Superconductivity

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
(Date: November 15, 2018)

First I tried to present just an introduction of the superconductivity including the BCS (Bardeen-Cooper-Schrieffer) theory in the Blackboard (this system is adopted in SUNY for the presentation of documents related to the Class). However, I realized that it was very difficult for me to write such a brief article on a variety of physics of superconductivity for a short time. The superconductivity is now one of the most important fields in the condensed matter physics. So I decided to write a longer review on the superconductivity, which may be useful for both the graduate students and undergraduate students who want to understand the physics. To this end, I spent a lot of times in reviewing many textbooks and papers (see the References), and my old lecture note of solid state physics after the Fall semester (2011) was over. Consequently I have finished writing this article, which is presented here.

After getting a Ph.D. in physics from the University of Tokyo, and doing a research at the Ochanomizu University (Tokyo, Japan), I came to the University of Illinois at Urbana-Champaign (UIUC). I did my research, collaborating with Prof. Hartmut Zabal (currently, Ruhr-Universität Bochum, Germany) in July 1984 (I stayed at UIUC for the period between 1984 and 1985). First I really realized that a brilliant theory of BCS was born from such a nice campus. I often saw Prof. Bardeen reading news paper at the library of the Physics Department. Prof. William McMillan had worked hard using his computers in basement of his home. At that times I was very interested in his ongoing model of the spin glass (so-called domain model or droplet model). I also saw Prof. David Pines (Bohm-Pines theory) and Prof. Anthony Leggett (Nobel laureate for superfluidity of liquid ^{3}He) in Physics Colloquium. In August, 1984, Prof. McMillan unfortunately died because of car accident. So I missed the opportunity to learn about his exciting model from him.

Recently I read a book titled "*Genius, The life and science of John Bardeen*, which was written by Hoddeson and Daitch. I was very impressed by this book. It vividly explained how the BCS theory was born from the hands of three distinguished scientists in the Department of Physics (UIUC). I also noticed what happened to the fate of the quantum theory of the charge density wave (CDW) proposed by Prof. Bardeen, around 1984 and 1985.

In 1986, the high temperature T_c superconductors were discovered by Dr. Georg Bednorz and Dr. Karl Műller (IBM, Zűrich). Many physicists were so excited at the big news. They attended the American Physical Society (APS) March Meeting 1987 (Spring) (Hilton Hotel, NY City), in order to catch up with the ongoing research on the high T_c superconductors such as $YBa_2Cu_3O_{7-\delta}$, so on, which cannot be explained by the BCS theory. This March Meeting was called a Woodstock of Physics in New York Times. Soon after I moved from Champaign (Illinois) to Binghamton (NY) (1986 Fall) as a professor of Physics, Prof. Ivar Giaever (Rensslaer Polytechnic Institute, Troy, NY) came to our campus (SUNY at Binghamton, NY) and gave an excellent and impressive talk on his discovery of electron tunneling in superconductors, evidence of energy gaps in superconductors (although his interest changed into the physics of brains). So I had a chance to see him in the Colloquium.

John Bardeen (May 23, 1908 – January 30, 1991) was an American physicist and electrical engineer, the only person to have won the Nobel Prize in Physics twice: first in 1956 with William Shockley and Walter Brattain for the invention of the transistor; and again in 1972 with Leon Neil Cooper and John Robert Schrieffer for a fundamental theory of conventional superconductivity known as the BCS theory.

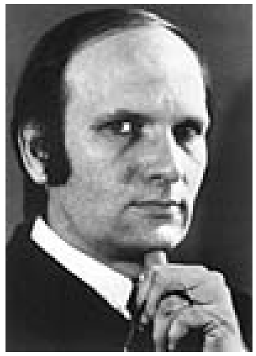
http://en.wikipedia.org/wiki/John_Bardeen

Leon N Cooper (born February 28, 1930) is an American physicist and Nobel Prize laureate, who with John Bardeen and John Robert Schrieffer, developed the BCS theory of superconductivity. He is also the namesake of the Cooper pair and co-developer of the BCM theory of synaptic plasticity.

http://en.wikipedia.org/wiki/Leon_Neil_Cooper

By late February or early March of 1956, it seemed clear that if somehow the entire ground state could be composed of such pairs, one would have a ground state with qualitatively different properties from the normal state. And this ground state - the state of superconductivity - would be separated from the excited states by an energy gap.
(from *True Genius, the life and science of John Bardeen*, by L. Hoddeson and V. Daitch)

John Robert Schrieffer (born May 31, 1931) is an American physicist and, with John Bardeen and Leon N Cooper, recipient of the 1972 Nobel Prize for Physics for developing the BCS theory, the first successful microscopic theory of superconductivity.



