Topics
Minimum deviation of angle in prism

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1. Introduction

According to the empirical law (Cauchy's equation), the index of refraction $n(\lambda)$ in a material decreases with increasing wavelength λ of light in the visible range. The index of refraction $n(\lambda)$ is related to the wave velocity as $n(\lambda) = c/v(\lambda)$ (>1), where c is the speed of light in vacuum and $v(\lambda)$ is the speed of light in the material. Snell's law indicates that the angle of refraction when the light enters the material depends on the index of refraction. The violet light ($\lambda \approx 400$ nm) refracts more than red light ($\lambda \approx 650$ nm) when passing from air into the material. This phenomenon is called the dispersion.

In order to understand the effects of dispersion, we consider what happens when light strikes a prism (see Figs. 1 and 2). A ray of light of a single wavelength that is incident on the first surface of the prism, emerges in a direction deviated from its original direction of travel by an angle of deviation δ. Suppose a beam of white light (a combination of all visible wavelength) is incident on a prism. Because of the dispersion, the different colors refract through different angles of deviation and the rays that emerges from the second surface of the prism, spread out in a series of colors known a visible spectrum;

Table 1

Red	780/630 nm
Orange	630/600 nm
Yellow	600/570 nm
Greenish yellow	570/550 nm
Green	550/520 nm
Blueish/green	520/500 nm
Blue	500/450 nm
Violet	450/380 nm

To measure the index of refraction $n(\lambda)$ for a specified wavelength λ, one use a goniometer. By determining the magnitude of the angle between the two sides of the prism (an apex angle α) and by measuring the angle of incidence (θ_{i1}) on to the first surface and the minimum deviation angle δ for the refraction at the second surface, one can obtain the value of $n(\lambda)$ by simple calculation.

The experiment of the minimum deviation angle in prism is one of the topics in the Sophomore laboratory (SUNY-Binghamton). We use the mercury and sodium lamps as light sources. These light sources have discrete characteristic wavelengths. We determine the index of refraction n(λ) for each wavelength λ.

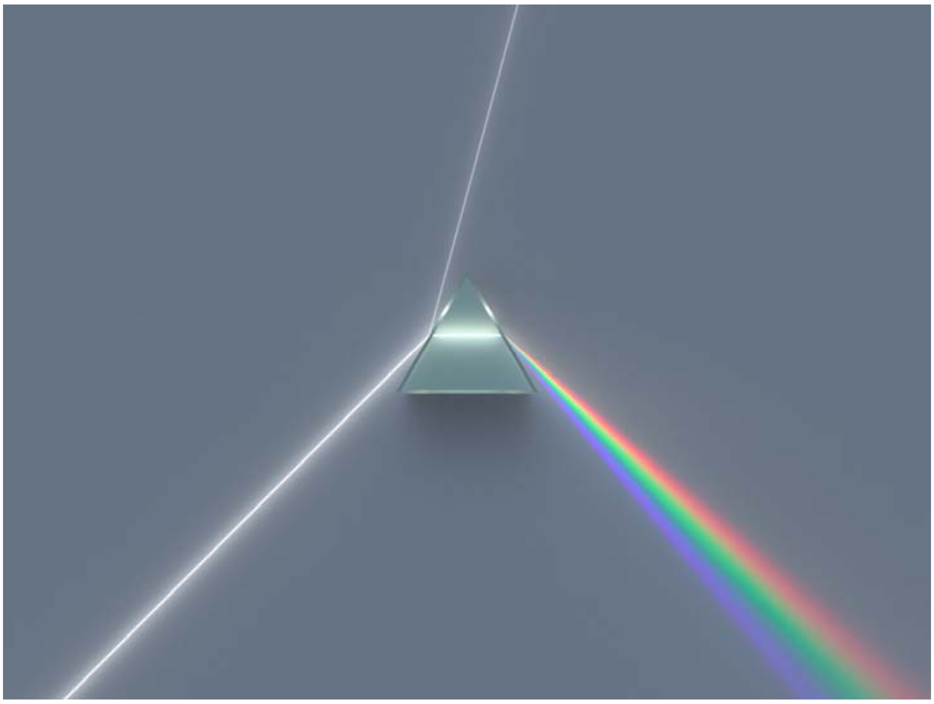

Fig.1 Refraction of light by a prism. A prism disperses rays of different colors. The ray of violet light is refracted more than the ray of red light.
(http://en.wikipedia.org/wiki/File:Dispersive_Prism_Illustration_by_Spigget.jpg)

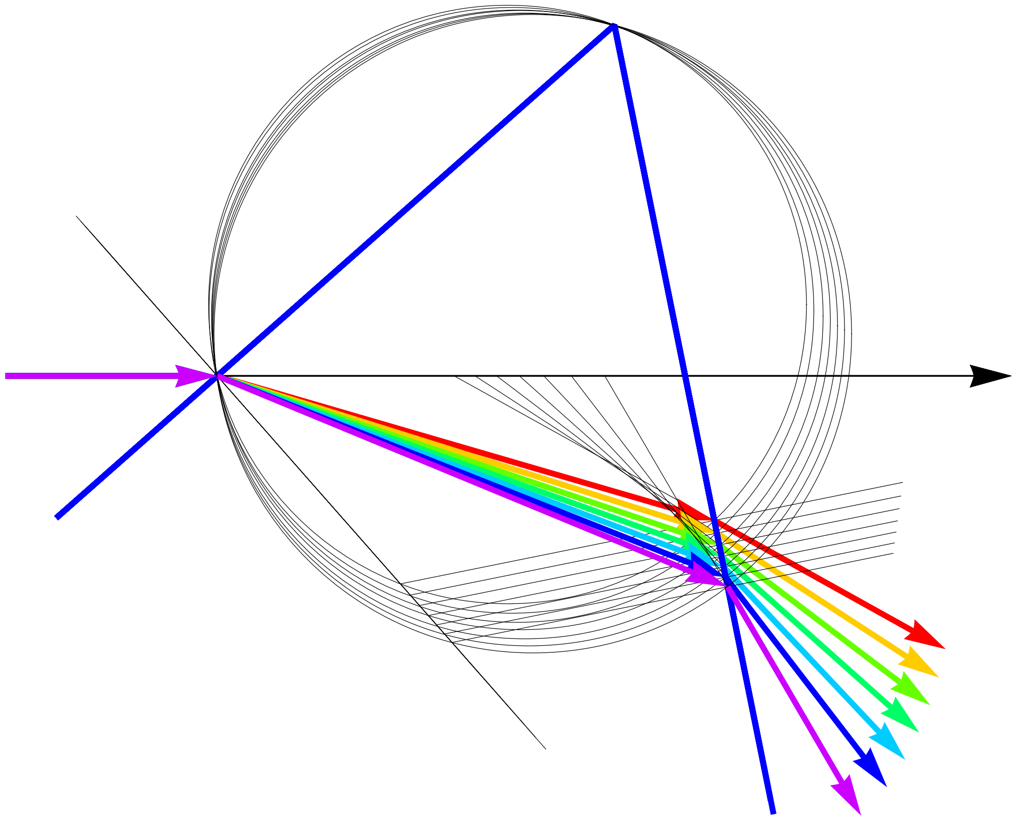

Fig.2 Refraction of a white light ray in a prism, which is simulated by using Mathematica (the detail how to draw this figure will be discussed later). The surfaces of the prisms are denoted by the blue lines. We choose α = 60° (the angle of apex of the prism) and the incident angle θ_{11} = 48.59°. The incident ray propagates along the positive x axis. We use n = 1.4 (red color), 1.45, 1.50 (green color), 1.55, 1.60, 1.65 (blue color), and 1.70 (pink). Note that the value of n used in the calculation is not related to the real colors of the visible spectrum.

