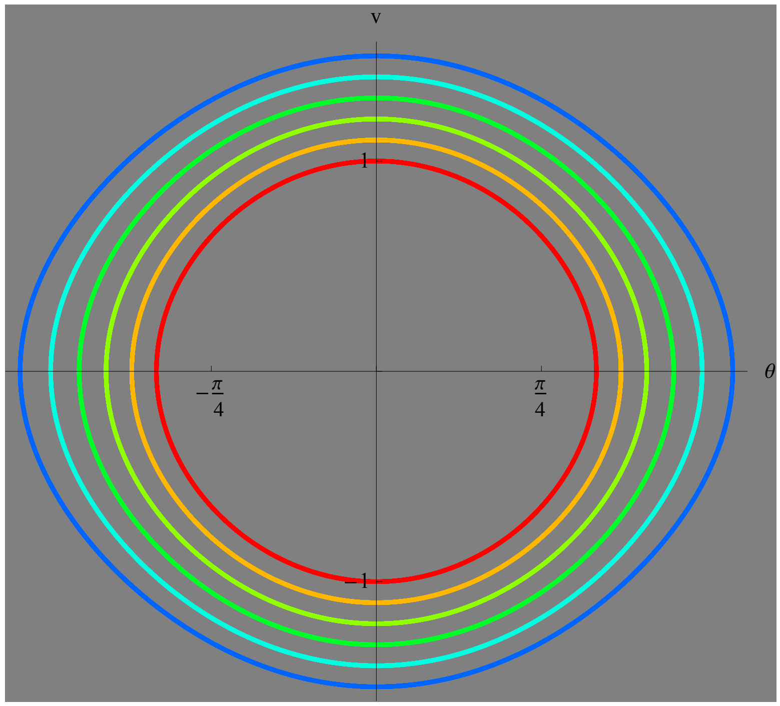
(j) v_0 = 2.10, 2.12, 2.14, 2.16, 2.18. and 2.20



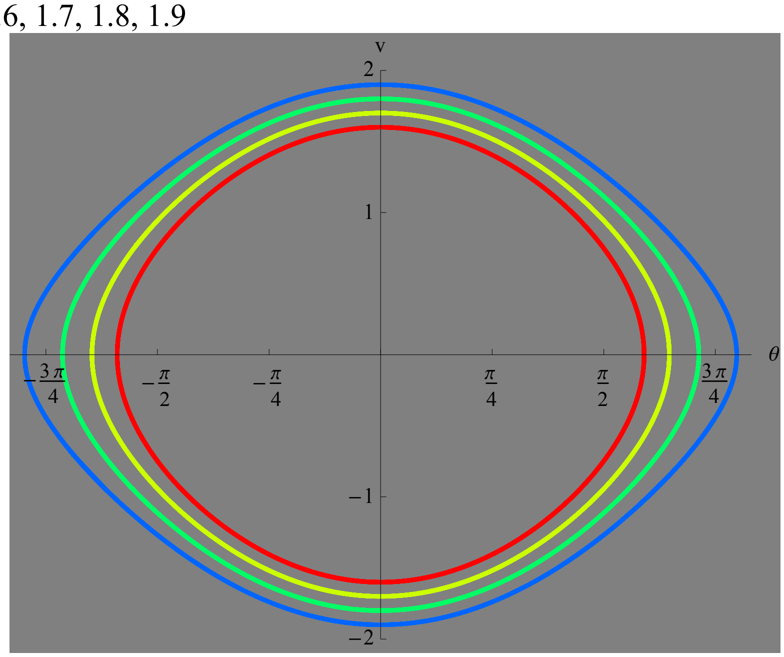
5.3 Trajectories in the phase space (*v* vs *θ*)

Figure shows the trajectories in the phase space **(*v* vs *θ*)** for $\varepsilon = 1$, when v_0 is changed as a parameter.

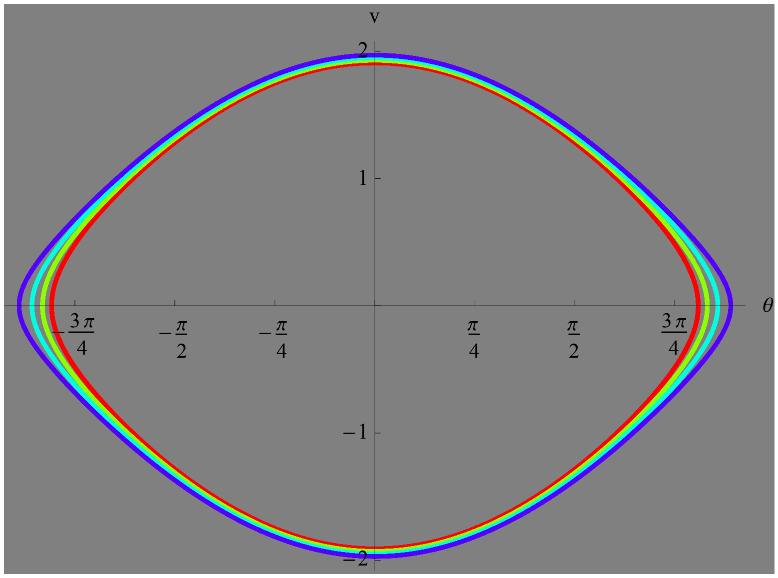
(a) v_0 = 1.0, 1.1, 1.2, 1.3, 1.4, 1.5



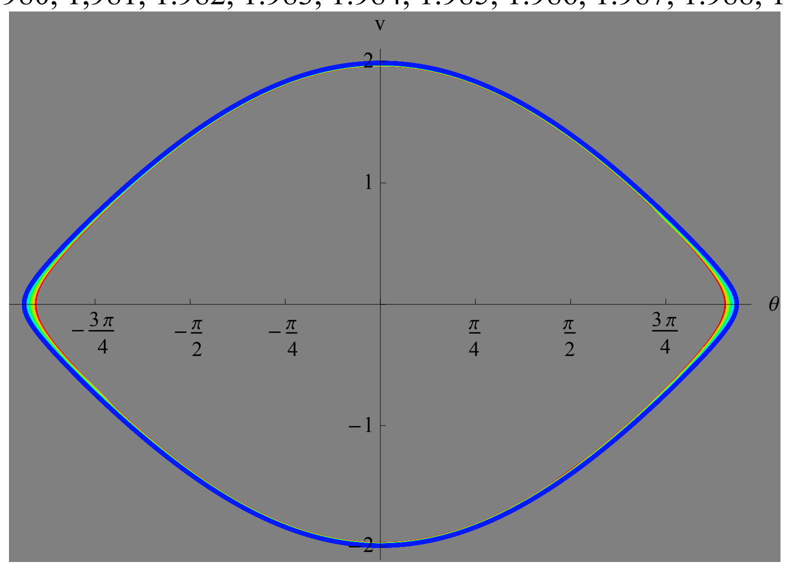
(b) v_0 = 1.6, 1.7, 1.8, 1.9



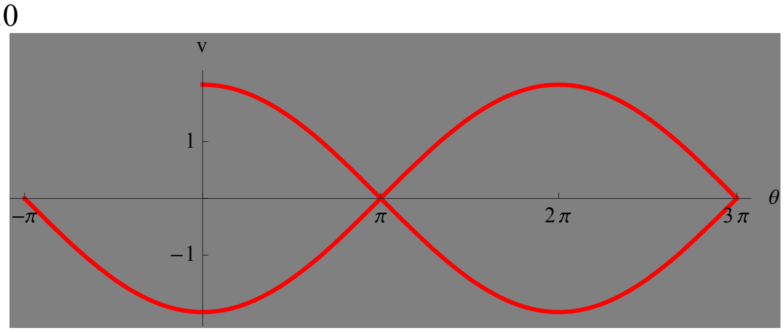
(c) v_0 = 1.91, 1.92, 1.93, 1.94, 1.95, 1.96, 1.97



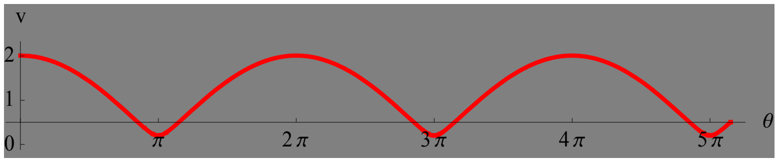
(d) v_0 = 1.980, 1,981, 1.982, 1.983, 1.984, 1.985, 1.986, 1.987, 1.988, 1.989, 1.990.



(e) $v_0 = 2.0$



(f) $v_0 = 2.01$



6. Fourier analysis using Fast Fourier transform (FFT)

The time dependence of q can be analyzed using the FFT analysis. The Fourier spectrum is the square of th absolute amplitude as a function of angular frequency ω. In the small limit of $v_o = 0$,

$$\omega_0 = \sqrt{\varepsilon} .$$

In the present case, $\omega_0 = 1$ since $\varepsilon = 1$.