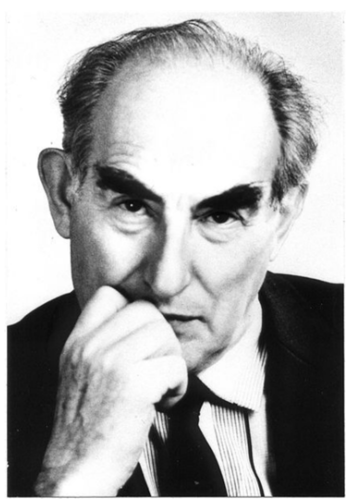
http://en.wikipedia.org/wiki/John_Robert_Schrieffer

Schrieffer worked more on the expression that night at his friend's house. In the morning, he did a variational calculation to determine the gap equation. "I solved the gap equation for the cut off potential. It was just a few hours work." Expanding the expression, he found he had written down a product of mathematical operators on the vacuum that expressed adding electrons to the vacuum. In his sum of a series of terms, each one corresponded to a different total number of pairs. He could hardly believe it. The expression "was really ordered in momentum space" and the ground state energy "was exponentially lower in energy," as required for the state to be stable.
(from *True Genius, the life and science of John Bardeen*, by L. Hoddeson and V. Daitch)

Lev Davidovich Landau (Russian language: Ле́в Дави́дович Ланда́у; January 22 [O.S. January 9] 1908– April 1, 1968) was a prominent Soviet physicist who made fundamental contributions to many areas of theoretical physics. His accomplishments include the co-discovery of the density matrix method in quantum mechanics, the quantum mechanical theory of diamagnetism, the theory of superfluidity, the theory of second-order phase transitions, the Ginzburg–Landau theory of superconductivity, the theory of Fermi liquid, the explanation of Landau damping in plasma physics, the Landau pole in quantum electrodynamics, and the two-component theory of neutrinos. He received the 1962 Nobel Prize in Physics for his development of a mathematical theory of superfluidity that accounts for the properties of liquid helium II at a temperature below 2.17 K.

http://en.wikipedia.org/wiki/Lev_Landau

Vitaly Lazarevich Ginzburg ForMemRS (Russian: Вита́лий Ла́заревич Ги́нзбург; October 4, 1916 – November 8, 2009) was a Soviet theoretical physicist, astrophysicist, Nobel laureate, a member of the Russian Academy of Sciences and one of the fathers of Soviet hydrogen bomb. He was the successor to Igor Tamm as head of the Department of Theoretical Physics of the Academy's physics institute (FIAN), and an outspoken atheist.



http://en.wikipedia.org/wiki/Vitaly_Ginzburg

((Note)) Here I present the topics of superconductivity (Josephson effect will be discussed in other chapter). I use the c.g.s. units. I use B for the magnetic induction (internal magnetic field) and H for the external magnetic field.

---

1. Disappearance of resistivity below T_c

We show that the Meissner effect (B = 0) is the fundamental properties of the superconductor. B = 0 cannot be derived from the property of ρ = 0.

$$\mathbf{E} = \rho\mathbf{J} \qquad \text{(Ohm's law).}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{J}$ is the current density, and ρ is the resistivity. When $\rho \to 0$ ($\boldsymbol{J}$ = constant), $\boldsymbol{E}$ must be zero. Using the Maxwell's equation, we have

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 .$$

The zero resistivity implies

$$\frac{\partial \mathbf{B}}{\partial t} = 0 ,$$

which is different from $\boldsymbol{B}$ = 0 (Meissner effect). In other words, if $\boldsymbol{B}$ = 0, we can say that $\frac{\partial \mathbf{B}}{\partial t} = 0$. But even if $\frac{\partial \mathbf{B}}{\partial t} = 0$, we cannot say that $\boldsymbol{B}$ = 0.

((**Persistent current**))