((**Note**)) Useful links are found in the following web site.

http://www.youtube.com/watch?v=nk_kZu23xiw
http://www.mtholyoke.edu/~mpeterso/classes/phys103/geomopti/MinDev.html

2. Theory

2.1 Definition

The ray emerges refracted from its original direction of travel by an angle δ, called the angle of deviation. δ depends on the apex angle α of the prism and the index of refraction n of the material. Since all the colors have different angles of deviation, white light will spread out into a spectrum.

(a) Violet deviates the most.
(b) Red deviates the least.
(c) The remaining colors are in between.

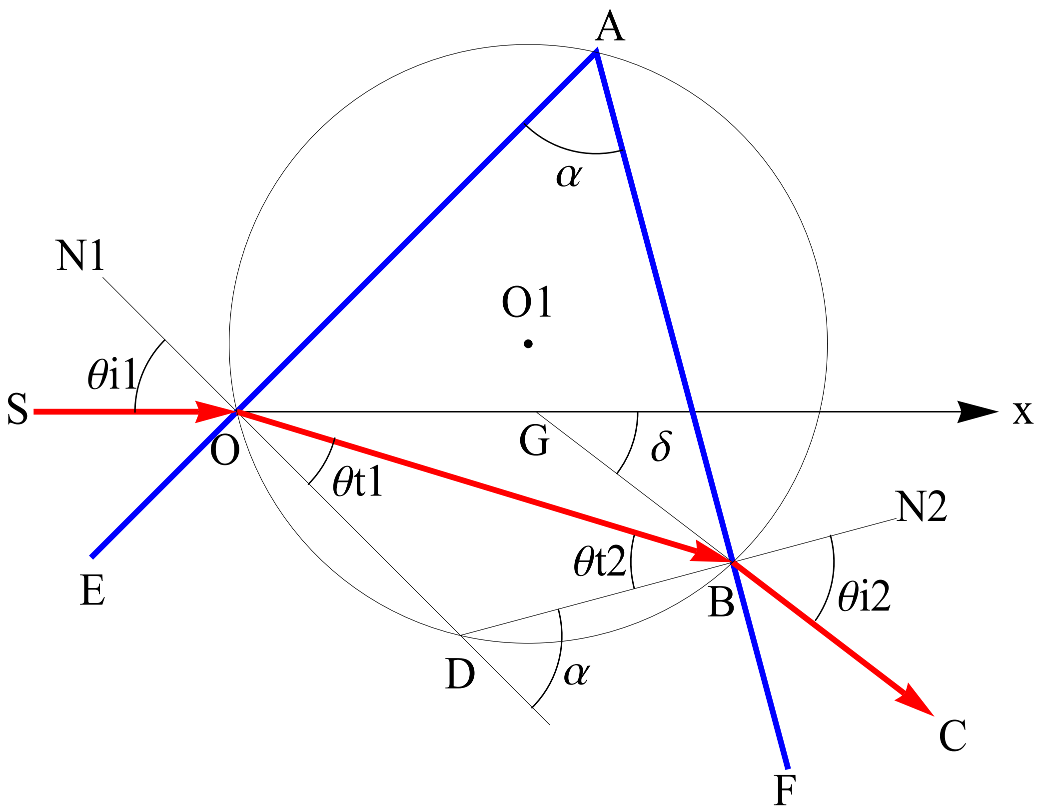

Fig.3 Refraction of a light ray by a prism with the angle of apex (α). The incident ray (denoted by the red line) propagates along the positive x axis and enters into the prism at the point O. The points A, O, D, and B are on the same circle with the center at the point O_1.

Suppose that the incident ray propagates along the positive x axis and enters into the prism at the point O. AE and AF are the surface of the prism. The point O is located on the first surface (AE) of the prism. α is the apex angle of the prism and δ is the deviation angle. From the geometrical consideration, the points O,A, B, and D are on the same circle. Then we have the following relations,

$$\alpha = \theta_{t1} + \theta_{t2}$$

$$\begin{aligned}\delta &= (\theta_{i1} - \theta_{t1}) + (\theta_{i2} - \theta_{t2}). \\ &= \theta_{i1} + \theta_{i2} - \alpha\end{aligned}$$

Note that

$$\angle BOG = \theta_{i1} - \theta_{t1}, \quad \text{and} \quad \angle GBO = \theta_{i2} - \theta_{t2}.$$

Snell's law:

$$\sin\theta_{i1} = n\sin\theta_{t1}, \qquad \sin\theta_{i2} = n\sin\theta_{t2},$$

where n is the index of refraction of the prism, which is dependent on the wavelength λ.

2.2 Minimum deviation of angle

Here we discuss the angle δ as a function of the incident angle θ_{i1}. From the Snell's law,

$$\sin\theta_{i2} = n\sin\theta_{t2} = n\sin(\alpha - \theta_{t1}) = n(\sin\alpha\cos\theta_{t1} - \cos\alpha\sin\theta_{t1}).$$

We note that

$$\cos\theta_{t1} = \sqrt{1-\sin^2\theta_{t1}} = \sqrt{1-\frac{1}{n^2}\sin^2\theta_{i1}}$$
$$\sin\theta_{t1} = \frac{\sin\theta_{i1}}{n}.$$

Then we get

$$\begin{aligned}\sin\theta_{i2} &= n\sin\alpha\sqrt{1-\frac{1}{n^2}\sin^2\theta_{i1}} - n\cos\alpha\frac{\sin\theta_{i1}}{n} \\ &= \sin\alpha\sqrt{n^2-\sin^2\theta_{i1}} - \cos\alpha\sin\theta_{i1}\end{aligned}.$$

The angle of deviation is obtained as

$$\delta = \theta_{i1} - \alpha + \arcsin(\sin\alpha\sqrt{n^2-\sin^2\theta_{i1}} - \cos\alpha\sin\theta_{i1}).$$

Here we assume that $\alpha = 60°$. We make a plot of the angle δ as a function of θ_{i1}, where the index of refraction n is changed as a parameter. It is found from Fig.4 that δ takes minimum at a characteristic angle (the minimum deviation angle),

$$\theta_{i1} = \arcsin[n\sin(\frac{\alpha}{2})].$$

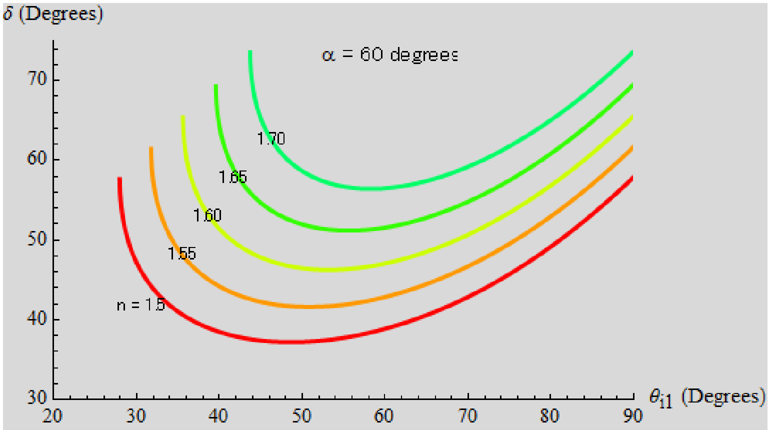

Fig.4 Deviation angle δ vs incident angle θ_{i1}, where n is changed as a parameter.