As v_o approaches $2\sqrt{\varepsilon}$, the time dependence of θ changes from sinusoidal wave to a square wave, indicating the enhancement of nonlinearity in the system. There appear 3ω, 5ω, 7ω, components. The frequency of the fundamental mode rapidly decreases.

$\varepsilon = 1$.

(a) $v_0 = 0.5$

ω (≈ 1) component is mainly observed.

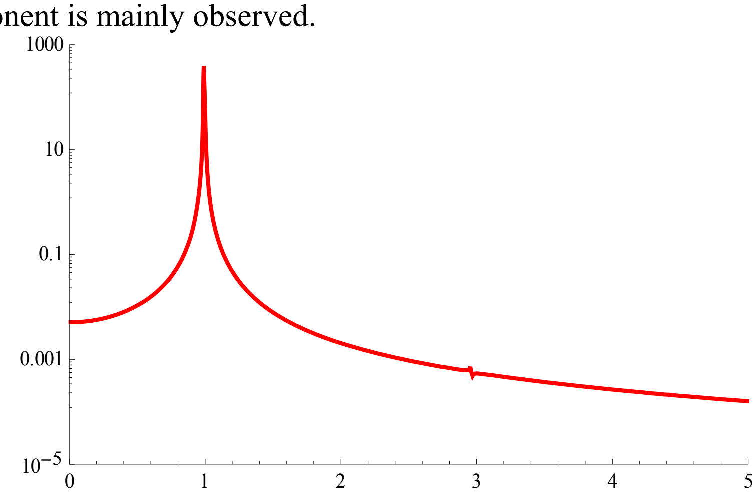
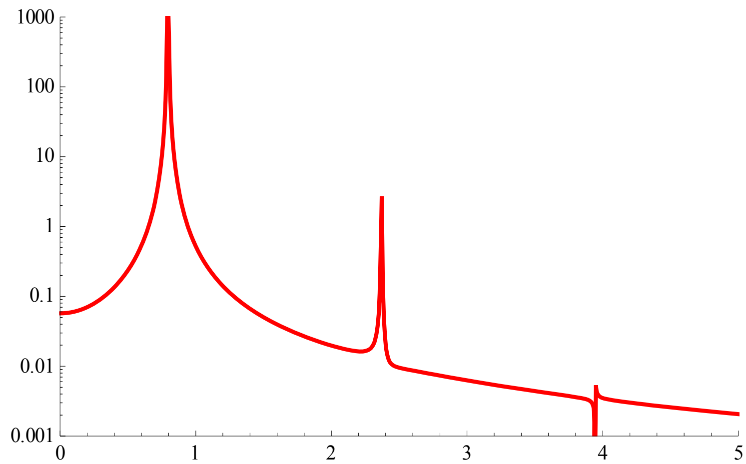


Fig. Fourier spectrum obtained from the FFT analysis of the θ vs t. The x axis is the angular frequency w and the y axis is the squares of the absolute value of complex amplitude. The fundamental harmonic is observed at $\omega = 1$.

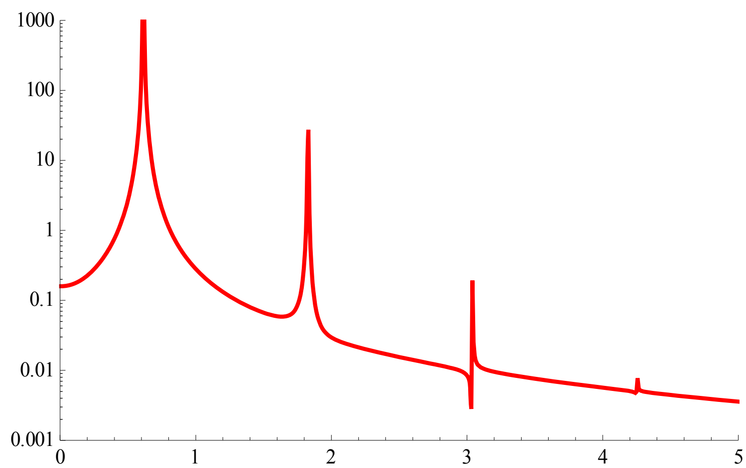
(b) $v_0 = 1.6$

ω and 3ω component are mainly observed.



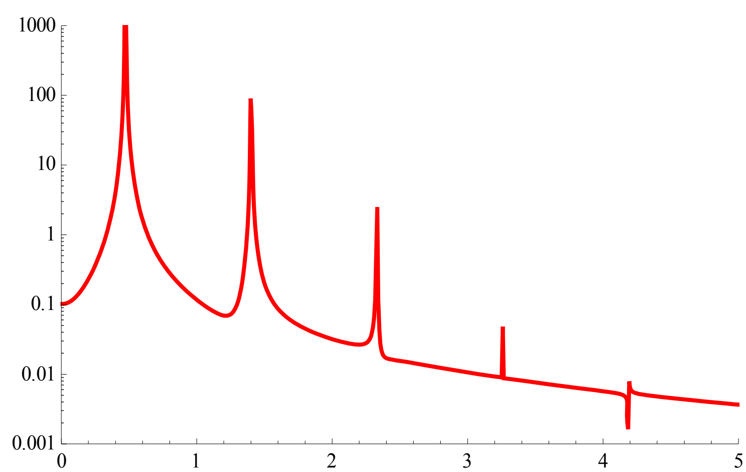
(c) $v_0 = 1.9$

ω, 3ω, and 5ω component are mainly observed.



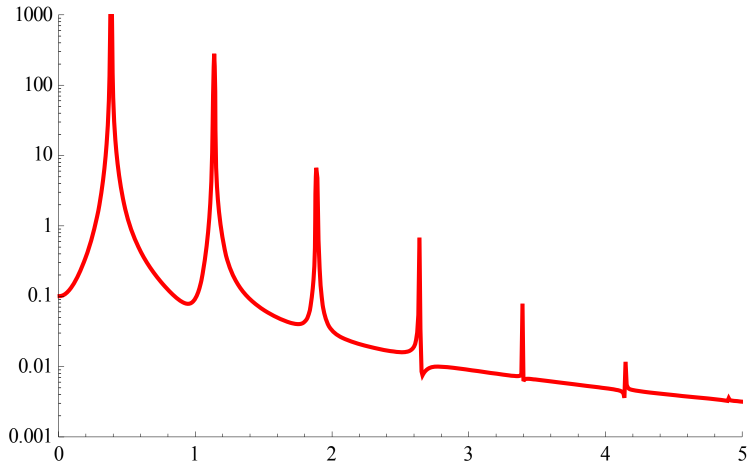
(d) $v_0 = 1.981$

ω, 3ω, 5ω, and 7ω component are mainly observed.



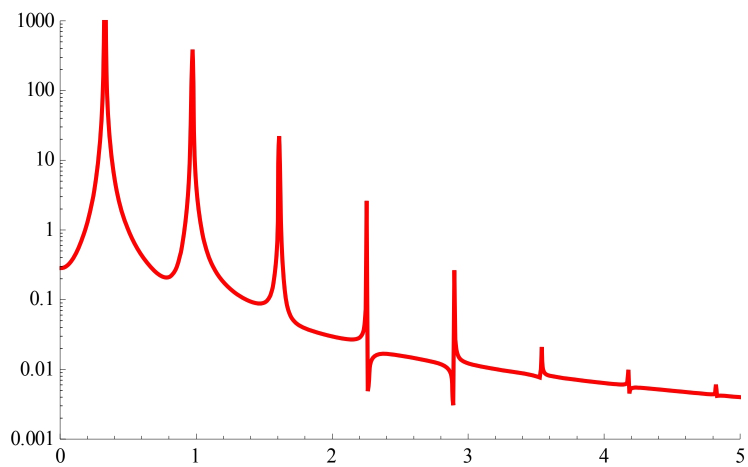
(e) $v_0 = 1.9962$

ω, 3ω, 5ω, 7ω, and 9ω component are mainly observed.



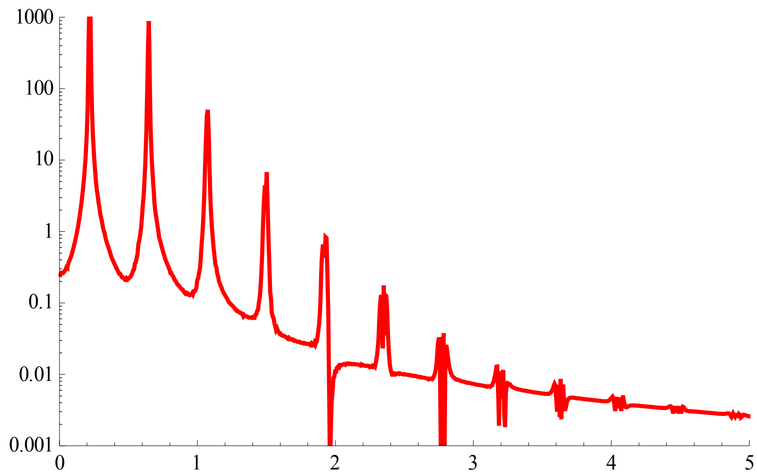
(f) $v_0 = 1.9991$

ω, 3ω, 5ω, 7ω, 9ω, 11ω, and 13ω component are mainly observed.

