# Optical pumping of $^{87}$Rb

**Masatsugu Sei Suzuki and Itsuko S. Suzuki**
**Department of Physics, State University of New York at Binghamton**
**Binghamton, NY 13902-6000**
**(Date: April 04, 2011)**

**Abstract**

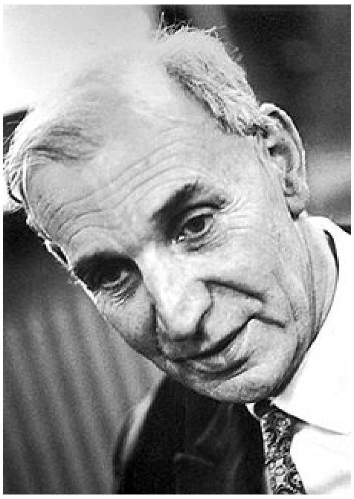
Optical pumping is to change the population number of atomic states by illumining of light beam to the system from ones in thermal equilibrium. The spin of the circularly polarized light is transferred to atoms, leading to the change of the angular momentum of atoms. This method was first proposed by Kastler in France in 1950. Although this method is very simple, one can make a surprisingly precise spectroscopy of atomic sublevels. This method makes it possible to do detailed studies for the interactions between atom and light. In 1966, Nobel prize was awarded to Kastler for his work on the method of optical pumping. The atomic operation by light is now the main topics of the atomic physics. We note that Cohen-Tannoudji got a Nobel prize in 1977 for the laser cooling. Cohen-Tannoudji was a student of Kasler.

In our Advanced Laboratory course (Senior Laboratory for undergraduate students and Graduate Laboratory for graduate students), we have an apparatus of optical pumping of Rb, (TeachSpin). Some students have a difficulty in understanding the physics of optical pumping. In this lecture note, we discuss the physics based on the atomic physics and quantum mechanics. The splitting of the energy levels in $^{87}$Rb (nuclear spin 3/2) will be discussed in terms of the eigenvalue problems with Mathematica. The transition probability for the absorption and emission of light due to the interaction with electric dipole moments will be discussed in terms of Wigner-Eckart theorem and Clebsch-Gordan coefficient. The physics of optical pumping in $^{85}$Rb (nuclear spin 5/2) will not be discussed here.

Atomic physics
Quantum mecahnics
Wigner-Eckart theorem
Clebsch-Gordan
Linear Zeeman effect
Quadratic Zeeman effect
Optical pumping

---

**Alfred Kastler** (May 3, 1902 – January 7, 1984) was a French physicist, and Nobel Prize laureate. Kastler was born in Guebwiller (Alsace) and later attended the Lycée Bartholdi in Colmar, Alsace, and École Normale Supérieure in Paris in 1921. After his studies, in 1926 he began teaching physics at the Lycée of Mulhouse, and then taught at the University of Bordeaux, where he was a university professor until 1941. Georges Bruhat asked him to come back to the École Normale Supérieure, where he finally obtained a chair in 1952. Collaborating with Jean Brossel, he researched quantum mechanics, the interaction between light and atoms, and spectroscopy. Kastler, working on combination of optical resonance and magnetic resonance, developed the technique of "optical pumping". Those works led to the completion of the theory of lasers and masers. He won the Nobel Prize in Physics in 1966 "for the discovery and development of optical methods for studying Hertzian resonances in atoms". He was president of the board of the Institut d'optique théorique et appliquée.



http://en.wikipedia.org/wiki/Alfred_Kastler

---

## 1. Introduction

The process of optical pumping is an excellent example for the interaction between light and matter. In our Advanced laboratory, one can use circularly polarized light to pump a particular level in Rb vapor. Then, using DC magnetic field up to 10 Oe and radio-frequency excitations, one manipulate the population of the pumped state. One will determine the energy separation between the adjacent Zeeman levels in Rb in a strong magnetic field as well as a weak magnetic field. Although the experiment is relatively simple to perform, one will need to understand a fair amount of atomic physics and quantum mechanics.

(i) Energy levels of $^{87}$Rb in the presence of spin-orbit interaction, hyperfine interaction, and magnetic field.
(ii) Selection rule for the absorption of the circular polarized ($\sigma^+$) photon of $D_1$ line, due to the interaction of light with electric dipole moment. The total angular momentum conservation of atom and light.
(iii) The increase in the population of the specific state using optical pumping.
(iv) The spacing of the Zeeman levels in the limits of weak and strong magnetic fields.
(v) Solving the eigenvalue problems.
(vi) Wigner-Eckart theorem and the Clebsch-Gordan coefficients

## 2. Overview on the optical pumping experiment

Optical pumping is a process in which absorption of light produces a population of the energy levels different from one in thermal equilibrium. In the present experiment, Rb atoms ($^{87}$Rb) in the presence of an external magnetic field are irradiated with circularly polarized photons in a narrow range of energies for the induction of 5 $^2S_{1/2} \rightarrow$ 5 $^2P_{1/2}$ (D$_1$

line) electric dipole transitions. The absorption can occur only if the total angular momentum of the incident photon and atom is conserved in the process. If the incident photon have angular momentum of $\hbar$ the only allowed transition are those in which $\Delta m_f = 1$. Thus every absorption produces an excited atom with one unit more of projected angular momentum just it had before the transition. On the other hand, the emission between the 5 $^2P_{1/2}$ and 5 $^2S_{1/2}$ level occur with only the restriction $\Delta m_f = -1, 0,$ and 1. The net result is a pumping of the atoms in the 5 S Zeeman levels toward the highest value of $m_f$.

### (a) Circularly polarized light

On the optical rail immediately after the lamp there is a Plano-convex lenses which serves to minimize spherical aberration and provide a more coherent incident beam. The focused beam then passes through an interference filter to isolate the 795 nm emission. The photons that are allowed past this filter then pass through a linear polarizer and then a quarter wave plate. The quarter wave plate is necessary to achieve a circularly polarized emission. For optical pumping to be achieved, an atom must absorb radiation resonant to that atom. Circularly polarized 795 nm light is a simple way to satisfy this criterion.

### (b) Rb chamber

At the heart of our optical rail lies our Rubidium chamber. In this chamber, we have Rb atoms and neon atoms. Neon is used specifically because it has no spin and as a noble gas it makes an excellent buffer. Due to the presence of this buffer gas, collisions between the rubidium atoms with each other or with the neon, occur frequently. If the buffer gas was removed, the rubidium atoms would frequently collide with the chamber walls and optical pumping would be unobtainable. Additionally, collisions between the neon and rubidium atoms increase the number of excited rubidium atoms.

### (c) DC magnetic field

Our Rb-Ne chamber is centered between three Helmholtz coils. In place is a horizontal field which acts as a static vertical magnetic field adjustment, and two vertical coils which provide horizontal magnetic fields. One of these vertical coils is used to provide a variable “sweeping field” while the other is used to provide a static horizontal field. Helmholtz coils are used to provide homogenous magnetic fields so it is not surprising that both vertical coils are wound among a common core although they are insulated from each other. Since there are three Helmholtz coils, using a common core for the two vertical coils reduces any inhomogeneity that may result from having to align another separate vertical coil. On opposite sides of the chamber, parallel to the optical path, there are two coils that provide an adjustable RF magnetic field. Situated inside the chamber is an adjustable oven which was usually set to 50°C.

### (d) Detector

The circularly polarized radiation then enters the rubidium chamber where it radiates in all directions. The photons which reach the inside of the chamber are absorbed by the atoms, while the remaining intensity is emitted through the end of the chamber opposite the incident beam. This is an important consideration because when optical pumping is achieved there will be maximum transmission and minimal absorption. When transitions between Zeeman levels occur, they are observable as the photon transmittance intensity decreases as absorption increases. Although the rubidium atoms emit photons as they decay to lower energy states, the photons are emitted in all directions so a negligible amount reaches the photo-detector.

If optical pumping is achieved and a weak magnetic field is present, excited atoms will decay to their ground state and spontaneously emit photons. At the end of the optical rail opposite the incident beam, the photons emitted from within the chamber are focused into a coherent beam by a second Plano-convex lens.

Before our beam passed through the interference filter, it is slightly pinkish in color. However, only 795nm radiation is allowed past the filter which means that our beam is no longer visible since 795nm is in the near-infrared region of the electromagnetic spectrum. The distance between the first Plano-convex lens and the rubidium lamp is adjusted until optimal coherence of the beam is achieved. This distance is measured and used to separate the detector from the second Plano-convex lens since visual adjustments are not possible as visible wavelengths are not present. Further precision can be achieved, however, by monitoring the detector for maximum output as the distance to the Plano-convex lens is adjusted. The heart of the detector is a simple photodiode which allows us to measures the relative magnitude of the radiation that is emitted by the excited atoms in the Rb chamber. Sensitivity settings are located on the detector’s housing.

Figure A

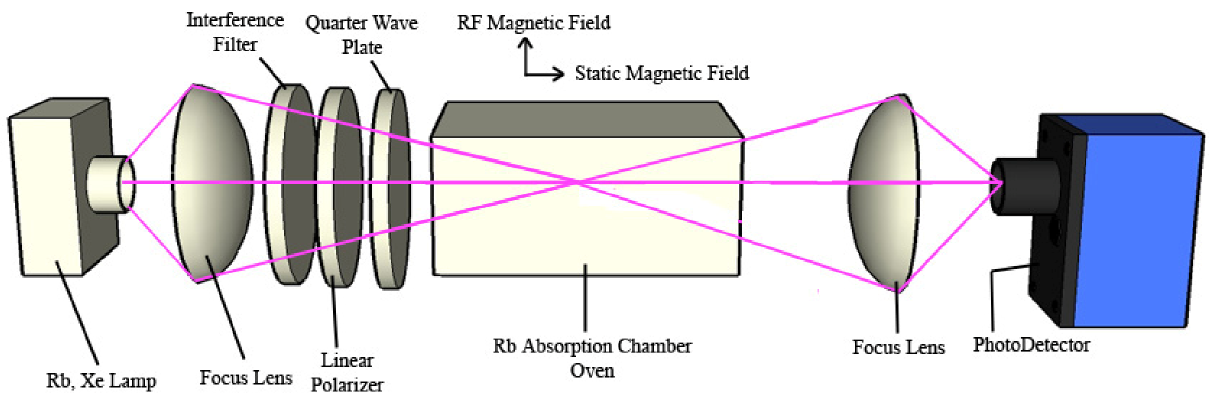


**Fig.1** Schematic diagram of the apparatus for a ribidium optical pumping experiment.

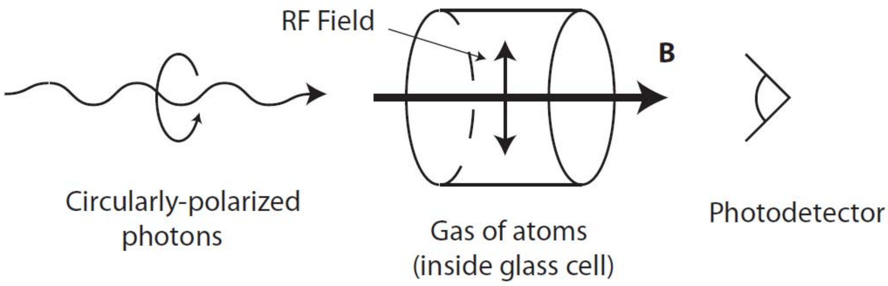



**Fig.2** The basic setup for the optical pumping. The direction of a magnetic field is the same as that of the propagation of the circularly-polarized light ($\sigma$+) with $D_1$ line. An AC magnetic field with radio frequency (rf), which is applied along a direction perpendicular to the propagation of the light, is applied to drive transition within Zeeman levels.

---

## 3. Level splitting due to the spin-orbit interaction

Alkali atoms possess an electron configuration that can be exploited to simplify them for angular momentum coupling and spectral analysis. Each of the alkali metals has an electronic configuration of a noble gas plus one valence electron. In the case of Rubidium, this is

$$1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^6 5s^1$$

$$\text{or } [\text{Kr}]5s^1.$$

$^{87}$Rb

(Z = 37, $N$ = 50)

Proton number = 37
Atomic mass = 85.4678
Nuclear spin $I$ = 3/2

The electron configuration of Rb is represented by

$$1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^6 5s^1$$

There is one electron outside the closed shell.
For $n$=5, we have $l_{max} = n - 1 = 5 - 1 = 4$

$l$ = 4 ($g$), 3($f$), 2($d$), 1($p$) and 0($s$).

(a) Spin-orbit coupling
$n = 5$

$l = 1$ and $s = 1/2$

$$D_1 \times D_{1/2} = D_{3/2} + D_{1/2},$$

leading to $j = 3/2$ and $j = 1/2$.

$D_{3/2}$ ($j = 3/2$, $m = 3/2, 1/2, -1/2, -3/2$)

or

$${}^{2s+1=2}P(l=1)_{j=3/2} = {}^2P_{3/2} \quad \rightarrow \quad 5^2P_{3/2}$$

$D_{1/2}$ ($j = 1/2$, $m = 1/2, -1/2$)

or

$${}^{2s+1=2}P(l=1)_{j=1/2} = {}^2P_{1/2} \quad \rightarrow \quad 5^2P_{1/2}$$

$l = 0$ and $s = 1/2$

$$D_0 \times D_{1/2} = D_{1/2},$$

leading to the state with $j = 1/2$.

$D_{1/2}$ ($j = 1/2$, $m = 1/2, -1/2$)

$${}^{2s+1=2}S(l=0)_{j=1/2} = {}^2S_{1/2}$$

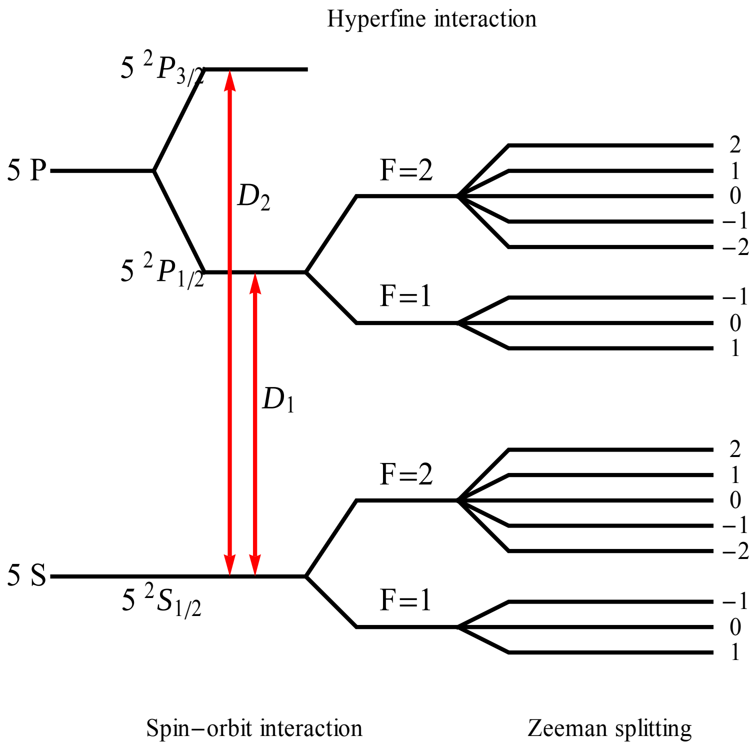



Fig.3 Level diagram of $^{87}$Rb with the nuclear spin $I$ = 3/2. The splittings are not to scale. $D_1$ line: ($5\ ^2S_{1/2} \rightarrow 5\ ^2P_{1/2}$). $\lambda$ = 794.978851156(23) nm. $D_2$ line: ($5\ ^2S_{1/2} \rightarrow 5\ ^2P_{3/2}$). $\lambda$ = 780.241209686(13) nm. $D_1$: 377.107463380 THz (794.978851156 nm).

## 4 Energy level splitting due to the hyperfine interaction

Nuclear spin 3/2 for $^{87}$Rb

(1) $j = 3/2$ ($5\ ^2P_{3/2}$) and $I$=3/2

$$D_{3/2} \text{ x } D_{3/2} = D_3 + D_2 + D_1 + D_0$$

leading to the magnetic substates

$F = 3$ ($m_f$ = -3, -2, -1, 0, 1, 2, 3), $F = 2$ ($m_f$ = -2, -1, 0, 1, 2)

$F = 1$ ($m_f$ = -1, 0, 1), ` $F = 0$ ($m_f$ = 0).

(2) $j = 1/2$. ($5\ ^2P_{1/2}$) and $I$=3/2

$$D_{3/2} \text{ x } D_{1/2} = D_2 + D_1$$

leading to the magnetic substates

$F = 2$ ($m_f = 2, 1, 0, -1, -2$), $F = 1$ ($m_f = 1, 0, -1$).

(3) $j = 1/2$ ($5\ ^2S_{1/2}$) and $I=3/2$.

$D_{3/2} \times D_{1/2} = D_2 + D_1$

leading to the magnetic substates

$F = 2$ ($m_f = 2, 1, 0, -1, -2$), $F = 1$ ($m_f = 1, 0, -1$).

The energy level diagrams of 87Rb is schematically shown below.

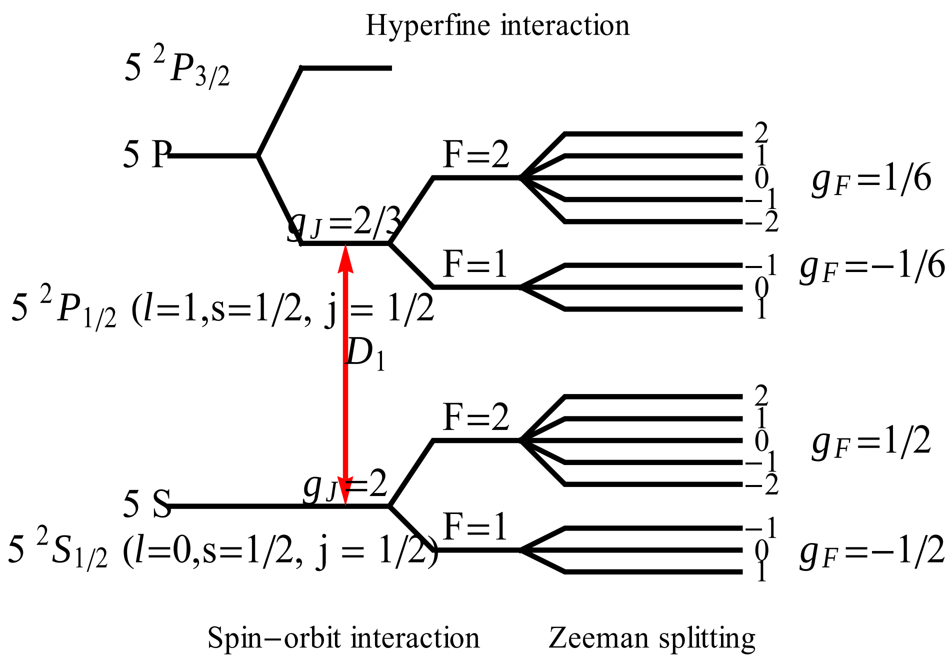



**Fig.4** Schematic energy level diagram of $^{87}$Rb in the presence of spin-orbit interaction, hyperfine interaction, and magnetic fied (Zeeman splitting). Note that the Landé $g$-factor for $F = 1$ is negative, implying that the highest energy level is $m_f = -1$ for $F = 1$. $D_1$ line ($\lambda = 794.978851156$ nm).

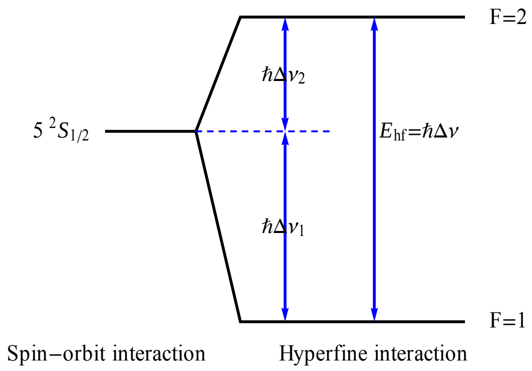



**Fig.5** Splitting of energy level $5\ ^2S_{1/2}$ due to the hyperfine interaction. $\Delta\nu_1$ = 4.271677 GHz, $\Delta\nu_2$ = 2.563006 GHz, and $\Delta\nu$ = 6.834682610904324 GHz.

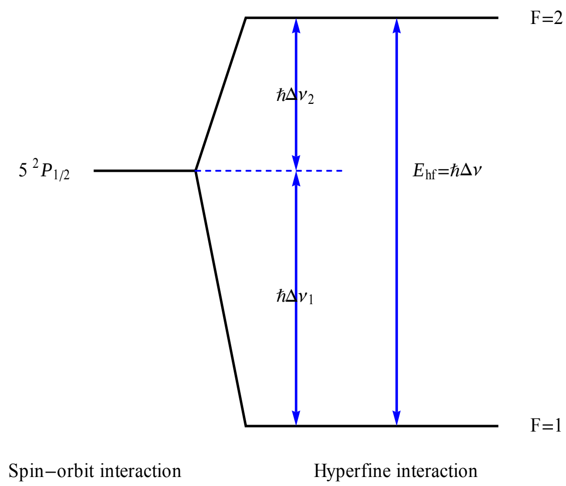



**Fig.6** Splitting of energy level $5\ ^2P_{1/2}$ due to the hyperfine interaction. $\Delta\nu_1$ = 0.50905 GHz, $\Delta\nu_2$ = 0.30543 GHz, and $\Delta\nu$ = 0.8145 GHz.