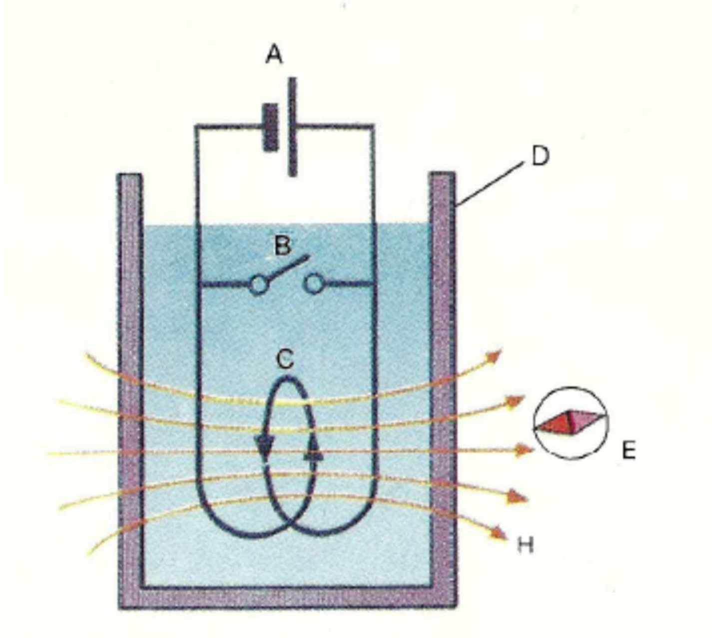
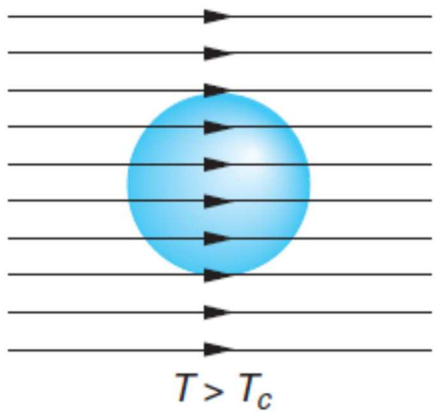
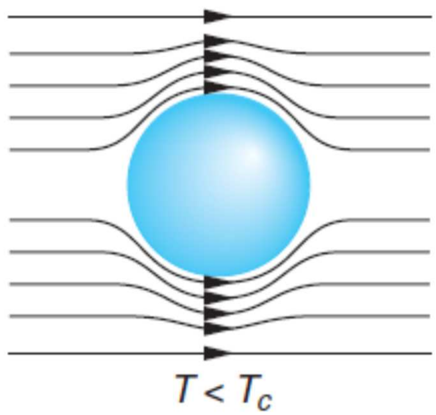


Fig. Schematic diagram for the persistent current. B is a thermal switch. It is closed so that the current circulates in the superconducting contour inside cryostat D, filled with liquid He. C is a superconducting ring which generates a magnetic field. The persistent current is used for the generation of an extremely stable magnetic field for the SQUID magnetometer.

2. Meissner effect: Meissner and Ochsenfeld (1933)

A bulk superconductor in a weak magnetic field will act as a perfect diamagnet with zero magnetic induction in the interior.





When a specimen is placed in a magnetic field and is then cooled through the critical temperature for superconductivity, the magnetic flux originally present is ejected from the specimen.

The demagnetiztion field contribution is negligible.

$$B = H + 4\pi M = 0,$$

where M (emu/cm^3) is the magnetization and H (Oe) is the external magnetic field.

$$H = -4\pi M.$$

((Note)) The unit of H is Oe. 1T = 10^4 Oe. 1 Gauss = 1 Oe.

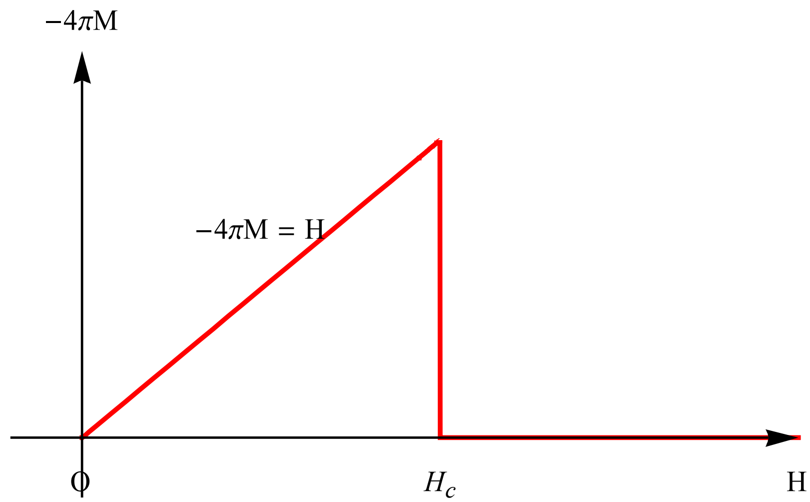


Fig. (-4πM) vs an external field H for the type-I superconductor. M is the magnetization

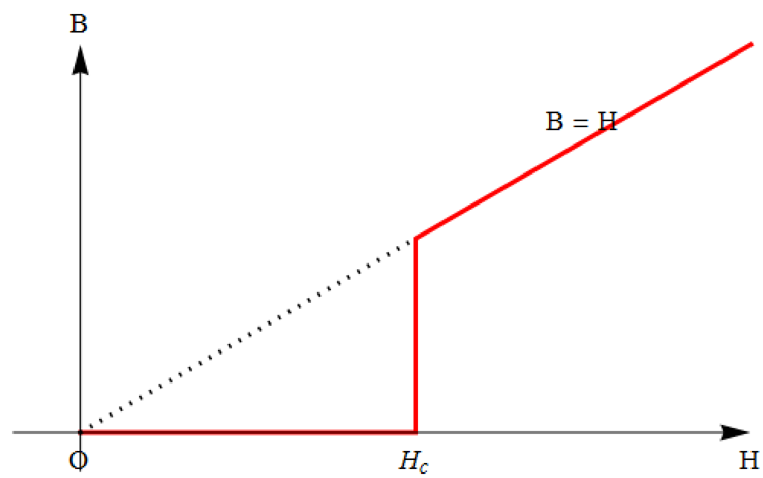


Fig. B vs an external field H for the type-I superconductor. B is the magnetic induction. $B = H + 4\pi M$.