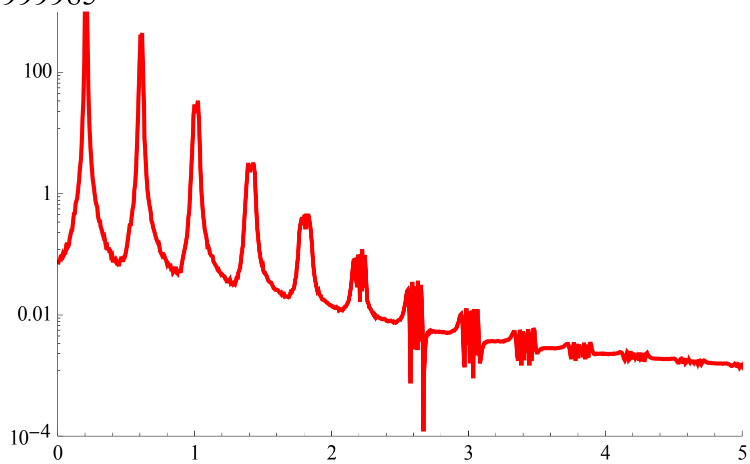


(g) $v_0 = 1.999995$

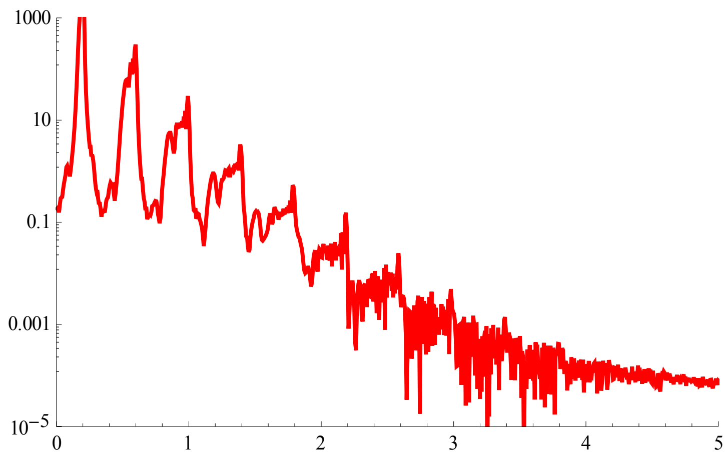
ω, 3ω, 5ω, 7ω, 9ω, 11ω, 13ω, 15ω, and 17ω component are mainly observed.



(h) $v_0 = 1.9999985$

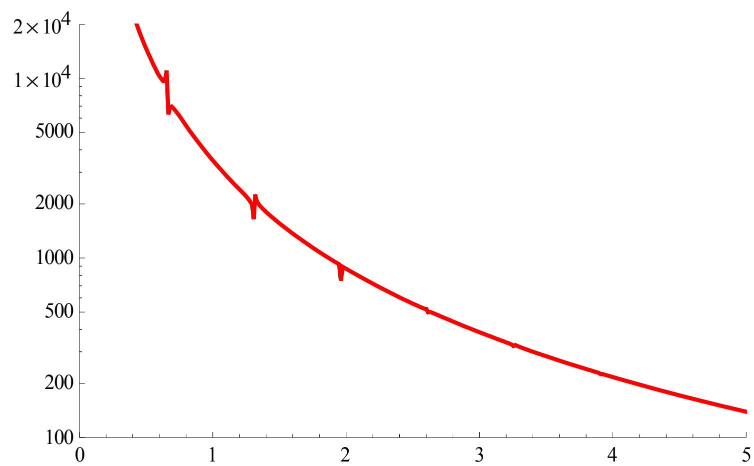


(i) $v_0 = 1.9999999$



The Fourier spectrum drastically changes at $v_0 = 2$. The transition of the behavior occurs from oscillatory to non-oscillatory.

(j) $v_0 = 2.001$



7. Resistive pendulum

7.1 Formulation

Here we consider the case of a simple pendulum with a resistive force. This may be realized in the case when the bob of the pendulum is immersed in liquid such as water. This force is assumed to a proportional to v $(= \dot{\theta})$. Then we have a second-order differential equation given by

$$\ddot{\theta} + \varepsilon \sin\theta + \beta\dot{\theta} = 0$$

where

$$\varepsilon = \frac{g}{l} = {\omega_0}^2 . \qquad \text{or} \qquad \omega_0 = \sqrt{\varepsilon}$$

Here we assume that β is very small positive value. The above differential equation is equivalent to a pair of the first-order differential equations, which are given by

$$\dot{\theta} = v$$
$$\dot{v} = -\varepsilon \sin\theta - \beta v$$

or

$$\frac{dv}{d\theta} = \frac{\dot{v}}{\dot{\theta}} = -\frac{(\varepsilon \sin\theta + \beta v)}{v}$$

In this Eq., both the numerator and the denominator of the right-hand side become zero when $v = 0$ and $\theta = n\pi$. This point in the trajectory in the v vs θ plane is called a singular point. In order to study the detail of the trajectory near the singular point, we introduce a new variable ϕ, which is defined by

$$\theta = n\pi + \phi$$

Then we get

$$\dot{\phi} = v$$
$$\dot{v} = -\varepsilon \sin(n\pi + \phi) - \beta v \approx (-1)^{n+1} \varepsilon\phi - \beta v$$

or the second-order differential equation

$$\ddot{\phi} + \beta\dot{\phi} + (-1)^n \varepsilon\phi = 0$$

When n = even, this equation is the same as one of simple harmonics with viscous damping;

$$\ddot{\phi} + \beta\dot{\phi} + \varepsilon\phi = 0$$

where $\beta < 2\sqrt{\varepsilon}$. Thus the points ($v$ = 0 ,θ = 0, 2π. 4π,…) are stable spiral points. The situation is rather different for n = odd,

$$\ddot{\phi} + \beta\dot{\phi} - \varepsilon\phi = 0$$

When $\phi \propto e^{\lambda t}$, the value of λ satisfies the quadratic equation given by

$$\lambda^2 + \beta\lambda - \varepsilon = 0$$

The roots of this equation are real. One root is a real positive and the other is a real negative. In this case the points (v = 0 ,θ = π, 3π. 5π,…) are saddle points.

7.2 Numerical calculation using Mathematica

7.2.1 Time dependence of θ

We calculate the time dependence of θ numerically. The Mathematica program with a NDSolve command, is used to solve the differential equations given by

$$\dot{\theta}(t) = v(t)$$
$$\dot{v}(t) = -\varepsilon \sin\theta(t) - \beta v(t)$$

with an initial conditions,

$$v(0) = v_0$$
$$\theta(0) = \theta_0$$

For simplicity, we choose ε = 1 and θ_0 = 0. Figure shows the plot of θ vs t for v_0 = 2.3, where β is changed as a parameter (0.01≤β≤0.16). Note that for β = 0, the pendulum continues to rotate around the pivot point for $v_0 > 2$. For finite values of β (see Fig.), however, the rotation angle θ increases with increasing t, and after a characteristic time depending on β, it starts to oscillate around a fixed value ($2n\pi$) with integer n and tends to reduce to the value $2n\pi$. The value of n is strongly dependent on the value of β; $n = 0$ for $\beta = 0.16$, $n = 1$ for $\beta = 0.12$ and 0.08, $n = 1$ for $\beta = 0.04$, $n = 3$ for $\beta = 0.02$, $n = 4$ for $\beta = 0.018$ and 0.016, $n = 5$ for $\beta = 0.014$, $n = 6$ for $\beta = 0.012$, and $n = 7$ for $\beta = 0.010$.

Figure shows the plot of v vs t for $v_0 = 2.3$, where β is changed as a parameter ($0.04 \leq \beta \leq 0.16$). The angular velocity v oscillates around $v = 0$ and tends to decay after sufficient long times. In other words, the average angular velocity (or the terminal angular velocity) is equal to zero, for any β (except for $\beta = 0$).