## 5. Absorption and emission of light due to the interaction with electric-dipole moment

We now consider the transition probability for the absorption and emission of the light due to the interaction with electric dipole moments. As will be shown in the Appendix (classical and quantum theory of radiation), the absorption cross section is obtained as

$$\sigma_{abs} = \frac{\hbar\omega \frac{2\pi}{\hbar} \frac{e^2}{m^2c^2} |\mathbf{A}_0|^2 \left| \langle \varphi_f | e^{i\mathbf{k}\cdot\mathbf{r}} \boldsymbol{\varepsilon} \cdot \hat{\mathbf{p}} | \varphi_i \rangle \right|^2 \delta(E_f - E_i - \hbar\omega)}{\frac{1}{2\pi} \frac{\omega^2}{c} |\mathbf{A}_0|^2}$$

$$= \frac{4\pi^2\hbar}{m^2\omega}\left(\frac{e^2}{\hbar c}\right) \left| \langle \varphi_f | e^{i\mathbf{k}\cdot\mathbf{r}} \boldsymbol{\varepsilon} \cdot \hat{\mathbf{p}} | \varphi_i \rangle \right|^2 \delta(E_f - E_i - \hbar\omega)$$

$$= \frac{4\pi^2}{m^2\omega} \frac{e^2}{\hbar c} \left| \langle \varphi_f | e^{i\mathbf{k}\cdot\mathbf{r}} \boldsymbol{\varepsilon} \cdot \hat{\mathbf{p}} | \varphi_i \rangle \right|^2 \delta(\omega_{fi} - \omega)$$

where $E_i$ and $E_f$ are the enegy levels of the system in the initial state and the final state, respectively. $\boldsymbol{\varepsilon}$ is the polarization vector, and $\hat{\mathbf{p}}$ is the quantum mechanical momentum. Here we assume that

$$e^{i\mathbf{k}\cdot\mathbf{r}} = e^{i\left(\frac{\omega}{c}\right)\mathbf{n}\cdot\mathbf{r}} = 1 + i\frac{\omega}{c}\mathbf{n}\cdot\mathbf{r} + \cdots \cong 1$$

and

$$\frac{\omega}{c} = k = \frac{2\pi}{\lambda}, \ \frac{\omega}{c}\mathbf{n}\cdot\mathbf{r} \cong \frac{2\pi}{\lambda}(\mathbf{n}\cdot\mathbf{r}) = \frac{\mathbf{n}\cdot\mathbf{r}}{\lambdabar} \cong \frac{r_{\text{atom}}}{\lambdabar} \ll 1.$$

where $\lambdabar = \lambda/(2\pi)$. This approximation is valid for $\lambda \gg r_{\text{atom}}$ (atomic dimension). Then we have

$$\sigma_{abs} = \frac{4\pi^2}{m^2\omega}\left(\frac{e^2}{\hbar c}\right) \left| \boldsymbol{\varepsilon} \cdot \langle \varphi_f | \hat{\mathbf{p}} | \phi_i \rangle \right|^2 \delta(\omega_{fi} - \hbar\omega)$$

where $\omega_{fi} = \dfrac{E_f - E_i}{\hbar}$

For simplicity we assume that

$$\boldsymbol{\varepsilon} = \mathbf{e}_x \qquad (\mathbf{n} = \mathbf{e}_z \text{; the propagating direction).}$$

We need to calculate

$$\langle \varphi_f | \hat{p}_x | \phi_i \rangle .$$

Since

$$[\hat{x}, \hat{H}_0] = \frac{i\hbar}{m}\hat{p}_x$$

and

$$\hat{H}_0|\varphi_i\rangle = E_i|\varphi_i\rangle, \qquad \hat{H}_0|\varphi_f\rangle = E_f|\varphi_f\rangle$$

we have

$$\langle\varphi_f|\hat{p}_x|\phi_i\rangle = \frac{m}{i\hbar}\langle\varphi_f|[\hat{x}, \hat{H}_0]|\phi_i\rangle == im\omega_{fi}\langle\varphi_f|\hat{x}|\phi_i\rangle$$

With the electric dipole approximation, we have

$$\begin{aligned}\sigma_{abs} &= \frac{4\pi^2}{m^2\omega}\alpha m^2\omega_{fi}^2\left|\langle\varphi_f|\hat{x}|\phi_i\rangle\right|^2\delta(\omega_{fi}-\omega)\\ &= 4\pi^2\alpha\omega_{fi}\left|\langle\varphi_f|\hat{x}|\phi_i\rangle\right|^2\delta(\omega_{fi}-\omega)\end{aligned}$$

In atomic physics, we define oscillator strength $f_{fi}$

$$f_{fi} \equiv \frac{2m\omega_{fi}}{\hbar}\left|\langle\varphi_f|\hat{x}|\phi_i\rangle\right|^2$$

Thomas-Reiche-Kuhn sum rule

$$\sum_f f_{fi} = 1.$$

Using the rule

$$\begin{aligned}\int\sigma_{abs}(\omega)d\omega &= 4\pi^2\alpha\sum_f\omega_{fi}\left|\langle\varphi_f|\hat{x}|\phi_i\rangle\right|^2\\ &== 4\pi^2\alpha\frac{\hbar}{2m} = 2\pi^2 c\left(\frac{e^2}{mc^2}\right)\end{aligned}$$

The Einstein's A and B coefficient:

$$
\begin{aligned}
W_{i\to f} &= \frac{4\pi^2 e^2}{\hbar^2 m^2 \omega_0^2}\overline{W}(\omega_0)\left|\left\langle \varphi_f \left| e^{i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\varepsilon}\cdot\hat{\mathbf{p}}\right|\varphi_i\right\rangle\right|^2 \\
&= \frac{4\pi^2 e^2}{\hbar^2 m^2 \omega_0^2}\overline{W}(\omega_0) m^2\omega_0^2\left|\left\langle \varphi_f \left|\hat{\mathbf{r}}_\varepsilon\right|\varphi_i\right\rangle\right|^2 \\
&= \frac{4\pi^2 e^2}{\hbar^2}\overline{W}(\omega_0)\left|\left\langle \varphi_f \left|\hat{\mathbf{r}}_\varepsilon\right|\varphi_i\right\rangle\right|^2 \\
&= B_{12}\overline{W}(\omega_0)
\end{aligned}
$$

where $\frac{\omega^2}{2\pi c^2}\left|\mathbf{A}_0\right|^2 = \overline{W}(\omega)d\omega = \frac{1}{c}I(\omega)d\omega$. The coefficients B12 and B21 are derived as

$$
\begin{aligned}
B_{12} = B_{21} &= \frac{4\pi^2 e^2}{\hbar^2}\left|\left\langle \varphi_f \left|\hat{\mathbf{r}}_\varepsilon\right|\varphi_i\right\rangle\right|^2 \\
&\cong \frac{4\pi^2 e^2}{3\hbar^2}\left|\left\langle \varphi_f \left|\hat{\mathbf{r}}\right|\varphi_i\right\rangle\right|^2
\end{aligned}
$$

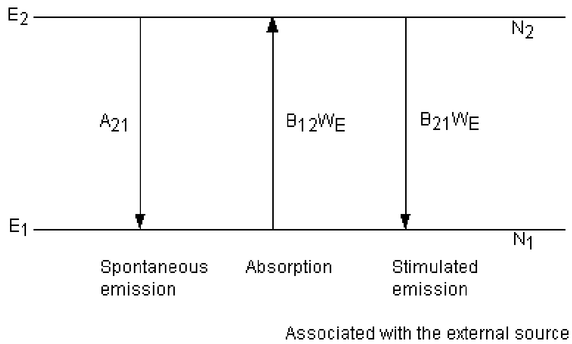



**Fig.7** The Einstein's A and B coefficients for the absorption, stimulated emission, and spontaneous emission.

## 6. Selection rule for the absorption and emission

We now calculate the matrix element

$$
I = \left|\left\langle \varphi_f \left|\hat{\mathbf{r}}\right|\phi_i\right\rangle\right|^2
$$

where

$$
\left\langle \varphi_f \left|\hat{\mathbf{r}}\right|\phi_i\right\rangle = \left\langle \varphi_f \left|\hat{x}\right|\phi_i\right\rangle\mathbf{e}_x + \left\langle \varphi_f \left|\hat{y}\right|\phi_i\right\rangle\mathbf{e}_y + \left\langle \varphi_f \left|\hat{z}\right|\phi_i\right\rangle\mathbf{e}_z
$$

is a vector. Then we have

$$I = \left|\left\langle \varphi_f \left|\hat{\mathbf{r}}\right|\phi_i \right\rangle\right|^2 = \left|\left\langle \varphi_f \left|\hat{x}\right|\phi_i \right\rangle\right|^2 + \left|\left\langle \varphi_f \left|\hat{y}\right|\phi_i \right\rangle\right|^2 + \left|\left\langle \varphi_f \left|\hat{z}\right|\phi_i \right\rangle\right|^2$$

Here we have

$$\left|\left\langle \varphi_f \left|\hat{x}+i\hat{y}\right|\phi_i \right\rangle\right|^2 = \left\langle \varphi_f \left|\hat{x}+i\hat{y}\right|\phi_i \right\rangle^* \left\langle \varphi_f \left|\hat{x}+i\hat{y}\right|\phi_i \right\rangle = \left\langle \varphi_i \left|\hat{x}-i\hat{y}\right|\phi_f \right\rangle \left\langle \varphi_f \left|\hat{x}+i\hat{y}\right|\phi_i \right\rangle$$

or

$$\left|\left\langle \varphi_f \left|\hat{x}+i\hat{y}\right|\phi_i \right\rangle\right|^2 = \left|\left\langle \varphi_i \left|\hat{x}\right|\phi_f \right\rangle\right|^2 + \left|\left\langle \varphi_f \left|\hat{y}\right|\phi_i \right\rangle\right|^2$$

Similarly we have

$$\left|\left\langle \varphi_f \left|\hat{x}-i\hat{y}\right|\phi_i \right\rangle\right|^2 = \left|\left\langle \varphi_i \left|\hat{x}\right|\phi_f \right\rangle\right|^2 + \left|\left\langle \varphi_f \left|\hat{y}\right|\phi_i \right\rangle\right|^2$$

Then we have

$$I = \left|\left\langle \varphi_f \left|\hat{\mathbf{r}}\right|\phi_i \right\rangle\right|^2 = \frac{1}{2}\left|\left\langle \varphi_f \left|\hat{x}+i\hat{y}\right|\phi_i \right\rangle\right|^2 + \frac{1}{2}\left|\left\langle \varphi_f \left|\hat{x}-i\hat{y}\right|\phi_i \right\rangle\right|^2 + \left|\left\langle \varphi_f \left|\hat{z}\right|\phi_i \right\rangle\right|^2$$

Spherical tensor of rank 1 is defined as

$$T_1^{(1)} = -\frac{\hat{x}+i\hat{y}}{\sqrt{2}}$$

$$T_0^{(1)} = \hat{z}$$

$$T_{-1}^{(1)} = \frac{\hat{x}-i\hat{y}}{\sqrt{2}}$$

From Wigner-Eckart theorem

$$\left\langle F', m_F' \left|\hat{T}_q^{(1)}\right| F, m_F \right\rangle \neq 0$$

for $m_F' = m_F + q$ and for $F' = F+1, F, F-1$, where $q$ = -1, 0, 1.

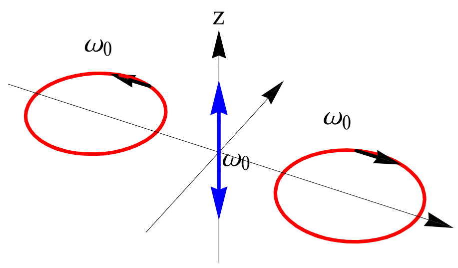



**Fig.8** Linearly polalized light and circularly polarized lights.

The linearly polarized wave ($\pi$-polarization) is expressed by

$$\begin{aligned}\mathbf{E}_\pi(t) &= E_0\mathbf{e}_z \cos\omega t \\ &= E_0\mathbf{e}_z[\frac{e^{i\omega t}+e^{-i\omega t}}{2}]\end{aligned}$$

The circularly polarized (counter clockwise, $\sigma^+$-polarization) in the *x*-*y* plane, is expressed by

$$\begin{aligned}\mathbf{E}_{\sigma^+}(t) &= \frac{E_0}{\sqrt{2}}(\mathbf{e}_x \cos\omega t + \mathbf{e}_y \sin\omega t) \\ &= \frac{E_0}{2\sqrt{2}}[\mathbf{e}_x\left(\frac{e^{i\omega t}+e^{-i\omega t}}{2}\right)+\mathbf{e}_y\left(\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\right)] \\ &= \frac{E_0}{2\sqrt{2}}(\mathbf{e}_x - i\mathbf{e}_y)e^{i\omega t} + \frac{E_0}{2\sqrt{2}}(\mathbf{e}_x + i\mathbf{e}_y)e^{-i\omega t}]\end{aligned}$$

We use the expression of $\mathbf{E}_{\sigma^+}(t)$ as

$$\mathbf{E}_{\sigma^+}(t) = \frac{E_0}{2\sqrt{2}}(\mathbf{e}_x + i\mathbf{e}_y)e^{-i\omega t}$$

The circularly polarized (clockwise, $\sigma^-$ polarization) in the *x*-*y* plane

$$\mathbf{E}_{\sigma^-}(t) = \frac{E_0}{\sqrt{2}}(\mathbf{e}_x \cos\omega t - \mathbf{e}_y \sin\omega t)$$
$$= \frac{E_0}{\sqrt{2}}[\mathbf{e}_x\left(\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right) - \mathbf{e}_y\left(\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right)]$$
$$= \frac{E_0}{2\sqrt{2}}(\mathbf{e}_x + i\mathbf{e}_y)e^{i\omega t} + \frac{E_0}{2\sqrt{2}}(\mathbf{e}_x - i\mathbf{e}_y)e^{-i\omega t}$$

We use

$$\mathbf{E}_{\sigma^-}(t) = \frac{E_0}{2\sqrt{2}}(\mathbf{e}_x - i\mathbf{e}_y)e^{-i\omega t}$$

Then the interaction between the electric field and atom is

$$H_{\text{int}} = -e(\hat{\mathbf{r}}\cdot\mathbf{E}_\pi + \hat{\mathbf{r}}\cdot\mathbf{E}_{\sigma^+} + \hat{\mathbf{r}}\cdot\mathbf{E}_{\sigma^-})$$
$$= (-e)\frac{E_0}{2}[\frac{(\hat{x}+i\hat{y})}{\sqrt{2}}e^{-i\omega t} + \frac{(\hat{x}-i\hat{y})}{\sqrt{2}} + \hat{z}]e^{-i\omega t}$$

The matrix elements are given by

$$\langle\phi_f|H_{\text{int}}|\phi_i\rangle = (-e)\frac{E_0}{2}[\langle\phi_f|\frac{(\hat{x}+i\hat{y})}{\sqrt{2}}|\phi_i\rangle + \langle\phi_f|\frac{(\hat{x}-i\hat{y})}{\sqrt{2}}|\phi_i\rangle + \langle\phi_f|\hat{z}|\phi_i\rangle],$$

corresponding to the contributions from the $\sigma$+ polarization, $\sigma$-polarization, and $\pi$-polarization, respectively.

## 7 Wigner-Eckart theorem

The electric dipole moment of the hyperfine structure between the initial state $|F_i, m_i\rangle$ and the final state $|F_f, m_f\rangle$ is represented by

$$e\langle F_f, m_f|T_q^{(k)}|F_i, m_i\rangle,$$

where the polarization state of the light is expressed by $|k=1, q\rangle$, where the angular momentum is $\hbar$ and $q$ = -1, 0, and 1. According to the Wigner-Eckart theorem,

$$\langle\alpha', F_f, m_f|T_q^{(k)}|\alpha, F_i, m_i\rangle,$$

is proportional to the Clebsch-Gordan coefficient.

$$\left\langle \alpha', F_f, m_f \left| \hat{T}_q^{(k)} \right| \alpha, F_i, m_i \right\rangle = \left\langle F_i, k; m_i, q \right| F_i, k; F_f, m_f \right\rangle \times$$ term independent of *m*', *m*, and *q*.

or

$$\left\langle \alpha', F, m_f \left| \hat{T}_q^{(k)} \right| \alpha, F_i, m_i \right\rangle = \left\langle F_i, k; m_i, q \right| F_i, k; F, m_f \right\rangle \frac{\left\langle \alpha', F \left\| \hat{T}^{(k)} \right\| \alpha, F_i \right\rangle}{\sqrt{2F_i + 1}}$$

where a and a, represent any additional quantum numbers required to specify the state. $\left\langle \alpha', F \left\| \hat{T}^{(k)} \right\| \alpha, F_i \right\rangle$ is called the reduced matrix element which is independent of $q$, $m_f$, and $m_i$. The numerical value of the Clebsch-Gordan coefficient is zero unless the triangular condition

$$F = F_i + k, F_i + k\text{-}1, ..., |F_i\text{-}k|,$$

is satisfied. Note that

$$D_{F_i} \otimes D_k = D_{F_i+k} + D_{F_i+k-1} + .... + D_{|F_i-k|}\,.$$

The Clebsch-Gordan coefficient is also zero unless the condition

$$m_f = m_i + q\,,$$

is satisfied. These two conditions ensure essentially the conservation of the total angular momentum of the system and of the component of this angular momentum on the axis of quantization. The transition probability is proportional to the square of the magnitude of the matrix element of the electric dipole moment, or is proportional to the absolute value of the Clebsch-Gordan coefficients.

## 8 Intrinsic angular momentum of photon ($\sigma$+-polarization)

The emission of radiation 5 $^2P_{1/2} \rightarrow$ 5 $^2S_{1/2}$ the angular momentum of the atom decreases by one unit. The principle of conservation of angular momentum therefore requires that the emitted photon shall have an intrinsic angular momentum of the angular momentum The emitted photon shall have an intrinsic angular momentum of $\hbar$ . Similarly in the decay of the

Whether *m* changes by +1, -1, or 0 depends on the nature of the photon. If the applied magnetic field is parallel to the direction of propagation of the photon, then a right-circularly polarized photon will always induce transition that have $\Delta m$ = 1. Left-circularly-polarized light produces $\Delta m$ = -1. The same thing is true for emission. An electron can fall from the 5 $^2P_{1/2}$ level to $^2S_{1/2}$ level and emit a photon with right or left circular polarization, depending on whether $\Delta m$ is +1 or -1.

The electric-dipole selection for circularly polarized light require either

$m_f = m_i + 1, \quad \text{or } m_f = m_i - 1.$

where $m_f$ and mi are the final and initial angular momentum-projection numbers along the direction of propagation of the light. The Hamiltonian for the circularly-polarized light as a spherical tensor operator. We use the Wigner-Eckart theorem to decide which transition probabilities are zero and which are not. Whether $m_f$ changes by +1, -1, or 0 depends on the nature of the photon. If the applied magnetic field is parallel to the direction of propagation of the photon, then a right-circularly polarized photon will always induce transition that have $\Delta m_f = 1$.

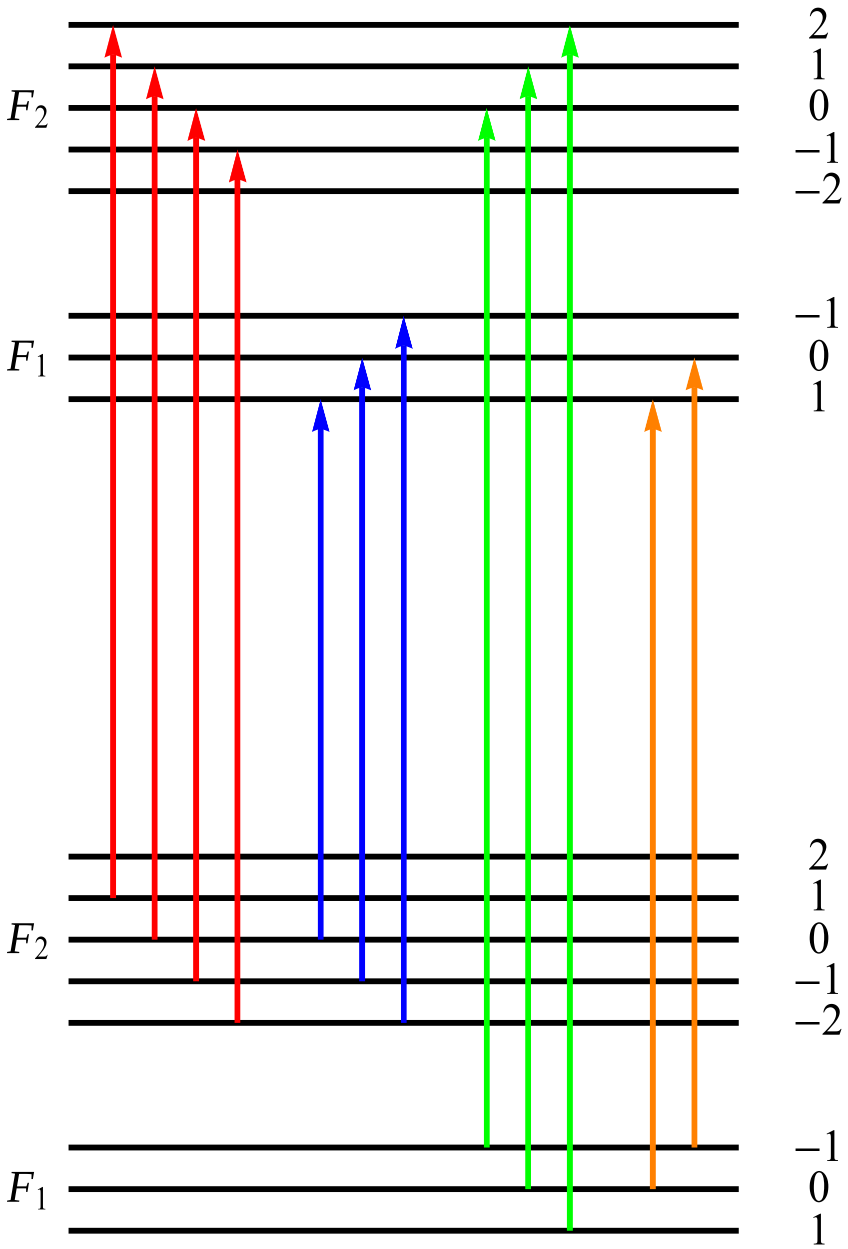



**Fig.9** Transition between the energy levels of $^{87}$Rb resulting from the effects of spin-orbit interaction, hyperfine interaction, and Zeeman effect (in the presence of magnetic field), caused by the circularly $\sigma^+$-polarization ($D_1$ lines). The transition processes are illustrated schematically in the figure, which depicts the histories of several atoms which are initially in various magnetic substates of a lower electronic state. Under irradiation by circulary polarized light, they make upward transitions to magnetic substates of an upper electronic state subject to the restriction $\Delta m_f = 1$.