4. The distribution of $B = 0$ in the superconducting sphere

We consider a superconducting sphere of radius R placed in a uniform external magnetic field H_0. If H_0 is small, the lines of force are expelled from the specimen. The field configuration external to the sphere is determined by the equations,

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \nabla \times \boldsymbol{B} = 0, \qquad \boldsymbol{B} \to \boldsymbol{H}_0 \qquad \text{as } r \to \infty$$

where r is the distance measured from the center of the sphere. The Meissner effect imposes the condition that no line of force can penetrate into the sphere. The normal component of $\boldsymbol{B}$ vanishes on the surface of the sphere,

$$(B_n)_{r=R} = 0$$

The appropriate solution of B in the exterior region is

$$\boldsymbol{B} = H_0\boldsymbol{e}_x + H_0\frac{R^3}{2}\nabla(\frac{\cos\theta}{r^2}) = -\nabla\varphi$$

(P.G. de Gennes, Superconductivity of metals and alloys, .M. Tinkham, Introduction to Superconductivity)

where

$$\boldsymbol{e}_x = \cos\theta\boldsymbol{e}_r - \sin\theta\boldsymbol{e}_\theta{}'$$

$$\nabla\varphi(r,\theta) = \boldsymbol{e}_r\frac{\partial}{\partial r}\varphi(r,\theta) + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\varphi(r,\theta) .$$

and

$$\phi(r,\theta) = -H_0(r + \frac{R^3}{2r^2})\cos\theta .$$

Note that

$$\nabla^2\phi = 0$$

The field $\boldsymbol{B}$ is expressed by

$$B = H_0(\cos\theta\boldsymbol{e}_r - \sin\theta\boldsymbol{e}_\theta) + H_0\frac{R^3}{2r^3}(-2\cos\theta\boldsymbol{e}_r - \sin\theta\boldsymbol{e}_\theta) .$$

When $r = R$, we get

$$\begin{aligned}\boldsymbol{B} &= H_0(\cos\theta\boldsymbol{e}_r - \sin\theta\boldsymbol{e}_\theta) + H_0\frac{1}{2}(-2\cos\theta\boldsymbol{e}_r - \sin\theta\boldsymbol{e}_\theta) \\ &= -\frac{3}{2}H_0\sin\theta\boldsymbol{e}_\theta\end{aligned}$$

Clearly the $\boldsymbol{e}_r$ component of $\boldsymbol{B}$ normal to the surface is equal to zero. We make a plot of the distribution of B using the Mathematica (ContourPlot and StreamPlot).