What is the condition for the angle of minimum deviation? The condition is derived as

$$\theta_{i1} = \theta_{i2}$$
$$\theta_{t1} = \theta_{t2} \qquad \text{(symmetric configuration)}$$

In other words, the ray OB should be parallel to the base of the prism (the isosceles triangle with the apex angle α) in the case of the angle of minimum deviation.

((**Proof**))

The angle δ has a minimum at the angle of minimum deviation,

$$\frac{d\delta}{d\theta_{i1}} = 0,$$

or

$$d\theta_{i2} = -d\theta_{i1}, \tag{1}$$

From $\alpha = \theta_{t1} + \theta_{t2}$, we have

$$d\theta_{t1} = -d\theta_{t2}. \tag{2}$$

From the Snell's law,

$$\begin{aligned} \cos\theta_{i1} d\theta_{i1} &= n\cos\theta_{t1} d\theta_{t1} \\ \cos\theta_{i2} d\theta_{i2} &= n\cos\theta_{t2} d\theta_{t2} \end{aligned}. \tag{3}$$

From Eqs.(1), (2), and (3), we have

$$\frac{\cos\theta_{i1}}{\cos\theta_{i2}} = \frac{\cos\theta_{t1}}{\cos\theta_{t2}} = \frac{\sqrt{1-\frac{1}{n^2}\sin^2\theta_{i1}}}{\sqrt{1-\frac{1}{n^2}\sin^2\theta_{i2}}},$$

$$\frac{\sqrt{1-\sin^2\theta_{i1}}}{\sqrt{1-\sin^2\theta_{i2}}} = \frac{\sqrt{n^2-\sin^2\theta_{i1}}}{\sqrt{n^2-\sin^2\theta_{i2}}}$$

which leads to the condition given by

$$\theta_{i1} = \theta_{i2}$$

$$\theta_{t1} = \theta_{t2} = \frac{\alpha}{2}.$$

Using this condition, the incident angle can be calculated as

$$\theta_{i1} = \arcsin[n\sin(\frac{\alpha}{2})].$$

When $\alpha = 60°$, we have

$\theta_{i1} = 48.59°$ for $n = 1.50$.
$\theta_{i1} = 53.13°$ for $n = 1.60$.
$\theta_{i1} = 58.21°$ for $n = 1.70$.

2.3 Index of refraction *n*, the angle of minimum deviation (symmetrical configuration)

In Fig.5, a ray is incident on one surface of a triangular glass prism in air. The angle of incidence θ_{i1} is chosen so that the emerging ray also makes the same angle θ_{t1} with the normal to the other surface. Show that the index of refraction n of the glass prism is given by

$$n = \frac{\sin\left(\frac{\delta+\alpha}{2}\right)}{\sin\frac{\alpha}{2}}.$$

where α is the vertex angle of the prism and δ is the deviation angle, the total angle through which the beam is turned in passing through the prism. (Under these conditions, the deviation angle δ has the smallest possible value, which is called the angle of minimum deviation).

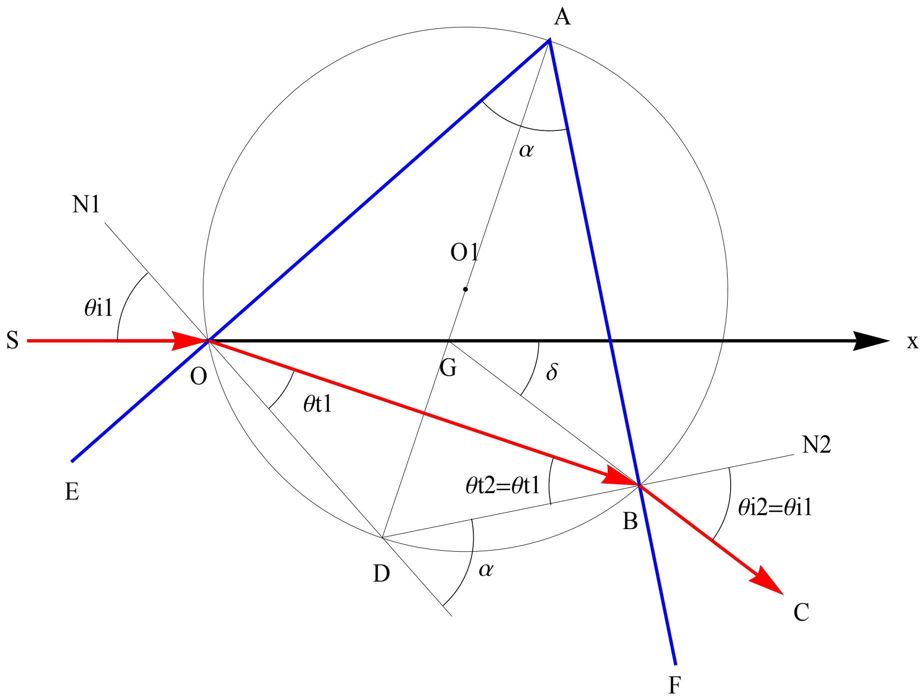

Fig.5 Symmetric configuration for the minimum deviation angle. $\alpha = 60°$. $n = 1.5$. $\theta_{i1} = 48.59°$. $\delta = 37.18°$. The vector $\overrightarrow{OB}$ is perpendicular to the vector $\overrightarrow{AD}$. Note that the line AD is the diameter of the circle, passing through the points O_1 and G.

