Physics of rainbow

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In May 2010, we heard from one of our friends about a lecture (youtube) by Dr Walter Lewin on the physics of rainbow. Note that a series of his lectures are presented in the Web site [MIT 8.02 Electricity and magnetism (Spring 2002)]. We saw his lecture in the Web site several times and were very impressed with his explanation of the physics on the rainbow. His lecture reminds us of the photo of rainbow over the Niagara Falls taken by one of the authors (I.S.S.) on June 1, 2002. The rainbow consisted of the primary rainbow, the secondary rainbow, and the Alexander's dark space between the primary and secondary rainbows. We realize that the nature of such a rainbow can be essentially explained in terms of the nature of waves, such as refraction and reflection (Snell's law), polarization (Brewster angle, Fresnel equation), and diffraction. Motivated by his lecture by Prof. Lewin, furthermore we have read several papers (including the paper of J.D. Jackson (his book on Classical Electrodynamics is very famous) and books (including the book of Boyer on the physics of rainbow). Thanks to these references, we understand the fundamental physics on the rainbow much more than previously.

In this note, we make a lot of figures using the Mathematica 7.0. We think that these figures make it much easier for students (undergraduate and graduate students) to understand the wave-nature of the rainbow. If one wants to study more advanced physics on the rainbow, it is suggested that one should read review articles such as the paper of Adam. In this note we discuss the fundamental physics on the rainbow. We need to point out that in this note, there is no physics which is newly discovered.

Content

REFERENCES

1. Overview

A rainbow is an optical phenomenon, which can be explained by the wave nature of light. A rainbow consists of the primary rainbow and the secondary rainbow (see Figs.1 - 3). The primary (main) rainbow is caused by two refractions and one internal reflection inside a rain droplet. The secondary rainbow, which appears outside the primary one, is caused by two refractions and two internal reflections inside the spherical water droplet. The colors of the secondary rainbow are reversed from the primary bow, and the secondary bow is twice as broad. There are many excellent references on the physics of the rainbow, including lectures, book, and papers in journals. The references which we use in this article are presented in REFERENCES.[1-9]

The dark band between the two rainbows (Alexander's dark space, named after Alexander of Aphrodisias, a follower of Aristotle and head of the Lyceum in Athens around AD 200) is a region of negligible scattering (from higher orders). The polarization of the rainbow is caused by the internal reflection. The rays strike the back surface of the drop close to the Brewster angle, so almost all the light reflected is polarized perpendicular to the incidence plane As the incidence plane is determined for each drop by the plane containing the sun, the drop, and the observer, the rainbow is polarized tangential to the arch. The primary rainbow is 94% polarized while the secondary is 90% polarized.

Figures 4 and 5 show the pattern of ray (red light) entering the upper and lower halves of the water droplet, where the incident angle θ_1 is changed from 0° to 90° as a parameter. The colors used for rays in this figure has nothing to do with the color of the lines for rainbow.

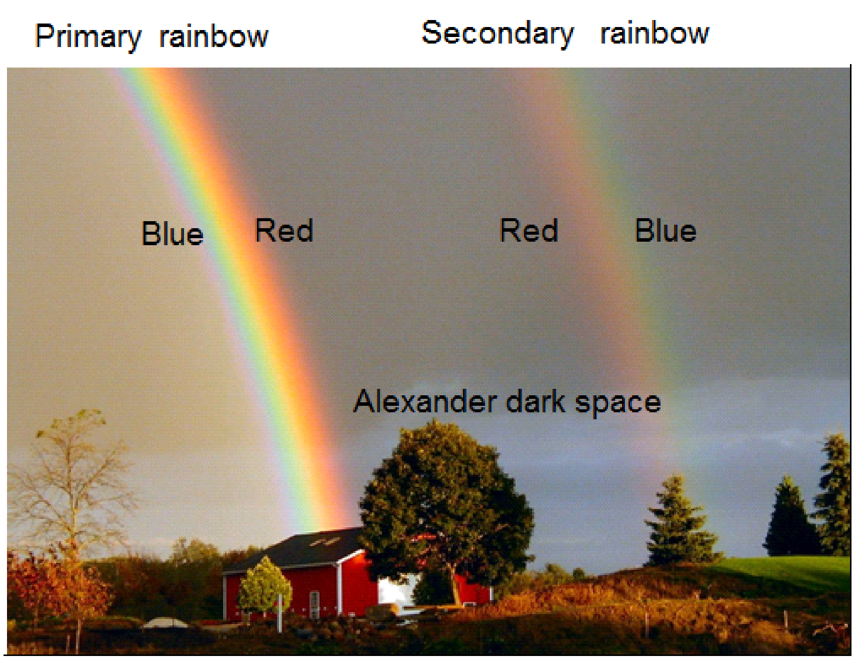

Fig.1(a) Typical rainbow. The rainbow consists of a primary rainbow, a secondary rainbow (note reversed colors), and an Alexander's dark space between the primary and secondary rainbows.
http://climate.met.psu.edu/data/frost/frosttraining.php

Fig.1(b) Rainbow over the Niagara Falls seen from Canadian side. This photo was taken by Itsuko Suzuki on June 1, 2002. One can see clearly both the primary rainbow and secondary rainbow. The inside of the primary rainbow is white in color, while the space between the primary and secondary rainbows (Alexander's dark space) is dark.

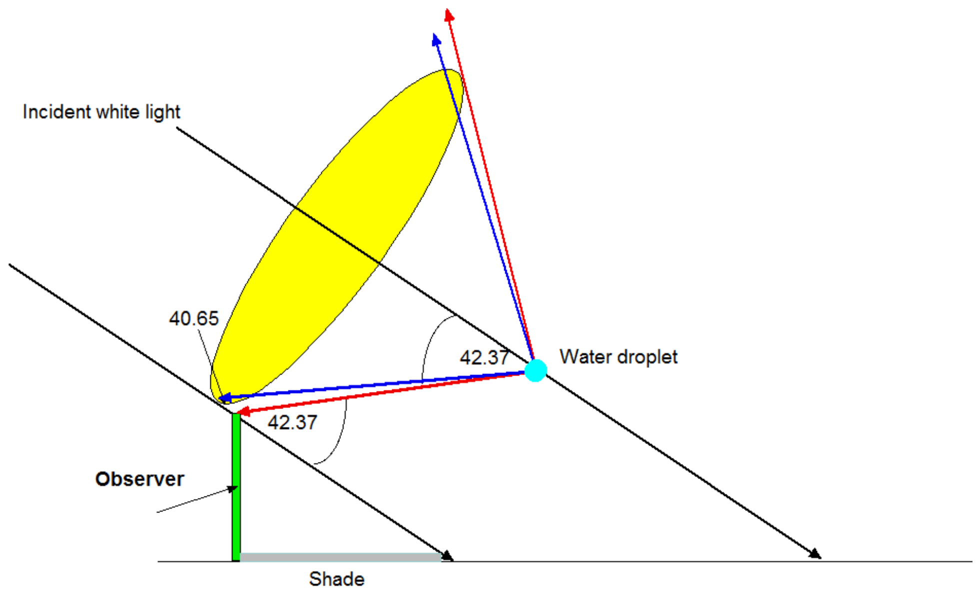

Fig.2 Forming of a primary rainbow. The sun is directly behind an observer. $\phi = 42.37°$ for the red light and 40.65° for the blue light.

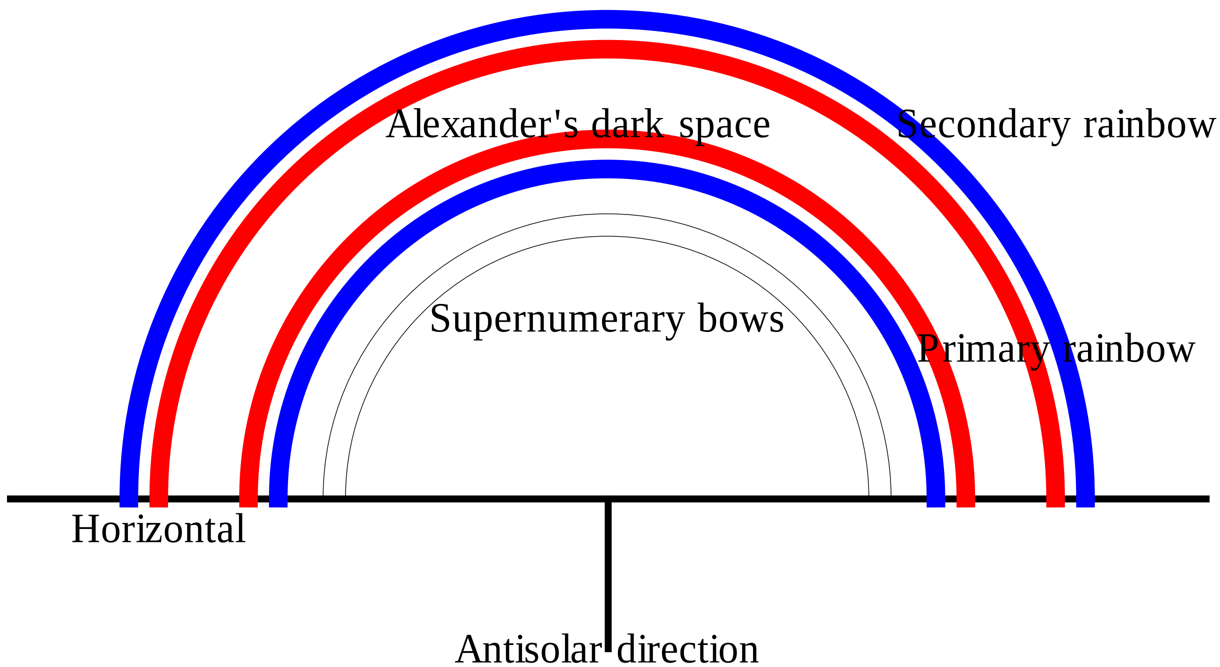

Fig.3 Structure of rainbow; a primary rainbow, a secondary rainbow (note, reversed colors), Alexander's dark space between the primary and secondary rainbows, and the supernumerary bows due to the interference.