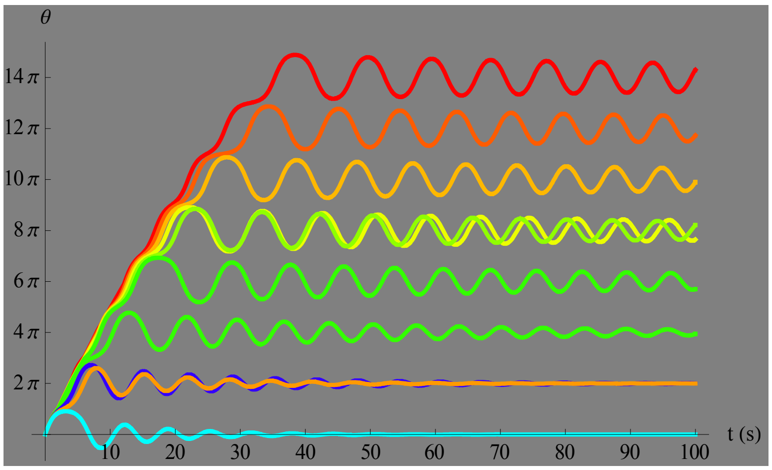


Fig. θ vs t for $\varepsilon = 1$. $v_0 = 2.3$. β is changed as a parameter. $\beta = 0.010$ (red), 0.012, 0.014, 0.016, 0.018, 0.020, 0.04, 0.08, 0.12, and 0.16 (blue)

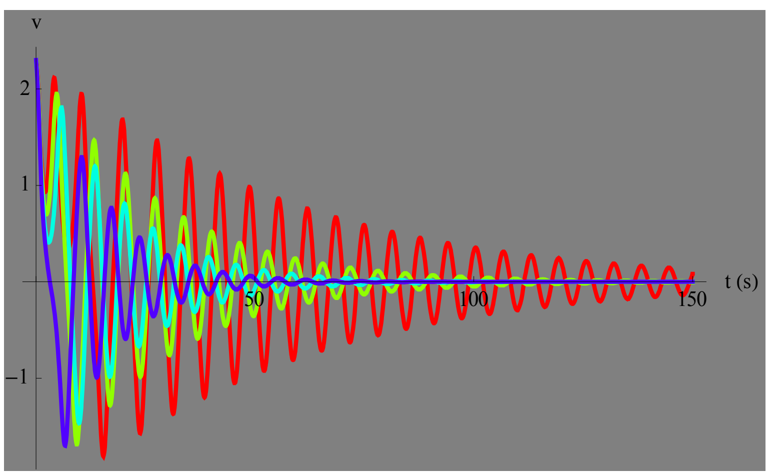


Fig. v vs t for $\varepsilon = 1$. $v_0 = 2.3$. β is changed as a parameter. $\beta = 0.04$ (red), 0.08 (light green), 0.12 (light blue), and 0.16 (dark blue)

(b) Trajectories in the phase space of v vs θ.

Figure shows the trajectories in the phase space of v vs θ, where $\varepsilon = 1$, $v_0 = 2.3$, and b is changed as a parameter. The spiral (stable) point are at $\theta = 2n\pi$ and $v = 0$, where n is an integer and the saddle points are at $\theta = (2n+1)\pi$ and $v = 0$. Each trajectories for finite β ($\beta \neq 0$) converges to the spiral points. The spiral points becomes far away from the origin with decreasing β. The terminal angular velocity v (the angular velocity at sufficiently long times) is equal to zero except for the case of $\beta = 0$.

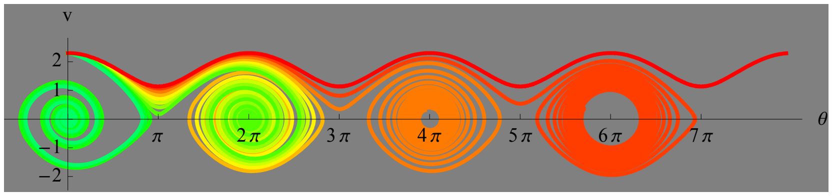


Fig. phase space of v ($= \dot{\theta}$)vs θ. $v_0 = 2.3$. $\beta = 0$, (red), 0.02, 0.04, 0.06, 0.08, 0.10, 0.12, 0.14, 0.16, 0.18, and 0.20 (green).

8. Analogy between simple rigid pendulum and Josephson current

8.1 Formulation

Finally we consider the motion of the pendulum with an applied torque T around the pivot point P.

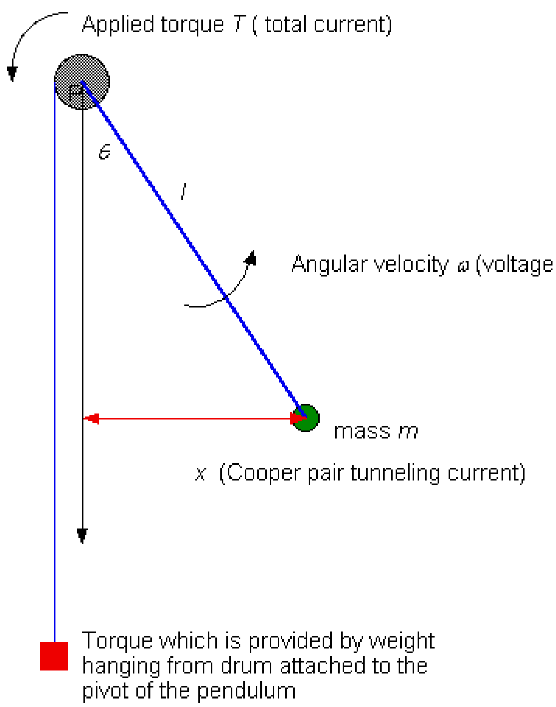


Fig.7 Simple pendulum with an applied torque.

The equation of motion for the pendulum is given by

$$I\ddot{\theta} = T - mgl\sin\theta - \eta\dot{\theta},$$

where T is an external torque, I is a moment of inertia around the pivot P and the third term of the right-hand side is a viscosity of air. Since $I = ml^2$, this equation can be rewritten as

$$\tau = \ddot{\theta} + \beta\dot{\theta} + \varepsilon\sin\theta$$

where

$$\beta = \frac{\eta}{ml^2} \qquad \text{and} \qquad \tau = \frac{T}{ml^2}$$

For simplicity we assume that $\varepsilon = 1$.

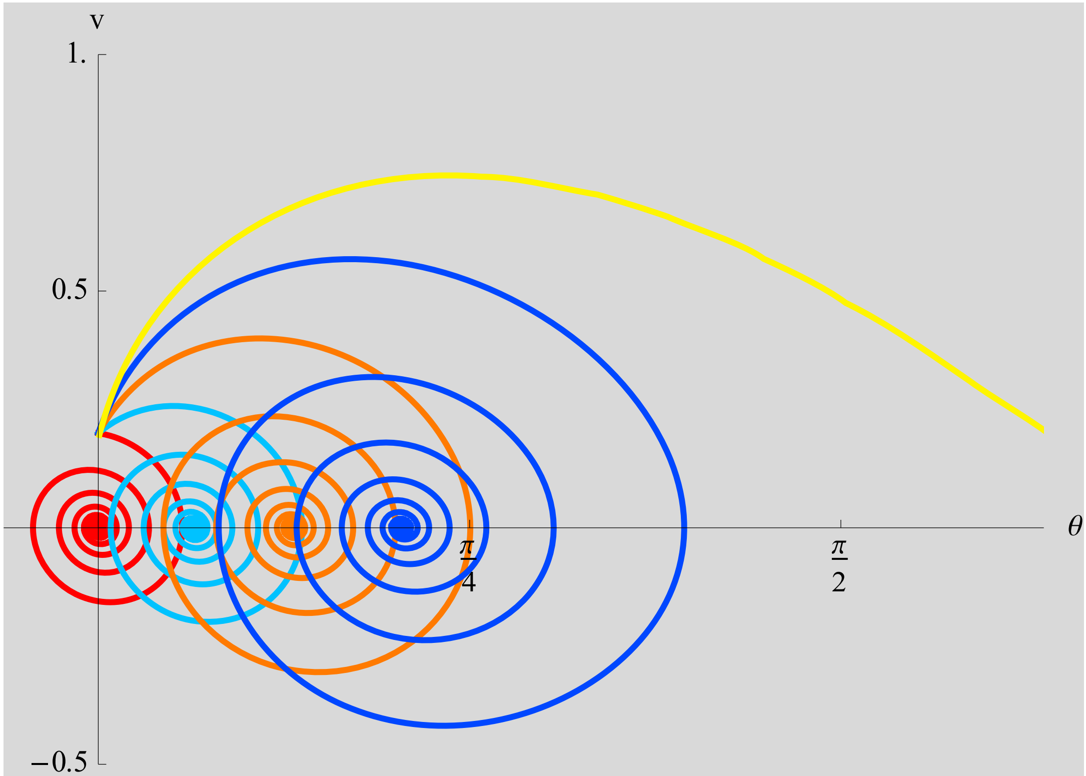
8.2 Numerical calculation for the simple case

8.2.1 β = 0.16 and v_0 = 0.2.
τ = 0 (red), 0.2, 0.4, 0.6 (blue), 0.8 (yellow)