## 9 $F_i = 1$ and $k = 1$ ($\sigma$+-polarization)

$D_1 \otimes D_1 = D_2 + D_1 + D_0$

**(i)** ***F* = 2** **($F_i$ = 1)**

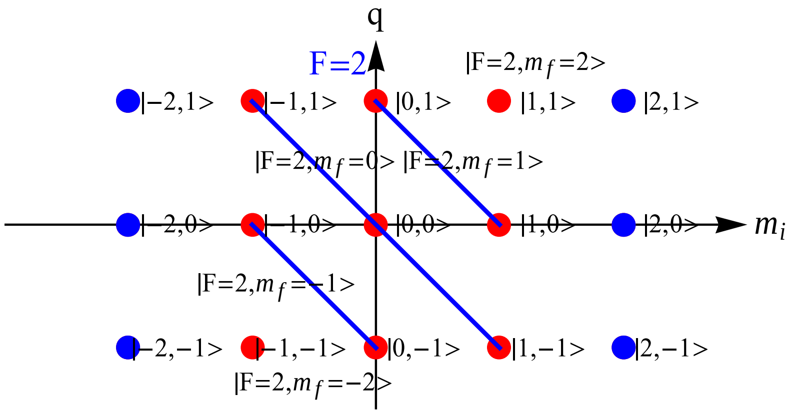



**Fig.10** $F = 2$, $m_f = -2, -1, 0, 1$ and $2$. $F_i = 1$ ($m_i = -1, 0, 1$)

$$\left|F=2, m_f=2\right\rangle = \left|F_i=1, m_i=1\right\rangle \left|k=1, q=1\right\rangle,$$

$$\left|F=2, m_f=1\right\rangle = \frac{\left|F_i=1, m_i=1\right\rangle \left|k=1, q=0\right\rangle}{\sqrt{2}} + \frac{\left|F_i=1, m_i=0\right\rangle \left|k=1, q=1\right\rangle}{\sqrt{2}},$$

$$\left|F=2, m_f=0\right\rangle = \frac{\left|F_i=1, m_i=1\right\rangle \left|k=1, q=-1\right\rangle}{\sqrt{6}} + \sqrt{\frac{2}{3}}\left|F_i=1, m_i=0\right\rangle \left|k=1, q=0\right\rangle + \frac{\left|F_i=1, m_i=-1\right\rangle \left|k=1, q=1\right\rangle}{\sqrt{6}}$$

$$\left|F=2, m_f=-1\right\rangle = \frac{\left|F_i=1, m_i=0\right\rangle \left|k=1, q=-1\right\rangle}{\sqrt{2}} + \frac{\left|F_i=1, m_i=-1\right\rangle \left|k=1, q=0\right\rangle}{\sqrt{2}}$$

$$\left|F=2, m_f=-2\right\rangle = \left|F_i=1, m_i=-1\right\rangle \left|k=1, q=-1\right\rangle$$

---

**(ii)** ***F* = 1** **($F_i$ = 1)**

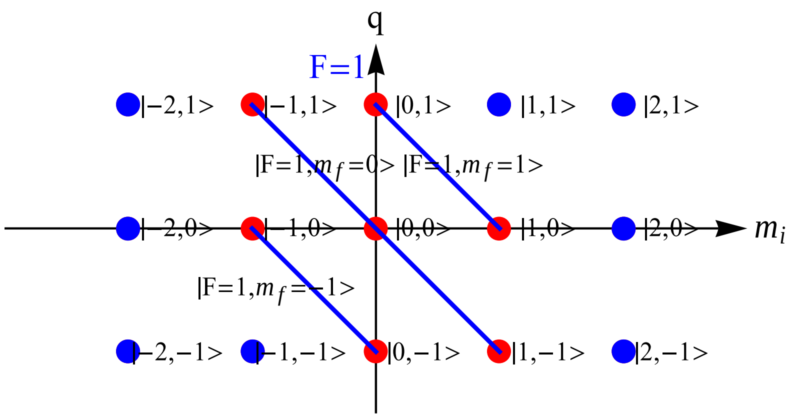



**Fig.11** $F = 1$, $m_f = -1$, 0, and 1. $F_i = 1$ ($m_i = -1$, 0, and 2)

$$\left|F=1, m_f=1\right\rangle = \frac{\left|F_i=1, m_i=1\right\rangle \left|k=1, q=0\right\rangle}{\sqrt{2}} - \frac{\left|F_i=1, m_i=0\right\rangle \left|k=1, q=1\right\rangle}{\sqrt{2}}$$

$$\left|F=1, m_f=0\right\rangle = \frac{\left|F_i=1, m_i=1\right\rangle \left|k=1, q=-1\right\rangle}{\sqrt{2}} - \frac{\left|F_i=1, m_i=-1\right\rangle \left|k=1, q=1\right\rangle}{\sqrt{2}}$$

$$\left|F=1, m_f=-1\right\rangle = \frac{\left|F_i=1, m_i=0\right\rangle \left|k=1, q=-1\right\rangle}{\sqrt{2}} - \frac{\left|F_i=1, m_i=-1\right\rangle \left|k=1, q=0\right\rangle}{\sqrt{2}}$$

---

**(iii)** ***F* = 0** **(*F*<sub></sub>)**

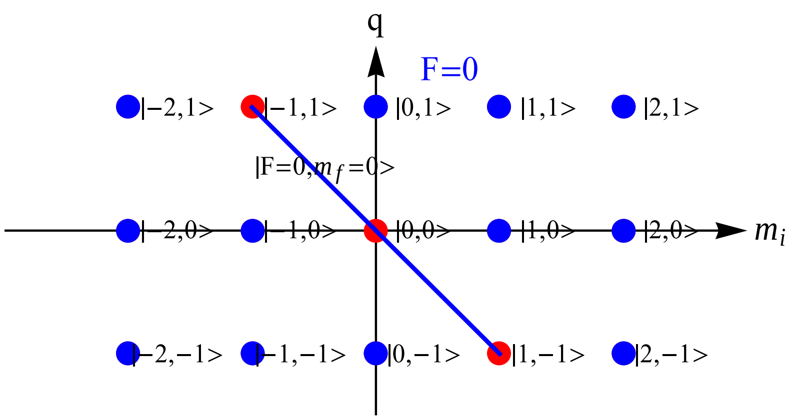



**Fig.12** $F = 0$, $m_f = 0$. $F_i = 1$ ($m_i = -1$, 0, 1)

$$\left|F=0,m_f=0\right\rangle = \frac{\left|F_i=1,m_i=1\right\rangle\left|k=1,q=-1\right\rangle}{\sqrt{3}} - \frac{\left|F_i=1,m_i=0\right\rangle\left|k=1,q=0\right\rangle}{\sqrt{3}} + \frac{\left|F_i=1,m_i=-1\right\rangle\left|k=1,q=1\right\rangle}{\sqrt{3}}$$

## 10. $F_i = 2$ and $k = 1$ ($\sigma$+-polarization)

$D_2 \otimes D_1 = D_3 + D_2 + D_1$

### (i) $F = 3$ ($F_i = 2$)

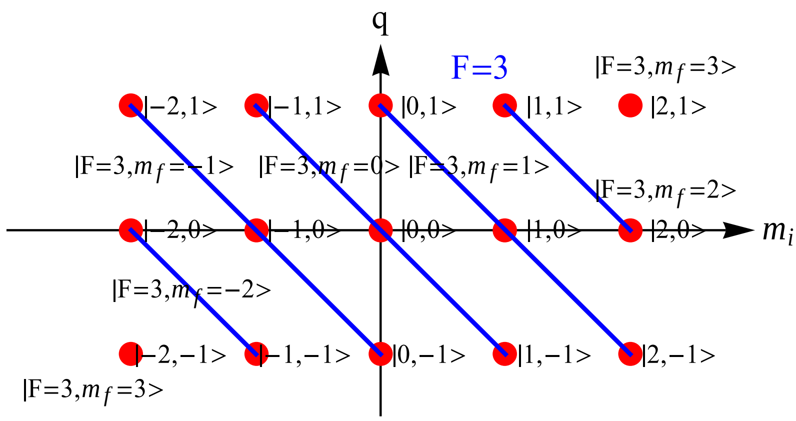



**Fig.13** $F$ = 3, $m_f$ = -3, -2, -1, 0, 1, 2, and 3. $F_i$ = 2 ($m_i$ = -2, -1, 0, 1, 2)

$$\left|F=3,m_f=3\right\rangle = \left|F_i=2,m_i=2\right\rangle\left|k=1,q=1\right\rangle,$$

$$\left|F=3,m_f=2\right\rangle = \frac{\left|F_i=2,m_i=2\right\rangle\left|k=1,q=0\right\rangle}{\sqrt{3}} + \sqrt{\frac{2}{3}}\,\frac{\left|F_i=2,m_i=1\right\rangle\left|k=1,q=1\right\rangle}{\sqrt{2}}$$

$$\left|F=3,m_f=1\right\rangle = \frac{\left|F_i=2,m_i=2\right\rangle\left|k=1,q=-1\right\rangle}{\sqrt{15}} + 2\sqrt{\frac{2}{15}}\left|F_i=2,m_i=1\right\rangle\left|k=1,q=0\right\rangle + \sqrt{\frac{2}{5}}\left|F_i=2,m_i=0\right\rangle\left|k=1,q=1\right\rangle$$

$$\left|F=3,m_f=0\right\rangle = \frac{\left|F_i=2,m_i=1\right\rangle\left|k=1,q=-1\right\rangle}{\sqrt{5}} + \sqrt{\frac{3}{5}}\left|F_i=2,m_i=0\right\rangle\left|k=1,q=0\right\rangle + \frac{\left|F_i=2,m_i=0\right\rangle\left|k=1,q=1\right\rangle}{\sqrt{5}}$$

$$\left|F=3,m_f=-1\right\rangle=\sqrt{\frac{2}{5}}\left|F_i=2,m_i=0\right\rangle\left|k=1,q=-1\right\rangle+2\sqrt{\frac{2}{15}}\left|F_i=2,m_i=-1\right\rangle\left|k=1,q=0\right\rangle+\frac{\left|F_i=2,m_i=-2\right\rangle\left|k=1,q=1\right\rangle}{\sqrt{15}}$$

$$\left|F=3,m_f=-2\right\rangle=\sqrt{\frac{2}{3}}\left|F_i=2,m_i=-1\right\rangle\left|k=1,q=-1\right\rangle+\frac{\left|F_i=2,m_i=-2\right\rangle\left|k=1,q=0\right\rangle}{\sqrt{3}}$$

$$\left|F=3,m_f=-3\right\rangle=\left|F_i=2,m_i=-2\right\rangle\left|k=1,q=-1\right\rangle$$

---

**(ii)** ***F* = 2 (*F*$_i$ = 2)**

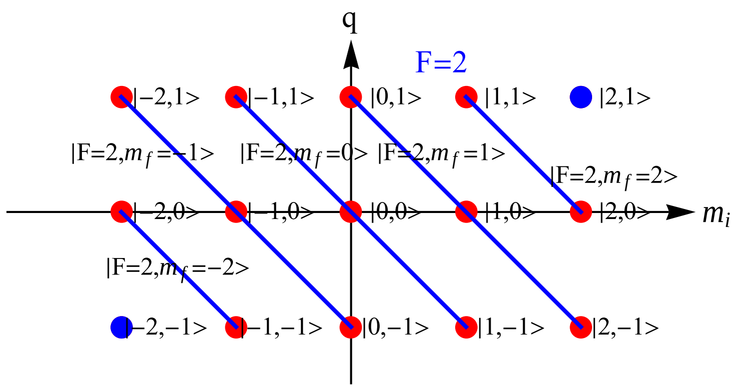



**Fig.14** $F = 2$, $m_f$ = -2, -1, 0, 1, and 2. $F_i = 2$ ($m_i$ = -2, -1, 0, 1, 2)

$$\left|F=2,m_f=2\right\rangle=\sqrt{\frac{2}{3}}\left|F_i=2,m_i=2\right\rangle\left|k=1,q=0\right\rangle-\frac{\left|F_i=2,m_i=1\right\rangle\left|k=1,q=1\right\rangle}{\sqrt{3}},$$

$$\left|F=2,m_f=1\right\rangle=\frac{\left|F_i=2,m_i=2\right\rangle\left|k=1,q=-1\right\rangle}{\sqrt{3}}+\frac{\left|F_i=2,m_i=1\right\rangle\left|k=1,q=0\right\rangle}{\sqrt{6}}-\frac{\left|F_i=2,m_i=0\right\rangle\left|k=1,q=1\right\rangle}{\sqrt{2}},$$

$$\left|F=2,m_f=0\right\rangle=\frac{\left|F_i=2,m_i=1\right\rangle\left|k=1,q=-1\right\rangle}{\sqrt{2}}-\frac{\left|F_i=2,m_i=-1\right\rangle\left|k=1,q=1\right\rangle}{\sqrt{2}}$$

$$\left|F=2,m_f=-1\right\rangle=\frac{\left|F_i=2,m_i=0\right\rangle\left|k=1,q=-1\right\rangle}{\sqrt{2}}-\frac{\left|F_i=2,m_i=-1\right\rangle\left|k=1,q=0\right\rangle}{\sqrt{6}}-\frac{\left|F_i=2,m_i=-2\right\rangle\left|k=1,q=1\right\rangle}{\sqrt{3}}$$

$$\left|F=2,m_f=-2\right\rangle=\frac{\left|F_i=2,m_i=-1\right\rangle\left|k=1,q=-1\right\rangle}{\sqrt{3}}-\sqrt{\frac{2}{3}}\frac{\left|F_i=2,m_i=-2\right\rangle\left|k=1,q=0\right\rangle}{\sqrt{6}}$$

**(vi) $F = 1$ ($F_i = 2$)**

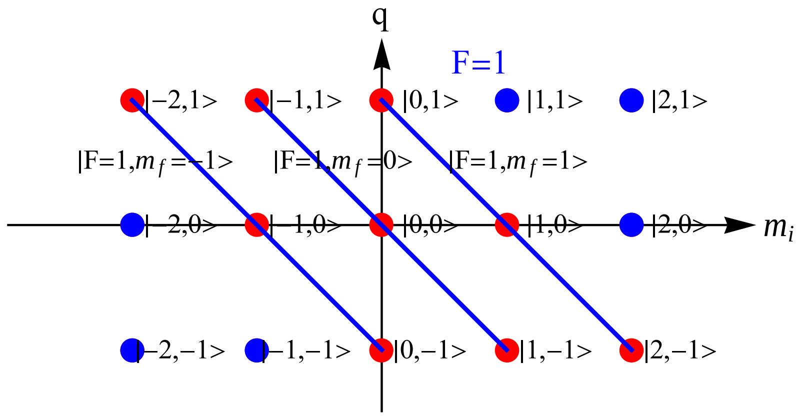



**Fig.15** $F = 1$, $m_f = -1, 0$, and $1$. $F_i = 2$ ($m_i = -2, -1, 0, 1, 2$)

$$\left|F=1,m_f=1\right\rangle=\sqrt{\frac{3}{5}}\left|F_i=2,m_i=2\right\rangle\left|k=1,q=-1\right\rangle-\sqrt{\frac{3}{10}}\left|F_i=2,m_i=1\right\rangle\left|k=1,q=0\right\rangle+\frac{\left|F_i=2,m_i=0\right\rangle\left|k=1,q=1\right\rangle}{\sqrt{10}}$$

$$\left|F=1,m_f=0\right\rangle=\sqrt{\frac{3}{10}}\left|F_i=2,m_i=1\right\rangle\left|k=1,q=-1\right\rangle-\sqrt{\frac{2}{5}}\left|F_i=2,m_i=0\right\rangle\left|k=1,q=0\right\rangle+\sqrt{\frac{3}{10}}\left|F_i=2,m_i=-1\right\rangle\left|k=1,q=1\right\rangle$$

$$\left|F=1,m_f=-1\right\rangle=\frac{\left|F_i=2,m_i=0\right\rangle\left|k=1,q=-1\right\rangle}{\sqrt{10}}-\sqrt{\frac{3}{10}}\left|F_i=2,m_i=-1\right\rangle\left|k=1,q=0\right\rangle+\sqrt{\frac{3}{5}}\left|F_i=2,m_i=-2\right\rangle\left|k=1,q=1\right\rangle$$

## 11. Clebsch-Gordan coefficients

The Clebsh-Gordan coefficients are shown in the following figures. The transition prbability for each process is proportional to the square of the Clebsch-Gordan coefficient.

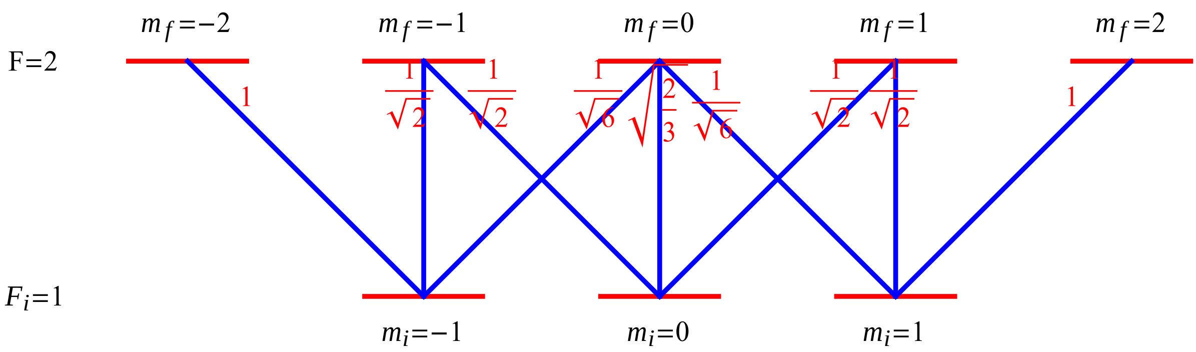



**Fig.16** Clebsch-Gordon coefficients for the transition between $F$ = 2 and $F_i$ = 1. The square of the Clebsch-Gordan coefficient corresponds to the probability.

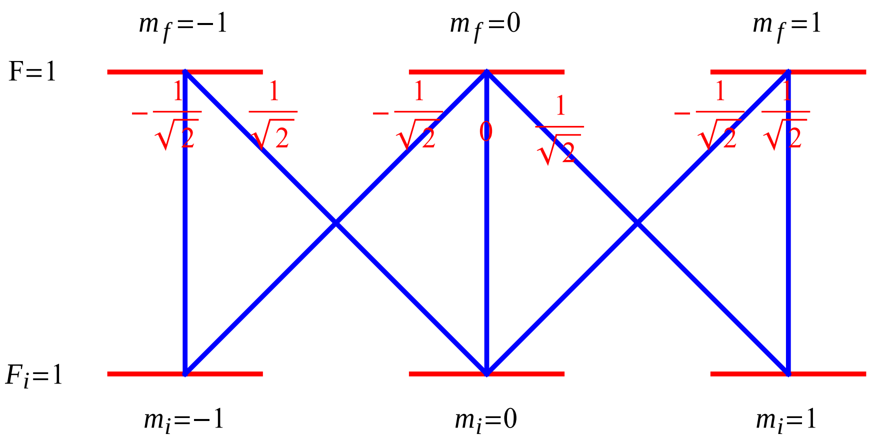



**Fig.17** Clebsch-Gordon coefficients for the transition between $F$ = 1 and $F_i$ = 1. The square of the Clebsch-Gordan coefficient corresponds to the probability.

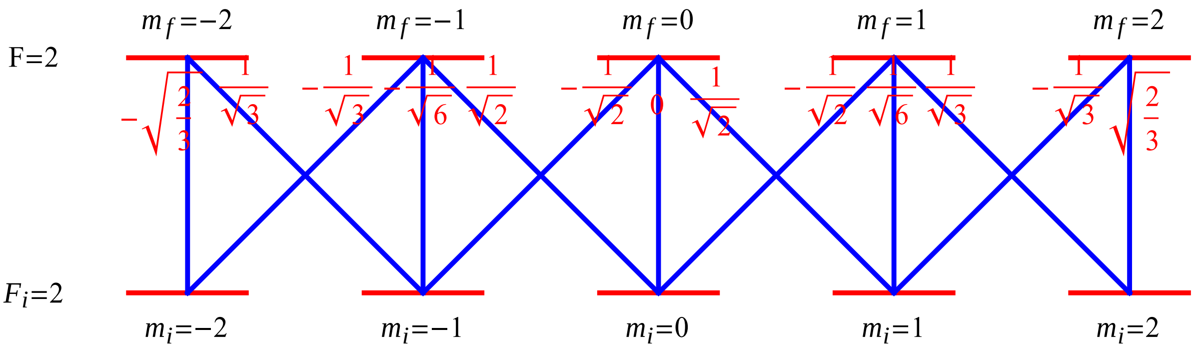



**Fig.18** Clebsch-Gordon coefficients for the transition between $F$ = 2 and $F_i$ = 2. The square of the Clebsch-Gordan coefficient corresponds to the probability.