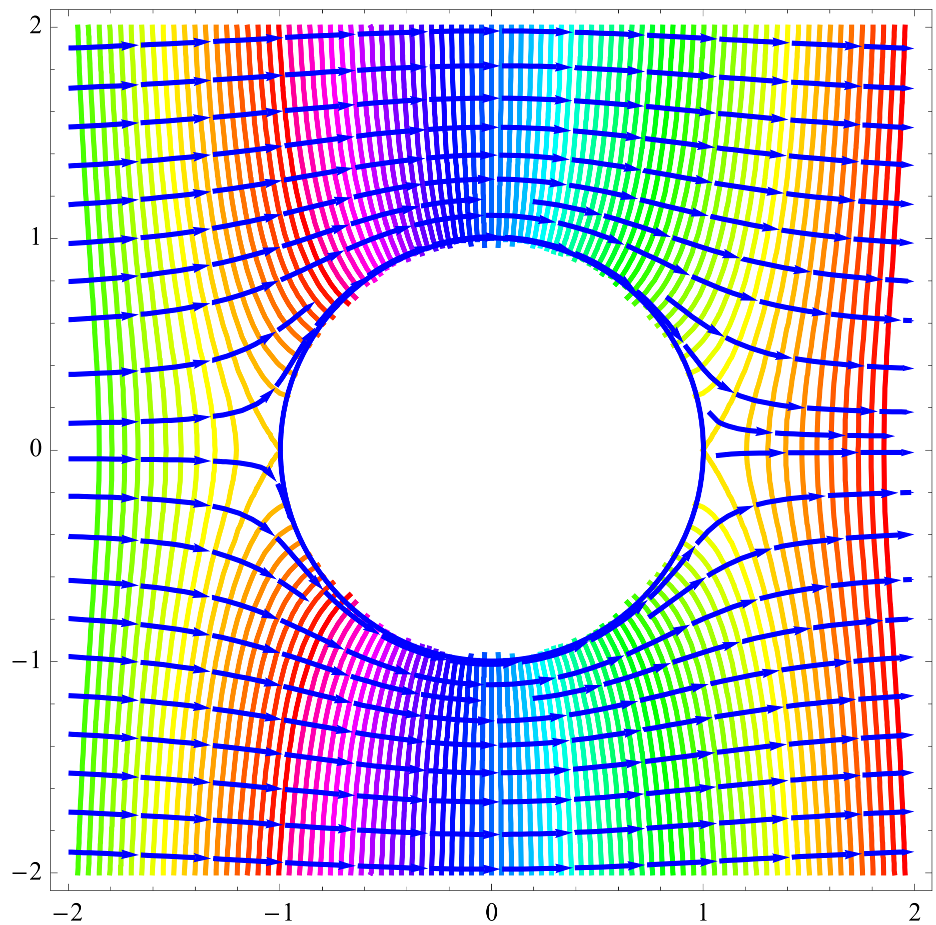


Fig. Magnetic field distribution around a superconducting sphere of radius *R*. For an external magnetic field H_0 which is relatively low, there is a complete Meissner effect. The length of arrows does not corresponds to the magnitude of $\boldsymbol{B}$.

((**Mathematica**))

```
Clear["Global`*"];

V[r_, θ_] := (- H0 r - H0 a^3/(2 r^2)) Cos[θ] ;

V2D = V[r, θ] /. {r → Sqrt[x^2 + y^2] , θ → ArcTan[x, y]} // Simplify;

rule1 = {a → 1, H0 → 1}; V2D1 = V2D /. rule1;

Hx = -D[V2D1, x] // Simplify; Hy = -D[V2D1, y] // Simplify;

g1 = ContourPlot[Evaluate[Table[V2D1 == α, {α, -2, 2, 0.05}]], {x, -2, 2}, {y, -2, 2},
   ContourStyle → Table[{Thick, Hue[0.015 i]}, {i, 0, 60}],
   RegionFunction → Function[{x, y}, x^2 + y^2 > 1]];

g2 = StreamPlot[Evaluate[{Hx, Hy}], {x, -2, 2}, {y, -2, 2}, StreamPoints → 100,
  StreamStyle → {Blue, Thick}, RegionFunction → Function[{x, y}, x^2 + y^2 > 1]];
g3 = Graphics[{Blue, Thick, Circle[{0, 0}, 1]}];

Show[g1, g2, g3, PlotRange → All]
```

((**Note**)) **Intermediate state**

On the equatorial circle ($\theta = \pm\frac{\pi}{2}$) on the surface of sphere), the tangential component is maximum and is $\frac{3}{2}H_0$, since the internal field *B* on the surface is given by

$$\boldsymbol{B} = -\frac{3}{2}H_0 \sin\theta \boldsymbol{e}_\theta = -\frac{3}{2}H_0 \boldsymbol{e}_\theta$$

For $B = 2H_c/3$, the field at the equatorial circle becomes equal to H_c. Thus For $B > 2H_c/3$, certain region of the sphere pass into the normal state. But there must still superconducting regions since $B < H_c$. In the domain

$$\frac{2H_c}{3} < B < H_c$$

there will be a co-existence of the normal and superconducting regions. This situation is called the intermediate state.

For more detail, see also the article (pages 42 - 45, the superconducting sphere) in More surprises in theoretical physics (R.E. Peierls, 1991, Princeton).

5. Parabolic law for critical field H_c vs T.

The relation between the critical magnetic field $H_c(T)$ vs T for the transition between the normal phase and the superconducting phase can be well described by a parabolic law,

$$H_c(T) = H_c(T = 0K)(1 - \frac{T^2}{T_c^{\,2}}),$$

for $T<T_c$. In fact this relation is experimentally confirmed from the measurement of $H_c(T)$ vs T for the type I superconductors. The data of $H_c(T)$ vs T for Sn is shown here.