The trajectory changes from the spiral orbits (small τ) to the open orbits (large τ) around τ = 0.6 – 0.8. The stable point of the spiral orbits is moved from the origin to an equilibrium point (= θ_s) on the positive θ axis (v = 0). The value of θ_s is related to τ as

$$\tau = \sin\theta_s$$

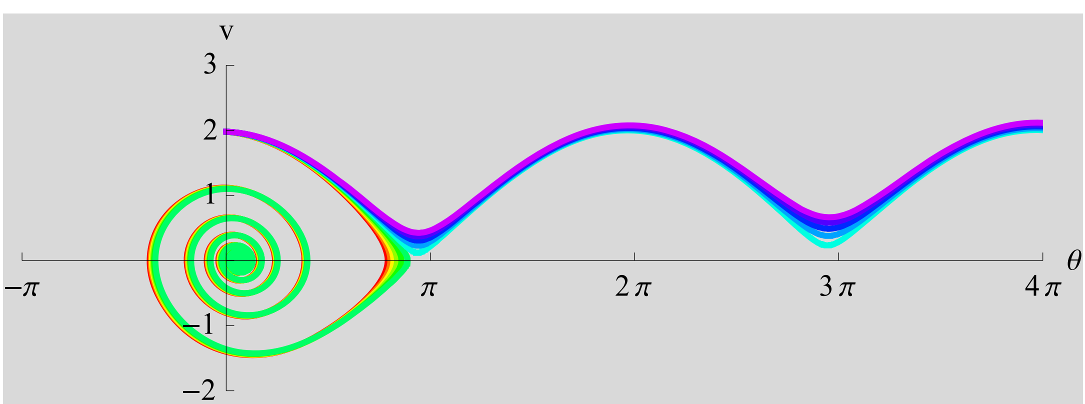
for $\tau \leq 0.6$. This relation is valid at least for small β and small v_0.



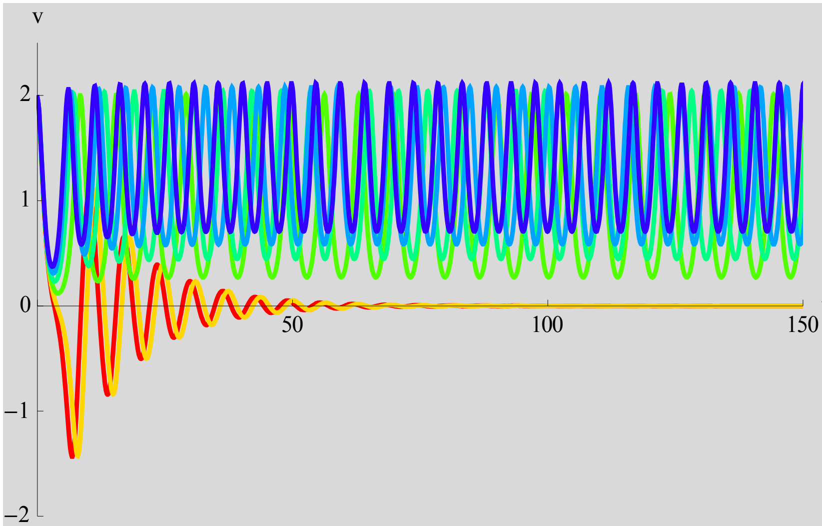
8.2.2 $\beta = 0.16$. $v_0 = 1.98$, $\tau = 0.19 - 0.25$.

The trajectory changes from the spiral orbits (small τ) to the open orbits (large τ) around $\tau = 0.21$. The stable point of the spiral orbits is moved from the origin to an equilibrium point (= θ_s) on the positive θ axis ($v = 0$). On the other hand, the value of v on the open orbit oscillates with increasing θ and reaches an equilibrium value v_o, which is not equal to zero. The time dependence of v drastically changed around $\tau = 0.21$. For $\tau = 0.20$, v oscillate around $v = 0$ with time t, while for τ>0.22,, v oscillates around a finite value of v (not equal to zero) 0 with time τ,

(a) Trajectories in the phase space



(b) The time dependence of v for t = 0.19, 0.20, 0.21, 0.22, 0.23, and 0.24.

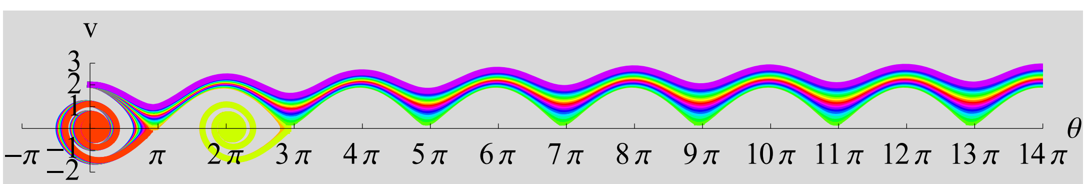


8.2. β = 0.16. τ = -0.01 – 0.40 ($\Delta\tau$ = 0.01)

Through the computer simulation we find some peculiar behavior in the trajectories in the phase space, where $\beta = 0.16$ and $\varepsilon = 1$. For each v_0 (= 2.01 – 3.15), the value of τ is changed as a parameter: typically τ = 0.01 – 0.40 with $\Delta\tau$ = 0.01. The trajectory changes from the spiral orbit for small τ to the open orbit for large τ.

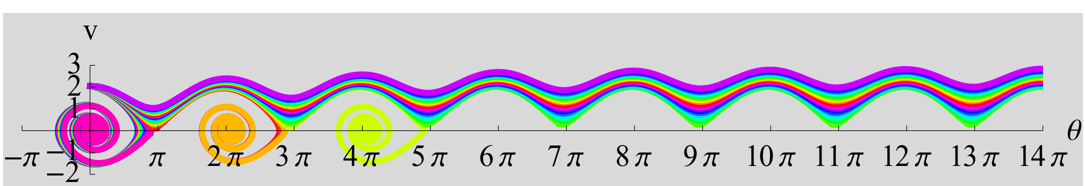
(a) $v_0 = 2.01$

The stable points of the spiral orbits are around $\theta = 0$ and 2π for small τ.



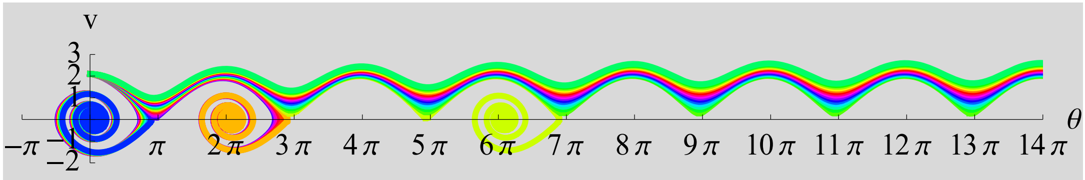
(b) $v_0 = 2.05$

The stable points of the spiral orbits are around $\theta = 0$, 2π, and 4π for small τ.



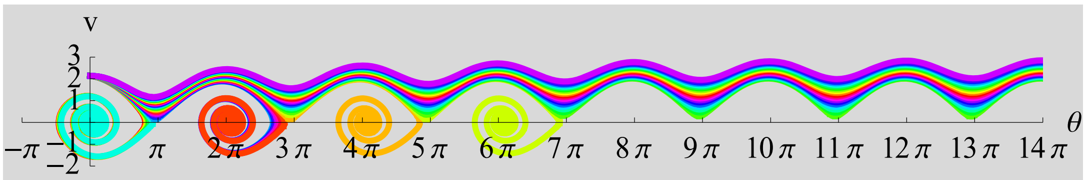
(c) $v_0 = 2.10$

The stable points of the spiral orbits are around $\theta = 0$, 2π, and 6π for small τ.



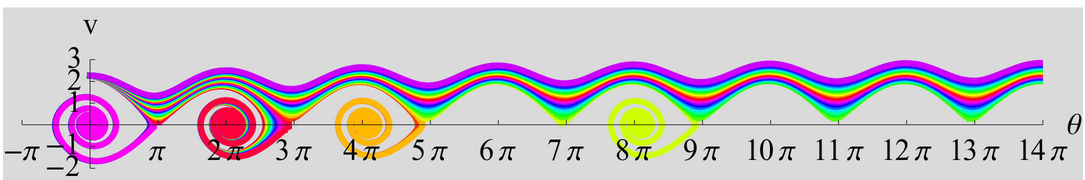
(d) $v_0 = 2.14$

The stable points of the spiral orbits are around $\theta = 0$, 2π, 4π, and 6π for small τ.



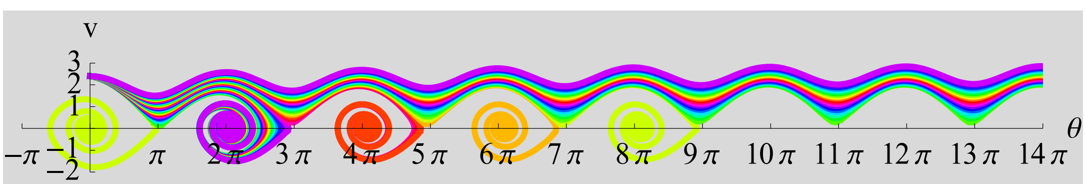
(e) $v_0 = 2.27$

The stable points of the spiral orbits are around $\theta = 0$, 2π, 4π, and 8π for small τ.