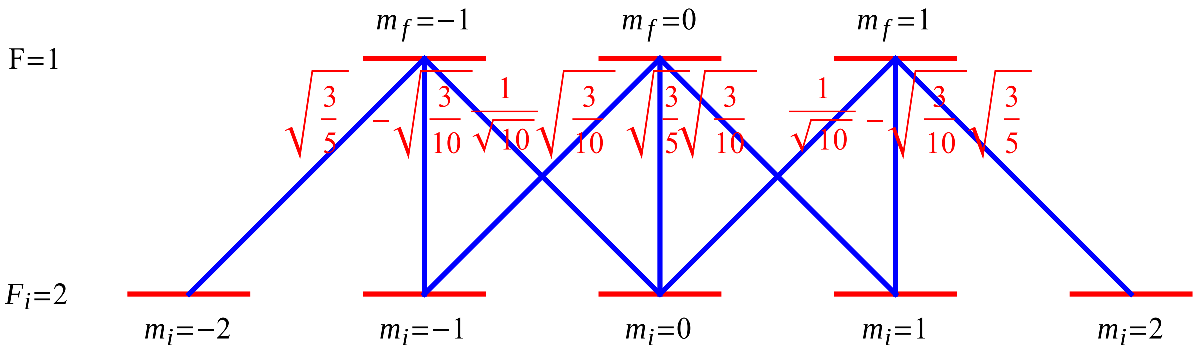



**Fig.19** Clebsch-Gordon coefficients for the transition between $F$ = 1 and $F_i$ = 2. The square of the Clebsch-Gordan coefficient corresponds to the probability.

## 12 Optical pumping: transition from $F = 2$ to $F = 1$ through absorption of light $D_1$ with $\sigma$+ polarization

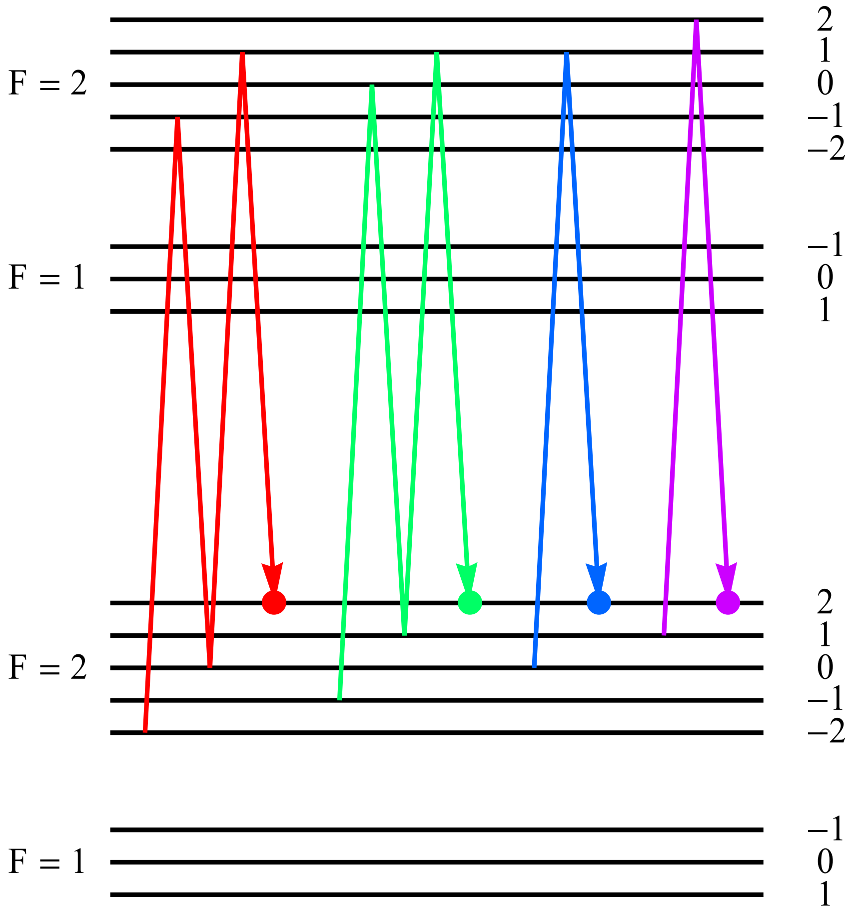



**Fig.20** Selection rule for the transition between the Zeeman levels for $F$ = 2 of 5 $^2S_{1/2}$ and the Zeeman levels for $F$ = 2 of 5 $^2P_{1/2}$. $\Delta m_f$ = 1 for the $\sigma$+ circularly polarization light. The population of the state $\left|F=2, m_f=2\right\rangle$ of 5 $^2P_{1/2}$ becomes maximum.

## 13 Optical pumping: transition from $F$ = 1 to $F$ = 1 through absorption of light $D_1$ with $\sigma$+ polarization

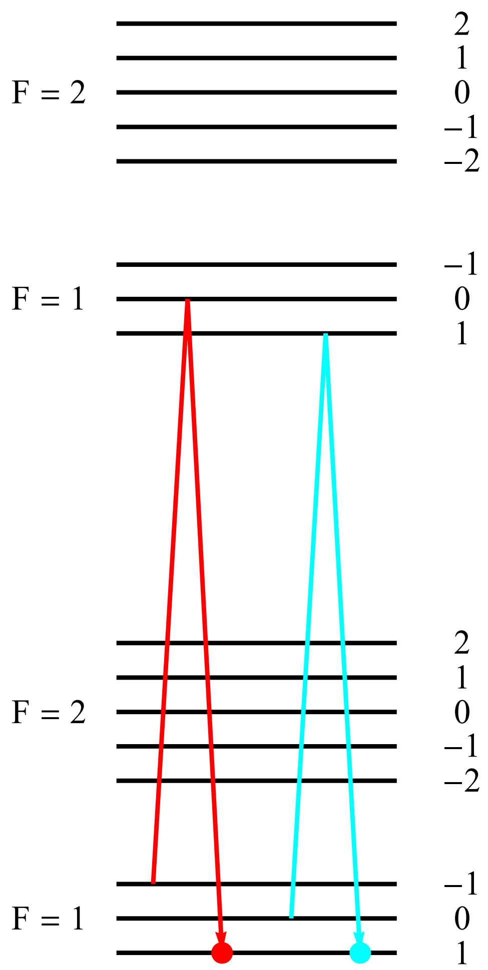



**Fig.21** Selection rule for the transition between the Zeeman levels for $F$ = 1 of 5 $^2S_{1/2}$ and the Zeeman levels for $F$ = 1 of 5 $^2P_{1/2}$. $\Delta m_f$ = 1 for the $\sigma^+$ circularly polarization light. The population of the state $|F = 1, m_f = 1\rangle$ of 5 $^2P_{1/2}$ becomes maximum.

## 14. The transition of *F* = 1 and *F* = 1.

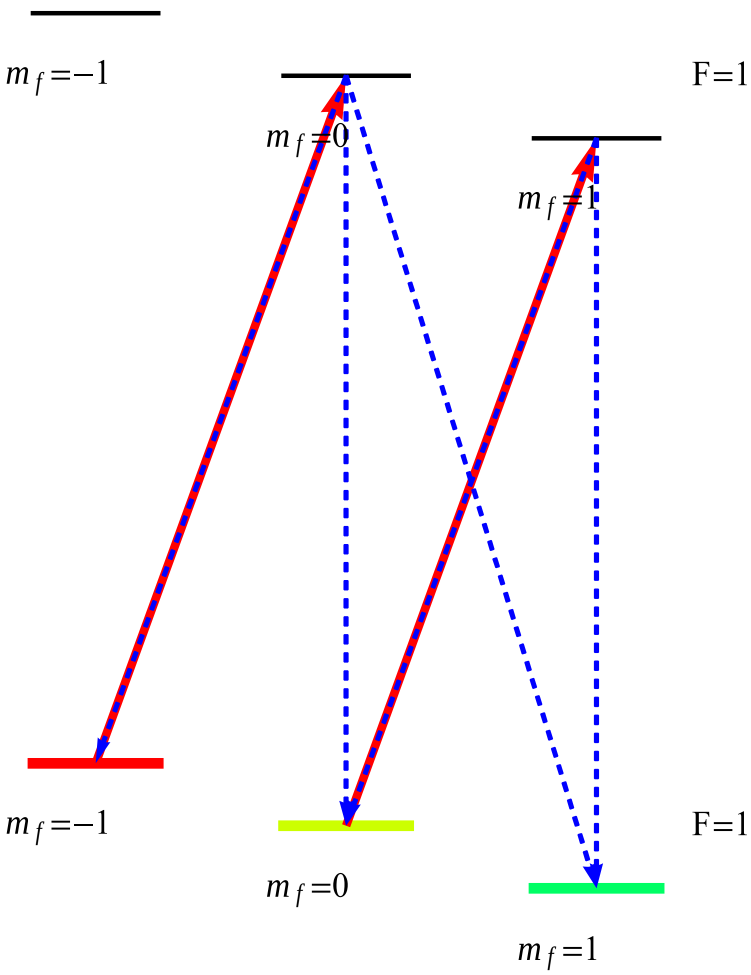



**Fig.22** Transition between $F$ = 1 and $F$ = 1. Population of the state $m_f$ = 1 is increased by the optical pumping.

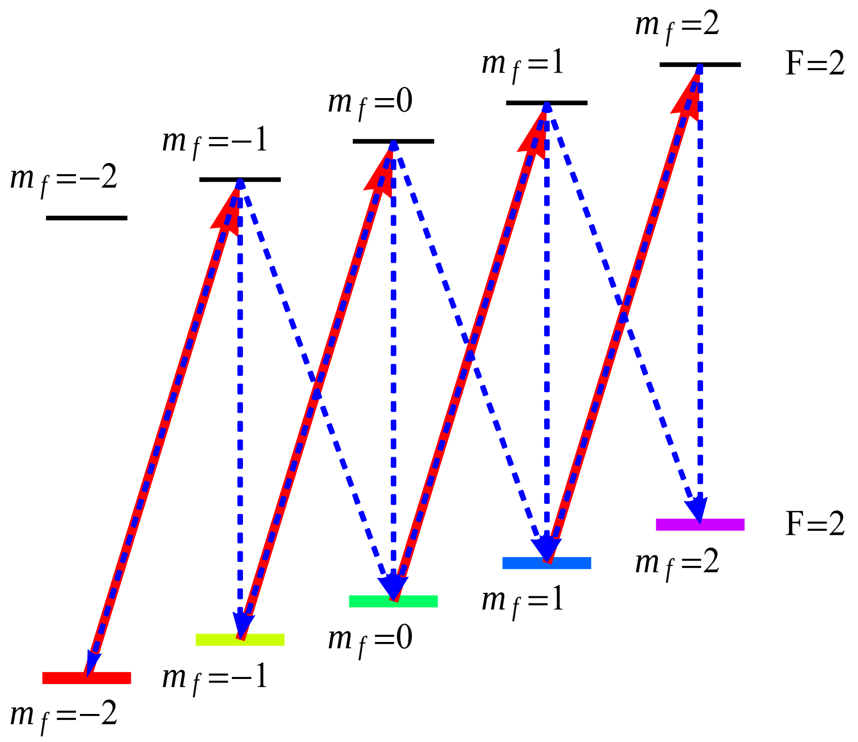



**Fig.23** Transition between $F = 2$ and $F = 2$.

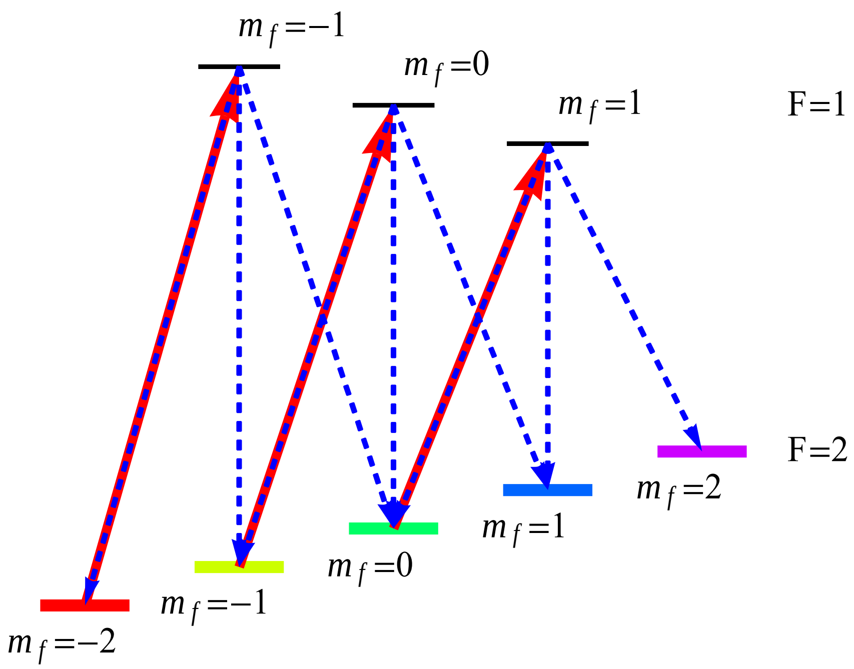



**Fig.24** Transition between $F = 2$ and $F = 1$.

## 15. Pumping process (simple model)

Before pumping, the atoms are divided evenly between the energy levels A and B.

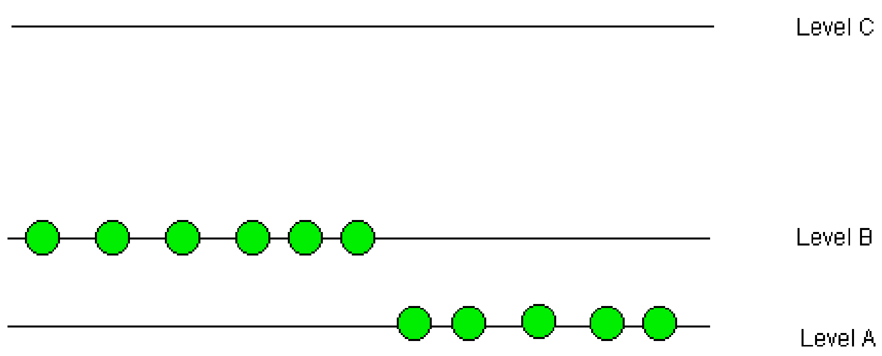



**Fig.25 (a)**

Suppose we irradiate a sample of these atoms with alight beam which the spectral line BC has been filtered. The beam contains photons that can excite atoms in the level A but not in level B. Atoms excited out of A absorb energy and rise to C. They will remain there for a short time (as little as a ten millionth of a second) and then emit energy, dropping back either to the A or B state. After absorbing photons from a beam of light (circularly polarized) and being raised to energy level C, atoms drop back in equal numbers to energy levels A and B.

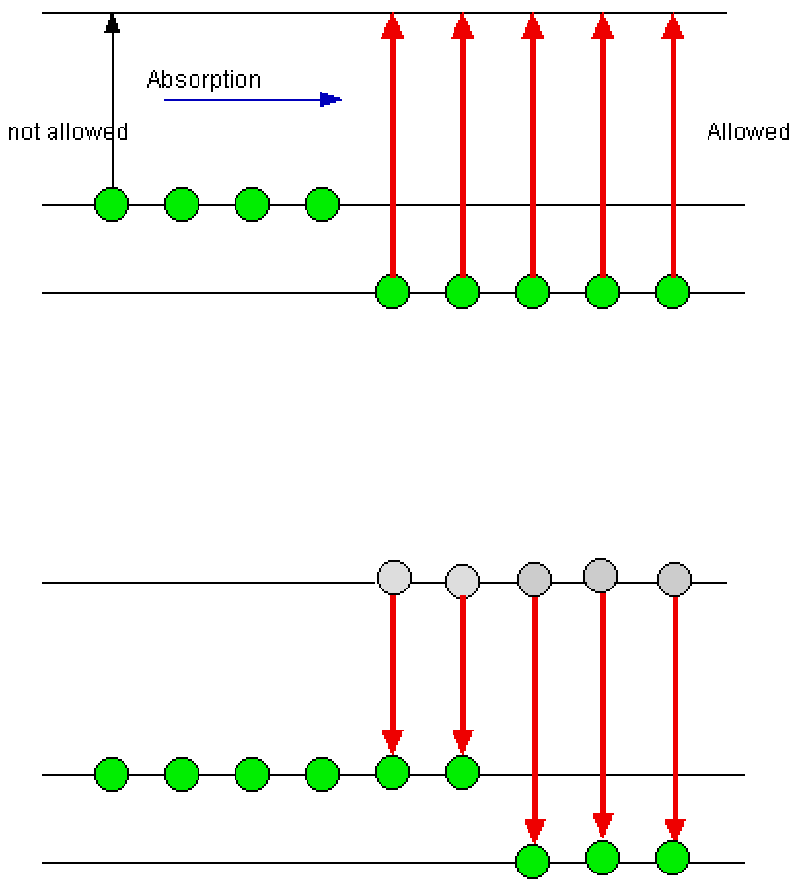



**Fig.25 (b) and (c)**

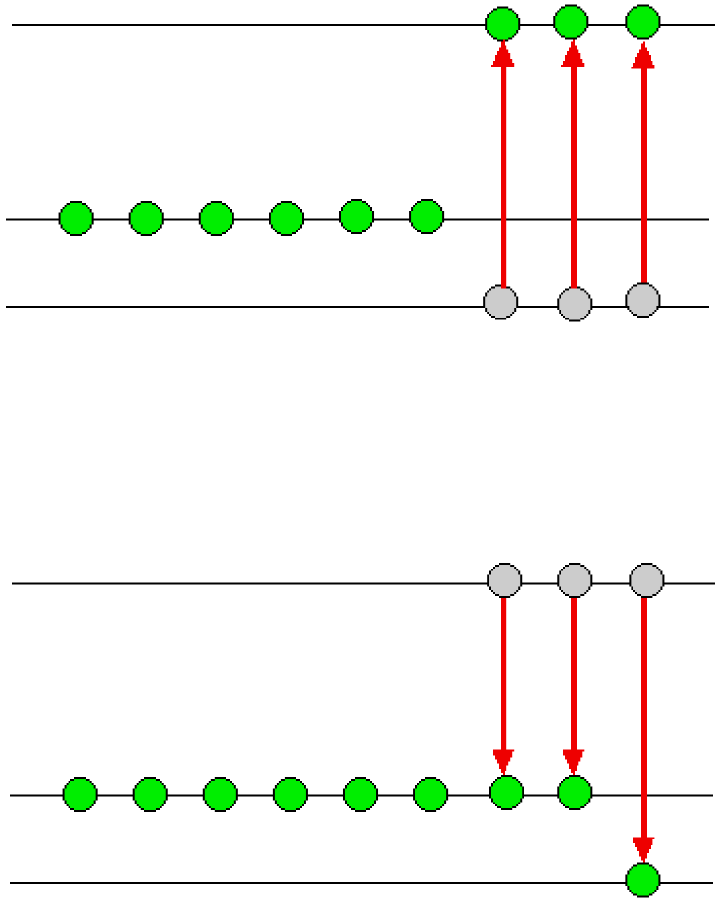


**Fig.25 (d) and (e)**

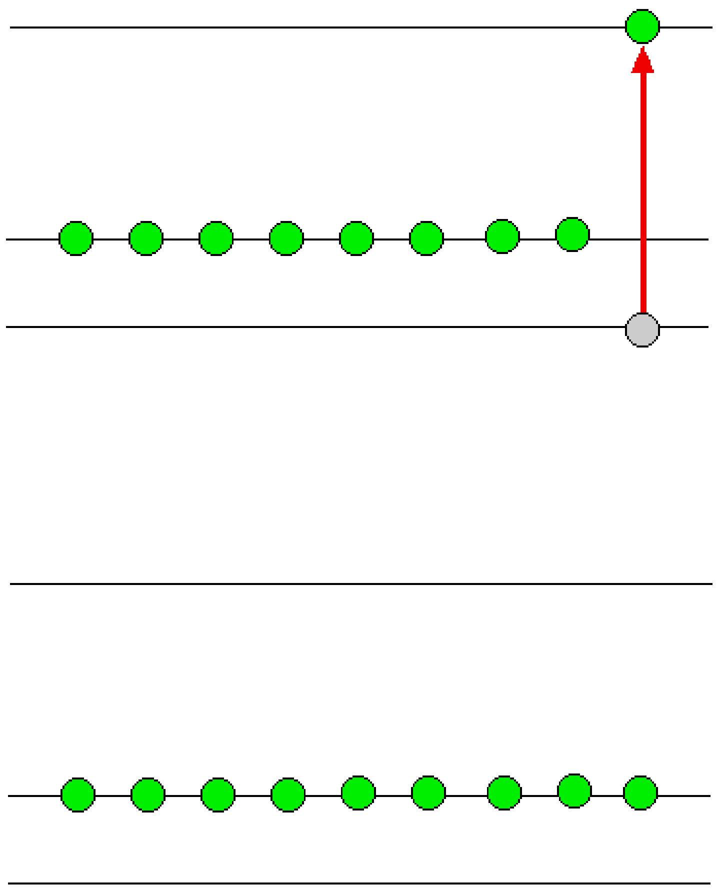


**Fig.25 (f) and (g)**

As the process continues all atoms are in the level B. The is no atom in the level A. In other words, given enough time, every atom must end up in the B state and the material is then completely pumped.

When an rf magnetic field is applied, electron precesses and acts as partially open shutter. If some atoms are suddenly returned to the A state, light will again be absorbed, and the brightness of the transmitted beam will drop sharply. Population of the state $m_f = 2$ is increased by the optical pumping.