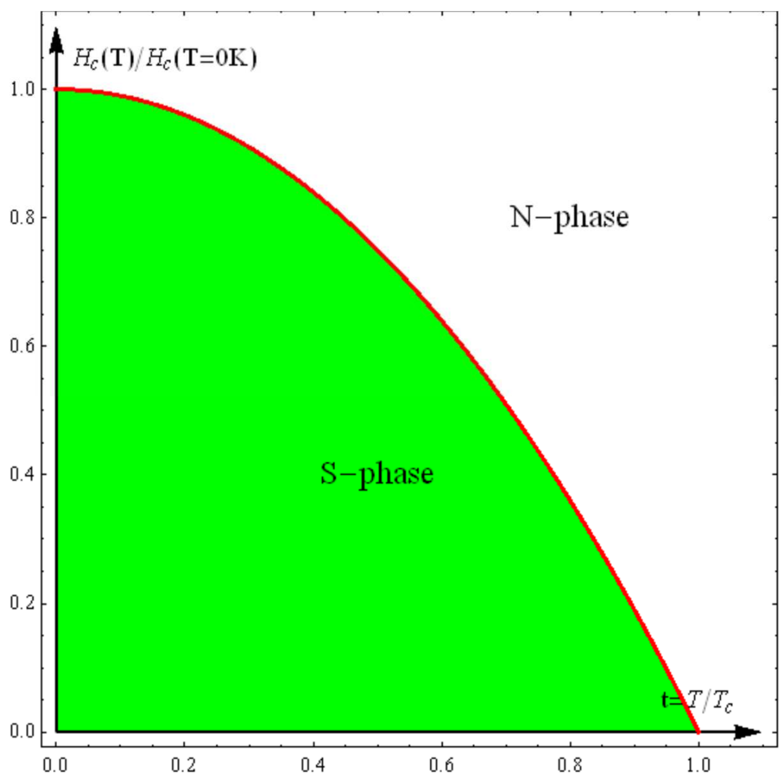


Fig. Phase diagram of a transition from the normal to superconducting phase. The critical field H_c vs T obeying a parabolic law

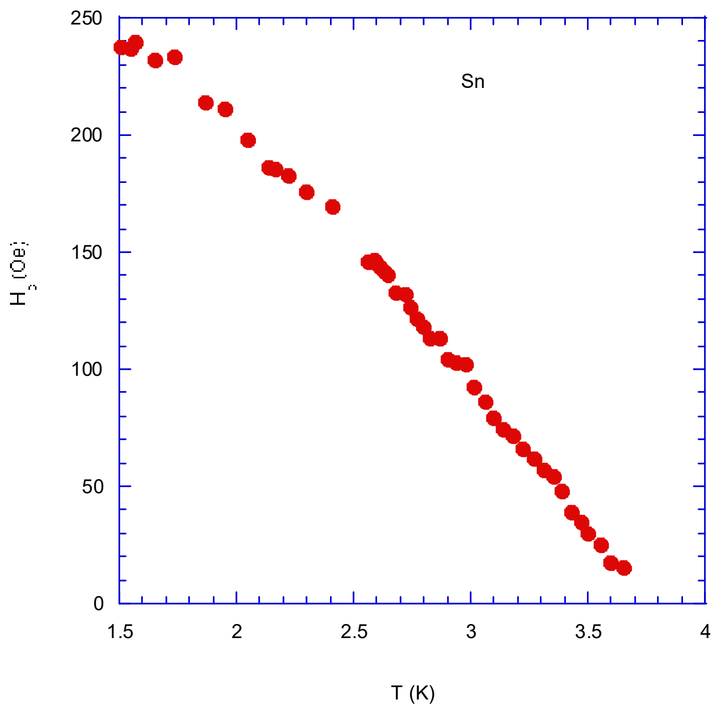


Fig. Critical field H_c of Sn (type-I superconductor) as a function of temperature (Advanced Lab. SUNY at Binghamton). H_c(0 K) = 260 Oe and T_c = 3.7 K. The data is fitted well to the parabolic law.

6. Experimental results from the Advanced Laboratory

In the Advanced laboratory (SUNY at Binghamton), we measured the magntization of superconductors Pb and Sn (type-I superconductor) using SQUID magnetometer.

(a) Magnetization (emu/cm^3) vs H (Oe) for Pb. T = 2.0 K. Lead (Pb): Type-I superconductor. T_c = 7.193 K. H_c = 803 Oe.

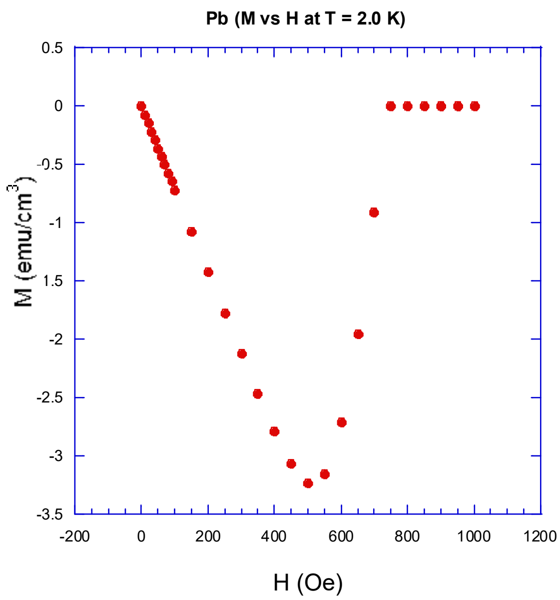


Fig. *M* vs *H* for Pb at $T = 2.0$ K. This is an original data obtained using SQUID magnetometer (M.S.) at the Binghamton University.

a. $T = 50$ K, $H = 0$ Oe, annealing for 200 sec
b. Cooling the sample from 50 K to 2.0 K in the absence of *H*
c. Aging the system at 2.0 K for 100 sec.
d. Measure the magnetization at $T = 2.0$ K with increasing *H* from 0 to 1000 Oe.