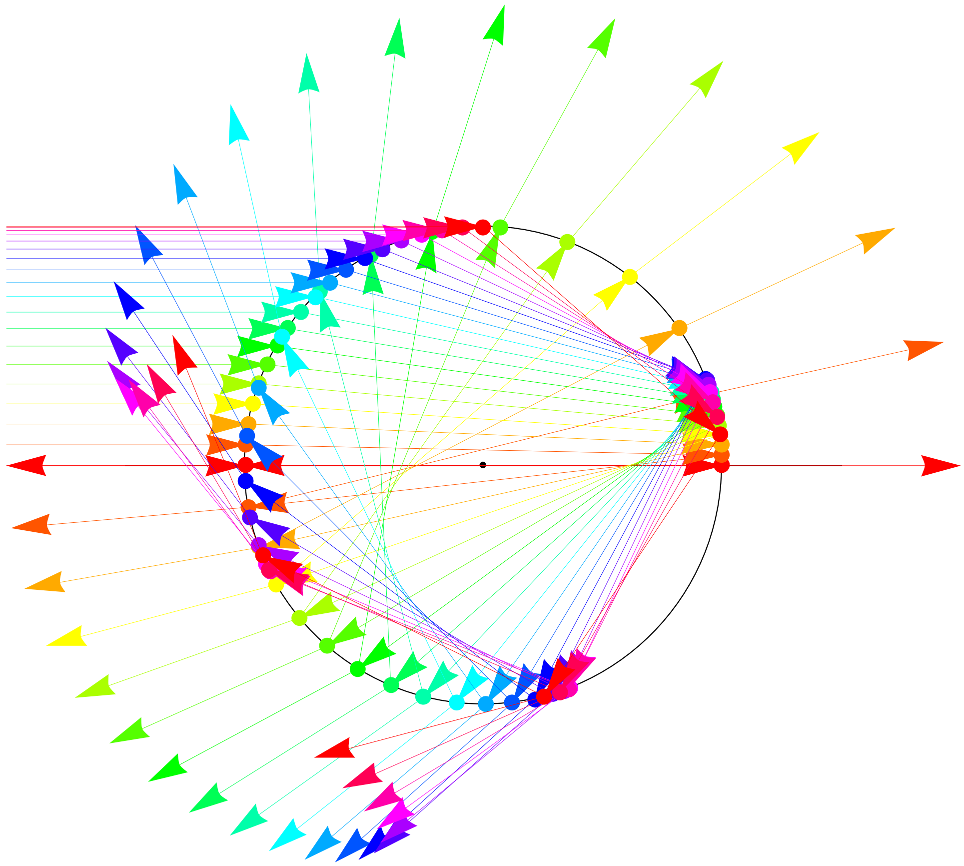

Fig.4 The pattern of rays entering the upper half of the water droplet. Note that the pattern of rays entering the lower half of the droplet is the same but flipped upside down. n_{water} = 1.331 for the red line. The incident angle θ_{i} is changed from 0° to 90° as a parameter. The colors used for rays in this figure has nothing to do with the color of the lines for rainbow.

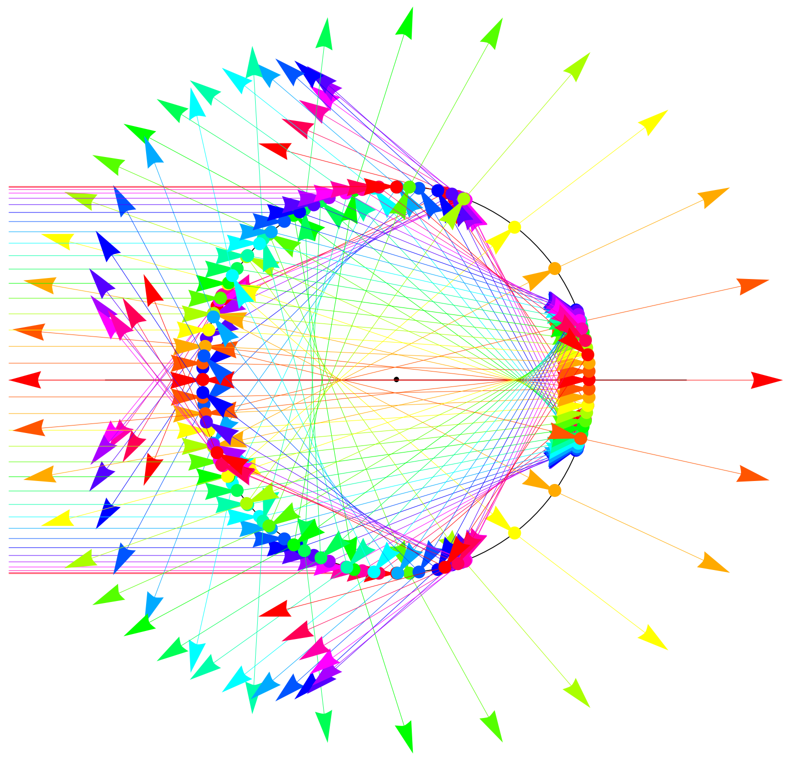

Fig.5(a) The pattern of rays entering both the upper and lower halves of the droplet. n_{water} = 1.331 for the red light. The incident angle θ_{i} is changed from -90° to 90° as a parameter ($\Delta\theta_{\mathrm{i}}$ = 5°). The colors used for rays in this figure has nothing to do with the color of the lines for rainbow. This figure is drawn using Mathematica 7.0.

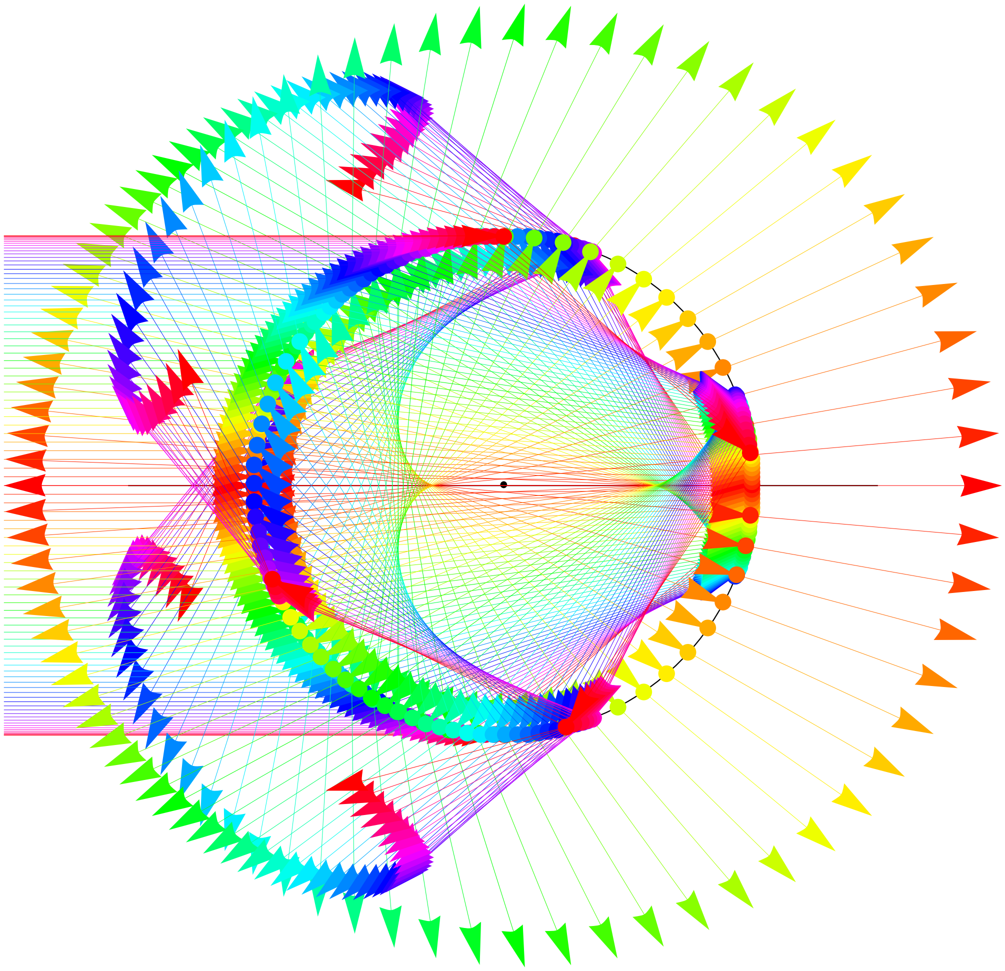

Fig.5(b) The pattern of rays entering both the upper and lower halves of the droplet. n_{water} = 1.331 for the red light. The incident angle θ_i is changed from -90° to 90° as a parameter ($\Delta\theta_i$ = 2°). The colors used for rays in this figure has nothing to do with the color of the lines for rainbow. This figure is drawn using Mathematica 7.0.

2. Index of refraction for water[7]

The beautiful color of the rainbow are a consequence of the variation of the index of refraction of water with wavelength. The dispersion is shown in Fig.6.