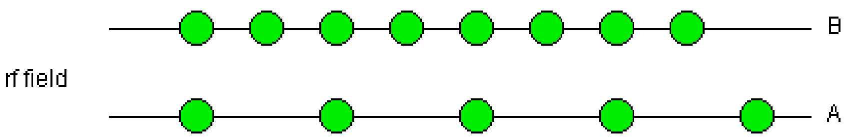



**Fig.25 (h)**

## 16. Linear Zeeman effect in a weak magnetic field

In a weak magnetic field limit, the spacing of the Zeeman splitting between the *m* levels of a given *F* state. In this case, the Hamiltonian (spin-orbit, hyperfine interaction, Zeeman (magnetic field)) is diagonal in the $|F, m_f\rangle$ basis. However, in the strong magnetic field limit, the spacing between the levels is not equal. Here we show the simple model for the Zeeman splitting in the weak magnetic field limit.

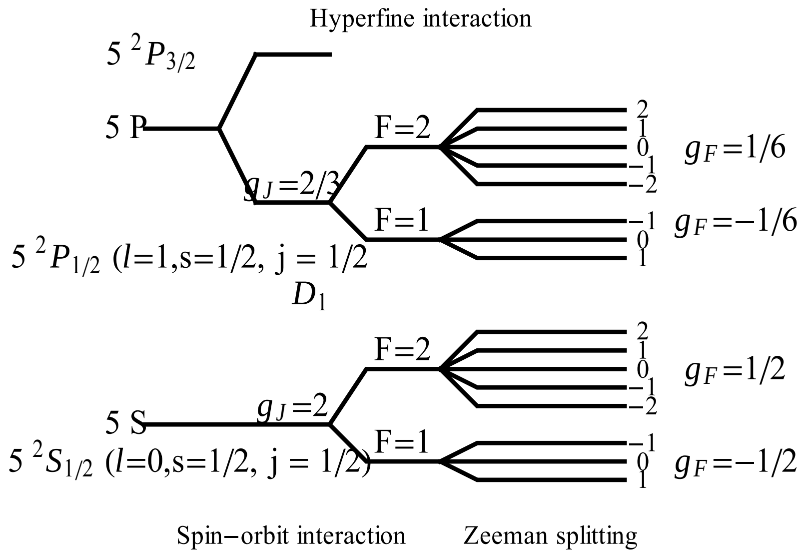



**Fig.26** Schematic energy levels of $^{87}$Rb ($I$ = 3/2) in the presence of spin-orbit interaction, hyperfine interaction, and the magnetic field.

If the magnetic field is relatively weak, the Zeeman energy is given by a simply expression

$$E(F, m_f) = g_F \mu_B B m_f + E(F, m_f = 0)$$

in the state $|F, m_f\rangle$ with $m_f = F$, $F$-1, ..., and -$F$. The spacing between the Zeeman levels in the presence of a magnetic field along the $z$ axis, is independent of $m_f$,

$$\Delta E = g_F \mu_B \Delta m_f B_z = g_F \mu_B B_z$$

since $\Delta m_f$ = 1. When $\Delta E = \hbar\omega = \hbar(2\pi\nu)$ [$\nu$ is the frequency, and $\omega$ is the angular frequency], we have

$$\frac{\hbar(2\pi\nu)}{B_z} = g_F \mu_B$$

or

$$\frac{\nu}{B_z} = \frac{g_F \mu_B}{2\pi\hbar} = 1.39962 g_F \text{ (MHz/Oe)}.$$

When $g_F = 1/2$

$$\frac{\nu}{B_z} = 1.39962 g_F = 0.69981 \text{ (MHz/Oe)}.$$

The linear relationship between energy levels and magnetic field only holds for small magnetic fields. When the Zeeman splitting grows relative to the hyperfine energy difference one has to take into account the quantum mixing of the states.

## (a) Landé g-factor $g_J$

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3 + \ldots, \ \mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \ldots.$$

The total angular momentum $\boldsymbol{J}$ is defined by

$$\mathbf{J} = \mathbf{L} + \mathbf{S},$$

where $\boldsymbol{L}$ is the orbital angular momentum $\boldsymbol{L}$ and S is the spin angular momentum. The total magnetic moment $\boldsymbol{\mu}$ is given by

$$\boldsymbol{\mu} = -\mu_B(\mathbf{L} + 2\mathbf{S}),$$

where $\mu_B$ is the Bohr magneton. The Landé g-factor is defined by

$$\boldsymbol{\mu}_J = -g_J \mu_B \mathbf{J},$$

Suppose that

$$\mathbf{L} = a\mathbf{J} + \mathbf{L}_\perp \text{ and } \mathbf{S} = b\mathbf{J} + \mathbf{S}_\perp,$$

where $a$ and $b$ are constants, and the vectors $\mathbf{S}_\perp$ and $\mathbf{L}_\perp$ are perpendicular to $\boldsymbol{J}$.
Here we have the relation $a + b = 1$, and $\mathbf{L}_\perp + \mathbf{S}_\perp = 0$. The values of $a$ and $b$ are determined as follows.

$$a = \frac{\mathbf{J}\cdot\mathbf{L}}{\mathbf{J}^2}, \qquad b = \frac{\mathbf{J}\cdot\mathbf{S}}{\mathbf{J}^2}.$$

Here we note that

$$\mathbf{J}\cdot\mathbf{S} = (\mathbf{L}+\mathbf{S})\cdot\mathbf{S} = \mathbf{S}^2 + \mathbf{L}\cdot\mathbf{S} = \mathbf{S}^2 + \frac{\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2}{2} = \frac{\mathbf{J}^2 - \mathbf{L}^2 + \mathbf{S}^2}{2},$$

or

$$\mathbf{J}\cdot\mathbf{S} = \frac{\mathbf{J}^2 - \mathbf{L}^2 + \mathbf{S}^2}{2} = \frac{\hbar^2}{2}[J(J+1) - L(L+1) + S(S+1)],$$

using the average in quantum mechanics. The total magnetic moment $\boldsymbol{\mu}$ is

$$\boldsymbol{\mu} = -\mu_B(\mathbf{L}+2\mathbf{S}) = -\mu_B[(a+2b)\mathbf{J} + (L_\perp + 2S_\perp)].$$

Thus we have the component of $\boldsymbol{\mu}$ along the direction of $\boldsymbol{J}$ as

$$\boldsymbol{\mu}_J = -\mu_B(a+2b)\mathbf{J} = -\mu_B(1+b)\mathbf{J} = -g_J\mu_B\mathbf{J},$$

with

$$g_J = 1 + b = 1 + \frac{\mathbf{J}\cdot\mathbf{S}}{\mathbf{J}^2} = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)}.$$

**(b) Landé g-factor $g_F$**

The total angular momentum $\boldsymbol{F}$ is defined by

$$\mathbf{F} = \mathbf{J} + \mathbf{I}.$$

where $\boldsymbol{I}$ is the nuclear spin. The total magnetic moment $\boldsymbol{\mu}$ is given by

$$\boldsymbol{\mu} = -\mu_B g_J\mathbf{J} + g_I\mu_N\mathbf{I} = -\mu_B(g_J\mathbf{J} - g_I\frac{\mu_N}{\mu_B}\mathbf{I})$$

where $g_I\mu_N\mathbf{I}$ is the nuclear magnetic moment. Note that $g_J>0$ and $g_I>0$. The direction of the nuclear magnetic moment is anti-parallel to that of the electron magnetic moments. Suppose that

$$\mathbf{J} = a\mathbf{F} + \mathbf{J}_\perp \text{ and } \mathbf{I} = b\mathbf{F} + \mathbf{I}_\perp,$$

where $a$ and $b$ are constants, and the vectors $\mathbf{J}_\perp$ and $\mathbf{I}_\perp$ are perpendicular to $\boldsymbol{F}$. Here we have the relation $a+b=1$, and $\mathbf{J}_\perp + \mathbf{I}_\perp = 0$. The values of $a$ and $b$ are determined as follows.

$$a = \frac{\mathbf{J}\cdot\mathbf{F}}{\mathbf{F}^2}, \qquad b = \frac{\mathbf{I}\cdot\mathbf{F}}{\mathbf{F}^2}.$$

Here we note that

$$\mathbf{J}\cdot\mathbf{F} = \mathbf{J}\cdot(\mathbf{J}+\mathbf{I}) = \mathbf{J}^2 + \mathbf{J}\cdot\mathbf{I} = \mathbf{J}^2 + \frac{\mathbf{F}^2 - \mathbf{I}^2 - \mathbf{J}^2}{2} = \frac{\mathbf{J}^2 - \mathbf{I}^2 + \mathbf{F}^2}{2},$$

since

$$\mathbf{F}^2 = \mathbf{J}^2 + \mathbf{I}^2 + 2\mathbf{J}\cdot\mathbf{I}$$

Then we have

$$\mathbf{J}\cdot\mathbf{F} = \frac{\mathbf{J}^2 - \mathbf{I}^2 + \mathbf{F}^2}{2} = \frac{\hbar^2}{2}[J(J+1) - I(I+1) + F(F+1)],$$

using the average in quantum mechanics. The total magnetic moment $\boldsymbol{\mu}$ is

$$\boldsymbol{\mu} = -\mu_B g_J \mathbf{J} + \mu_N g_I \mathbf{I} = -\mu_B[g_J(a\mathbf{F} + \mathbf{J}_\perp) - g_I \frac{\mu_N}{\mu_B}(b\mathbf{F} + \mathbf{I}_\perp)$$

$$= -\mu_B[(ag_J - b\frac{\mu_N}{\mu_B}g_I)\mathbf{F} + (g_J\mathbf{J}_\perp - \frac{\mu_N}{\mu_B}g_I\mathbf{I}_\perp)]$$

The Landé $g$-factor $g_F$ for $\boldsymbol{F}$ is defined by

$$\boldsymbol{\mu}_F = -g_F\mu_B\mathbf{F} = -\mu_B(ag_J - b\frac{\mu_N}{\mu_B}g_I)\mathbf{F} \approx -\mu_B a g_J \mathbf{F}$$

Here we neglect the contribution from the magnetic moment from the nuclear spin. Thus we get

$$g_F = ag_J = \frac{\mathbf{J}\cdot\mathbf{F}}{\mathbf{F}^2}g_J = g_J[\frac{J(J+1) - I(I+1) + F(F+1)}{2F(F+1)}].$$

---

## 17 Quadratic Zeeman effect

### 17.1 Clebsch-GFordan coefficients

(a) spin-orbit interaction ($j = 1/2$, $l = 0$, $s = 1/2$)

$$\left|j = \frac{1}{2}, m_j = \frac{1}{2}\right\rangle = |l = 0, m_l = 0\rangle|s = 1/2, m_s = -1/2\rangle$$

$$\left|j = \frac{1}{2}, m_j = -\frac{1}{2}\right\rangle = |l = 1, m_l = 0\rangle|s = 1/2, m_s = -1/2\rangle$$

(b) spin-orbit + hyperfine interaction

$F = 2$ ($j = 1/2$, $l = 0$, $s = 1/2$; $I = 3/2$)

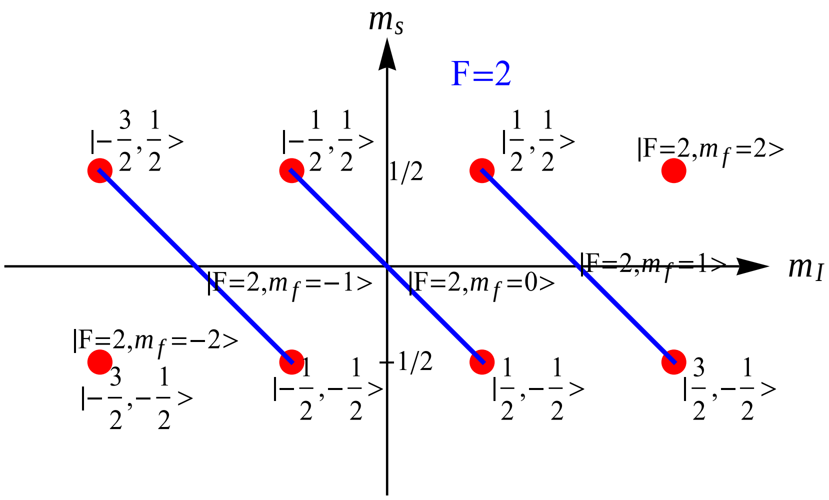



**Fig.27**

$$\left|F=2,m_f=2\right\rangle=\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$

$$\left|F=2,m_f=1\right\rangle=\frac{\sqrt{3}}{2}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle+\frac{1}{2}\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle$$

$$\left|F=2,m_f=0\right\rangle=\frac{1}{\sqrt{2}}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle+\frac{1}{\sqrt{2}}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle$$

$$\left|F=2,m_f=-1\right\rangle=\frac{1}{2}\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle+\frac{\sqrt{3}}{2}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle$$

$$\left|F=2,m_f=-2\right\rangle=\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle$$

---

$F = 1$ ($j = 1/2$, $l = 0$, $s = 1/2$; $I = 3/2$)

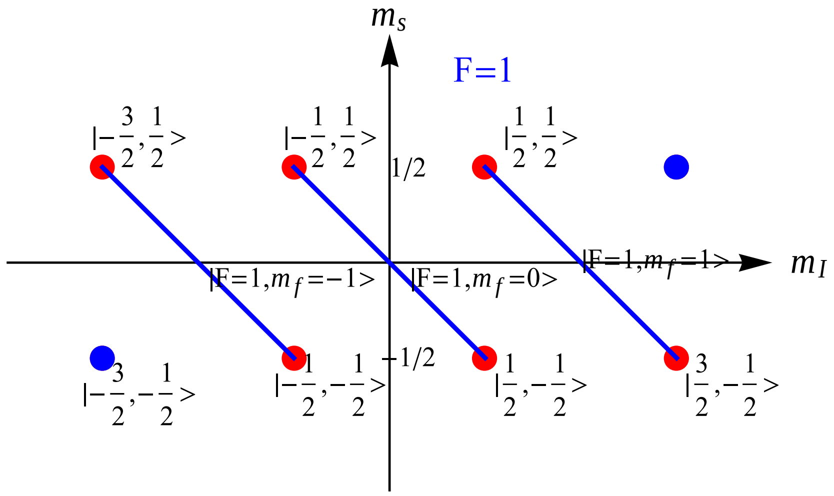



**Fig.28**

$$\left|F=1,m_F=1\right\rangle=\frac{1}{2}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle-\frac{\sqrt{3}}{2}\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle$$

$$\left|F=1,m_F=0\right\rangle=\frac{1}{\sqrt{2}}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle-\frac{1}{\sqrt{2}}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle$$

$$\left|F=1,m_F=-1\right\rangle=\frac{\sqrt{3}}{2}\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle-\frac{1}{2}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle$$

## 18. Hamiltonian

We need to calculate the matrix elements of the Hamiltonian defined by

$$H=H_{hf}+H_{Zeeman}$$

where

$$\boldsymbol{\mu}=-\mu_B g_J \mathbf{J}+\mu_N g_I \mathbf{I}$$

where $\mu_B$ is the Bohr magneton and $\mu_N$ is the nuclear magneton.

$$H_{Zeeman}=-\boldsymbol{\mu}\cdot\mathbf{B}=\mu_B g_J J_z B-\mu_N g_I I_z B$$

where

$$\mu_J=-g_J\mu_B\mathbf{J}$$

$$\mu_I = g_I \mu_B \mathbf{I}$$

$$\mu_F = -g_F \mu_B \mathbf{F}$$

where

$$\mathbf{F} = \mathbf{J} + \mathbf{I}$$

$$H_{hf} = A\mathbf{I} \cdot \mathbf{J}$$

where

$$\begin{aligned}\mathbf{I} \cdot \mathbf{J} &= I_x J_x + I_y J_y + I_z J_z \\ &= \frac{1}{4}(J^+ + J^-)(I^+ + I^-) - \frac{1}{4}(J^+ - J^-)(I^+ - I^-) + I_z J_z\end{aligned}$$

or

$$H_{hf} = A\mathbf{I} \cdot \mathbf{J} = \frac{A}{2}(\mathbf{F}^2 - \mathbf{I}^2 - \mathbf{J}^2) \to \frac{A}{2}[F(F+1) - I(I+1) - j(j+1)]$$

where

$$A = \frac{E_{hf}}{2} = 3.417341305 \text{ GHz.}$$

with

$E_{hf}$ = 6.834682610 GHz for $^{87}$Rb

# 18 Calculation of matrix elements for the Hamiltonian

$$H_{hf}\left|F=2,m_F=2\right\rangle=\frac{A}{2}[F(F+1)-I(I+1)-j(j+1)]\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$
$$=\frac{3A}{4}\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$
$$H_{Zeeman}\left|F=2,m_F=2\right\rangle=(\mu_B g_J J_z B-\mu_N g_I I_z B)\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$
$$=(\mu_B g_J mB-\mu_N g_I m_I B)\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$
$$=(\mu_B g_J \frac{1}{2}B-\mu_N g_I \frac{3}{2}B)\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$

---

$$H_{hf}\left|F=2,m_F=1\right\rangle=\frac{3A}{4}[\frac{\sqrt{3}}{2}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle+\frac{1}{2}\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$
$$H_{Zeeman}\left|F=2,m_F=1\right\rangle=(\mu_B g_J J_z B-\mu_N g_I I_z B)[\frac{\sqrt{3}}{2}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$
$$+\frac{1}{2}\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$
$$=(\mu_B g_J \frac{1}{2}B-\mu_N g_I \frac{1}{2}B)\frac{\sqrt{3}}{2}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$
$$+(-\mu_B g_J \frac{1}{2}B-\mu_N g_I \frac{3}{2}B)\frac{1}{2}\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$

---

$$H_{hf}\left|F=2,m_F=0\right\rangle=\frac{3A}{4}[\frac{1}{\sqrt{2}}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$
$$+\frac{1}{\sqrt{2}}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$
$$H_{Zeeman}\left|F=2,m_F=0\right\rangle=(\mu_B g_J J_z B-\mu_N g_I I_z B)[\frac{1}{\sqrt{2}}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$
$$+\frac{1}{\sqrt{2}}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$
$$=(\mu_B g_J \frac{1}{2}B+\mu_N g_I \frac{1}{2}B)[\frac{1}{\sqrt{2}}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$
$$+(-\mu_B g_J \frac{1}{2}B-\mu_N g_I \frac{1}{2}B)\frac{1}{\sqrt{2}}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$

$$H_{hf}\left|F=2,m_F=-1\right\rangle=\frac{3A}{4}[\frac{1}{2}\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$
$$+\frac{\sqrt{3}}{2}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$
$$H_{Zeeman}\left|F=2,m_F=-1\right\rangle=(\mu_B g_J J_z B-\mu_N g_I I_z B)[\frac{1}{2}\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$
$$+\frac{\sqrt{3}}{2}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$
$$=(\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{3}{2}B)\frac{1}{2}\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$
$$+(-\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{1}{2}B)\frac{\sqrt{3}}{2}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]]$$

---

$$H_{hf}\left|F=2,m_F=-2\right\rangle=\frac{3A}{4}\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$
$$H_{Zeeman}\left|F=2,m_F=-2\right\rangle=(\mu_B g_J J_z B-\mu_N g_I I_z B)\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle$$
$$=(-\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{3}{2}B)\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle$$

---

$$H_{hf}\left|F=1,m_F=1\right\rangle=\frac{A}{2}[F(F+1)-I(I+1)-j(j+1)][\frac{1}{2}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$
$$-\frac{\sqrt{3}}{2}\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$
$$=-\frac{5}{4}A[\frac{1}{2}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$
$$-\frac{\sqrt{3}}{2}\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$
$$H_{Zeeman}\left|F=1,m_F=1\right\rangle=(\mu_B g_J\frac{1}{2}B-\mu_N g_I\frac{1}{2}B)\frac{1}{2}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$
$$-(-\mu_B g_J\frac{1}{2}B-\mu_N g_I\frac{3}{2}B)\frac{\sqrt{3}}{2}\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$

---

$$H_{hf}\left|F=1,m_F=0\right\rangle=-\frac{5}{4}A[\frac{1}{\sqrt{2}}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$

$$-\frac{1}{\sqrt{2}}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$

$$H_{Zeeman}\left|F=1,m_F=0\right\rangle=(\mu_B g_J J_z B-\mu_N g_I I_z B)[\frac{1}{\sqrt{2}}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$

$$-\frac{1}{\sqrt{2}}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$

$$=(\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{1}{2}B)[\frac{1}{\sqrt{2}}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$

$$-(-\mu_B g_J\frac{1}{2}B-\mu_N g_I\frac{1}{2}B)\frac{1}{\sqrt{2}}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$

---

$$H_{hf}\left|F=1,m_F=-1\right\rangle=-\frac{5}{4}A[\frac{\sqrt{3}}{2}\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$

$$-\frac{1}{2}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$

$$H_{Zeeman}\left|F=1,m_F=-1\right\rangle=(\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{3}{2}B)\frac{\sqrt{3}}{2}\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$

$$-(-\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{1}{2}B)\frac{1}{2}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$