From the geometry for the symmetric configuration

$$\theta_{t1} = \theta_{t2}, \qquad \text{and} \qquad \theta_{i1} = \theta_{i2}.$$

Snell's law:

$$\sin\theta_{i1} = n\sin\theta_{t1}.$$

We also have the relations

$$\delta = 2(\theta_{i1} - \theta_{t1}),$$

and

$$\theta_{t1} = \frac{\alpha}{2},$$

which leads to the expression given by

$$\theta_{i1} = \frac{\delta + \alpha}{2}.$$

Then the index of refraction n is derived as

$$n = \frac{\sin\theta_{i1}}{\sin\theta_{t1}} = \frac{\sin\left(\dfrac{\delta+\alpha}{2}\right)}{\sin\left(\dfrac{\alpha}{2}\right)}.$$

3. Formulation for the calculation of minimum deviation of angle

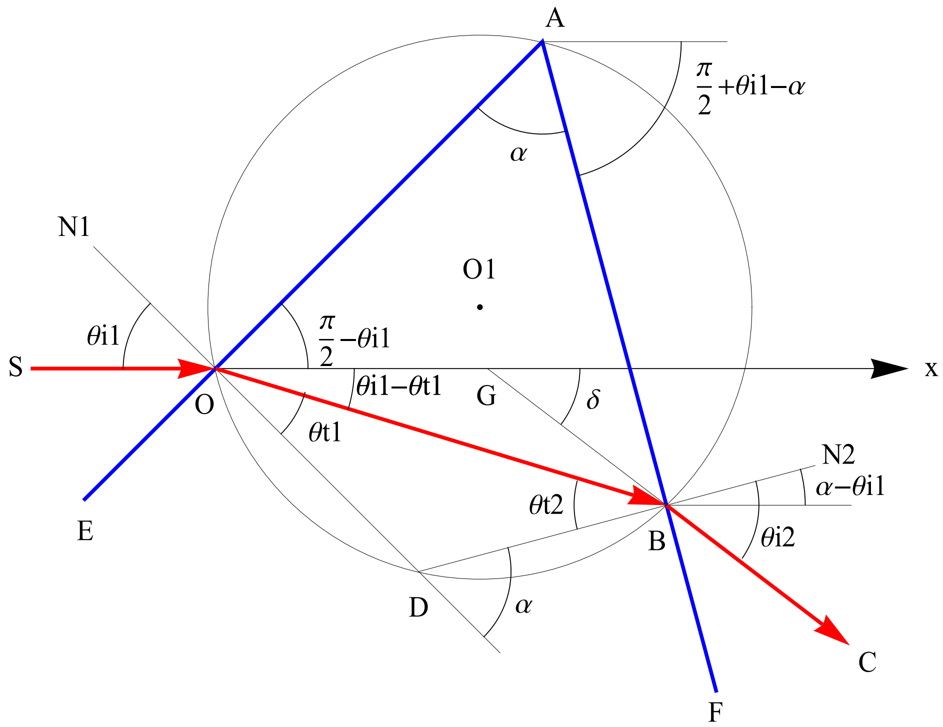

Fig.6 The geometry for the refraction of a light ray by a prism (in general case). The coordinates of all the points are expressed in terms of θ_{i1}, θ_{i2}, θ_{t1}, θ_{t2}, and α. The line N_1O is normal to the first surface of the prism, and the line N_2B is normal to the second surface of the prism. The angle between the positive x axis and the line GC is the deviation angle of the prism.

We consider the incident ray propagating along the positive x axis (the direction SO). The surfaces of the prism are denoted by blue lines. We assume that the prism is rotated around the origin O. The length OA (= 1) is fixed for simplicity. O = {0, 0} is the origin. From the geometry, the points O, A, B, and D are located on the same circle (the center position and radius will be discussed later). O_1 is the center of the circle. From the geometry, we have

$$\theta_{t1} + \theta_{t2} = \alpha ,$$

$$(\theta_{i1}-\theta_{t1})+(\theta_{i2}-\theta_{t2})=\delta\,,$$

or

$$\alpha-(\theta_{t1}+\theta_{t2})=\delta\,.$$

The Snell's law yields the relations as follows,

$$\sin\theta_{i1}=n\sin\theta_{t1}\,,\qquad \text{and} \qquad \sin\theta_{i2}=n\sin\theta_{t2}\,.$$

(a) The position vector of the point A is given by

$$\begin{aligned}\overrightarrow{OA}&=(A_x,A_y)\\&=(\sin\theta_{i1},\cos\theta_{i1})\end{aligned},$$

The unit vector of $\overrightarrow{BC}$ is given by

$$\frac{\overrightarrow{BC}}{BC}=\{\cos(\theta_{i1}+\theta_{i2}-\alpha),-\sin(\theta_{i1}+\theta_{i2}-\alpha)\}\,.$$

(b) The position vector of the point B is given by

$$\begin{aligned}\overrightarrow{OB}&=(B_x,B_y)\\&=\{\cos(\theta_{i1}-\theta_{t1})\sec(\alpha-\theta_{t1})\sin\alpha,-\sec(\alpha-\theta_{t1})\sin\alpha\sin(\theta_{i1}-\theta_{t1})\}\end{aligned}.$$

The point B is the intersection of the line OB given by

$$y=-\tan(\theta_{i1}-\theta_{t1})x\,, \tag{1}$$

and the line AB given by

$$y-A_y=-\tan(\frac{\pi}{2}-\alpha+\theta_{i1})(x-A_x)\,. \tag{2}$$

(c) The position vector of the point D is given by

$$\overrightarrow{OD}=\{\cos\theta_{i1}\tan(\alpha-\theta_{t1}),-\sin\theta_{i1}\tan(\alpha-\theta_{t1})\}\,.$$

The point D is the intersection of the line OD denoted by

$$y=-\tan\theta_{i1}x\,, \tag{3}$$

and the line BD denoted by

$$y - B_y = \tan(\alpha - \theta_{i1})(x - B_x) \quad (4)$$

(d) The position vector of the point G is given by

$$\overrightarrow{OG} = \{\csc(\alpha - \theta_{i1} - \theta_{i2})\sec(\alpha - \theta_{t1})\sin\alpha\sin(\alpha - \theta_{i2} - \theta_{t1}), 0\}.$$

The point G is the intersection of the x axis and the line BC denoted by

$$y - B_y = -\tan(\theta_{i1} + \theta_{i2} - \alpha)(x - B_x) \quad (3)$$

(e) The position vector of the center O_1 of the circle is given by

$$\overrightarrow{OO_1} = \{\frac{1}{2}[\sin\theta_{i1} + \cos\theta_{i1}\tan(\alpha - \theta_{t1})], \frac{1}{2}\cos(\alpha + \theta_{i1} - \theta_{t1})\sec(\alpha - \theta_{t1})\}.$$