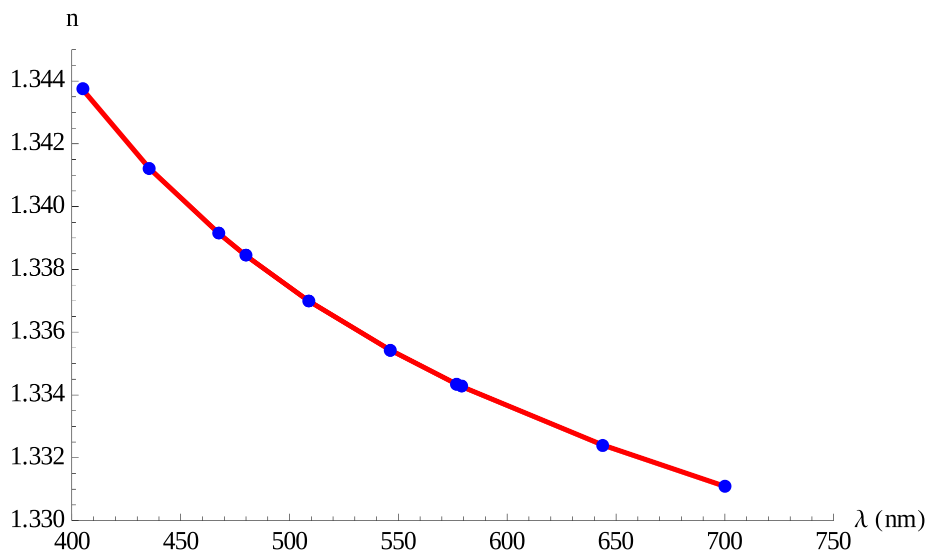

Fig.6 Index of refraction of water as a function of wavelength. The visible light interval is between 400 and 700 nm. n = 1.331 for the red light (n = 1.331, $\lambda \cong$ 700 nm). n = 1.343 for the blue light ($\lambda \cong$ 420 nm). n = 1.344 for the violet light ($\lambda \cong$ 400 nm).

3. A primary rainbow

A rainbow is produced by the reflection of sunlight by spherical drops of water in the air. As shown in Fig. 7, a ray that refracts into a drop at point A, is reflected from the back surface of drop at the point B, and refracts back into the air at the point C. The angles of incidence (θ_i) and refraction (θ_r) are shown at points A, B, C, and D. We define an angle of deflection ϕ (= $\angle$HOG),

$$\phi = -2\theta_i + 4\theta_r , \qquad (2)$$

where the direction of $\overrightarrow{HO}$ is that of the incident ray, and the direction of $\overrightarrow{OG}$ is that of the outgoing ray (see Fig.7). At the refraction at the point A of the water droplet, we have a Snell's law,

$$n_{air} \sin(\theta_i) = n_{water} \sin(\theta_r) . \qquad (3)$$

or

$$\theta_r = \arcsin[\frac{n_{air}}{n_{water}} \sin(\theta_i)] . \qquad (4)$$

Note that n_{water} is the index of refraction for water; n_{water} = 1.331 for the red light and 1.343 for the blue light (dispersion). Then ϕ can be rewritten as

$$\phi = 4\theta_r - 2\theta_i = 4\arcsin[\frac{n_{air}}{n_{water}}\sin(\theta_i)] - 2\theta_i \quad (5)$$

,

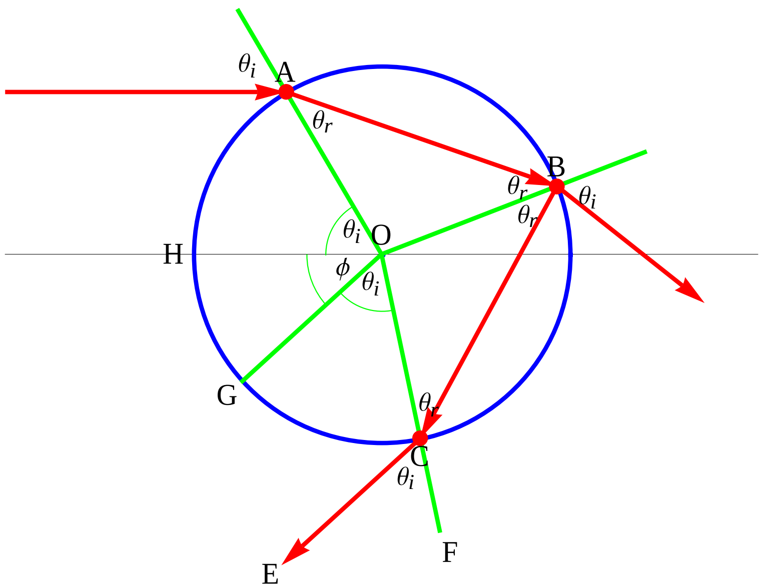

Fig.7 Primary rainbow. The blue circle denotes the spherical surface of water droplet. There is one reflection at the point B, and two refractions at the point A and C on the surface of the water droplet. The index of refraction of water depends on the wavelength. The angle of deflection ϕ is equal to $\angle$HOG. Note that $\overrightarrow{OG}$ is parallel to $\overrightarrow{CE}$. $\overrightarrow{HO}$ is parallel to the direction of the incident ray. θ_i = 59.5267° and $\phi = \phi_{\max}$ = 42.3698°. n_w = 1.331 for the red light.

We calculate the derivative of ϕ with respect to θ_i.

$$\frac{d\phi}{d\theta_i} = -2 + \frac{4\cos\theta_i}{\sqrt{n_{water}{}^2 - \sin^2\theta_i}} . \quad (6)$$

We find that $\frac{d\phi}{d\theta_i} = 0$ when

$$\cos^2\theta_i = \frac{1}{3}(n_{water}{}^2 - 1) \qquad (7)$$

Note that θ_i = 59.9267° for n_{water} = 1.331 (red light, see Fig.8) and θ_i = 58.8303° for n_{water} = 1.343 (blue light, see Fig.9). A rainbow will form when the angular deflection ϕ is stationary in the incident angle θ_i, that is, when $\frac{d\phi}{d\theta_i} = 0$ .If this condition is satisfied, all the rays with incident angle close to θ_i will be sent back in the same direction, producing a bright zone in the sky (see Figs. 4 and 5).

Figure 10 shows the ideal path of the red light and blue light through a spherical water droplet for the primary rainbow, where θ_i = 59.5267° and ϕ_{max}= 42.3698° for the red light (n_{water} = 1.331) and θ_i = 58.8303° and ϕ_{max} = 40.6459° for the blue light (n_{water} = 1.343). In summary, the different colors of light correspond to different wavelengths of light, which are refracted at slightly different angles, thus splitting the white sunlight into mainly red and blue lights (*dispersion*).

The color of the primary rainbow are spread over about 2° out of the 42° away from the anti-solar point. The viewer sees the rainbow with the red at the outer side of the arc and the blue on the inner side.

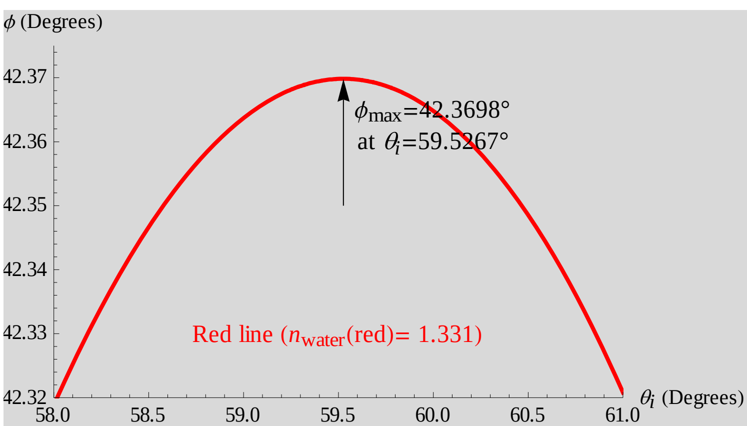

Fig.8 Plot of the angle of deflection, ϕ, as a function of θ_i (the primary rainbow) where n_{water} = 1.331 for the red light. The angle ϕ has a local maximum (ϕ_{max} = 42.3698°) at θ_i = 59.5267°.

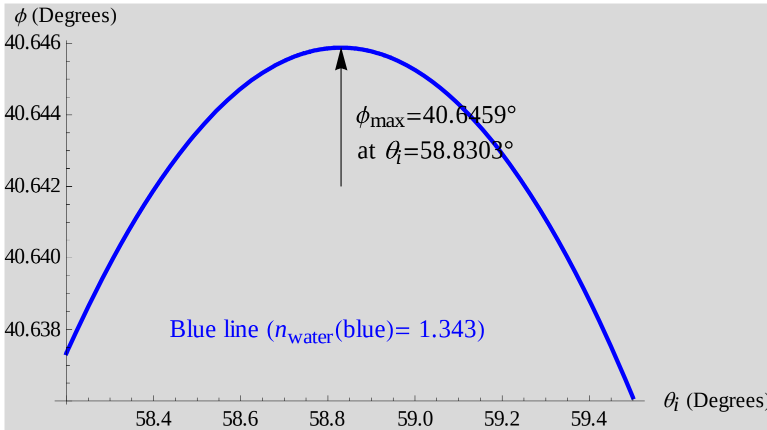

Fig.9 Plot of the angle of deflection, ϕ, as a function of θ_i (for the primary rainbow) where n_{water} = 1.343 for the blue light. The angle ϕ has a local maximum (ϕ_{max} = 40.6459°) at θ_i = 58.8303°.