(b) Zero-field cooled magnetization (emu/cm^3) vs temperature *T* (K) for Pb. H is changed as a parameter. Lead (Pb): Type-I superconductor. $T_c = 7.2$ K. $H_C = 803$ Oe.

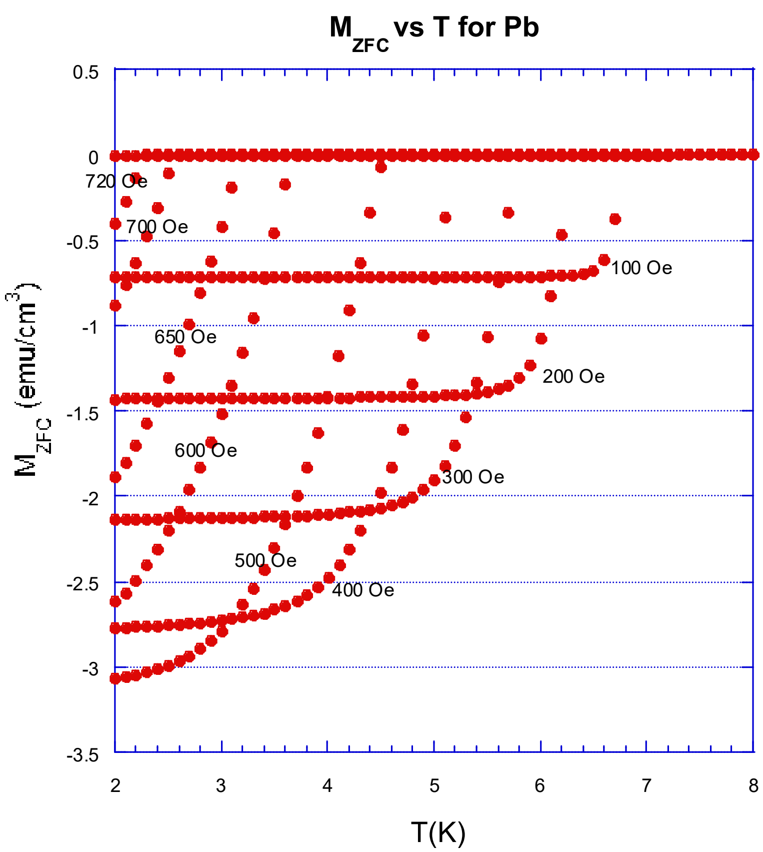


Fig. The zero-field cooled magnetization M_{ZFC} (emu/cm^3) vs T (K) for Pb for each H, where H = 100 - 720 O2. The measurement was carried out as follows.

a. $T = 50$ K, $H = 0$ Oe, annealing for 200 sec

b. Cooling the sample from 50 K to 2 K in the absence of H

c. Aging the system at 2 K for 100 sec.

d Switch on the magnetic field H at 2 K.

e. Measure the ZFC magnetization with increasing T.