The radius of the circle is

$$R = \frac{1}{2}|\sec(\alpha - \theta_{t1})|.$$

4. Minimum deviation of angle (Mathematica)

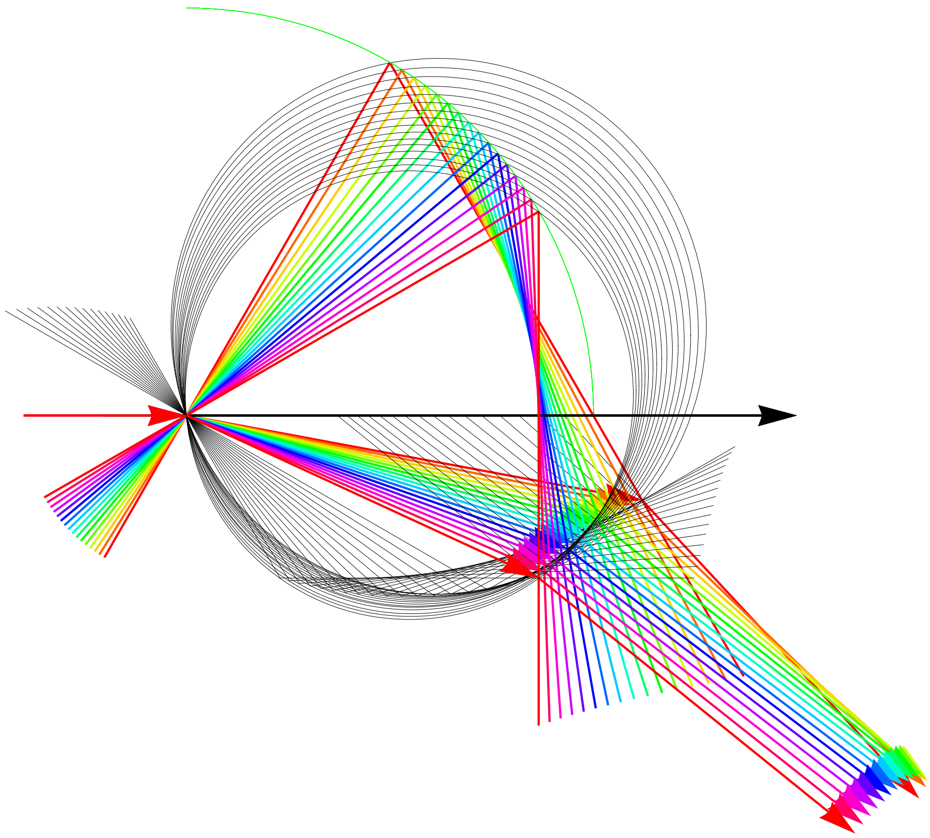

Fig.7 The point of apex in the prism is rotated around the origin (the fixed point) along the green line. The color of the surfaces of the prism changes on each rotation. The incident ray propagates along the x axis. The apex angle of the prism is $\alpha = 60°$. $n = 1.50$. As the incident angle increases, the deviation angle decreases and reaches a minimum value. With further increasing the incident angle, the deviation angle starts to increase (minimum deviation of angle in prism). θ_{i1} = 30º - 60º ($\Delta\theta_{i1}$ = 2º).

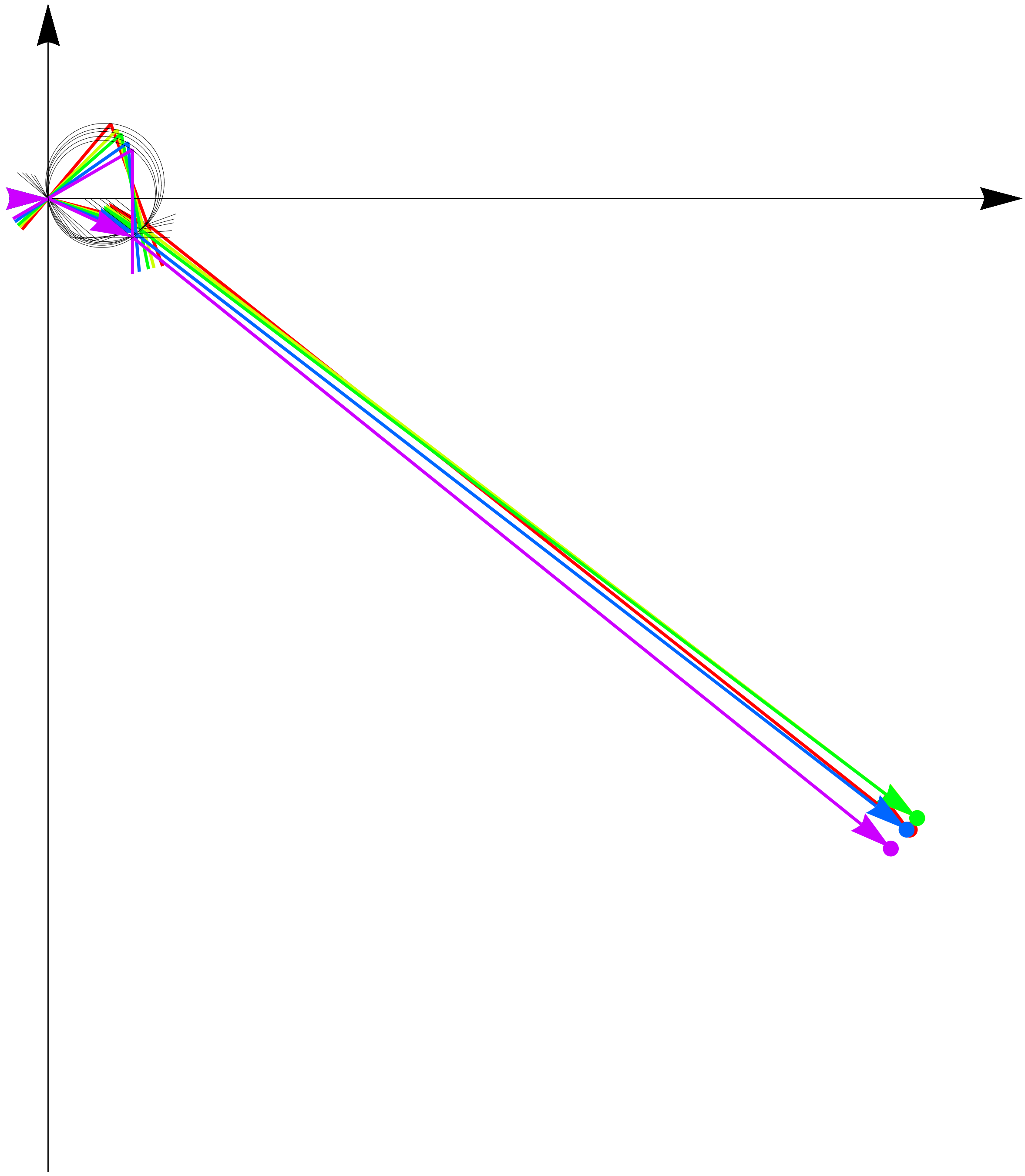

Fig.8 n =1.5. α = 60°. θ_{i1} = 40° (red). θ_{i1} = 45° (yellow). θ_{i1} = 48.59° (green, minimum deviation of angle), θ_{i1} = 55° (blue), and θ_{i1} = 60° (purple). This figure is made by the Mathematica (see Appendix).