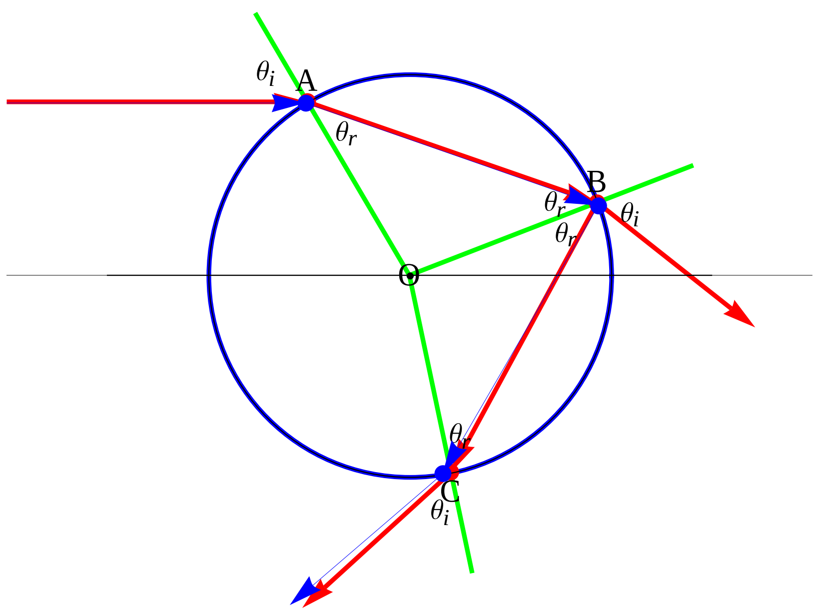

Fig.10 Ideal path of a light ray through a spherical water droplet for the primary rainbow. The primary rainbow is formed by rays that undergo two refractions at the points A and C, and internal reflection at the point B. Red light: θ_i = 59.5267°, n_{water} = 1.331. Blue light: θ_i = 58.8303°, n_{water} = 1.343. The angle of defection as a function of θ_i has a local maximum; ϕ_{max} = 42.3698° for the red light and ϕ_{max} = 40.6459° for the blue light. The angle ϕ_{max} for the red light is larger than that for the blue light.

4. Secondary rainbow

In Fig.11, we show the path of the light in the secondary rainbow. There are two refractions (A and D) and two internal reflections (B and C). The angle of deflection ϕ is defined by given by $\angle$FOG. Note that the direction of $\overrightarrow{GO}$ is that of the incident ray, and the direction of $\overrightarrow{OF}$ is that of the outgoing ray (see Fig.11). Noting that $\angle$FOD = $\angle$AOG = θ_i and $\angle$AOF = $\theta_i - \phi$, we have

$$\phi = 2\theta_i - 6\theta_r + \pi = 2\theta_i - 6\arcsin[\frac{n_{air}}{n_{water}}\sin(\theta_i)] + \pi \tag{8}$$

where $n_{air}\sin(\theta_i) = n_{water}\sin(\theta_r)$ (Snell's law).

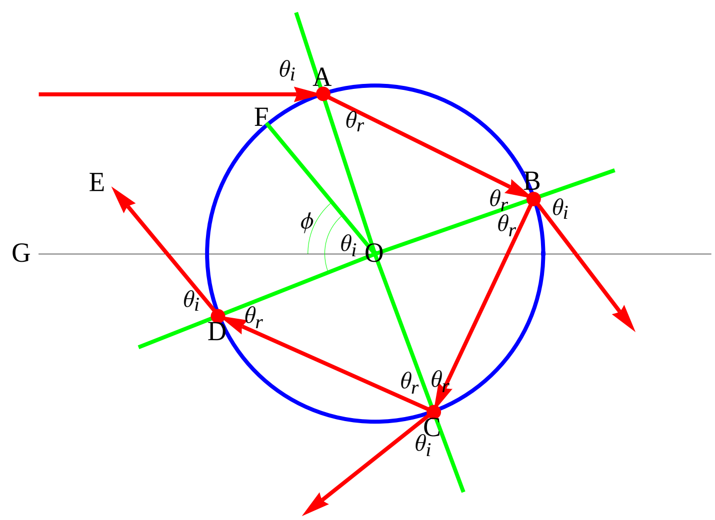

Fig.11 Secondary rainbow, there are two reflections at the points B and C, and two refractions at the point A and D. n_{water} = 1.331 for the red line. θ_i = 71.9072° and $\phi = \phi_{min}$ = 50.3651°. The angle of deflection f is defined by $\angle$FOG. $\overrightarrow{DE} // \overrightarrow{OF}$. $\angle$FOD = $\angle$AOG = θ_i. The angle ϕ as a function of θ_i has a local minimum (ϕ_{min}).

We calculate the derivative of ϕ with respect to θ_i.

$$\frac{d\phi}{d\theta_i} = -2 + \frac{4\cos\theta_i}{\sqrt{n_{water}^2 - \sin^2\theta_i}} . \tag{9}$$

We find that $\frac{d\delta}{d\theta_i} = 0$ when

$$\cos\theta_i = \frac{\sqrt{n_{water}{}^2 - 1}}{2\sqrt{2}}. \qquad (10)$$

Note that ϕ_{min} = 50.3657° at θ_i = 71.9072° for n_{water} = 1.331 (red light) and ϕ_{min}= 53.4778° at θ_i = 71.5215° for n_{water} = 1.343 (blue light). A rainbow will form when the angular deflection ϕ is stationary in the incident angle θ_i, that is, when $\frac{d\phi}{d\theta_i} = 0$. If this condition is satisfied, all the rays with incident angle close to θ_i will be sent back in the same direction, producing a bright zone in the sky.

Figure 14 shows the ideal path of the red and blue lights through a spherical water droplet for the secondary rainbow, where θ_i = 71.9072° and ϕ_{min} = 50.3651° for the red light (n_{water} = 1.331) and θ_i = 71.5215° and ϕ_{min} = 53.478°. for the blue light (n_{water} = 1.343). In summary, the secondary rainbow is about 10° further out from the anti-solar point than the primary bow, is about twice as wide, and has its colors reversed.

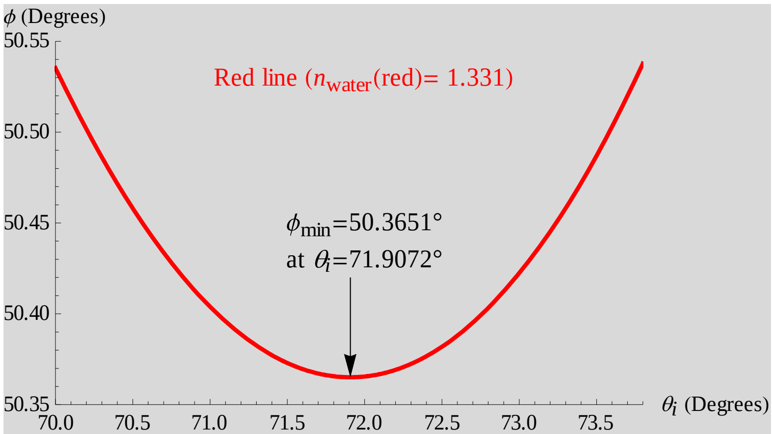

Fig.12 Plot of the angle of deflection, ϕ, as a function of θ_i (for the secondary rainbow), where n_{water} = 1.331 for the red light. The angle ϕ has a local minimum (ϕ_{min} = 50.3651°) at θ_i = 71.9072°.

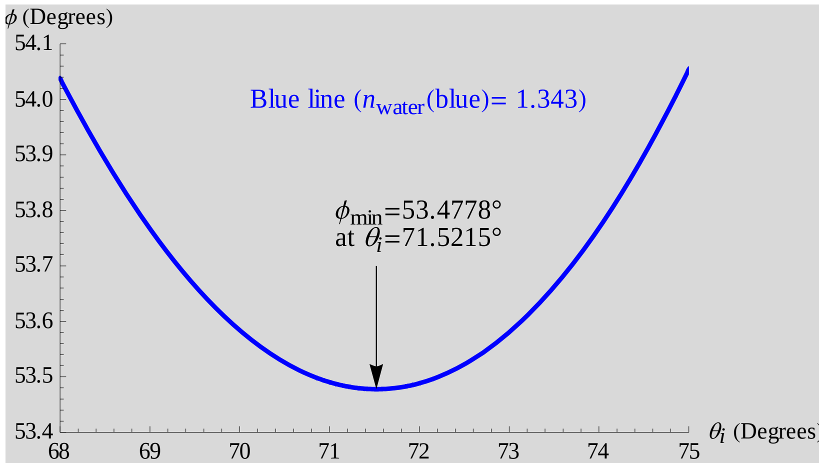

Fig.13 Plot of the angle of deflection, ϕ, as a function of θ_i (for the secondary rainbow) where n_{water} = 1.343 for the blue light. The angle ϕ has a local minimum (ϕ_{min} = 53.4778°) at θ_i = 71.5215°.

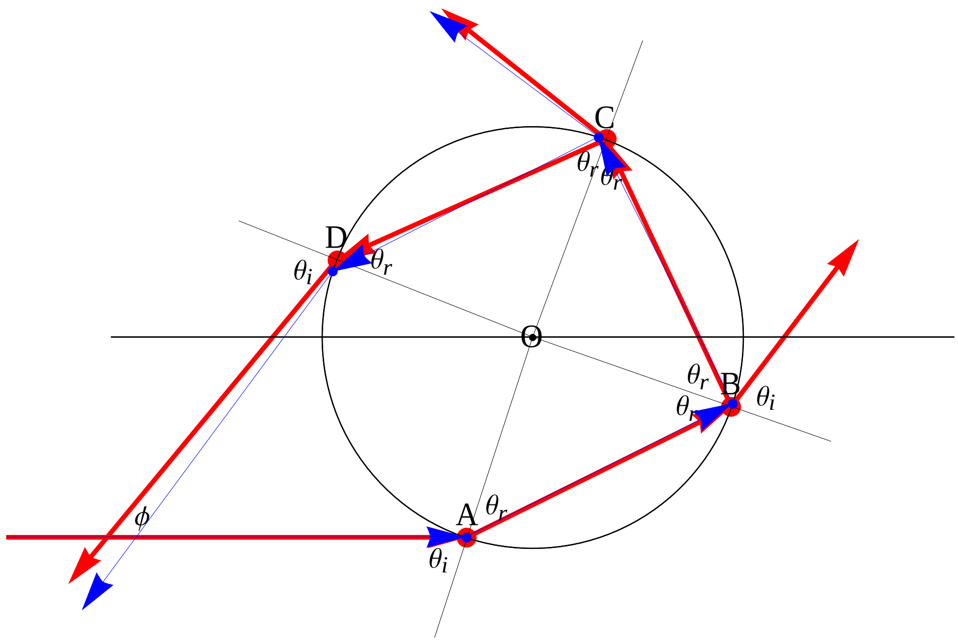

Fig.14 The light path for the secondary rainbow. The red light: θ_i = 71.9072°, n_{water} = 1.331. ϕ_{min} = 50.3651°. The blue light: θ_i = 71.5215°, n_{water} = 1.343. ϕ_{min} = 53.478°. ϕ_{min} is the minimum angle of the deflection (for the secondary rainbow).