## 19 The matrix of Hamiltonian under the basis of $\left|F,m_f=2\right\rangle$

$$H\left|F=2,m_f=2\right\rangle=$$

$$=(\frac{3}{4}A+\mu_B g_J\frac{1}{2}B-\mu_N g_I\frac{3}{2}B)\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$

$$=A_{11}\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$

$$H\left|F=2,m_f=1\right\rangle=$$

$$=[\frac{3}{4}A+(\mu_B g_J\frac{1}{2}B-\mu_N g_I\frac{1}{2}B)]\frac{\sqrt{3}}{2}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$

$$+[\frac{3}{4}A-(\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{3}{2}B)]\frac{1}{2}\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$

$$=A_{22}\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle+A_{23}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle+$$

$$H\left|F=2,m_f=0\right\rangle=[\frac{3}{4}A+(\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{1}{2}B)]\frac{1}{\sqrt{2}}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$

$$+[\frac{3}{4}A-(\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{1}{2}B)]\frac{1}{\sqrt{2}}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$

$$=A_{34}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle+A_{35}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle+$$

$$H\left|F=2,m_f=-1\right\rangle=[\frac{3}{4}A+(\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{3}{2}B)]\frac{1}{2}\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$

$$+[\frac{3}{4}A+(-\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{1}{2}B)]\frac{\sqrt{3}}{2}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$

$$=A_{46}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle+A_{47}\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$

$$H\left|F=2,m_f=-2\right\rangle=[\frac{3}{4}A+(-\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{3}{2}B)]\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$

$$=A_{58}\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle$$

$$H\left|F=1,m_f=1\right\rangle=[-\frac{5}{4}A+(\mu_B g_J\frac{1}{2}B-\mu_N g_I\frac{1}{2}B)]\frac{1}{2}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$

$$+[\frac{5}{4}A+(\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{3}{2}B)]\frac{\sqrt{3}}{2}\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$

$$=A_{62}\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle+A_{63}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$

$$H\left|F=1,m_f=0\right\rangle=[-\frac{5}{4}A+(\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{1}{2}B)]\frac{1}{\sqrt{2}}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$

$$+[\frac{5}{4}A+(\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{1}{2}B)\frac{1}{\sqrt{2}}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$

$$=A_{74}\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle+A_{75}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$

$$H\left|F=1,m_f=-1\right\rangle=[-\frac{5}{4}A+(\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{3}{2}B)]\frac{\sqrt{3}}{2}\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$$

$$+[\frac{5}{4}A+(\mu_B g_J\frac{1}{2}B-\mu_N g_I\frac{1}{2}B)]\frac{1}{2}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle]$$

$$=A_{86}\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle+A_{87}\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle+$$

where

$$A_{11}=\frac{3}{4}A+(\mu_B g_J\frac{1}{2}-\mu_N g_I\frac{3}{2})B$$

$$A_{22}=[\frac{3}{4}A-(\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{3}{2}B)]\frac{1}{2}$$

$$A_{23}=[\frac{3}{4}A+(\mu_B g_J\frac{1}{2}B-\mu_N g_I\frac{1}{2}B)]\frac{\sqrt{3}}{2}$$

$$A_{34}=[\frac{3}{4}A-(\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{1}{2}B)]\frac{1}{\sqrt{2}}$$

$$A_{35}=[\frac{3}{4}A+(\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{1}{2}B)]\frac{1}{\sqrt{2}}$$

$$A_{46}=[\frac{3}{4}A+(-\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{1}{2}B)]\frac{\sqrt{3}}{2}$$

$$A_{47}=[\frac{3}{4}A+(\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{3}{2}B)]\frac{1}{2}$$

$$A_{58}=[\frac{3}{4}A+(-\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{3}{2}B)]$$

$$A_{62} = [\frac{5}{4}A + (\mu_B g_J \frac{1}{2}B + \mu_N g_I \frac{3}{2}B)]\frac{\sqrt{3}}{2}$$

$$A_{63} = [-\frac{5}{4}A + (\mu_B g_J \frac{1}{2}B - \mu_N g_I \frac{1}{2}B)]\frac{1}{2}$$

$$A_{74} = [\frac{5}{4}A + (\mu_B g_J \frac{1}{2}B + \mu_N g_I \frac{1}{2}B)\frac{1}{\sqrt{2}}$$

$$A_{75} = [-\frac{5}{4}A + (\mu_B g_J \frac{1}{2}B + \mu_N g_I \frac{1}{2}B)]\frac{1}{\sqrt{2}}$$

$$A_{86} = [\frac{5}{4}A + (\mu_B g_J \frac{1}{2}B - \mu_N g_I \frac{1}{2}B)]\frac{1}{2}$$

$$A_{87} = [-\frac{5}{4}A + (\mu_B g_J \frac{1}{2}B + \mu_N g_I \frac{3}{2}B)]\frac{\sqrt{3}}{2}$$

Then we get

$$H\begin{pmatrix} \left|F=2, m_f=2\right\rangle \\ \left|F=2, m_f=1\right\rangle \\ \left|F=2, m_f=0\right\rangle \\ \left|F=2, m_f=-1\right\rangle \\ \left|F=2, m_f=-2\right\rangle \\ \left|F=1, m_f=1\right\rangle \\ \left|F=1, m_f=0\right\rangle \\ \left|F=1, m_f=-1\right\rangle \end{pmatrix} = \begin{pmatrix} A_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{22} & A_{23} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{34} & A_{35} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{46} & A_{47} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{58} \\ 0 & A_{62} & A_{63} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{74} & A_{75} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{86} & A_{87} & 0 \end{pmatrix} \begin{pmatrix} \left|I=\frac{3}{2}, m_I=\frac{3}{2}\right\rangle \left|j=\frac{1}{2}, m=\frac{1}{2}\right\rangle \\ \left|I=\frac{3}{2}, m_I=\frac{3}{2}\right\rangle \left|j=\frac{1}{2}, m=-\frac{1}{2}\right\rangle \\ \left|I=\frac{3}{2}, m_I=\frac{1}{2}\right\rangle \left|j=\frac{1}{2}, m=\frac{1}{2}\right\rangle \\ \left|I=\frac{3}{2}, m_I=\frac{1}{2}\right\rangle \left|j=\frac{1}{2}, m=-\frac{1}{2}\right\rangle \\ \left|I=\frac{3}{2}, m_I=-\frac{1}{2}\right\rangle \left|j=\frac{1}{2}, m=\frac{1}{2}\right\rangle \\ \left|I=\frac{3}{2}, m_I=-\frac{1}{2}\right\rangle \left|j=\frac{1}{2}, m=-\frac{1}{2}\right\rangle \\ \left|I=\frac{3}{2}, m_I=-\frac{3}{2}\right\rangle \left|j=\frac{1}{2}, m=\frac{1}{2}\right\rangle \\ \left|I=\frac{3}{2}, m_I=-\frac{3}{2}\right\rangle \left|j=\frac{1}{2}, m=-\frac{1}{2}\right\rangle \end{pmatrix}$$

Here we note that

$$
\begin{pmatrix}
\left|F=2,m_f=2\right\rangle \\
\left|F=2,m_f=1\right\rangle \\
\left|F=2,m_f=0\right\rangle \\
\left|F=2,m_f=-1\right\rangle \\
\left|F=2,m_f=-2\right\rangle \\
\left|F=1,m_f=1\right\rangle \\
\left|F=1,m_f=0\right\rangle \\
\left|F=1,m_f=-1\right\rangle
\end{pmatrix}
=
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0
\end{pmatrix}
\begin{pmatrix}
\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle \\
\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle \\
\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle \\
\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle \\
\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle \\
\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle \\
\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle \\
\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle
\end{pmatrix}
$$

or

$$
\begin{pmatrix}
\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle \\
\left|I=\frac{3}{2},m_I=\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle \\
\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle \\
\left|I=\frac{3}{2},m_I=\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle \\
\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle \\
\left|I=\frac{3}{2},m_I=-\frac{1}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle \\
\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle \\
\left|I=\frac{3}{2},m_I=-\frac{3}{2}\right\rangle\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle
\end{pmatrix}
=
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \\
0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\left|F=2,m_f=2\right\rangle \\
\left|F=2,m_f=1\right\rangle \\
\left|F=2,m_f=0\right\rangle \\
\left|F=2,m_f=-1\right\rangle \\
\left|F=2,m_f=-2\right\rangle \\
\left|F=1,m_f=1\right\rangle \\
\left|F=1,m_f=0\right\rangle \\
\left|F=1,m_f=-1\right\rangle
\end{pmatrix}
$$

The final form of the eigenvalue problem is as follows.

with

$$M_1 = \begin{pmatrix} A_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{22} & A_{23} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{34} & A_{35} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{46} & A_{47} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{58} \\ 0 & A_{62} & A_{63} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{74} & A_{75} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{86} & A_{87} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

or

$$M_1 = \begin{pmatrix} A_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{A_{22}+\sqrt{3}A_{23}}{2} & 0 & 0 & 0 & \frac{-\sqrt{3}A_{22}+A_{23}}{2} & 0 & 0 \\ 0 & 0 & \frac{A_{34}+A_{35}}{\sqrt{2}} & 0 & 0 & 0 & \frac{-A_{34}+A_{35}}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}A_{46}+A_{47}}{2} & 0 & 0 & 0 & \frac{-A_{46}+\sqrt{3}A_{47}}{2} \\ 0 & 0 & 0 & 0 & A_{58} & 0 & 0 & 0 \\ 0 & \frac{A_{62}+\sqrt{3}A_{63}}{2} & 0 & 0 & 0 & \frac{-\sqrt{3}A_{62}+A_{63}}{2} & 0 & 0 \\ 0 & 0 & \frac{A_{74}+A_{75}}{\sqrt{2}} & 0 & 0 & 0 & \frac{-A_{74}+A_{75}}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}A_{86}+A_{87}}{2} & 0 & 0 & 0 & \frac{-A_{86}+\sqrt{3}A_{87}}{2} \end{pmatrix}$$

## 20. Eigenvalue problem

$E=A_{11}$ is the eigenvalue of Hamiltonian with the eigenvalue; eigenstate: $\left|F=2, m_f=2\right\rangle$.

$$A_{11} = \frac{3}{4}A + (\mu_B g_J \frac{1}{2} - \mu_N g_I \frac{3}{2})B .$$

$E=A_{58}$ is the eigenvalue of Hamiltonian with the eigenvalue; eigenstate: $\left|F=2, m_f=-2\right\rangle$.

$$A_{58}=\frac{3}{4}A+(-\mu_B g_J\frac{1}{2}B+\mu_N g_I\frac{3}{2}B)\,.$$

The basis of $\{F=2, m_F=1>$ and $F=1, m_f=1>\}$;

$$H_{subset1}=\begin{pmatrix}\frac{A_{22}+\sqrt{3}A_{23}}{2} & \frac{-\sqrt{3}A_{22}+A_{23}}{2}\\ \frac{A_{62}+\sqrt{3}A_{63}}{2} & \frac{-\sqrt{3}A_{62}+A_{63}}{2}\end{pmatrix}$$

$$\begin{aligned}\lambda_{11}&=\frac{1}{4}(-4g_I\mu_N B-A-2\sqrt{(g_J\mu_B+g_I\mu_N)^2B^2+2A(g_J\mu_B+g_I\mu_N)B+4A^2})\\ &\approx-\frac{5}{4}A-\frac{1}{4}(g_J\mu_B+5g_I\mu_N)B+\frac{3}{32}\frac{(g_J\mu_B+g_I\mu_N)^2B^2}{A}+\frac{3}{128}\frac{(g_J\mu_B+g_I\mu_N)^3B^3}{A^2}+O(B)^4\end{aligned}$$

$$\begin{aligned}\lambda_{12}&=\frac{1}{4}(-4g_I\mu_N B-A+2\sqrt{(g_J\mu_B+g_I\mu_N)^2B^2+2A(g_J\mu_B+g_I\mu_N)B+4A^2})\\ &\approx\frac{3}{4}A+\frac{1}{4}(g_J\mu_B-3g_I\mu_N)B+\frac{3}{32}\frac{(g_J\mu_B+g_I\mu_N)^2B^2}{A}-\frac{3}{128}\frac{(g_J\mu_B+g_I\mu_N)^3B^3}{A^2}+O(B)^4\end{aligned}$$

The basis of $\{F=2, m_f=0>$ and $F=1, m_f=0>\}$;

$$H_{subset2}=\begin{pmatrix}\frac{A_{34}+A_{35}}{\sqrt{2}} & \frac{-A_{34}+A_{35}}{\sqrt{2}}\\ \frac{A_{74}+A_{75}}{\sqrt{2}} & \frac{-A_{74}+A_{75}}{\sqrt{2}}\end{pmatrix}.$$

The eigenvalues are

$$\begin{aligned}\lambda_{21}&=\frac{1}{4}(-A-2\sqrt{(g_J\mu_B+g_I\mu_N)^2B^2+4A^2})\\ &\approx-\frac{5}{4}A-\frac{(g_J\mu_B+g_I\mu_N)^2B^2}{8A}+O(B)^4\end{aligned}$$

and

$$\begin{aligned}\lambda_{22}&=\frac{1}{4}(-A+2\sqrt{(g_J\mu_B+g_I\mu_N)^2B^2+4A^2})\\ &\approx\frac{3}{4}A+\frac{(g_J\mu_B+g_I\mu_N)^2B^2}{8A}+O(B)^4\end{aligned}$$

The basis of { $F = 2, m_f = -1>$ and $F = 1, m_f = -1>$ }

$$H_{subset3} = \begin{pmatrix} \frac{\sqrt{3}A_{46} + A_{47}}{2} & \frac{-A_{46} + \sqrt{3}A_{47}}{2} \\ \frac{\sqrt{3}A_{86} + A_{87}}{2} & \frac{-A_{86} + \sqrt{3}A_{87}}{2} \end{pmatrix}$$

The eigenvalues are obtained as

$$\lambda_{31} = -\frac{1}{4}A + g_I\mu_N B - \frac{1}{2}\sqrt{(g_J\mu_B + g_I\mu_N)^2 B^2 - 2A(g_J\mu_B + g_I\mu_N)B + 4A^2}$$
$$\approx -\frac{5}{4}A + \frac{1}{4}(g_J\mu_B + 5g_I\mu_N)B - \frac{3}{32}\frac{(g_J\mu_B + g_I\mu_N)^2 B^2}{A}$$
$$-\frac{3}{128}\frac{(g_J\mu_B + g_I\mu_N)^3 B^3}{A^2} + O(B)^4$$

$$\lambda_{32} = -\frac{1}{4}A + g_I\mu_N B + \frac{1}{2}\sqrt{(g_J\mu_B + g_I\mu_N)^2 B^2 - 2A(g_J\mu_B + g_I\mu_N)B + 4A^2}$$
$$\approx \frac{3}{4}A + \frac{1}{4}(-g_J\mu_B + 3g_I\mu_N)B + \frac{3}{32}\frac{(g_J\mu_B + g_I\mu_N)^2 B^2}{A}$$
$$+\frac{3}{128}\frac{(g_J\mu_B + g_I\mu_N)^3 B^3}{A^2} + O(B)^4$$

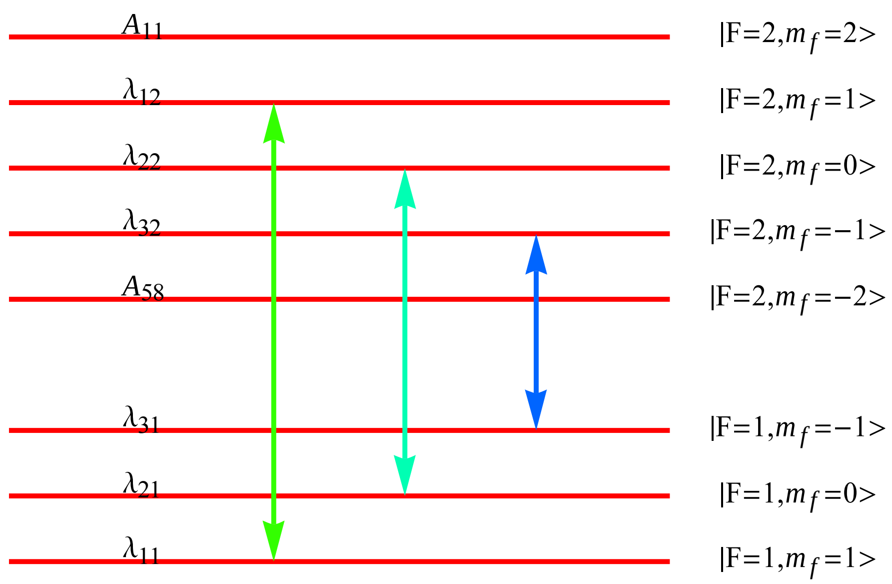



**Fig.29** In the presence of a strong magnetic field, there occur mixed states. (i) Mixed states of $|F = 2, m_f = 1\rangle$ and $|F = 1, m_f = 1\rangle$. (ii) Mixed states of

$\left|F=2, m_f=0\right\rangle$ and $\left|F=1, m_f=0\right\rangle$. (iii) Mixed state of $\left|F=2, m_f=-1\right\rangle$ and $\left|F=1, m_f=-1\right\rangle$.

## 21. Simulation for the quadratic Zeeman effect

The nuclear magneton $\mu_N$ is

$$\mu_N = 5.05078324\times 10^{-24} \text{ emu (erg/Oe).}$$

The Bohr magneton $\mu_B$ is

$$\mu_B = 9.27400915\times 10^{-21} \text{ emu.}$$

The mass of proton is

$$m_p = 1.6726231 \text{ x } 10^{-27} \text{ kg}$$

The mass of electron is

$$m_e = 9.1093897 \text{ x } 10^{-31} \text{ kg}$$

The nucleus has a magnetic moment $\boldsymbol{\mu}_I$ that is related to the nuclear spin $I$ by

$$\boldsymbol{\mu}_I = g_I \mu_N \mathbf{I}$$

We assume that $g_I = 1$.

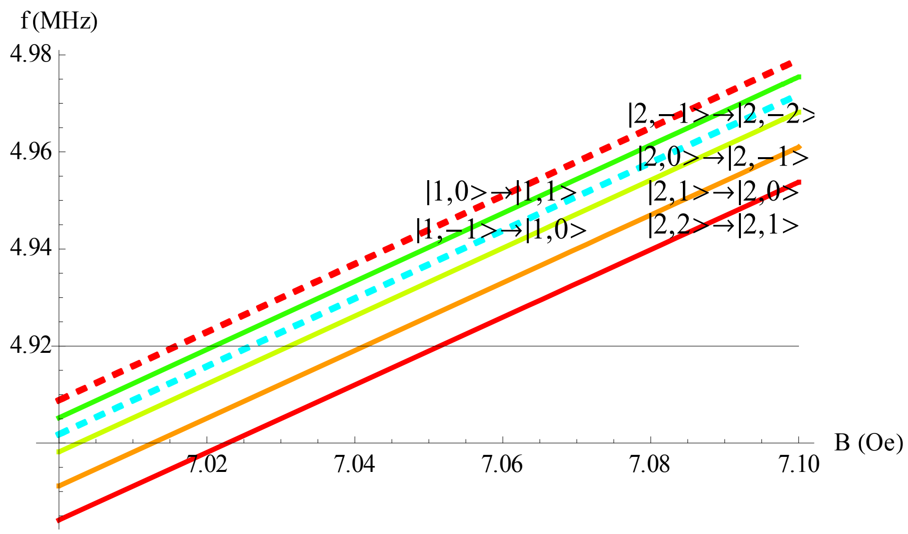


**Fig.30** The magnetic field dependence of the separation of the Zeeman levels, $f$ [MHz]. (i) $|F=2, m_f=2\rangle$ and $|F=2, m_f=1\rangle$. (ii) $|F=2, m_f=1\rangle$ and $|F=2, m_f=0\rangle$. (iii) $|F=2, m_f=0\rangle$ and $|F=2, m_f=-1\rangle$. (iv) $|F=2, m_f=-1\rangle$ and $|F=2, m_f=-2\rangle$. (v) $|F=1, m_f=-1\rangle$ and $|F=1, m_f=0\rangle$ (blue dashed line). (vi) $|F=1, m_f=0\rangle$ and $|F=1, m_f=1\rangle$ (red dashed line). $g_I = 1$ (assumed).