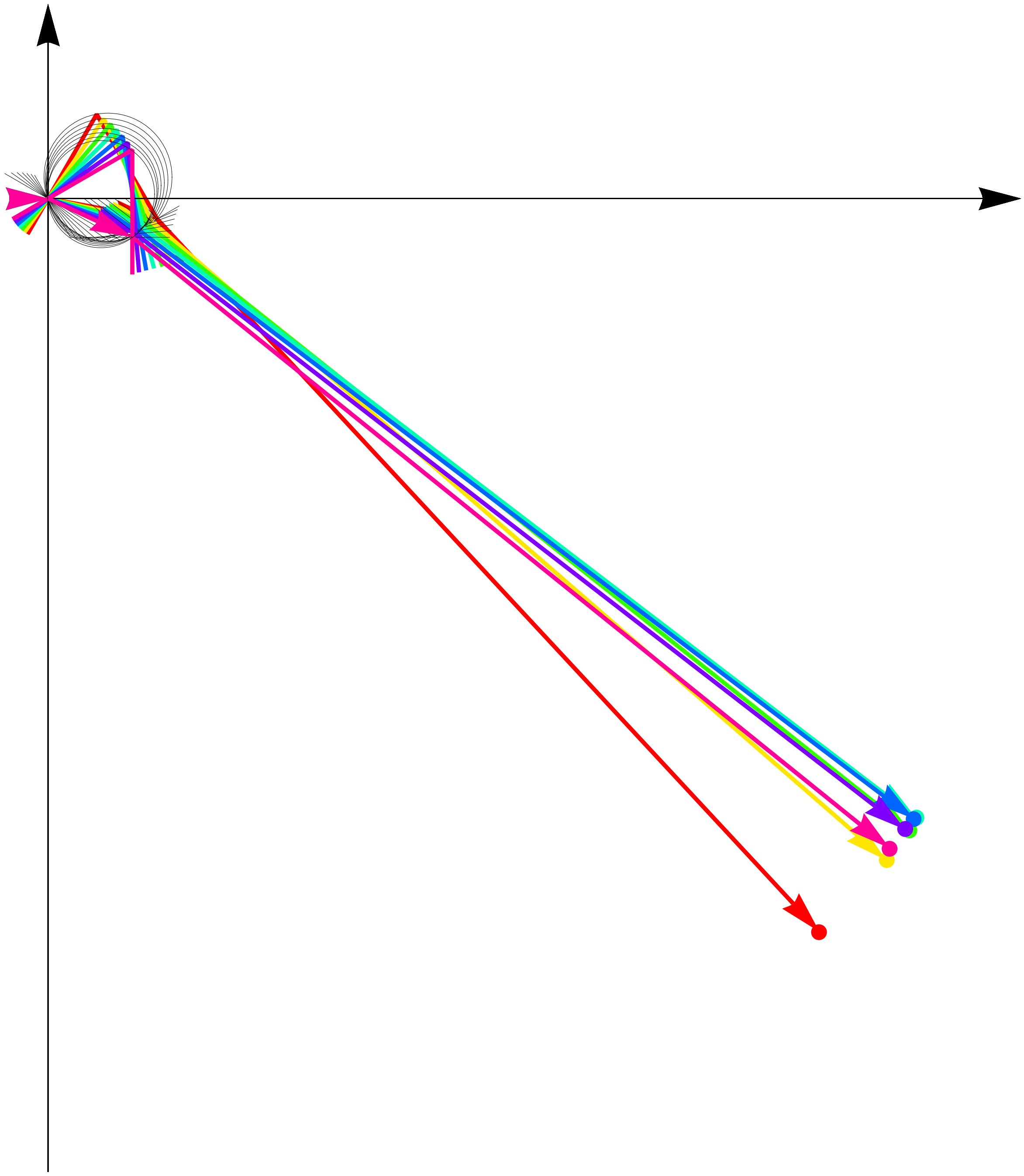

Fig.9 n =1.5. α = 60°. θ_{i1} = 30° (red). θ_{i1} = 35° (yellow). θ_{i1} = 40° (green), θ_{i1} = 45° (dark green), θ_{i1} = 50° (blue), θ_{i1} = 55° (purple), and θ_{i1} = 60° (pink). This figure is made by the Mathematica.

6. Cauchy's equation

Cauchy's equation is an empirical relationship between the refractive index and wavelength of light for a particular transparent material. It is named for the mathematician Augustin Louis Cauchy, who defined it in 1836. The most general form of Cauchy's equation is

$$n(\lambda) = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \ldots$$

where n is the refractive index, λ is the wavelength, A, B, C, and so on., are coefficients that can be determined for a material by fitting the equation to measured refractive indices at known wavelengths. The coefficients are usually quoted for λ as the vacuum wavelength in μm.
Usually, it is sufficient to use a two-term form of the equation:

$$n(\lambda) = A + \frac{B}{\lambda^2}.$$

where the coefficients A and B are determined specifically for this form of the equation. The theory of light-matter interaction on which Cauchy based this equation was later found to be incorrect. In particular, the equation is only valid for regions of normal dispersion in the visible wavelength region. In the infrared, the equation becomes inaccurate, and it cannot represent regions of anomalous dispersion. Despite this, its mathematical simplicity makes it useful in some applications. The Sellmeier equation is a later development of Cauchy's work that handles anomalously dispersive regions, and more accurately models a material's refractive index across the ultraviolet, visible, and infrared spectrum.

7. Experiment

The index of refraction for any wavelength can be determined from a measurement of the minimum angle of deviation in refraction of light by a prism. The deviation produced by a prism depends on the angle of incidence (θ_{1i}), the apex angle of the prism (α), and the index of refraction. The angle of deviation is a minimum in the symmetric situation when the angles of the incoming and outgoing rays make equal angles with the prism surfaces.

In this symmetric case, the index of refraction is given by the relation

$$n(\lambda) = \frac{\sin[\frac{\alpha + \delta(\lambda)}{2}]}{\sin(\frac{\alpha}{2})}$$

where δ is the angle of minimum deviation for the particular wavelength used. The value of n(λ) can thus be obtained experimentally for any λ at which we have a spectral line available.

8. Spectrum of mercury light source

The prominent mercury lines are at 404.6563 nm, 407,8980 nm (violet), 435.835 nm (blue) and 546.074 nm (green).

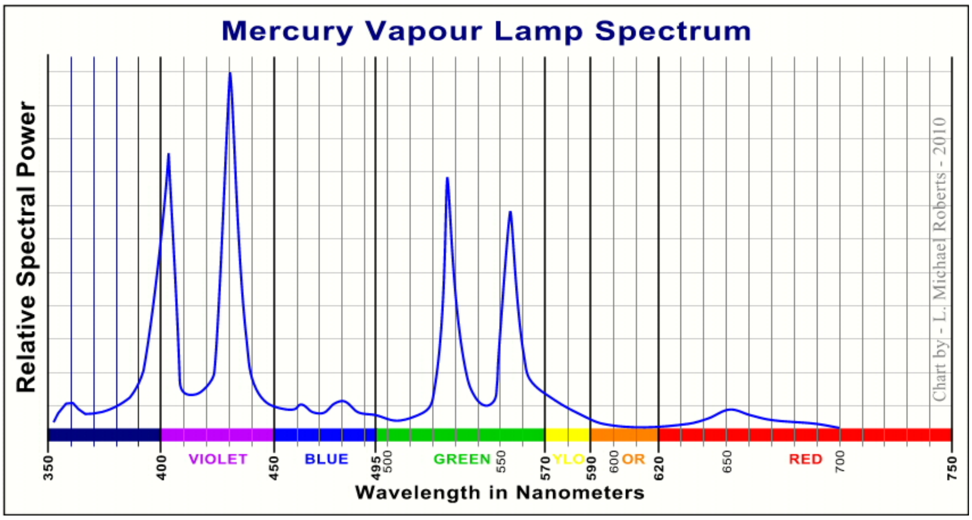

Fig.10 Mercury vapor lamp spectrum
http://upload.wikimedia.org/wikipedia/en/9/94/Mercury_Vapour_Lamp_Spectrum.jpg

Persistent Lines of Neutral Mercury (Hg I)