5. Polarization[4]

5.1 Brewster angle

David Brewster showed in 1812 that the scattered rainbow light was almost completely polarized, confirming earlier observations of Biot (of the Biot-Savart law in magnetism). More brilliant photographs of rainbows may be obtained by using a polarizer on the camera lens, because one finds that the intensity of the rainbow colors is affected by the rotation of the polarizer. The plane of polarization of the primary rainbow is tangent to the rainbow arc (see Fig.15). This polarization arises at the internal reflection in the water drop which is near the Brewster angle.

Here we apply the Snell's law to the reflection at the point B (see Fig.7)

$$n_{water} \sin\theta_r = n_{air} \sin\theta_i \, .$$

The condition for the Brewster angle is

$$\theta_i + \theta_r = \frac{\pi}{2} \, . \tag{11}$$

Then we have

$$n_{water} \sin\theta_r = n_{air} \sin(\frac{\pi}{2} - \theta_r) = n_{air} \cos\theta_r \, , \tag{12}$$

or

$$\tan\theta_r = \frac{n_{air}}{n_{water}} \, . \tag{13}$$

When n_{water} = 1.331 (the red light), the Brewster angle is evaluated as θ_{rB} = 36.918°. When n_{water} = 1.343 (the blue light), the Brewster angle is evaluated as θ_{rB} = 36.671°.

(a) For n_{water} = 1.331 (the red light) and θ_{i} = 59.5267°, we get

$$\sin\theta_r = \frac{n_{air}}{n_{water}} \sin\theta_i = \frac{n_{air}}{n_{water}} \sin(59.5267°) = 0.64753 \, ,$$

or

$$\theta_r = 40.356° \, .$$

which is close to the Brewster angle (θ_{rB} = 36.918°).

(b) For n_{water} = 1.343 (the blue light) and θ_i = 58.8303°, we get

$$\sin\theta_r = \frac{n_{air}}{n_{water}}\sin\theta_i = \frac{n_{air}}{n_{water}}\sin(58.8303°) = 0.64753,$$

or

$$\theta_r = 39.576°.$$

which is also close to the Brewster angle (θ_{rB} = 36.918°).

5.2 Degree of polarization

The polarization of the rainbow is caused by the internal reflection at the point B. The rays strike the back surface of the droplet close to the Brewster angle, so almost all the light reflected is polarized perpendicular to the incidence plane. As the incidence plane is determined for each drop by the plane containing the sun, the drop, and the observer, the rainbow is polarized tangential to the arc (see Fig.15).

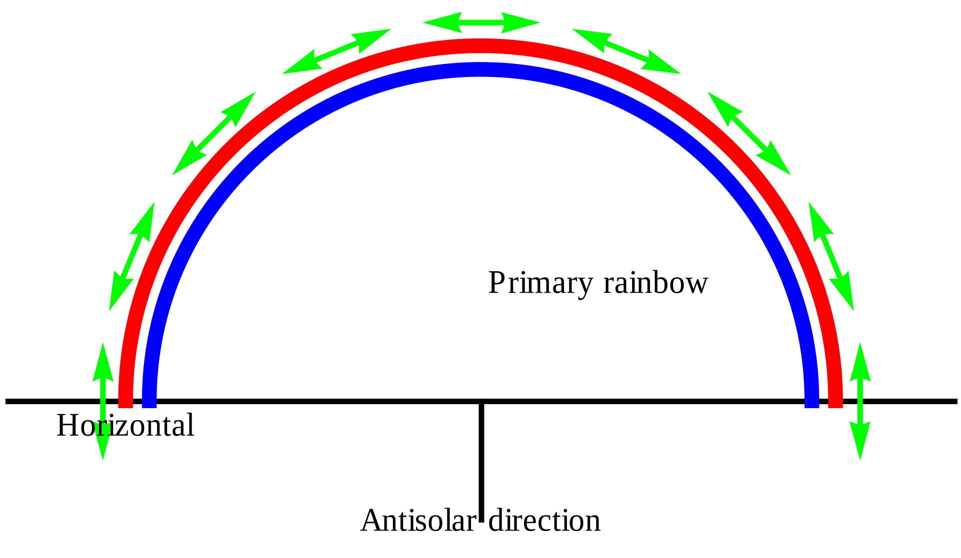

Fig.15 The primary rainbow which is almost linearly polarized. The green arrows indicate the direction of electric fields, which is tangentially polarized.

The primary rainbow is 94 % polarized while the secondary rainbow is 90% polarized. The extra brightness of the sky inside the primary rainbow (and outside the secondary rainbow) is also polarized tangentially (but to a lesser degree) as it has the same origin as the bows.

5.3 Fresnel's equation[4]

The degree of polarization can be calculated from Fresnel's equations which relate reflected and transmitted amplitudes to incident amplitude as a function of angle of incidence (see Fig.16).

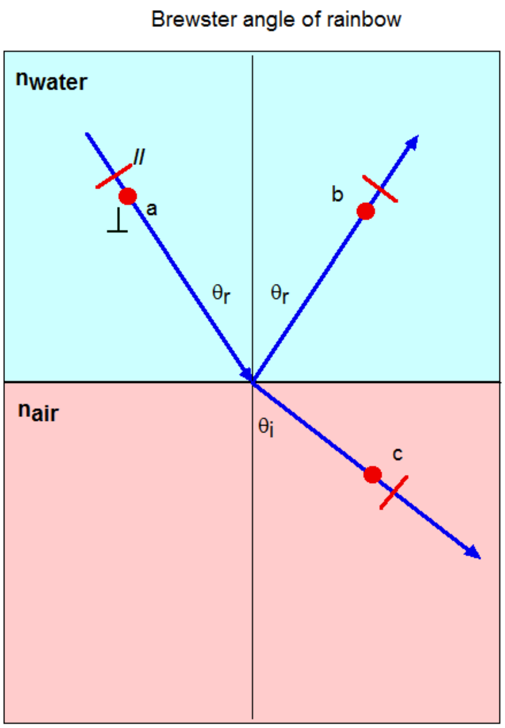

Fig.16 The configuration of the Brewster angle. When $\theta_r + \theta_i = 90°$ is satisfied, the electric field *E* parallel to the plane of incidence vanishes, leading to the linearly polarized ray with *E* perpendicular to the plane of incidence.

We have

$$n_{air} \sin\theta_i = n_{water} \sin\theta_r \qquad \text{(Snell's law).}$$

The TE polarization (*s*-polarization). ***E*** is normal to the plane of incidence (⊥). The TM polarization (*p*-polarization). ***E*** lies in the plane of incidence (//).

$$R_\perp = \left(\frac{E_{0b}}{E_{0a}}\right)_\perp^2 = \frac{\sin^2(\theta_i - \theta_r)}{\sin^2(\theta_i + \theta_r)}, \qquad (14)$$

$$R_{//} = \left(\frac{E_{0b}}{E_{0a}}\right)_{//}^2 = \frac{\tan^2(\theta_r - \theta_i)}{\tan^2(\theta_r + \theta_i)}. \tag{15}$$

Observe that while $R_{\perp}$ can never be zero, $R_{//}$ is indeed zero when the denominator is infinite, i.e., when $\theta_r + \theta_i = 90°$. The reflectance, for linear light with E parallel to the plane of incidence, thereupon vanishes. The degree of the polarization is given by

$$f = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{R_{\perp} - R_{//}}{R_{\perp} + R_{//}}. \tag{16}$$

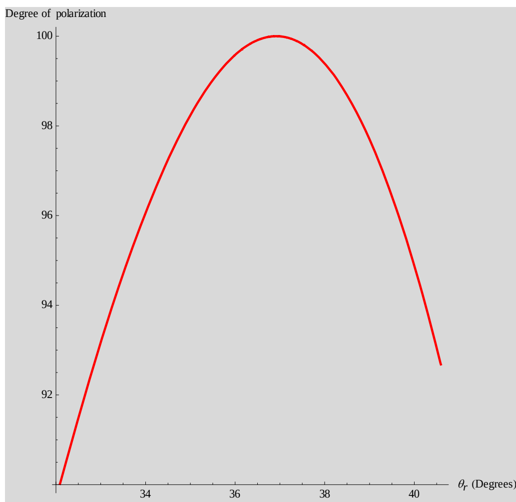

Fig.17 Degree of polarization as a function of θ_r for the red light. $n_{water} = 1.331$.

The degree of polarization has a maximum at $\theta_i = 53.0819°$ ($\theta_r = 36.9181°$). When $\theta_i = 59.5627°$, the degree of polarization for the red light (the primary rainbow) is 93.6% (see Figs. 17 and 18).