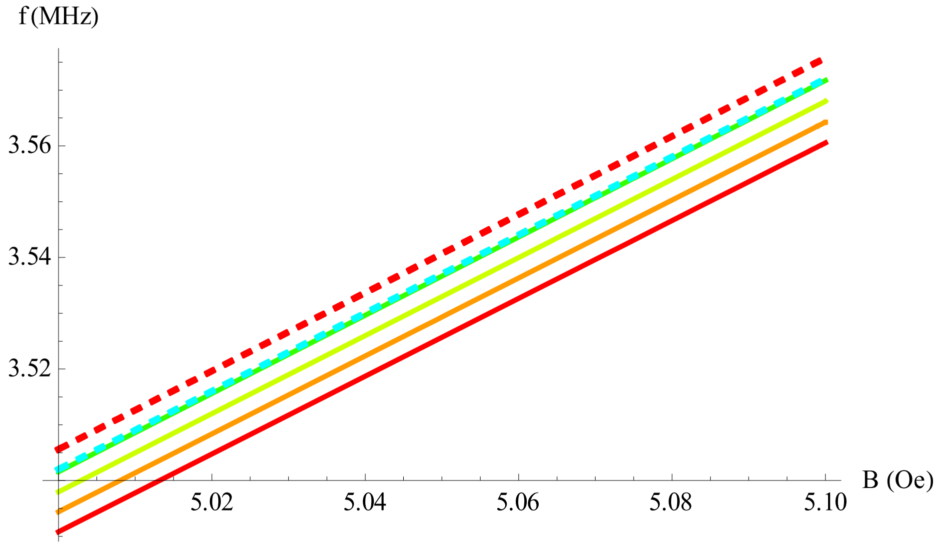



**Fig.31** The separation of the Zeeman levels, $f$ [MHz] as a function of $B$ around $B$ = 5.0 Oe. $g_I = 1$ (assumed).

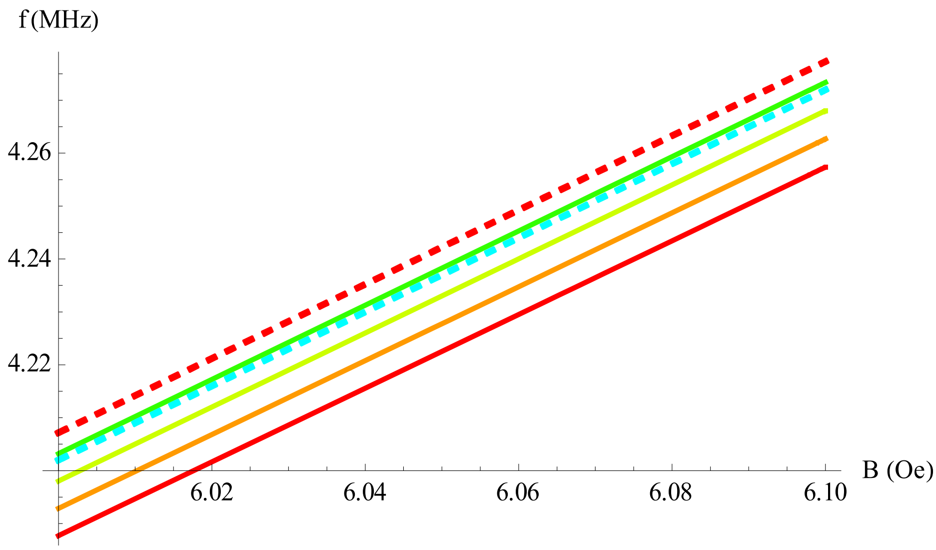


**Fig.32** The separation of the Zeeman levels, $f$ [MHz] as a function of $B$ around $B$ = 6.0 Oe. $g_I$ = 1 (assumed).

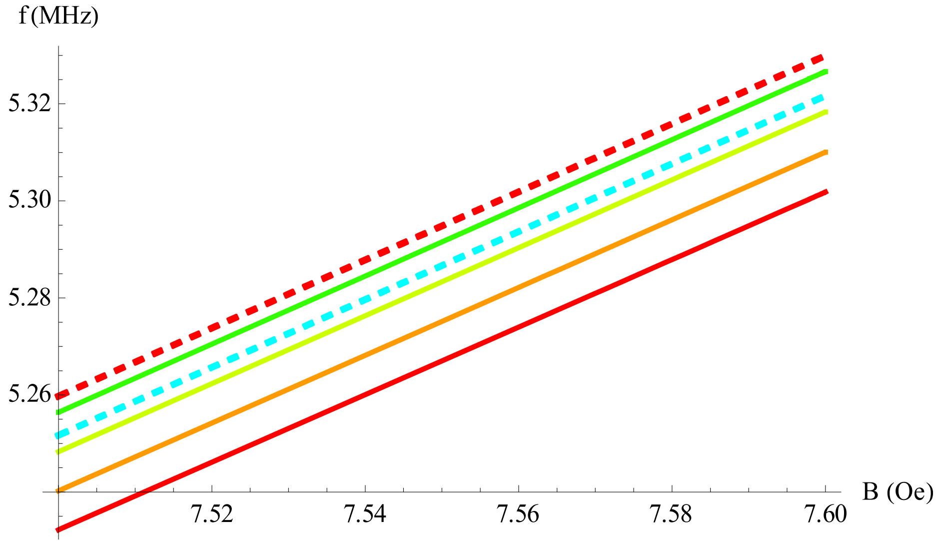



**Fig.33** The separation of the Zeeman levels [MHz] as a function of $B$ around $B$ = 7.5 Oe. $g_I$ = 1 (assumed).

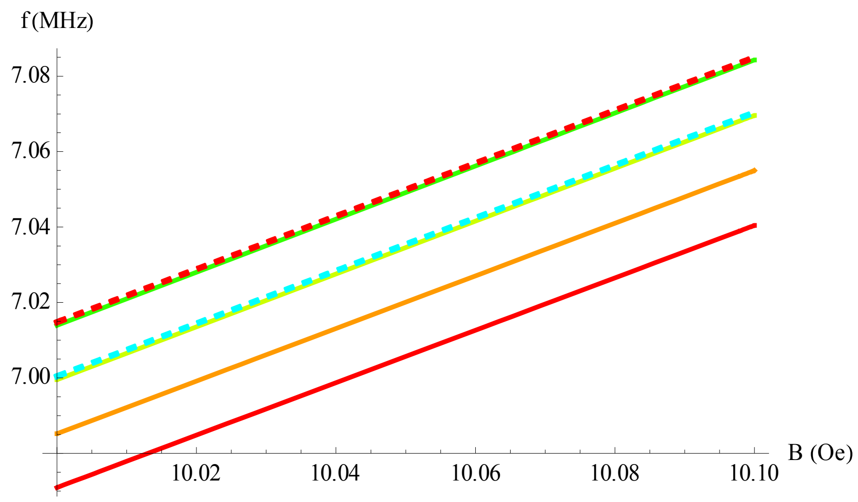



**Fig.34** The separation of the Zeeman levels, $f$ [MHz] as a function of $B$ around $B$ = 10.0 Oe. $g_I$ = 1 (assumed).

## 22. Energy levels of $^{87}$Rb (simulation)

We calculate the Zeeman splitting in $^{87}$Rb from the theory.

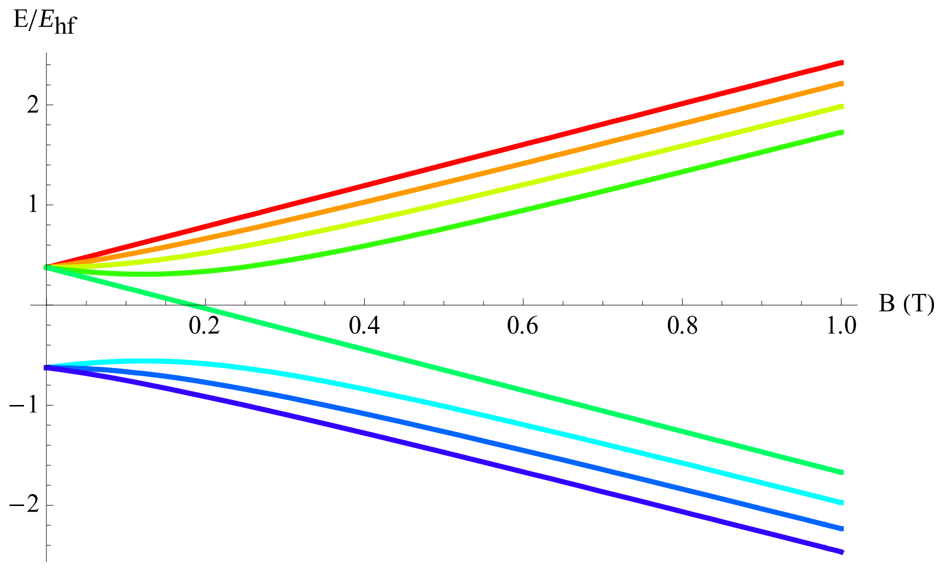



**Fig.35** Energy levels of $^{87}$Rb. $g_I = 1$ (assumed). $B = 0$ - 1.0 T.

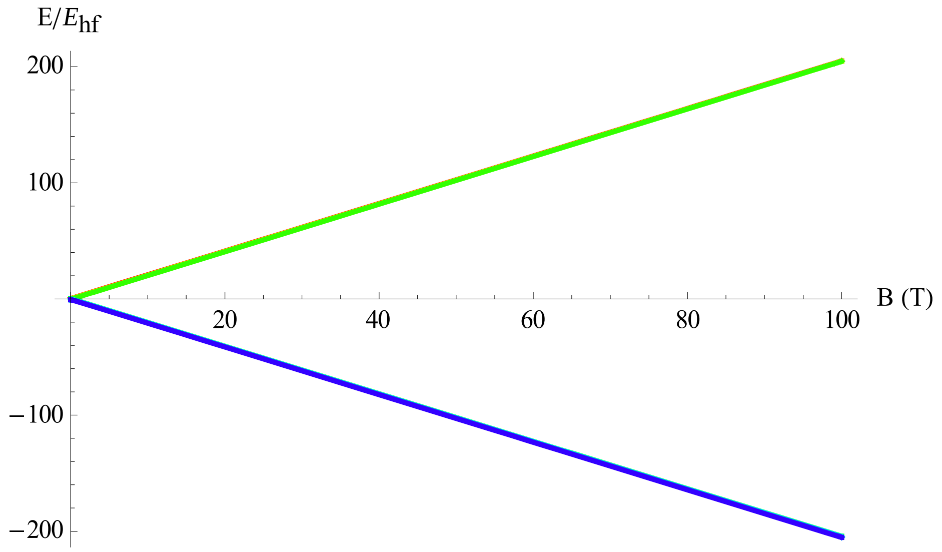



**Fig.36** Energy levels of $^{87}$Rb. $g_I = 1$ (assumed). $B = 0$ - 100 T. Paschen-Back effect.

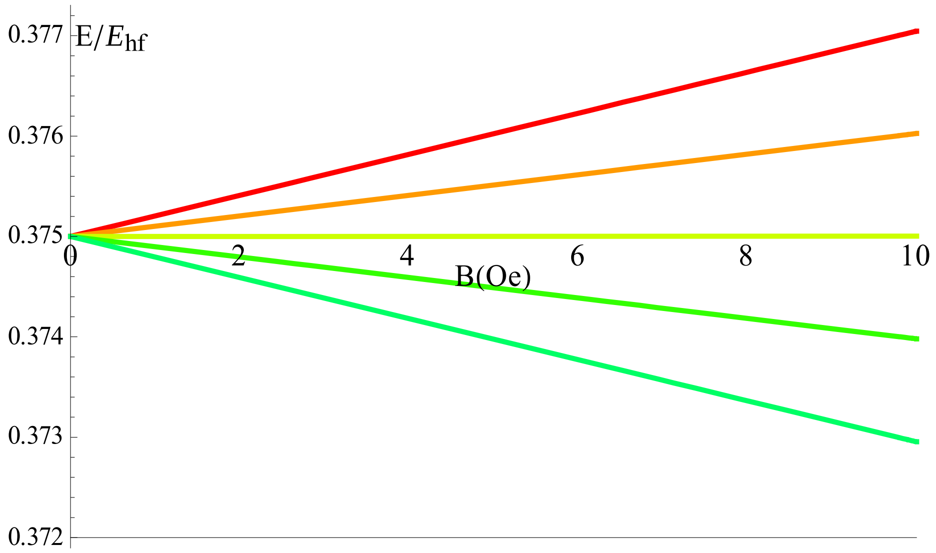



**Fig.37** Energy levels of $^{87}$Rb around 5 $^2P_{1/2}$. $g_I = 1$ (assumed). $B = 0$ - 10 Oe.

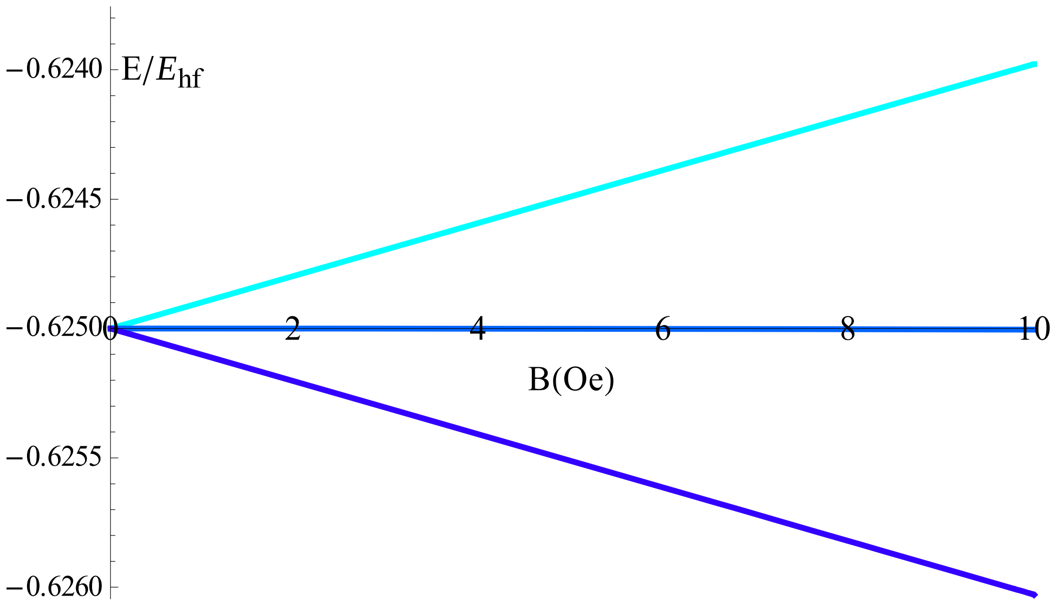



**Fig.38** Energy levels of $^{87}$Rb around 5 $^2S_{1/2}$. $g_I = 1$ (assumed). $B = 0$ - 10 Oe.

## 23. Frequency vs Earth magnetic field (at Binghamton): *B* = 0.3 Oe.

The frequencies (MHz) of the six separations between adjacent Zeeman levels for B ≈ 0.3 Oe at Binghamton, NY.

Table 1

B (Oe)

| | | | | | | |
|---|---|---|---|---|---|---|
| 0.3 | 0.209753 | 0.209766 | 0.209779 | 0.209792 | 0.210236 | 0.210223 |
| 0.31 | 0.216744 | 0.216758 | 0.216771 | 0.216785 | 0.217244 | 0.21723 |
| 0.32 | 0.223735 | 0.22375 | 0.223764 | 0.223779 | 0.224252 | 0.224237 |
| 0.33 | 0.230726 | 0.230742 | 0.230757 | 0.230773 | 0.23126 | 0.231245 |
| 0.34 | 0.237717 | 0.237734 | 0.23775 | 0.237767 | 0.238268 | 0.238252 |
| 0.35 | 0.244708 | 0.244725 | 0.244743 | 0.244761 | 0.245277 | 0.245259 |
| 0.36 | 0.251699 | 0.251717 | 0.251736 | 0.251754 | 0.252285 | 0.252266 |
| 0.37 | 0.25869 | 0.258709 | 0.258729 | 0.258748 | 0.259293 | 0.259273 |
| 0.38 | 0.26568 | 0.265701 | 0.265722 | 0.265742 | 0.266301 | 0.26628 |
| 0.39 | 0.272671 | 0.272693 | 0.272715 | 0.272737 | 0.273309 | 0.273287 |
| 0.4 | 0.279662 | 0.279685 | 0.279708 | 0.279731 | 0.280318 | 0.280295 |
| 0.41 | 0.286652 | 0.286677 | 0.286701 | 0.286725 | 0.287326 | 0.287302 |
| 0.42 | 0.293643 | 0.293668 | 0.293694 | 0.293719 | 0.294334 | 0.294309 |
| 0.43 | 0.300634 | 0.30066 | 0.300687 | 0.300713 | 0.301342 | 0.301316 |
| 0.44 | 0.307624 | 0.307652 | 0.30768 | 0.307708 | 0.308351 | 0.308323 |
| 0.45 | 0.314615 | 0.314644 | 0.314673 | 0.314702 | 0.315359 | 0.31533 |
| 0.46 | 0.321605 | 0.321636 | 0.321666 | 0.321696 | 0.322367 | 0.322337 |
| 0.47 | 0.328596 | 0.328627 | 0.328659 | 0.328691 | 0.329375 | 0.329344 |
| 0.48 | 0.335586 | 0.335619 | 0.335652 | 0.335685 | 0.336384 | 0.336351 |
| 0.49 | 0.342576 | 0.342611 | 0.342645 | 0.34268 | 0.343392 | 0.343358 |
| 0.5 | 0.349567 | 0.349602 | 0.349638 | 0.349674 | 0.3504 | 0.350365 |

**REFERENCES**


A. Kastler, J. Phys. Radium **11**, 225 (1950).
A. Kastler, Compt. Rend. **232**, 953 (1951).
C.J. Foot, Atomic Physics (Oxford University Press, Oxford, 2005).
A. Corney, Atomic and Laser Spectroscopy (Clarendon Press, Oxford, 1977).
W. Happer, Rev. Mod. Phys. **44**, 169 (1972).
R. Benumof, Am. J. Phys. **33**, 151 (1965).
N. F. Ramsey, Nuclear Moments (Wiley, New York,1953).
N. F. Ramsey, Molecular Beams (Clarendon Press, Oxford, 1956).
N.F. Ramsey, Phys. Rev. **76**, 996 (1949).
T.R. Carver, Science **141**, 599 (1963).
G. Breit and I.I. Rabi, Phys. Rev. **38**, 2082 (1931).
G. Breit, Rev. Mod. Phys. **5**, 91 (1933).
D.A. Steck, "Rubidium 87 D Line Data," available online at http://steck.us/alkalidata (revision2.0.1, 2, May 2008).
R.L. de Zafra, Am. J. Phys. **28**, 646 (1960).
A.L. Bloom, Sci. Am. October 1960, p.72.
A.L. Bloom, Appl. Opt. **1**, 61 (1962).
C. Cohen-Tannoudji, J. Physique **24**, 653 (1963).
TeachSpin, Inc. “Optical Pumping of Rubidium”
Stuart Deutsch, unpublished report (Phys.429, Senior Laboratory, Binghamton University, 2005).
David Boyle, unpublished report (Phys.429 Senior Laboratory, Binghamton University, 2006)
Colin Edmunds, unpublished report (Phys.429, Senior Laboratory, Binghamton University, 2009).

## APPENDIX

### A1. Experimental Results obtained by David Boyle (Senior Lab, Binghamton University, 2006)

### (a) Low field Zeeman effect

To determine the $g_F$-factors of each Rubidium isotope, it was necessary to acquire resonance transition data. Using the geometry of the coils, the value of the Bohr magneton and the measured frequency, a plot of magnetic field as a function of current could be used to determine the $g_F$-factors. The RF coils were activated at 10 kHz intervals between 100 kHz and 200 kHz. At each of these frequencies, the sweep field was manually adjusted to center on one resonance peak for each isotope and the current generating the field was recorded. Figure 9 depicts the zero field peak along with the resonance peaks for the two Rubidium isotopes.

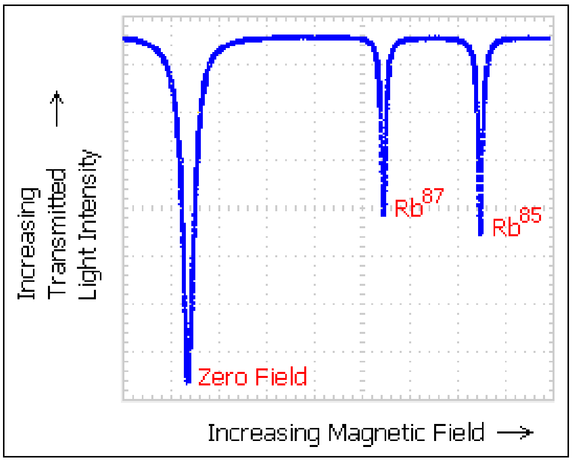


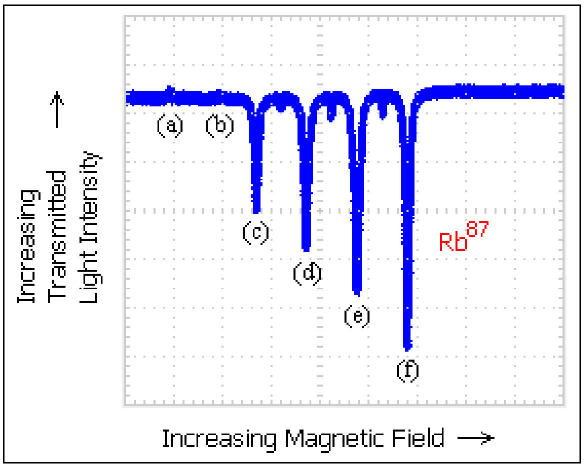

**Fig. A-1** – Zero field and resonance transitions for $Rb^{87}$ and $Rb^{85}$ at 120 ± 5 kHz

### (c) Quadratic Zeeman effect

The perturbation applied was larger than the linear Zeeman effect, and the splitting of the $2F + 1$ magnetic levels was no longer linear in the magnetic field (hence the spacing between the levels was no longer equal). $2F$ resonances (with $\Delta F = 0$, $\Delta m_f = \pm 1$) could now be observed for each atom, which translated to six for $^{87}$Rb and ten for $^{85}$Rb. To observe the resonances, the intensity of transmitted light and magnetic field current were again monitored on an oscilloscope. $^{87}$Rb resonances were investigated in the frequency range of 4.70 ± 0.01 MHz to 5.30 ± 0.02 MHz in 0.1 MHz increments. To find the grouping of resonances at each frequency, the sweep field was first used to center the magnetic field intensity at the zero field peak (which is independent of RF frequency). The main field was then increased until the first resonance peak was observed, then the sweep field would be slowly increased to trace out the changing transmittance on the oscilloscope. A resonance plot for $^{87}$Rb is shown in Fig. A-2. The transitions can be represented in Dirac notation as

$$|F, m_f\rangle \rightarrow |F, m_f - 1\rangle$$

---



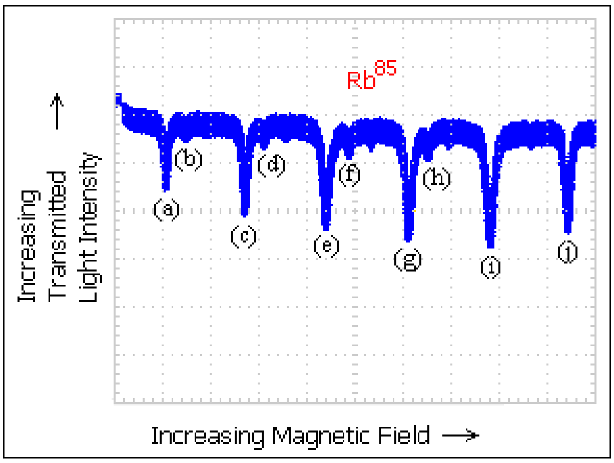

**(a)** $|1,0\rangle \rightarrow |1,-1\rangle$ **(d)** $|2,0\rangle \rightarrow |2,-1\rangle$

**(b)** $|1,1\rangle \rightarrow |1,0\rangle$ **(e)** $|2,1\rangle \rightarrow |2,0\rangle$

**(c)** $|2,-1\rangle \rightarrow |2,-2\rangle$ **(f)** $|2,2\rangle \rightarrow |2,1\rangle$

**Fig.A-2** Resonance transitions for $^{87}$Rb under the quadratic Zeeman effect.

$^{85}$Rb resonances were investigated in the frequency range of 3.30 ± 0.02 MHz to 3.70 ± 0.02 MHz in 0.1 MHz increments (Fig. A-3). The same procedure was followed as for $^{87}$Rb and the resulting trace can be seen in Fig. The small peaks between the resonance peaks for both isotopes correspond to double quantum transitions.