Intensity	Wavelength (Å)	A_{ki} ($10^8 s^{-1}$)	Energy Levels (cm^{-1})	Configurations	Terms	J	Line Ref.	A_{ki} Ref.
1000	1849.499	7.46	0.000 54068.781	$5d^{10}(^1S)6s^2$ $5d^{10}(^1S)6s6p$	^1S $^1P°$	0 1	WA63	M00
1000	2536.517	0.080	0.000 39412.300	$5d^{10}(^1S)6s^2$ $5d^{10}(^1S)6s6p$	^1S $^3P°$	0 1	BAL50	FW96
250	2967.280	0.45	37645.080 71336.164	$5d^{10}(^1S)6s6p$ $5d^{10}(^1S)6s6d$	$^3P°$ $(^1/_2,^3/_2)$	0 1	BAL50	FW96
600	3650.153	1.3	44042.977 71431.311	$5d^{10}(^1S)6s6p$ $5d^{10}(^1S)6s6d$	$^3P°$ $(^1/_2,^5/_2)$	2 3	BAL50	FW96
400	4046.563	0.21	37645.080 62350.456	$5d^{10}(^1S)6s6p$ $5d^{10}(^1S)6s7s$	$^3P°$ ^3S	0 1	BAL50	FW96
1000	4358.328	0.557	39412.300 62350.456	$5d^{10}(^1S)6s6p$ $5d^{10}(^1S)6s7s$	$^3P°$ ^3S	1 1	BAL50	FW96
500	5460.735	0.487	44042.977 62350.456	$5d^{10}(^1S)6s6p$ $5d^{10}(^1S)6s7s$	$^3P°$ ^3S	2 1	BAL50	FW96
200	10139.76	0.271	54068.781 63928.243	$5d^{10}(^1S)6s6p$ $5d^{10}(^1S)6s7s$	$^1P°$ ^1S	1 0	BAL50	FW96

Table 2 Wavelength of the mercury vapor lamp spectrum

9. Spectrum of sodium light source

Experimentally, we observe two closely separated yellow lines - known as the sodium D lines - one at 589.592 nm and other at 588.995 nm. The sodium D lines correspond to the 3p → 3s transition.

D_1 line (589.6 nm): $\quad {}^2P_{1/2} \rightarrow {}^2S_{1/2}$

D_2 line (589.0 nm): $\quad {}^2P_{3/2} \rightarrow {}^2S_{1/2}$

We consider a sodium atom. From standard atomic spectroscopy, the ground-state configuration is $(1s)^2(2s)^2(2p)^6(3s)$. The inner 10 electrons can be visualized to form a spherically symmetrical electron cloud. We are interested in the excitation of the eleventh electron from 3s to a possible higher state. the nearest possibility is excitation to 3p. Because the central potential is no longer of the pure Coulomb form, 3s and 3p are now split. The fine structure brought by spin orbit coupling (V_{LS}) refers to even a finer split within 3p, between ${}^2P_{1/2}$ and ${}^2P_{3/2}$, where the subscript refers to the j (= l + 1/2 and l - 1/2 with l = 1 for orbital and s = 1/2 for spin). The lower ${}^2S_{1/2}$ has no spin-orbit interaction.

(a) For the electron with 3s state (l = 0, s = 1/2)

$$D_0 \times D_{1/2} = D_{1/2}$$

Thus we have j = 1/2. The state is described by ${}^2S_{1/2}$.

$$|j = 1/2, m = 1/2\rangle = |m_l = 0, m_s = 1/2\rangle$$
$$|j = 1/2, m = -1/2\rangle = |m_l = 0, m_s = -1/2\rangle$$

(b) For the electron with 3p state (l = 1, s = 1/2)

$$D_1 \times D_{1/2} = D_{3/2} + D_{1/2}$$

Thus we have j = 3/2 and j = 1/2. The state is described by ${}^2P_{3/2}$ and ${}^2P_{1/2}$

$$|j = 3/2, m = -3/2\rangle = |m_l = -1, m_s = -1/2\rangle$$
$$|j = 3/2, m = -1/2\rangle = \sqrt{\frac{2}{3}}|m_l = 0, m_s = -1/2\rangle + \frac{1}{\sqrt{3}}|m_l = -1, m_s = 1/2\rangle$$
$$|j = 3/2, m = 1/2\rangle = \frac{1}{\sqrt{3}}|m_l = 1, m_s = -1/2\rangle + \sqrt{\frac{2}{3}}|m_l = 0, m_s = 1/2\rangle$$
$$|j = 3/2, m = 3/2\rangle = |m_l = 1, m_s = 1/2\rangle$$
$$|j = 1/2, m = -1/2\rangle = \frac{1}{\sqrt{3}}|m_l = 0, m_s = -1/2\rangle - \sqrt{\frac{2}{3}}|m_l = -1, m_s = 1/2\rangle$$
$$|j = 1/2, m = 1/2\rangle = \sqrt{\frac{2}{3}}|m_l = 1, m_s = -1/2\rangle - \frac{1}{\sqrt{3}}|m_l = 0, m_s = 1/2\rangle$$

Table 2

Term	j	l	s
$^2P_{3/2}$	3/2	1	1/2
$^2P_{1/2}$	1/2	1	1/2
$^2S_{1/2}$	1/2	0	1/2

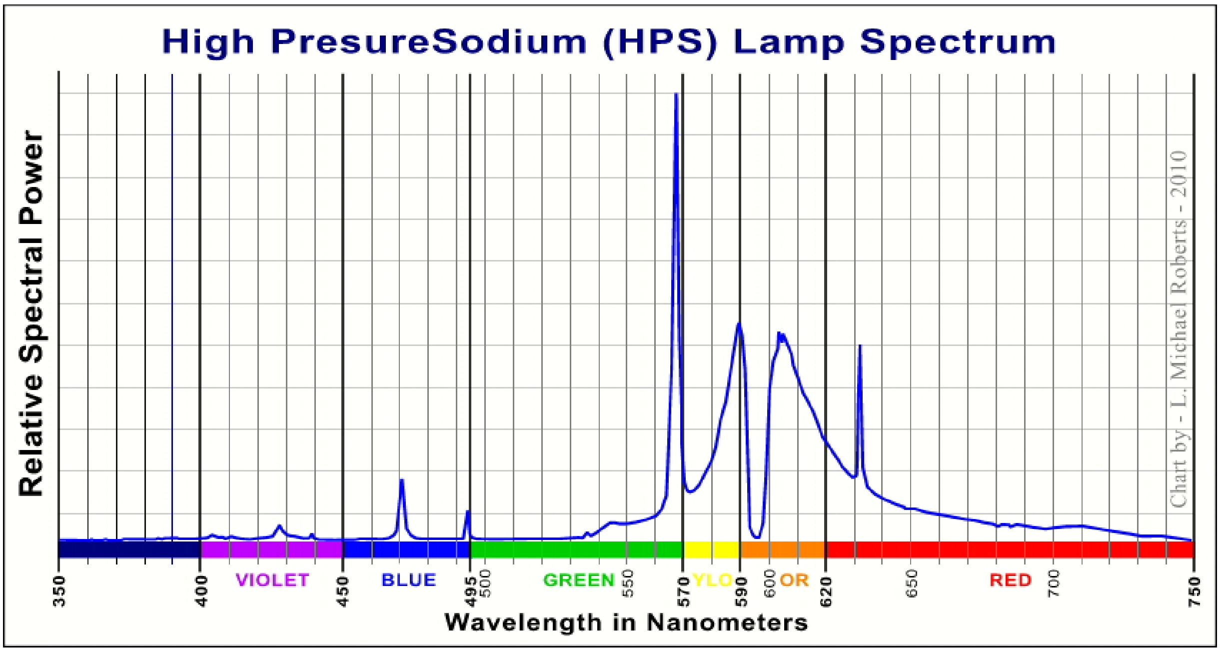

Fig.11 Sodium vapor lamp spectrum
http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/modpic/mertube.jpg