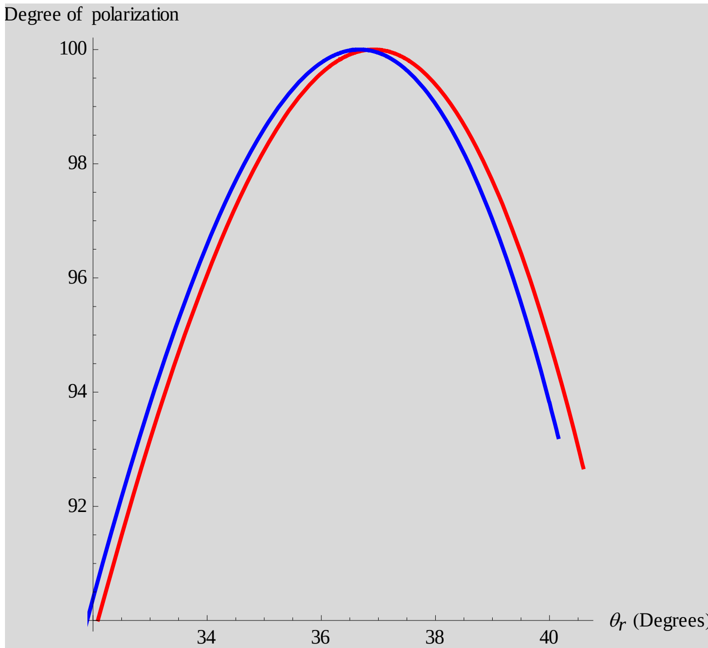

Fig.18 Degree of polarization for the red light (n_{water} = 1.331) and blue light (n_{water} = 1.341).

6. Supernumerary rainbows

6.1 Interference phenomena[2, 6, 9]

A series of fine weakly colored bows (so called supernumerary, see Fig.19) that can frequently be seen just inside the primary rainbow. These arise from the interference of two parallel rays with different paths. The supernumerary bows are the repeated green and purple bands just inside the primary bow.

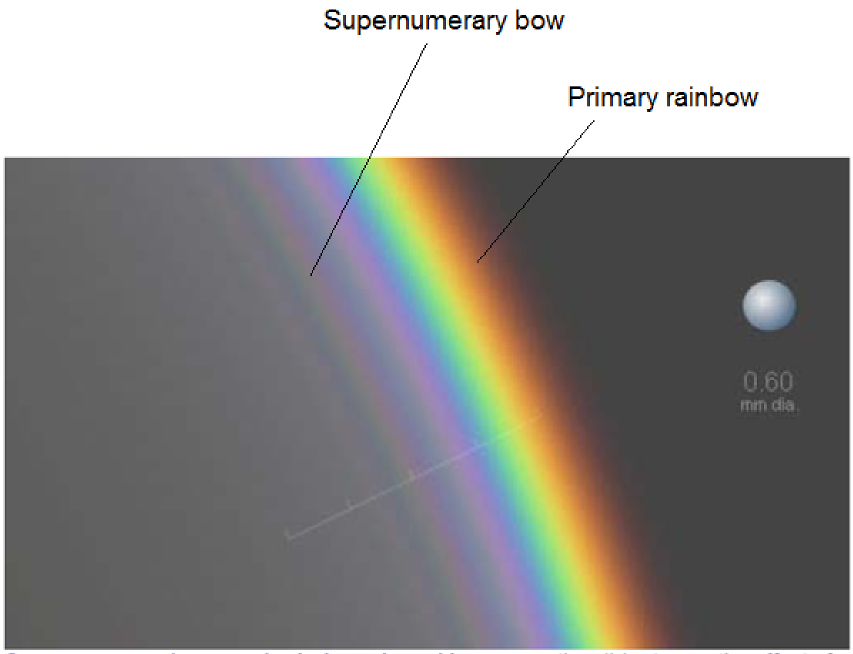

Fig.19 The supernumeraries are the closely spaced greenish purple arcs on the inner (blue) side of the primary bow. Supernumerary bow arises from the interference of two rays having the same angle of deflection.
http://www.atoptics.co.uk/rainbows/

Figure 20 shows the angle of deflection for the primary rainbow as a function of the incident angle θ_i for the red light (n_{water} = 1.331). The angle ϕ has a local maximum (ϕ_{max} = 42.3698º at θ_i = 59.5267º. When $\phi < \phi_{max}$, there are two intersection points of θ_i; for example, θ_i = 58.8592° and θ_i = 60.1895° for ϕ = 42.36º. In other words, for each angle of deviation there are two rays at different incident angles. The interference between two rays emerging at the same angle, but travelling different optical paths within the water droplets explain for the existence of supernumeraries. Figure 21 shows the light path of the light with such values of θ_i. Because of the same value of ϕ, the outgoing rays are parallel to each other. Since the values of θ_i are different for these two rays, there is a path difference between these two rays, leading to the interference. The constructive interference occurs when the path difference is given by $\Delta\varphi = n\lambda$, where λ is the wavelength and N is an integer. The destructive interference occurs when $\Delta\varphi = (N + 1/2)\lambda$. Because of coherence in the path differences within the water droplet, the supernumeraries may be created by small, uniform sized raindrops.

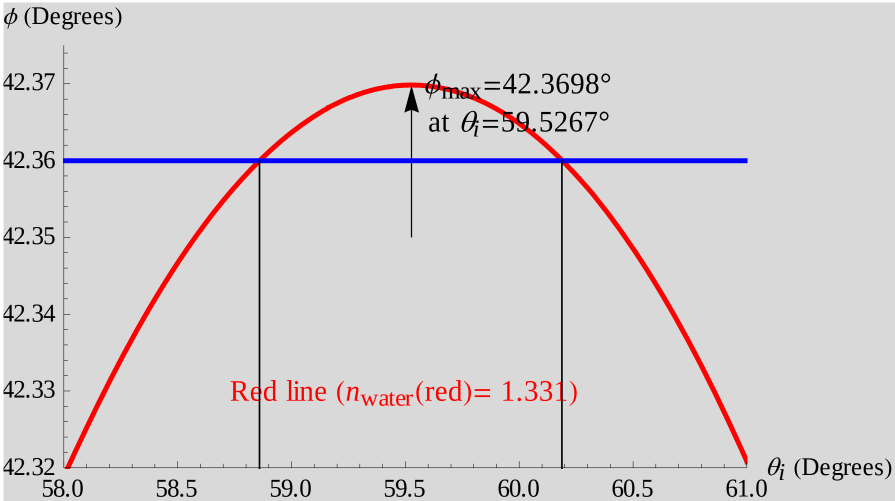

Fig.20 Plot of the angle of deflection ϕ as a function of θ_i for n_{water} = 1.331 (for the primary rainbow). The angle ϕ has a local maximum (ϕ_{max} = 42.3698°) at θ_i = 59.5267°. There is one intersection point of θ_i when $\phi = \phi_{max}$, but there are two intersection points of θ_i when $\phi<\phi_{max}$; for example, θ_i = 58.8592° and θ_i = 60.1895° for ϕ = 42.36°.

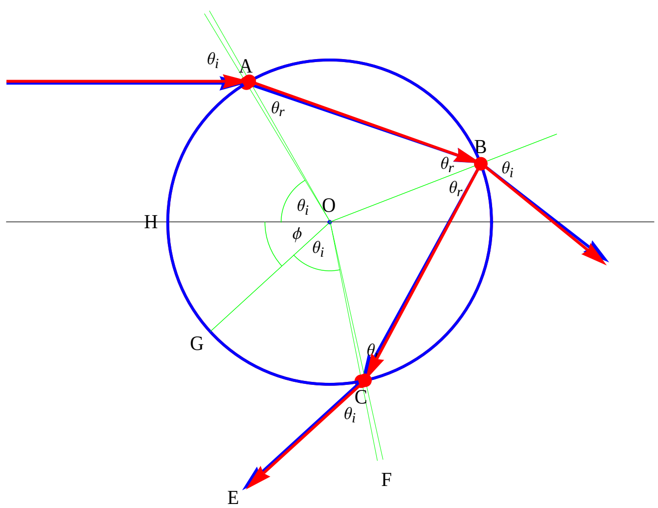

Fig.21(a) Interference of two rays with θ_i = 58.8592° and θ_i = 60.1895° in the primary rainbow**.** Two rays emerge at the angle of deflection (ϕ = 42.36°) (near the primary rainbow) for the index of refraction, n = 1.331 (the red light). $\overrightarrow{OG}$ is parallel to $\overrightarrow{CE}$. $\overrightarrow{HO}$ is parallel to the direction of the incident ray. The interference of two rays leads to the supernumerary inside the primary rainbow.

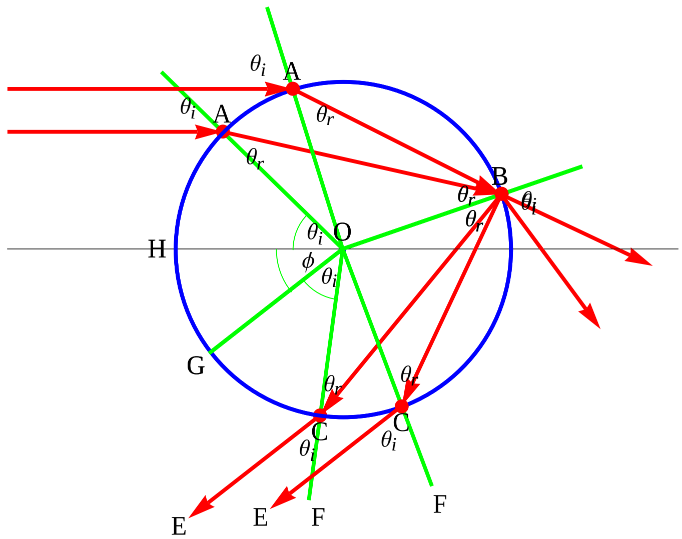

Fig.21(b) Interference of two rays with θ_i = 72.6112° and θ_i = 44.2996° in the primary rainbow**.**

Figure 22 shows the angle of deflection for the secondary rainbow as a function of the incident angle θ_i for the red light (n_{water} = 1.331). The angle ϕ has a local minimum ϕ_{min} = 50.3651° for θ_i = 71.9072°). When $\phi > \phi_{min}$, there are two intersection points of θ_i; for example, θ_i = 68.1956° and 75.5141° for ϕ = 51.0°. Figure 23 shows the light path of the light with such values of θ_i. Because of the same value of ϕ, the outgoing rays are parallel to each other. Since the values of θ_i are different for these two rays, there is a path difference between these two rays, leading to the interference. The secondary supernumerary is the faint and broad arc outside the secondary bow. The secondary supernumerary is very rare, while the supernumeraries inside the primary bow are not unusual.