**(a)** $|3,-2\rangle \rightarrow |3-3\rangle$ **(f)** $|2,1\rangle \rightarrow |2,0\rangle$

**(b)** $|2,-1\rangle \to |2,-2\rangle$ **(g)** $|3,1\rangle \to |3,0\rangle$

**(c)** $|3,-1\rangle \to |3,-2\rangle$ **(h)** $|2,2\rangle \to |2,1\rangle$

**(d)** $|2,0\rangle \to |2,-1\rangle$ **(i)** $|3,2\rangle \to |3,1\rangle$

**(e)** $|3,0\rangle \to |3,-1\rangle$ **(j)** $|3,3\rangle \to |3,2\rangle$

**Fig.A-3** Resonance transitions for $^{85}$Rb under the quadratic Zeeman effect.

---

## A2 Interaction with radiation field

### A2.1. Hamiltonian

The classical radiation field ($\hat{p}$: operator of the system, quantum mechanical operator)

$$\begin{aligned}\hat{H} &= \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2 \\ &= \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\cdot\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right) \\ &= \frac{1}{2m}\left[\hat{\mathbf{p}}^2 + \frac{e^2}{c^2}\mathbf{A}^2 - \frac{e}{c}(\mathbf{A}\cdot\hat{\mathbf{p}} + \hat{\mathbf{p}}\cdot\mathbf{A})\right]\end{aligned}$$

where $e$ is the charge of electron $e = -|e|$. We note that

$$\begin{aligned}(\mathbf{A}\cdot\hat{\mathbf{p}} + \hat{\mathbf{p}}\cdot\mathbf{A})\psi(\mathbf{r}) &= \mathbf{A}\cdot\frac{\hbar}{i}\nabla\psi(\mathbf{r}) + \frac{\hbar}{i}\nabla\cdot(\mathbf{A}\psi(\mathbf{r})) \\ &= \mathbf{A}\cdot\frac{\hbar}{i}\nabla\psi(\mathbf{r}) + \frac{\hbar}{i}\nabla\cdot(\mathbf{A}\psi(\mathbf{r})) \\ &= \mathbf{A}\cdot\frac{\hbar}{i}\nabla\psi(\mathbf{r}) + \frac{\hbar}{i}(\nabla\psi(\mathbf{r})\cdot\mathbf{A} + \psi(\mathbf{r})\nabla\cdot\mathbf{A}) \\ &= \frac{2\hbar}{i}\mathbf{A}\cdot\nabla\psi(\mathbf{r}) + \frac{\hbar}{i}\psi(\mathbf{r})(\nabla\cdot\mathbf{A})\end{aligned}$$

Thus

$$\hat{H} = \frac{1}{2m}\left[\hat{\mathbf{p}}^2 + \frac{e^2}{c^2}\mathbf{A}^2 - \frac{2e}{c}\mathbf{A}\cdot\hat{\mathbf{p}} - \frac{e\hbar}{ic}(\nabla\cdot\mathbf{A})\right]$$

Here we use the Coulomb gauge $\nabla\cdot\mathbf{A} = 0$. Then

$$\begin{cases} \hat{H}' = -\dfrac{e}{mc}\mathbf{A}\cdot\hat{\mathbf{p}} = \dfrac{|e|}{mc}\mathbf{A}\cdot\hat{\mathbf{p}} \\ \hat{H}'' = \dfrac{e^2}{2mc^2}\mathbf{A}^2 \end{cases}$$

## A2.2. Classical radiation field

Maxwell's equation

$$\begin{cases} \nabla\cdot\mathbf{E} = 4\pi\rho \\ \nabla\times\mathbf{E} = -\dfrac{1}{c}\dfrac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{B} = \dfrac{4\pi}{c}\mathbf{j} + \dfrac{1}{c}\dfrac{\partial\mathbf{E}}{\partial t} \end{cases}$$

$$\nabla\cdot\mathbf{j} + \frac{\partial\rho}{\partial t} = 0$$

$$\begin{cases} \mathbf{B} = \nabla\times\mathbf{A} \\ \mathbf{E} = -\dfrac{1}{c}\dfrac{\partial\mathbf{A}}{\partial t} - \nabla\phi \end{cases}$$

where $\boldsymbol{A}$ is a vector potential and $\phi$ is a scalar potential.

$$\begin{cases} \nabla^2\phi + \dfrac{1}{c}\dfrac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -4\pi\rho \\ \nabla^2\mathbf{A} - \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left(\nabla\cdot\mathbf{A} + \dfrac{1}{c}\dfrac{\partial\phi}{\partial t}\right) = -\dfrac{4\pi\mathbf{j}}{c} \end{cases}$$

**Coulomb gauge**

We start any pair of $\boldsymbol{A}$ and $\phi$. Using the Gauge transformation

$$\begin{cases} \mathbf{A}' = \mathbf{A} - \nabla\chi \\ \phi' = \phi + \dfrac{1}{c}\dfrac{\partial\chi}{\partial t} \end{cases}$$

we have a pair of $\boldsymbol{A}'$and $\phi'$, where

$$\nabla\cdot\mathbf{A}' = 0$$

or

$$\nabla\cdot(A - \nabla\chi) = 0$$

or

$$\nabla^2 \chi = \nabla \cdot \mathbf{A}$$

This is a Poisson equation with known value of $\nabla \cdot A$. The solution of χ is uniquely determined. Therefore we can always choose the Coulomb gauge with $\nabla \cdot \mathbf{A}' = 0$. Here we assume that

$$\nabla \cdot \mathbf{A} = 0 \text{ (Coulomb gauge)}$$

In the vacuum, we have

$$\rho = 0, \ \mathbf{j} = 0$$

$$\begin{cases} \nabla^2 \phi = 0 \\ \nabla^2 \mathbf{A} - \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\dfrac{1}{c}\dfrac{\partial \phi}{\partial t}\right) = 0 \end{cases}$$

From the first equation, we have $\phi = 0$

or

$$(\nabla \cdot \mathbf{A} = 0, \ \phi = 0)$$

Then we have

$$\nabla^2 \mathbf{A} - \frac{1}{c}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0, \text{ with } \nabla \cdot \mathbf{A} = 0$$

The solution is

$$\mathbf{A} = 2\mathbf{A}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t)'$$

where

$$\omega^2 = k^2 c^2 \text{ or } \omega = ck \text{ (Dispersion relation)}$$

Since

$$\nabla \cdot \mathbf{A} = -2(\mathbf{k} \cdot \mathbf{A}_0)\sin(\mathbf{k} \cdot \mathbf{r} - \omega t) = 0$$

we have

$$\mathbf{k} \cdot \mathbf{A}_0 = 0$$

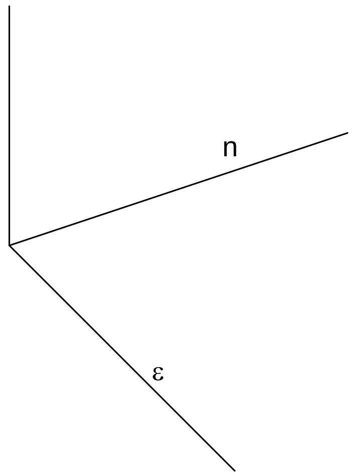



**Fig.A3** ***n***: unit vector of the propagating wavevector and $\boldsymbol{\varepsilon}$ is the polarization vector.

***A*** must lie in a plane perpendicular to the direction of the propagation vector.

$$\mathbf{A} \equiv 2|\mathbf{A}_0|\boldsymbol{\varepsilon}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t) = |\mathbf{A}_0|\boldsymbol{\varepsilon}\left[e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]$$

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = 2|\mathbf{A}_0|\boldsymbol{\varepsilon}\left(-\frac{1}{c}\right)\omega\sin(\mathbf{k}\cdot\mathbf{r}-\omega t) = -2|\mathbf{A}_0|\frac{\omega}{c}\boldsymbol{\varepsilon}\sin(\mathbf{k}\cdot\mathbf{r}-\omega t)$$

and

$$\mathbf{B} = \nabla\times\mathbf{A} = -2|\mathbf{A}_0|\frac{\omega}{c}(\mathbf{n}\times\boldsymbol{\varepsilon})\sin(\mathbf{k}\cdot\mathbf{r}-\omega t).$$

where $\hat{n}$ is the unit vector defined by $\mathbf{n} = \mathbf{k}/|\mathbf{k}|$. The electromagnetic energy is

$$\begin{aligned}\varepsilon &= \frac{1}{8\pi}\left(\mathbf{E}^2+\mathbf{B}^2\right) = \frac{1}{8\pi}\left(4|\mathbf{A}_0|^2\frac{1}{c^2}\omega^2 + 4|\mathbf{A}_0|^2\frac{1}{c^2}\omega^2\right)\sin^2(\mathbf{k}\cdot\mathbf{r}-\omega t)\\ &= \frac{\omega^2|\mathbf{A}_0|^2}{2\pi c^2}\left[1-\cos(2\mathbf{k}\cdot\mathbf{r}-2\omega t)\right]\end{aligned}$$

The time average of $\varepsilon$ over a period $T$ (= $2\pi/\omega$) is

$$\frac{1}{T}\int_0^T \frac{1}{8\pi}\left(\mathbf{E}^2+\mathbf{B}^2\right)dt = \frac{\omega^2|\mathbf{A}_0|^2}{2\pi c^2} = u\ (\mathrm{erg/cm^3}).$$

The Poynting vector ***S*** is defined by

$$\mathbf{S} = \frac{c}{4\pi}(\mathbf{E}\times\mathbf{B})$$
$$= 2|\mathbf{A}_0|\frac{\omega}{c}\hat{\varepsilon}\times\left[2|\mathbf{A}_0|\frac{\omega}{c}(\hat{n}\times\hat{\varepsilon})\right]\sin^2(\mathbf{k}\cdot\mathbf{r}-\omega t)$$
$$= 4|\mathbf{A}_0|^2\frac{\omega^2}{c^2}\hat{n}\frac{1}{2}[1-\cos(2\mathbf{k}\cdot\mathbf{r}-2\omega t)]$$

The time average of ***S*** over a period $T$ (= $2\pi/\omega$) is

$$s = \frac{1}{T}\int_0^T \mathbf{S}dt = \frac{|\mathbf{A}_0|^2\omega^2}{2\pi c}\hat{n} = cu \ (\text{erg/s}\cdot\text{cm}^2)\,.$$

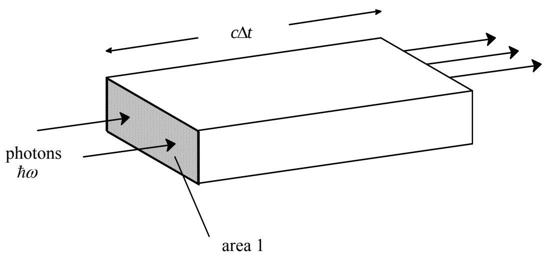



**Fig.A4** Poynting vector.

In summary we have

Energy density $u$;

$$u = \frac{\omega^2|\mathbf{A}_0|^2}{2\pi c^2} \ (\text{erg/cm}^3).$$

The intensity $s$; the energy flow per unit area per unit time.

$$s = cu = \frac{\omega^2|\mathbf{A}_0|^2}{2\pi c} \ (\text{erg /s cm}^2).$$

The flux of photons (the number of photons per unit area per unit time)

$$f = \frac{s}{\hbar\omega} = \frac{\omega|\mathbf{A}_0|^2}{2\pi\hbar c}\,.$$

## A2.3 Application — interaction with the classical radiation field

Classical radiation field

$\Rightarrow$ electric or magnetic field derivable from a classical radiation field as opposed to quantized field

$$\hat{H} = \frac{1}{2m}\hat{\mathbf{p}}^2 - e\phi(\hat{\mathbf{r}}) + \frac{|e|}{mc}\mathbf{A}\cdot\hat{\mathbf{p}}$$

$(e > 0) \Leftarrow$ We use $q$=-|$e$| (|e| > 0), which is justified if

$$\nabla\cdot\mathbf{A} = 0.$$

We work with a monochromatic field of the plane wave

$$\mathbf{A} = 2|\mathbf{A}_0|\boldsymbol{\varepsilon}\cos(\mathbf{k}\cdot\mathbf{r} - \omega t)$$

with

$$\mathbf{k} = \frac{\omega}{c}\mathbf{n}, \qquad \boldsymbol{\varepsilon}\cdot\mathbf{k} = 0$$

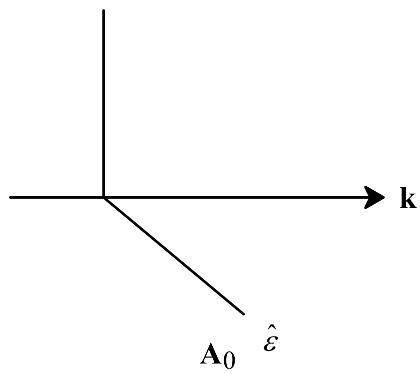



**Fig.A5** The direction of the vector potential A0 which is the same as that of the polarization vector $\boldsymbol{\varepsilon}$.

Then ***A*** can be rewritten as

$$\mathbf{A} = |\mathbf{A}_0|\boldsymbol{\varepsilon}\left[e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]$$

$$\Rightarrow \qquad \hat{H} = \hat{H}_0 + \hat{H}_1$$

$\hat{H}_1$: time dependent perturbation

$$\hat{H}_1 = \frac{e}{mc}|\mathbf{A}_0|\boldsymbol{\varepsilon}\cdot\hat{\mathbf{p}}\left[e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]$$

$$= \underset{\substack{\uparrow \\ \text{responsible} \\ \text{for} \\ \text{stimulated} \\ \text{emission}}}{\hat{H}_1^+ e^{-i\omega t}} + \underset{\substack{\uparrow \\ \text{responsible} \\ \text{for} \\ \text{absorption}}}{\hat{H}_1 e^{i\omega t}}$$

## A2.4 Stimulated emission and absorption

Using the matrix element given by

$$\left(\hat{H}_1^+\right)_{fi} = \frac{e|\mathbf{A}_0|}{mc}\left\langle\varphi_f\left|e^{i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\varepsilon}\cdot\hat{\mathbf{p}}\right|\varphi_i\right\rangle,$$

we have the Fermi's golden rule,

$$W_{i\to f} = \frac{2\pi}{\hbar}\frac{e^2}{m^2c^2}|\mathbf{A}_0|^2\left|\left\langle\varphi_f\left|e^{i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\varepsilon}\cdot\hat{\mathbf{p}}\right|\varphi_i\right\rangle\right|^2\delta\left(E_f - E_i \pm \hbar\omega\right)$$

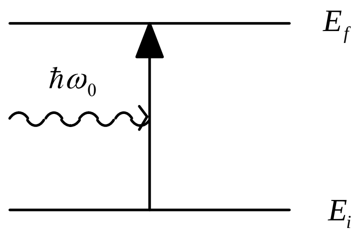
$$\begin{cases} u = \dfrac{\omega^2}{2\pi c^2}|\mathbf{A}_0|^2 \ (\text{erg/cm}^3) \to \overline{W}(\omega)d\omega\left(\text{erg}\dfrac{\text{s}}{\text{cm}^3}\dfrac{1}{\text{s}}\right) \\ s = cu\left(\text{erg}\dfrac{\text{s}}{\text{cm}^3}\right) \to I(\omega)d\omega \quad \left(I(\omega) = c\overline{W}(\omega)\right) \quad I = \left[\dfrac{\text{erg}\cdot\text{s}}{\text{cm}^3}\cdot\dfrac{\text{cm}}{\text{s}}\right] = \left[\text{erg}\dfrac{1}{\text{cm}^2}\right] \end{cases}$$

$$\frac{\omega^2}{2\pi c^2}|\mathbf{A}_0|^2 \to \overline{W}(\omega)d\omega = \frac{1}{c}I(\omega)d\omega$$

$$W_{i\to f} = \frac{2\pi}{\hbar}\int\frac{e^2}{m^2c^2}\left[\frac{2\pi c^2}{\omega^2}\overline{W}(\omega)\right]d\omega\left|\left\langle\varphi_f\left|e^{i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\varepsilon}\cdot\hat{\mathbf{p}}\right|\varphi_i\right\rangle\right|^2\delta\left(E_f - E_i \pm \hbar\omega\right)$$

Since $\delta\left(E_f - E_i \pm \hbar\omega\right) = \frac{1}{\hbar}\delta\left(\omega_0 \pm \omega\right)$,

$$W_{i\to f}^{(a)} = \frac{4\pi^2e^2}{\hbar^2m^2\omega_0^2}\overline{W}(\omega_0)\left|\left\langle\varphi_f\left|e^{i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\varepsilon}\cdot\hat{\mathbf{p}}\right|\varphi_i\right\rangle\right|^{22} \qquad \text{(absorption)}$$

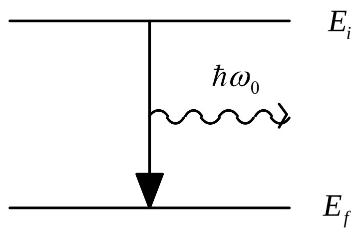



(absorption)

**Fig.A6** Absorption from the ground state ($E_i$) to the excited state ($E_f$) in a system with two-levels.

Similarly

$$W_{i\to f}^{(e)} = \frac{4\pi^2 e^2}{\hbar^2 m^2 \omega_0^2} \overline{W}(\omega_0) \left| \left\langle \varphi_f \left| e^{i\mathbf{k}\cdot\mathbf{r}} \boldsymbol{\varepsilon} \cdot \hat{\mathbf{p}} \right| \varphi_i \right\rangle \right|^2 \qquad \text{(stimulated emission)}$$




(stimulated emission)

**Fig.A7** Stimulated emission from the excited state to the ground state in a system with two-levels.

---

## A.3 Clebsch-Gordan coefficients using Mathematica

The Clebsch-Gordan coefficients can be easily evaluated using the Mathematica program. We show the example for the case of $j_1$=1 and $j_2$ = 1. Since $D_1 \times D_1 = D_2 + D_1 + D_0$, we have the three cases; $j$ = 2 ($m$ = -2, -1, 0, 1, and 2), $j$ = 1 ($m$ = -1, 0, and 1), and $j$ = 0 ($m$ = 0).

■

**j1=1 and j2=1**

```
Clear["Global`*"];

CG[j_, m_, j1_, j2_] :=
 Sum[ClebschGordan[{j1, m1}, {j2, m - m1}, {j, m}] a[j1 , m1]
   b[j2, m - m1], {m1, -j1, j1}]

CG[2, 2, 1, 1]
```

ClebschGordan::phy : ThreeJSymbol[{1, −1}, {1, 3}, {2, −2}] is not physical. ≫

ClebschGordan::phy : ThreeJSymbol[{1, 0}, {1, 2}, {2, −2}] is not physical. ≫

$$a[1,1]\,b[1,1]$$

```
CG[2, 1, 1, 1]
```

ClebschGordan::phy : ThreeJSymbol[{1, −1}, {1, 2}, {2, −1}] is not physical. ≫

$$\frac{a[1,1]\,b[1,0]}{\sqrt{2}}+\frac{a[1,0]\,b[1,1]}{\sqrt{2}}$$

```
CG[2, 0, 1, 1]
```

$$\frac{a[1,1]\,b[1,-1]}{\sqrt{6}}+\sqrt{\frac{2}{3}}\,a[1,0]\,b[1,0]+\frac{a[1,-1]\,b[1,1]}{\sqrt{6}}$$

```
CG[2, -1, 1, 1]
```

ClebschGordan::phy : ThreeJSymbol[{1, 1}, {1, −2}, {2, 1}] is not physical. ≫

$$\frac{a[1,0]\,b[1,-1]}{\sqrt{2}}+\frac{a[1,-1]\,b[1,0]}{\sqrt{2}}$$

```
CG[2, -2, 1, 1]
```

ClebschGordan::phy : ThreeJSymbol[{1, 0}, {1, −2}, {2, 2}] is not physical. ≫

ClebschGordan::phy : ThreeJSymbol[{1, 1}, {1, −3}, {2, 2}] is not physical. ≫

$$a[1,-1]\,b[1,-1]$$

```
CG[1, 1, 1, 1]
```

ClebschGordan::phy : ThreeJSymbol[{1, −1}, {1, 2}, {1, −1}] is not physical. ≫

$$\frac{a[1,1]\,b[1,0]}{\sqrt{2}}-\frac{a[1,0]\,b[1,1]}{\sqrt{2}}$$

```
CG[1, -1, 1, 1]
```

ClebschGordan::phy : ThreeJSymbol[{1, 1}, {1, −2}, {1, 1}] is not physical. ≫

$$\frac{a[1,0]\,b[1,-1]}{\sqrt{2}}-\frac{a[1,-1]\,b[1,0]}{\sqrt{2}}$$

```
CG[0, 0, 1, 1]
```

$$\frac{a[1,1]\,b[1,-1]}{\sqrt{3}}-\frac{a[1,0]\,b[1,0]}{\sqrt{3}}+\frac{a[1,-1]\,b[1,1]}{\sqrt{3}}$$