APPENDIX
Animation program for the minimum deviation of angle in prism by Mathematica

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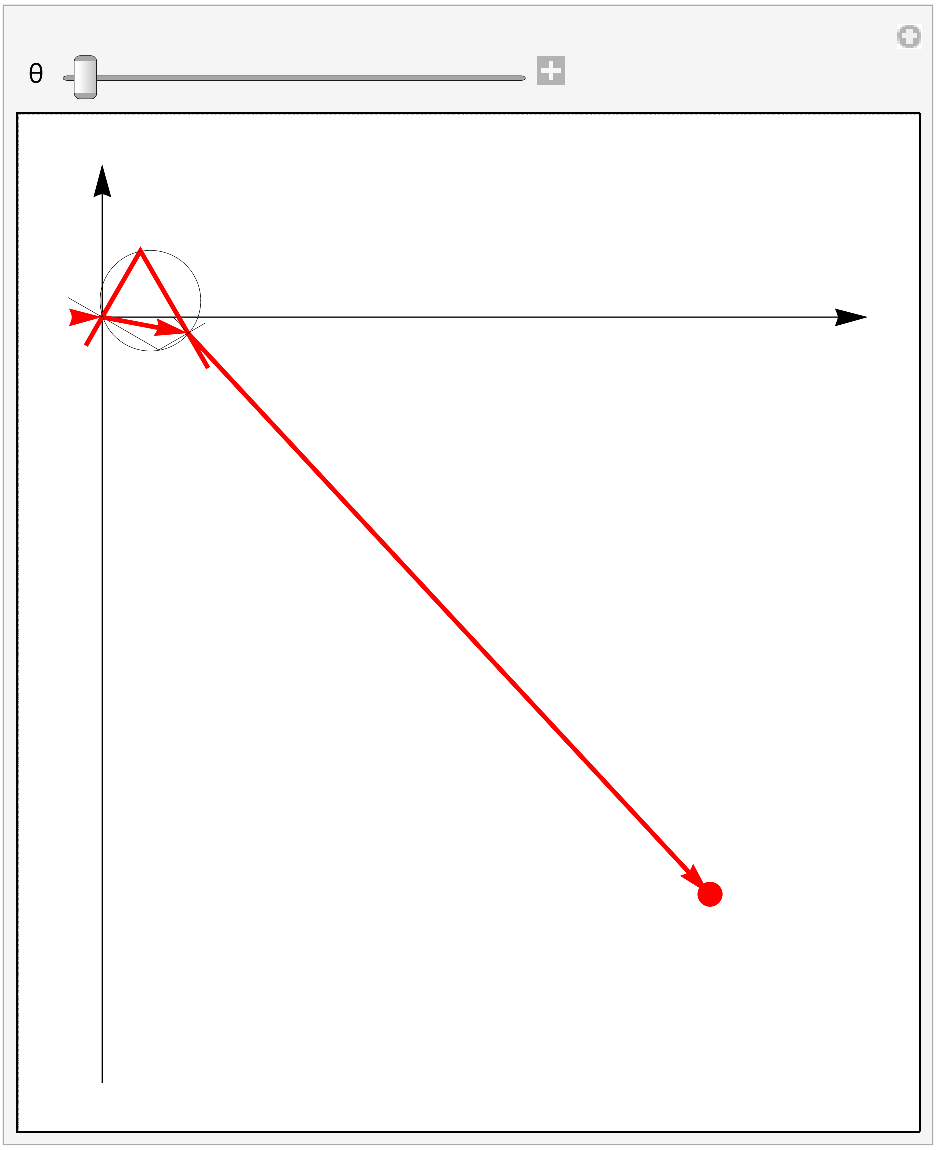
A1 = {Cos[π/2 - θi1], Sin[π/2 - θi1]}; O1 = {0, 0}; AM1 = -0.4 A1;
B11 = {Cos[θi1 - θt1] Sec[α - θt1] Sin[α], -Sec[α - θt1] Sin[α] Sin[θi1 - θt1]};
D11 = {Cos[θi1] Tan[α - θt1], -Sin[θi1] Tan[α - θt1]}; BD11 = B11 - D11; AD11 = A1 - D11;
R1 = 1/2 Sqrt[AD11.AD11] // Simplify; CN1 = 1/2 (A1 + D11) // Simplify;
F11 = {Csc[α - θi1 - θi2] Sec[α - θt1] Sin[α] Sin[α - θi2 - θt1], 0};
BC1 = 10 {Cos[θi1 + θi2 - α], -Sin[θi1 + θi2 - α]}; C1 = B11 + BC1 // Simplify;
rule1 = {θt1 → ArcSin[1/ng Sin[θi1]]}; rule2 = {θi2 → ArcSin[ng Sin[θt2]]};
rule3 = {θt2 → α - ArcSin[1/ng Sin[θi1]]}; A11 = A1 /. rule2 /. rule3 /. rule1;
AM11 = AM1 /. rule2 /. rule3 /. rule1; BC2 = BC1 /. rule2 /. rule3 /. rule1;
C12 = C1 /. rule2 /. rule3 /. rule1; B12 = B11 /. rule2 /. rule3 /. rule1;
D12 = D11 /. rule2 /. rule3 /. rule1; CN2 = CN1 /. rule2 /. rule3 /. rule1;
BD12 = BD11 /. rule2 /. rule3 /. rule1; F12 = F11 /. rule2 /. rule3 /. rule1;
R2 = R1 /. rule2 /. rule3 /. rule1;
f[θ_] :=
  Graphics[{Arrow[{{0, -10}, {0, 2}}], Arrow[{{-0.2, 0}, {10, 0}}], Hue[(θi1 - π/6)/(π/6)], Thick,
     Line[{O1, A11}], Line[{A11, B12 + 0.4 (B12 - A11)}], Line[{O1, AM11}], Black,
     Thin, Line[{D12, B12}], Line[{B12, B12 + 0.6 BD12}], Line[{O1, D12}], Line[{-0.6 D12, O1}],
     Line[{F12, B12}], Hue[(θi1 - π/6)/(π/6)], Thick, Arrow[{{-0.4, 0}, {0, 0}}], Arrow[{O1, B12}],
     Arrow[{B12, C12}] , Black, Thin, Circle[CN2, R2], PointSize[0.03], Hue[0], Point[C12]}] /.
   θi1 → θ;

ng = 1.5; α = 60 °; Manipulate[Show[f[θ], PlotRange → All, AspectRatio → Automatic],
 {θ, π/6, 1.2 π/3}]
```


θ