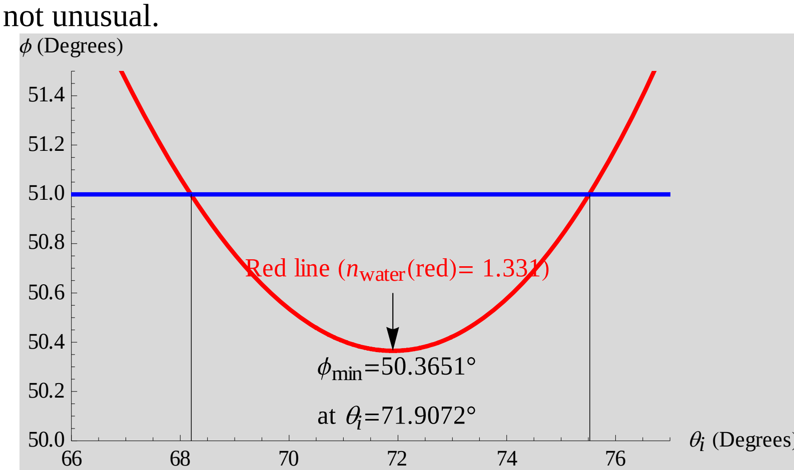

Fig.22 Plot of the angle of deflection ϕ as a function of θ_i for n_{water} = 1.331 (for the secondary rainbow). The angle ϕ has a local minimum ϕ_{min} = 50.3651° for θ_i = 71.9072°. There are one intersection point of θ_i when ϕ = ϕ_{min}, but two intersection points of θ_i when $\phi < \phi_{max}$. For a pair of θ_i = 68.1956° and 75.5141°,

the angle of deflection is the same (ϕ = 51.0°), leading to the parallel rays. The path difference of two rays contributes to the interference.

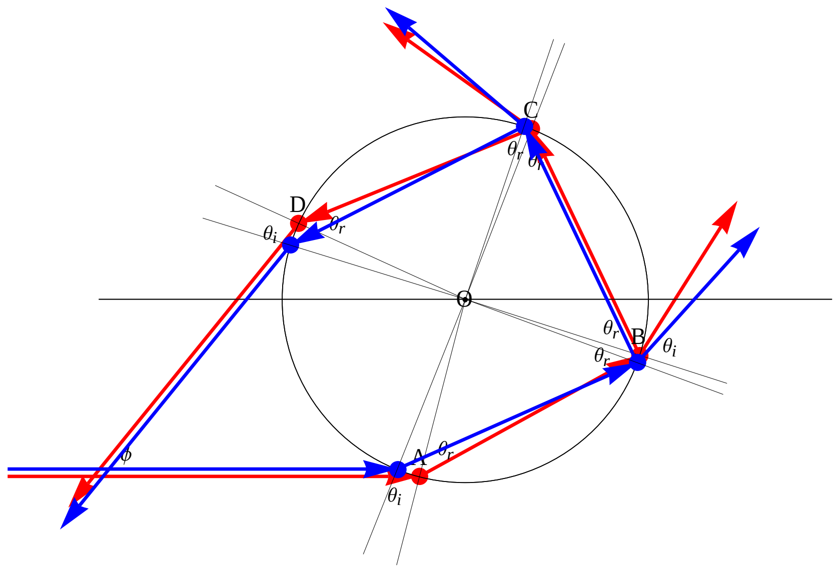

Fig.23 Interference of two rays (which are parallel) with θ_i = 75.5141° and θ_i = 68.1956° in the secondary rainbow. ϕ = 51.0°. This interference leads to the secondary supernumerary outside the secondary rainbow.

6.2 Optical path calculation[6, 9]

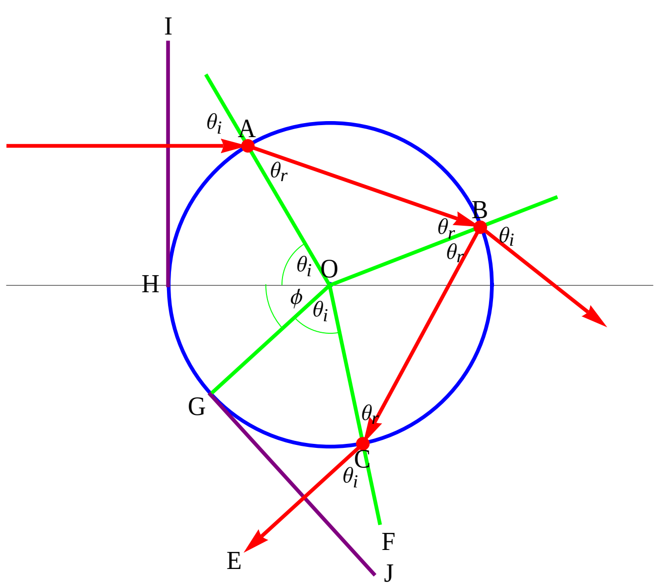

Fig.24 Geometrical optics of a primary rainbow for the calculation of optical path. $\overrightarrow{GJ}$ is perpendicular to $\overrightarrow{CE}$. $\phi = \phi_{max}$ = 42.3698° and θ_i = 59.5267°. n_{water} = 1.331.

Referring to Fig.24, the lines HI and GJ are convenient ones for defining the optical path of a ray in the neighborhood of the critical ray (ϕ = $\phi_{\max}$). According to Jackson, The phase ψ is evaluated by

$$\psi = 2ka(1 - \cos\theta_i + 2n_{water}\cos\theta_r), \qquad (17)$$

with

$$\theta_r = \arcsin[\frac{n_{air}}{n_{water}}\sin(\theta_i)] = \arcsin[\frac{1}{n_{water}}\sin(\theta_i)], \qquad (18)$$

where a is the radius of the water droplet, the $2a(1-\cos\theta_i)$ represents the sum of the distance from the line HI to the drop's surface and the line GJ to the drop's surface, and $4an\cos\theta_r$ is the length (times n) of the path interior to the drop. k is the wavenumber and is given by $2\pi/\lambda$. The angle of deflection ϕ is given by

$$\phi = -2\theta_i + 4\theta_r. \qquad (19)$$

For simplicity we use $\sin\theta_i = x$. Then the phase ψ and f can be rewritten using x as

$$\chi = 2ka(1 - \sqrt{1-x^2} + 2\sqrt{n^2 - x^2}), \qquad (20)$$

and

$$\phi = -2\arcsin(x) + 4\arcsin(\frac{x}{n}), \qquad (21)$$

where $n = n_{\mathrm{water}}$ for the sake of simplicity. The derivatives of χ and ϕ with respect to x are given by

$$\frac{d\chi}{dx} = 2kax(\frac{1}{\sqrt{1-x^2}} - \frac{2}{\sqrt{n^2-x^2}}),$$

and

$$\frac{d\phi}{dx} = -2\left(\frac{1}{\sqrt{1-x^2}} - \frac{2}{\sqrt{n^2-x^2}}\right).$$

From these expressions, we get a relation given by

$$\frac{d\chi}{dx} = -kax\frac{d\phi}{dx}. \tag{22}$$

When $x = x_0 + \xi$ in the limit of very small ξ,

$$\frac{d\chi}{d\xi} = -ka\left(x_0\frac{d\phi}{d\xi} + \xi\frac{d\phi}{d\xi}\right). \tag{23}$$

Integration on both sides from 0 to ξ yields

$$\begin{aligned}\chi(\xi) - \chi(\xi=0) &= -ka[x_0(\phi-\phi_{\max}) + \int_0^{\xi}\xi'\frac{d\phi}{d\xi'}d\xi'] \\ &= -ka[x_0(\phi-\phi_{\max}) + \xi\phi - \int_0^{\xi}\phi(\xi')d\xi']\end{aligned}. \tag{24}$$

Here we assume that

$$\phi(\xi) = \phi_{\max} + \phi''(0)\frac{\xi^2}{2} + O(\xi^3), \tag{25}$$

where $\phi(\xi)$ has a local maximum at $\xi = 0$; $\phi'(0) = 0$ and $\phi''(0) < 0$. Inserting the form of $\phi(\xi)$ in the integral, we have

$$\begin{aligned}\chi(\xi) - \chi(\xi=0) &= -ka[x_0(\phi-\phi_{\max}) + \xi\phi - (\xi\phi_{\max} + \frac{\xi^3}{6}\phi''(0)] \\ &= -ka[(x_0+\xi)(\phi-\phi_{\max}) - \frac{\xi^3}{6}\phi''(0)] \\ &= -ka[(x_0+\xi)\frac{\xi^2}{2}\phi''(0) - \frac{\xi^3}{6}\phi''(0)]\end{aligned}, \tag{26}$$

or

$$\chi(\xi) - \chi(\xi = 0) = ka[-\phi''(0)](x_0 \frac{\xi^2}{2} + \frac{1}{3}\xi^3) \quad . \tag{27}$$

Note that $\phi'(0) = 0$ and $\phi''(0) < 0$ for the local maximum. With equal and opposite small ξ, the phase difference $\Delta\varphi$ is obtained as

$$\Delta\varphi = \chi(\xi) - \chi(-\xi) = ka[-\phi''(0)]\frac{2}{3}\xi^3 \quad . \tag{28}$$

The constructive interference occurs when $\Delta\varphi$ is equal to $2\pi N$ (N; integer);

$$\xi_n = [-\phi''(0)]^{-1/3}(\frac{3\pi N}{ka})^{1/3} \,. \tag{29}$$

Then the angle of deflection for the constructive interference is given by

$$\phi(\xi_n) - \phi_{\max} = -\frac{1}{2}[-\phi''(0)]^{1/3}(\frac{3\pi N}{ka})^{2/3} \quad . \tag{30}$$

(Actually, a more correct procedure has N + ¼ replacing N). The angles of constructive interference mark the positions of supernumerary rainbows. They lie at smaller angles than $\phi_{\max}$. The angle $\phi(\xi_n) - \phi_{\max}$ depends on the droplet size, varying as $(ka)^{-2/3}$. For small drops, the angle becomes large. In order to see clearly the supernumerary, one needs small drops, uniform in size.