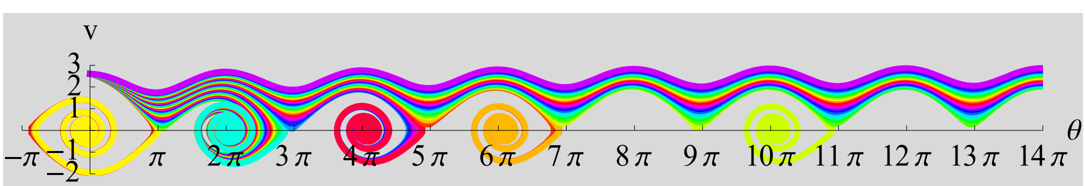
(e) $v_0 = 2.40$

The stable points of the spiral orbits are around $\theta = 0$, 2π, 4π, 6π, and 8π for small τ.



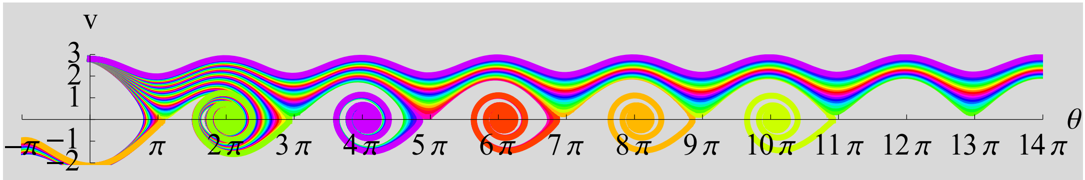
(f) $v_0 = 2.60$

The stable points of the spiral orbits are around $\theta = 0$, 2π, 4π, 6π, and 10π.



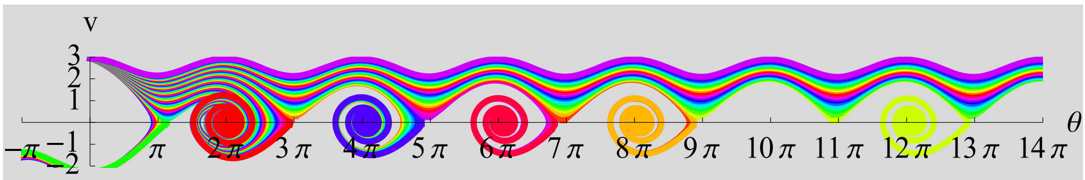
(g) $v_0 = 2.80$

The stable points of the spiral orbits are around $\theta = 0$, 2π, 4π, 6π, 8π and 10π.



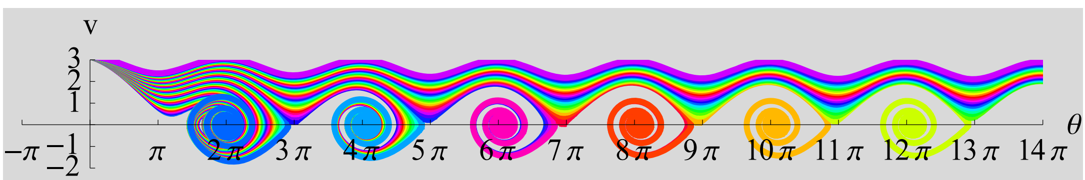
(h) $v_0 = 2.95$

The stable points of the spiral orbits are around $\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi$ and 12π.



(i) $v_0 = 3.15$

The stable points of the spiral orbits are around $\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi, 10\pi$, and 12π.



8.3 Analogy with the Josephson junction

We show that this differential equation is essentially the same as that of the Josephson current of superconductors. The angle θ for the pendulum corresponds to the phase difference of two superconductors in the Josephson junction. The corresponding relations between two models are as follows.

$$T = I\ddot{\theta} + +\eta\dot{\theta} + mgl\sin\theta \ \text{(pendulum).}$$

$$I = \frac{\hbar C}{2e}\ddot{\phi} + \frac{\hbar}{2eR}\dot{\phi} + I_c\sin\phi \ \text{(Josephson juncton).}$$

Josephson junction	pendulum
phase difference ϕ	deflection θ
total current across junction I	applied torque T
Capacitance C	Moment of inertia
normal tunneling conductance $1/R$	viscous damping η
Josephson current $I_c\sin\phi$	horizontal displacement of bob $x = l\sin\theta$
voltage across junction V	angular velocity $\omega = \dot{\theta}$

(1) $T = mgl\sin\theta$, $\omega = \dfrac{d\theta}{dt} = 0$ (or $\tau = \varepsilon\sin\theta$, $v = 0$)

When a small torque is applied, the pendulum finally settles down at a constant angle of deflection θ. No angular velocity (ω = 0, or v = 0) corresponds to no voltage across a junction (V = 0). The junction is superconducting. The Josephson current is $I_c\sin\phi$.

(2) If the torque is gradually increased, the pendulum deflects to a greater but steady angle. We can pass more current through a junction without any voltage appearing.

(3) Critical torque T_c = mgl (or τ_c = ε). This is the torque which deflects the pendulum through a right angle so that it is horizontal.

(4) For $T>T_c$ (or $\tau>\varepsilon$), the pendulum cannot remain at rest but rotates continuously. As the pendulum rotates, the horizontal deflection x oscillates. The angular velocity is always in the same direction. This corresponds to the case of $I>I_c$ and $V\neq 0$. A DC voltage will appear across the junction if the current passed through it exceeds a critical value.

The detail of the Josephson junction and DC SQUID will be found in the Lecture Note on the Solid State Physics (M. Suzuki and I.S. Suzuki). All the calculations are made using the NDSolve command in the Mathematica.

9. CONCLUSION

The physics of a simple pendulum for large angles is much more complicated than that for small angle. In this note we show two methods. The time dependence of the angle can be exactly described by a Jacobi elliptic function. Thanks to Mathematica, this function can be easily used like sine and cosine functions. We also use the numerical calculation of the original differential equation with appropriate initial conditions. As is expected, the results of numerical calculation are in excellent agreement with the exact solutions from the theory without any approximation. Since the numerical method is much simpler than the theoretical calculations. The numerical method is not so powerful for a simple pendulum since the exact solution is given by a Jacobi elliptic function. However, this method is very useful when the perturbation effect such as a resistive pendulum and the Josephson current of the superconductivity. The trajectories in the phase space in the resistive pendulum with an external torques are dependent on the damping factor, the initial velocity, and the external torque. It is interesting to study systematically the behavior of the trajectories. This may help our understanding the nonlinear behavior of the Josephson junction.

REFERENCES

1. A.B. Pippard, The Physics of Vibrartion 1, Cambridge University Press 1978
2. Morikazu Toda, Theory of Oscillation Shokabo, Tokyo 1975 [in Japanese].
3. M. Suzuki and I.S. Suzuki, Lecture Note on Solid State Physics Josephson effect and DC SQUID
4. Wikipedia, http://en.wikipedia.org/wiki/Pendulum

APPENDIX

A. Elliptic Integral of the First Kind

Let the elliptic modulus k satisfy $0<k^2<1$, and the Jacobi amplitude be given by $\phi =$ am u. The incomplete elliptic integral of the first kind is then defined as

$$u = F(\phi, k) = \int_0^{\phi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}.$$

The incomplete elliptic integral of the first kind is implemented in Mathematica as EllipticF[ϕ, m (= k^2)]. Note the use of the parameter $m = k^2$ instead of the modulus k. Letting

$$\xi = \sin\varphi, \qquad d\xi = \cos\varphi d\varphi = \sqrt{1-\xi^2}\, d\varphi,$$

$F(\phi, k)$ can be rewritten as

$$u = F(\phi, k) = \int_0^{z=\sin\phi} \frac{d\xi}{\sqrt{1-\xi^2}\sqrt{1-k^2\xi^2}} \qquad (=\text{EllipticF}[\phi, m\ (= k^2)])$$

This integral satisfies

$$F(-\phi, k) = -F(\phi, k)$$

Special values of $F(\phi, k)$include

$$F(0, k) = 0$$

$$F(\frac{1}{2}\pi, k) = K(k)$$

where $K(k)$ is the complete elliptic integral of the first kind,

$$K(k) = \int_0^1 \frac{d\xi}{\sqrt{1-\xi^2}\sqrt{1-k^2\xi^2}}$$

The complete elliptic integral of the first kind is implemented in Mathematica as EllipticK[m (= k^2)]. Note the use of the parameter $m = k^2$ instead of the modulus k. The period T of the simple pendulum

$$T = \frac{4}{\sqrt{\varepsilon}} K(k) \qquad (= \frac{4}{\sqrt{\varepsilon}} \text{EllipticK}[m\ (= k^2)])$$

B. Jacobi elliptic function

The Jacobi elliptic functions are standard form of elliptic functions, The three basic functions are denoted sn(u, k), cn(u, k), and dn(u, k), where k is known as the elliptic modulus. These functions arise from the inversion of the elliptic integral of the first kind,

$$\phi = F^{-1}(u,k) = am(u,k) = am(u)$$

where ϕ is the Jacobi amplitude. From this, it follows that

$$\sin\phi = \sin(am(u,k)) = sn(u,k)$$
$$\cos\phi = \cos(am(u,k)) = cn(u,k)$$
$$\sqrt{1-k^2\sin^2\phi} = \sqrt{1-k^2 sn^2(u,k)} = dn(u,k)$$

Note that sn(u, k) and cn(u, k) is a periodic function in u with period 4 K(k). dn(u, k) is a periodic function of 2 K(k). The Jacobi elliptic functions sn(u, k), cn(u, k), and dn(u, k) are implemented in Mathematica as JacobiSN[u, m (= k^2)], JacobiCN[u, m (= k^2)], and JacobiDN[u, m (= k^2)], and Note the use of the parameter $m = k^2$ instead of the modulus k.

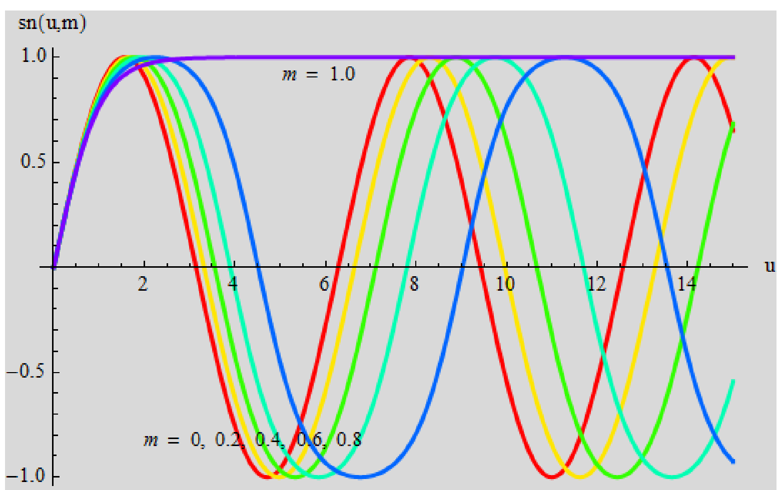


Fig. Plot of sn(u, m) as a function of u, where m is changed as a parameter. m = 0 (red), 0.2, 0.4 0.6, 0.8 (blue), and 1.0 (purple).

C Commands of mathematica

We use the following Mathematica commands in the present calculations.

Fourier

Fourier[list] finds the discrete Fourier transform of a list of complex numbers.

NDSolve

NDSolve[eqns, y, {x, x_{min}, x_{max}}] finds a numerical solution to the ordinary differential equations eqns for the function y with the independent variable x in the range x_{min} to x_{max}.

EllipticK[m (= k^2)]

EllipticK[$m = k^2$] gives the complete elliptic integral of the first kind $K(k = \sqrt{m})$.

JacobiSN[u, m (= k^2)] and **JacobiCN[u, m (= k^2)]**

JacobiSN[u, m], JacobiCN[u, m], etc. give the Jacobi elliptic functions,

$sn(u, k = \sqrt{m})$ and $cn(u, k = \sqrt{m})$.

ContourPlot

ContourPlot[$f == g$, {x, x_{min}, x_{max}}, {y, y_{min}, y_{max}}] plots contour lines for which $f = g$, in the x-y plane

Graphics

Graphics[primitives,options] represents a two-dimensional graphical image.

ParametricPlot

ParametricPlot[{f_x(u), $f_y(u)$}, {u, u_{min}, u_{max}}] generates a parametric plot of a curve with x and y coordinates and f_x(u) and $f_y(u)$} are a function of u.

D. FFT analysis using Fourier (Mathematica program) (Section 2.4 and Chapter 6)

FFT analysis of the wave form for Simple Pendulum

We analyze the Fourier conmonent by using FFT (*Mathematica* , Fourier).

1. We choose the N (= 2^n) data, typically the data between 0 and T (=tmax).
2. The minimum time division is $\Delta = \frac{T}{N}$, which corresponds to the maximum ωmax = $\frac{2\pi}{\Delta} = \frac{2\pi}{T}$ N
3. Th Fourier spectrum is plotted as a function of the channel numer multiplied by $\frac{2\pi}{T}$.

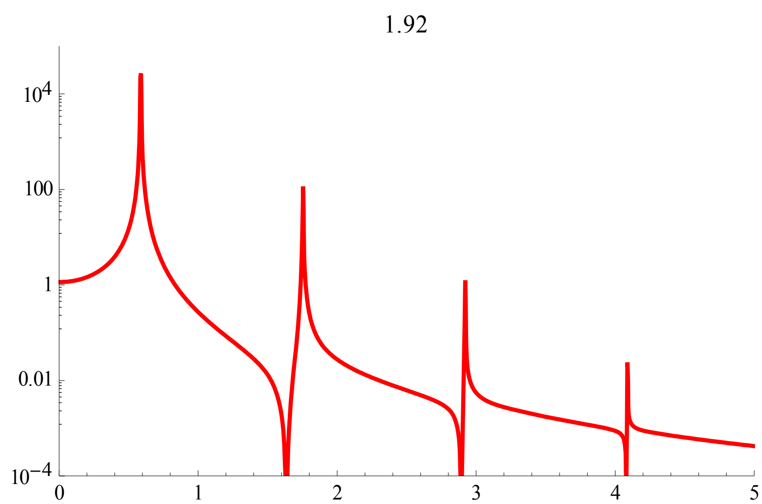
In[1]:=
```
Clear["Global`*"]
```

In[2]:=
```
G1[t_] := 2 ArcSin[k JacobiSN[t, k^2]] /. k → v0/2
```

In[3]:=
```
pentime[{θ0_, v1_}, ϵ_, tmax_, N1_] :=
 Module[{numtable},
  numtable = Table[{t, G1[t] /. v0 → v1}, {t, 0, tmax, tmax/N1}]]
```

In[4]:=
```
eq1 = pentime[{0, 1.92}, 1, 1600, 32768]; d1 = Dimensions[eq1][[1]];
list1 = Table[eq1[[i, 2]], {i, 1, d1 - 1}]; list2 = Fourier[list1] // Chop;
list3 = Table[{2 π/1600 n, Abs[list2[[n]]]^2}, {n, 0, 32768}];
ListLogPlot[list3, Joined → True,
 PlotRange → {{0, 5}, {0.0001, 100000}}, PlotStyle → {Thick, Red},
 PlotLabel → 1.92]
```

Out[4]=

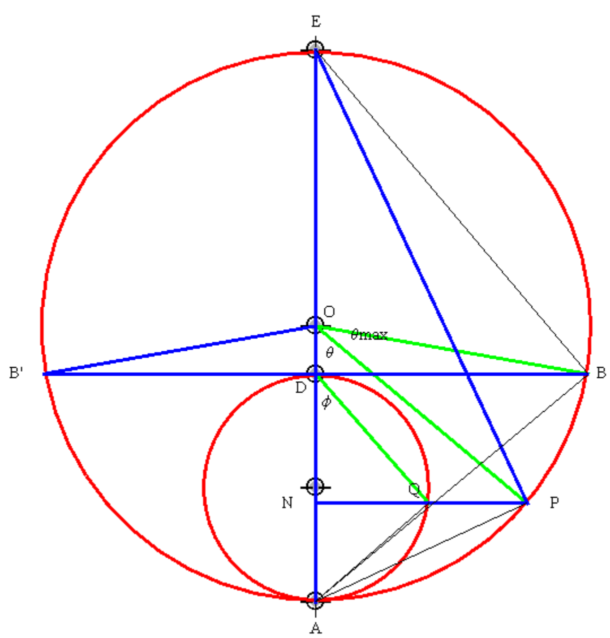


E. Geometry from Graphics (Chapter 4)

In[1]:=
```
rule1 = {θmax → 80 °, θ → 50 °}
```
Out[1]=
```
{θmax → 80 °, θ → 50 °}
```
In[2]:=
```
O1 = {0, 0}; E1 = {0, 1}; A1 = {0, -1}; B1 = {Sin[θmax], -Cos[θmax]};
B2 = {-Sin[θmax], -Cos[θmax]}; P1 = {Sin[θ], -Cos[θ]}; N1 = {0, - Cos[θ]};
O2 = {0, -((1 + Cos[θmax])/2)}; D1 = {0, -Cos[θmax]}; R2 = (1 - Cos[θmax])/2;
eq1 = x^2 + (y + (1 + Cos[θmax])/2)^2 == ((1 - Cos[θmax])/2)^2 /. y → -Cos[θ] // Simplify;
eq2 = Solve[eq1, x] // FullSimplify;
Q1 = {x /. eq2[[2]], -Cos[θ]}
```
Out[2]=

$$\left\{\sqrt{2}\sqrt{(\mathrm{Cos}[\theta]-\mathrm{Cos}[\theta\mathrm{max}])\,\mathrm{Sin}\left[\frac{\theta}{2}\right]^2},\ -\mathrm{Cos}[\theta]\right\}$$