6.3 Huygens' construction, Airy integral[6, 9]

Consider the line GJ in Fig.25. A wave along this line will have the form of the wave function,

$$\psi \propto \exp[i(k_{//}z + \mathbf{k}_\perp \cdot \mathbf{r}_\perp + \chi(\xi)] \,, \tag{31}$$

where z is the direction of scattering at $\phi_{\max}$ (the direction of $\overrightarrow{CE}$) and $\mathbf{r}_\perp$ is measured along $\overrightarrow{GJ}$, with value of $R\, x_0 = R \sin\theta_i$ for $\phi = \phi_{\max}$. If the wave is propagating in the direction ϕ (the direction of $\overrightarrow{OL}$), then we have

$$\mathbf{k}_\perp \cdot \mathbf{r}_\perp = kR(\phi - \phi_{\max})x \,, \tag{32}$$

where $|\boldsymbol{r}_\perp| = Rx$ and $|\boldsymbol{k}_\perp| = k(\phi_{\max} - \phi)$ (see Fig.25).

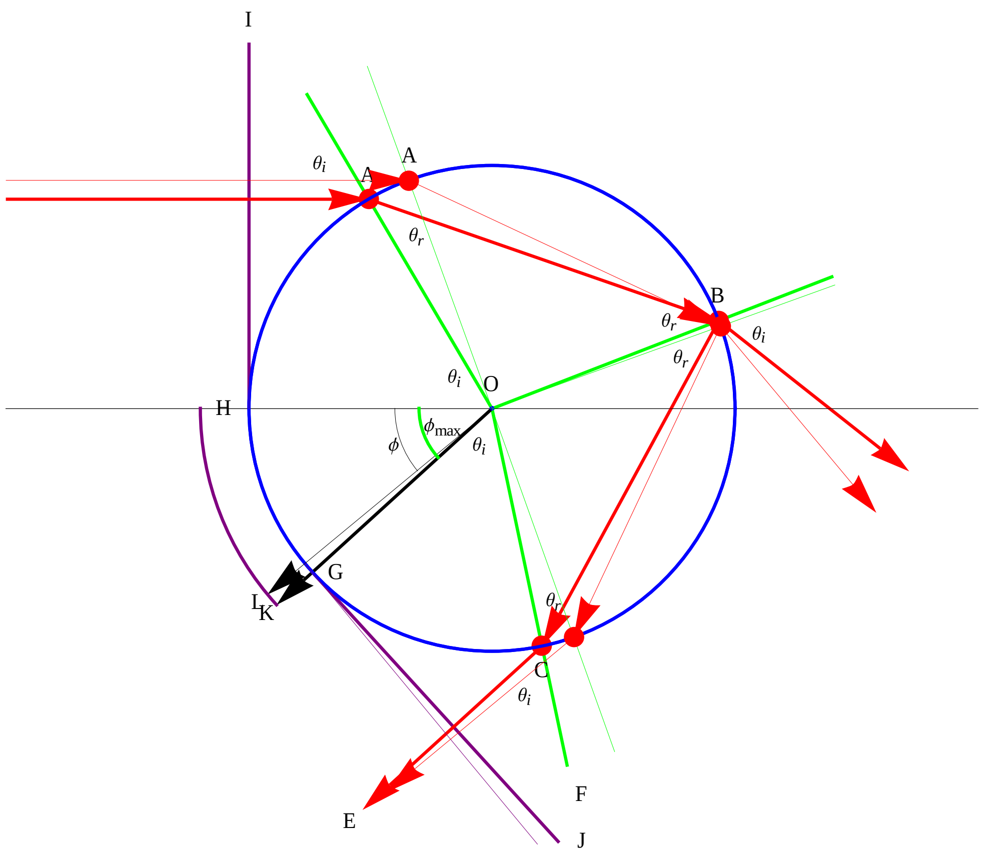

Fig.25 Geometrical optics of a primary rainbow. $n_{water} = 1.331$. $\theta_i = 59.5267°$ and $\phi_{max} = 42.3698°$. $\theta_i = 70°$ and $\phi = 39.6432°$ (as an example). KL $= -k(\phi - \phi_{max})$. The direction of $\overrightarrow{KL}$ is antiparallel to that of $\overrightarrow{GJ}$ (the direction of $\boldsymbol{r}_\perp$).

Since $\phi < \phi_{max}$, $\mathbf{k}_\perp$ and $\mathbf{r}_\perp$ are antiparallel. Since z is constant on the line $\overrightarrow{GJ}$, the relevant parts of the wave's overall phase are

$$\begin{aligned}\mathbf{k}_\perp \cdot \mathbf{r}_\perp + \chi(\xi) &= ka[(x_0+\xi)(\phi-\phi_{max})] + \chi(\xi=0) + ka[-\phi''(0)](x_0\frac{\xi^2}{2}+\frac{1}{3}\xi^3) \\ &= ka[(x_0+\xi)\phi''(0)\frac{\xi^2}{2}] + \chi(\xi=0) + ka[-\phi''(0)](x_0\frac{\xi^2}{2}+\frac{1}{3}\xi^3) \\ &= \chi(\xi=0) + ka\phi''(0)(\frac{1}{6}\xi^3)\end{aligned}$$

or

$$\mathbf{k}_\perp \cdot \mathbf{r}_\perp + \chi(\xi) - \chi(\xi = 0) = ka\phi''(0)(\frac{1}{6}\xi^3) \,. \qquad (33)$$

Along the line $\overrightarrow{GJ}$, the wave amplitude in the vicinity of $\xi = 0$ has the approximate form

$$\psi(\xi) = \exp[-ikR(-\phi''(0))\frac{\xi^3}{6}] \,. \qquad (35)$$

Following Jackson,[6] it can be shown using the Kirchoff integral for the diffraction that the amplitude of the scattered wave near $\phi = \phi_{\max}$ is

$$\begin{aligned}\psi_{scatt} &\propto \int_{-\infty}^{\infty} d\xi \exp[-ika\{(\phi - \phi_{\max})\xi - \phi''(0)\frac{\xi^3}{6}\}] \\ &= 2\int_0^{\infty} d\xi \cos[ka\{(\phi - \phi_{\max})\xi - \phi''(0)\frac{\xi^3}{6}\}] \,.\end{aligned}$$

This function is proportional to the Airy integral defined by

$$Ai(-\eta) = \frac{1}{\pi}\int_0^{\infty} \cos(\frac{t^3}{3} - \eta t)dt$$

(see Fig.26), where

$$\eta = [\frac{2k^2a^2}{-\phi''(0)}]^{1/3}(\phi_{\max} - \phi) \,.$$

Since

$$\phi(\xi_n) - \phi_{\max} = -\frac{1}{2}[-\phi''(0)]^{1/3}[\frac{3\pi(N + 1/4)}{ka}]^{2/3}$$

we have

$$\eta = [\frac{3\pi(N + 1/4)}{2}]^{2/3}$$

The maxima occur at

η = 1.01879 (N = 0, 1.11546), 4.8201(N = 2, 4.82632), 7.37218 (N = 4, 7.37485), 9.53545 (N = 6, 9.93705), 11.4751(N = 8, 11.4762), 13.2622 (N = 10, 13.263), 14.9359 (N = 12, 14.9366).

The minima occur at
η = 3.2482 (N = 1, 3.26163), 6.16331 (N = 3, 6.16713), 8.48849 (N = 5, 8.49051), 10.5277 (N = 7, 10.529), 12.3848 (N = 9, 12.3857), 14.1115 (N = 11, 14.1122), 15.7382 (N = 13, 15.7388).

The number of parentheses are values of $\eta = [\frac{3\pi(N+1/4)}{2}]^{2/3}$ for N = 0, 1, 2,...For larger N values, the agreement is excellent.

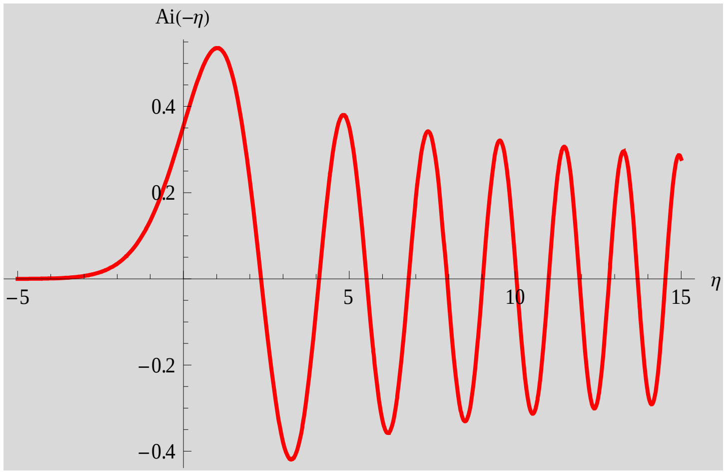

Fig.26 Plot of the Airy integral Ai(-η) as a function of η.

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