In[3]:=
```
B11 = B1 /. rule1 // N; B21 = B2 /. rule1 // N; P11 = P1 /. rule1 // N; O21 = O2 /. rule1 // N;
N11 = N1 /. rule1 // N; Q11 = Q1 /. rule1 // N; R21 = R2 /. rule1 // N; B111 = B11 + {0.05, 0};
D11 = D1 /. rule1 // N; O111 = O1 + {0.05, 0.05}; O112 = O1 + {0.055, -0.1}; O113 = O1 + B11/5;
D111 = D11 + {-0.05, -0.05}; D112 = D11 + {0.04, -0.1}; E111 = E1 + {0, 0.1};
N111 = N11 + {-0.1, 0}; Q111 = Q11 + {-0.05, 0.05}; P111 = P11 + {0.1, 0}; A111 = A1 + {0, -0.1};
B211 = B21 + {-0.1, 0}
```
Out[3]=
```
{-1.08481, -0.173648}
```
In[4]:=
```
Graphics[{Locator[E1], Locator[O1], Locator[O21], Locator[A1], Locator[D11],
  Thick, Red, Circle[O1, 1], Circle[O21, R21], Thick, Green, Line[{O1, P11}],
  Line[{Q11, D11}], Line[{O1, B11}], Blue, Thick, Line[{A1, E1}], Line[{B21, B11}],
  Line[{O1, B21}], Line[{E1, P11}], Line[{P11, N11}], Thin, Black, Line[{E1, B11}],
  Line[{A1, P11}], Line[{A1, B11}], Line[{Q11, A1}], Text["E" , E111], Text["B", B111],
  Text["O" , O111], Text["D", D111], Text["N", N111], Text["Q", Q111], Text["P", P111],
  Text["A", A111], Text["ϕ", D112], Text["θ", O112], Text["θmax", O113], Text["B'", B211]}]
```
Out[4]=



F. Numerical method of solving differential equation (NDSolve) (Chapters 5, 7, and 8)

Initial condition

$\theta(t=0)=0$, $\theta'(t=0)=v0$

The total energy E is conserved. $E=\frac{1}{2}v0^2$. The maximum of the potential energy is 2ϵ.

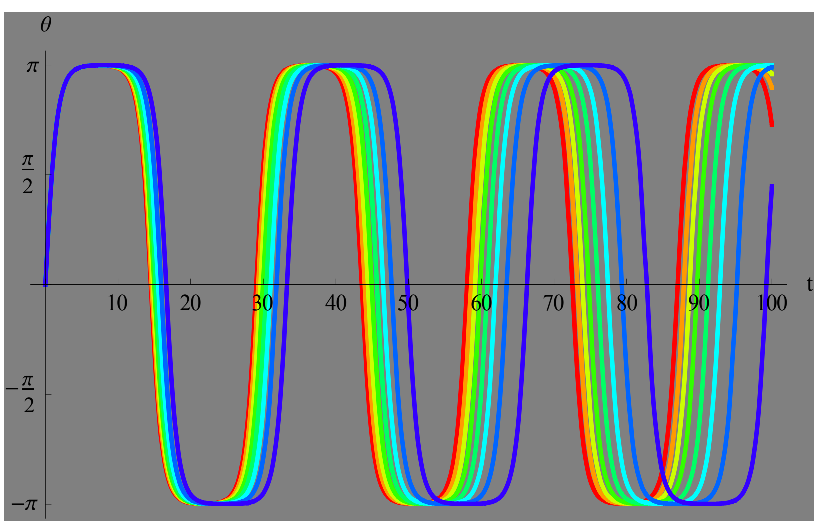
$\epsilon=\omega 0^2$. The character of the motion drastically changes when E is larger than 2ϵ. In other words, The critical value of v0 is $v0=2\sqrt{\epsilon}$.

```
phase[{θ0_, v0_}, ω0_, tmax_, opts__] :=
 Module[{numso1, numgraph},
  numso1 =
   NDSolve[{ ω0^2 Sin[θ[t]] + v'[t] == 0, v[t] == θ'[t],
      θ[0] == θ0, v[0] == v0}, {θ[t], v[t]}, {t, 0, tmax}] // Flatten;
  numgraph = Plot[Evaluate[θ[t] /. numso1], {t, 0, tmax},
    opts, DisplayFunction → Identity]]
```

For simplicity we assume that $\omega 0 = 1$ (ϵ=1). The critical value of v0 is 2 in this case.

(a) v0 = 1.999992, 1.999993, 1.999994, 1.999995, 1.999996, 1.999997, 1.999998, 1.999999

```
phlist =
 phase[{0, #}, 1.0, 100,
    PlotStyle → {Thick, Hue[100 000 (# - 1.999992)]},
    AxesLabel → {"t", "θ"}, Prolog → AbsoluteThickness[2],
    Background → GrayLevel[0.5], PlotRange → All,
    Ticks → { Range[0, 100, 10], π Range[-3/2, 3/2, 1/2]},
    DisplayFunction → Identity] & /@
  Range[1.999992, 1.999999, 0.000001];
Show[phlist, DisplayFunction → $DisplayFunction]
```

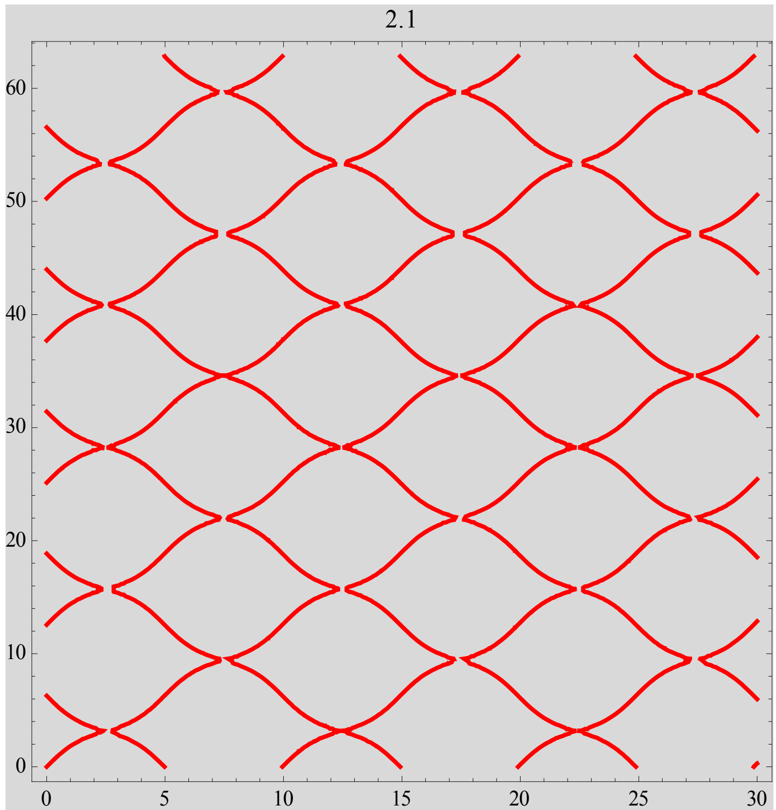


G. ContourPlot (Section 3.1)

In[1]:=
```
G1[v0_] := Sin[θ/2] == JacobiSN[t/κ, κ^2] /. κ → 2/v0
```

In[2]:=
```
f2 = ContourPlot[Evaluate[G1[2.1]], {t, 0, 30}, {θ, 0, 20 π},
  ContourStyle → {Thick, Red}, Background → LightGray,
  PlotPoints → 70, PlotLabel → 2.1]
```

Out[2]=



In[3]:=

H. ParametricPlot: Trajectories in the phase space (Section 8.2)

```
In[1]:= Clear["Global`*"]

In[2]:= phase[{θ0_, τ_}, {β_, v0_}, tmax_, opts__] :=
         Module[{numsol, numgraph},
          numsol =
           NDSolve[{ v'[t] + β v[t] + Sin[θ[t]] == τ, θ'[t] == v[t], θ[0] == θ0,
              v[0] == v0}, {v[t], θ[t]}, {t, 0, tmax }] // Flatten;
          numgraph = ParametricPlot[
             {Evaluate[θ[t] /. numsol], Evaluate[v[t] /. numsol]},
             {t, 0, tmax }, opts, DisplayFunction → Identity]]

In[3]:= phlist =
         phase[{0, #}, {0.16, 2.1}, 100,  PlotStyle → {Thick, Hue[8 (# - 0.05)]},
            AxesLabel → {"θ", "v"}, Background → LightGray,
            PlotRange → {{- π, 14 π}, {-2, 3}},
            Ticks → { π Range[-10, 14], Range[-2, 4]},
            DisplayFunction → Identity] & /@ Range[0.01, 0.35, 0.01];
        Show[phlist, DisplayFunction → $DisplayFunction]
```

Out[3]=