H = 1 Oe, 100, 200, 300, 400, 500, 600, 650, 700, 720 Oe

(c). Sn

Type-I superconductor: $T_c = 3.722$ K, $H_c = 309$ Oe

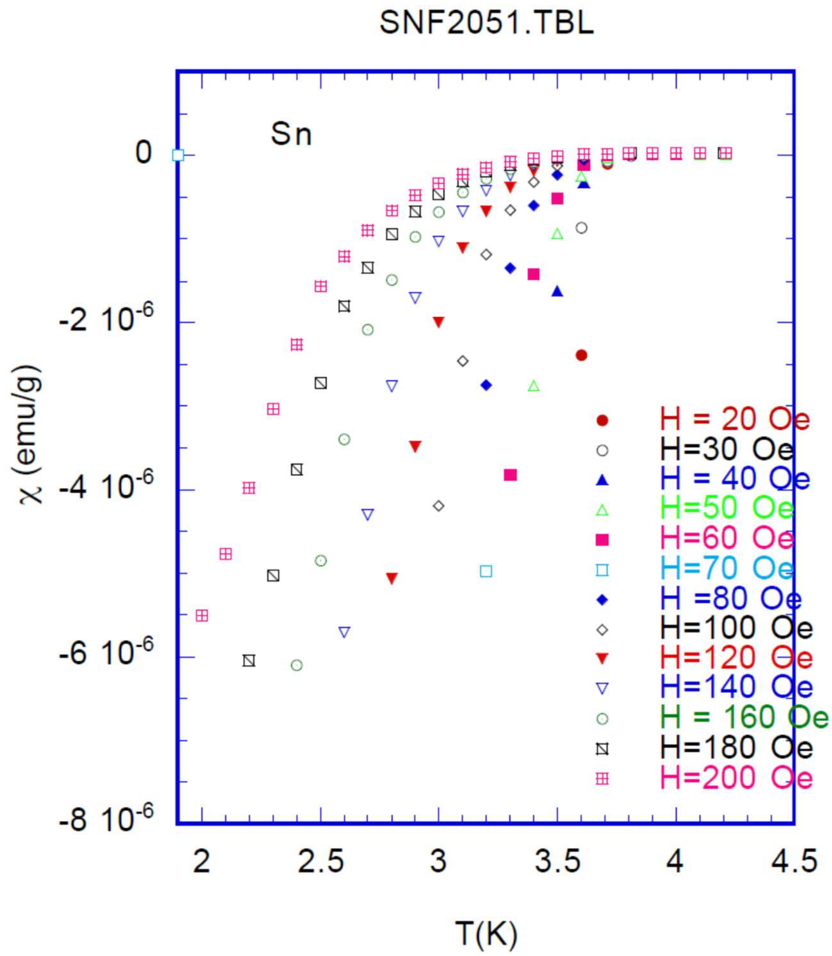


Fig. ZFC (zero-field cooled) magnetization M (emu/g) of Sn. T_c = 3.7 K. The critical field is H_c = 300 Oe.

7. FC and ZFC susceptibility of superconductor

Recently, one can measure easily magnetic susceptibility of superconductor by using SQUID magnetometer. There are two kinds of measurement of the susceptibility; zero-field cooled (ZFC) susceptibility and field cooled (FC) susceptibility.

(i) ZFC susceptibility measurement.
The system is rapidly cooled from high temperatures well above the critical temperature T_c to the lowest temperature T_0 below T_c. After an external magnetic field is applied to the superconductor at $T = T_0$, the susceptibility is measured in the presence of magnetic field with increasing temperature.

(ii) FC susceptibility
The FC susceptibility is measured in the presence of an external magnetic field as the temperature is decreased from high temperature well above T_c to the lowest temperature T_0 well below T_c, in the presence of magnetic field.

If the system exhibits a Meissner effect below T_c, the T dependence of ZFC susceptibility is exactly the same as that of the FC susceptibility below Tc. They exhibits a complete diamagnetic behavior,

$$\chi_{ZFC} = \chi_{FC} = -\frac{1}{4\pi}.$$

Figure shows a typical example of the ZFC and FC susceptibility for the superconductor Rb_3C_{60}. The ZFC susceptibility (diamagnetic) is much smaller than the FC susceptibility below T_c (= 28 K).

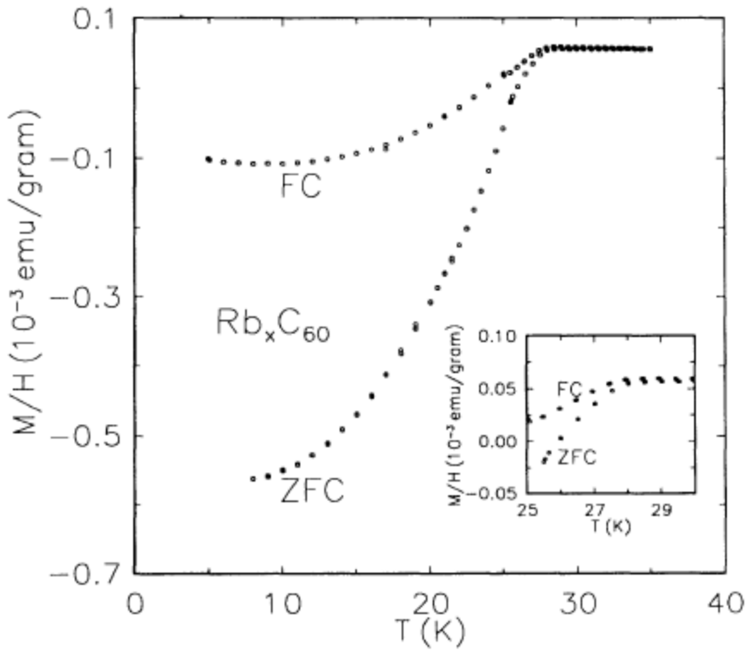


Fig. Magnetization of a sample of nominal composition Rb_3C_{60}. The data labeled ZFC were obtained upon warming in a field of 2 Oe, after cooling the sample in zero applied field. The FC data were obtained by cooling the sample in 2 Oe, illustrating flux expulsion.
[M.J. Rosseinsky et al, Phys. Rev. Lett. **66**, 2830 (1991); "Superconductivity at 28 K in Rb_xC60."].

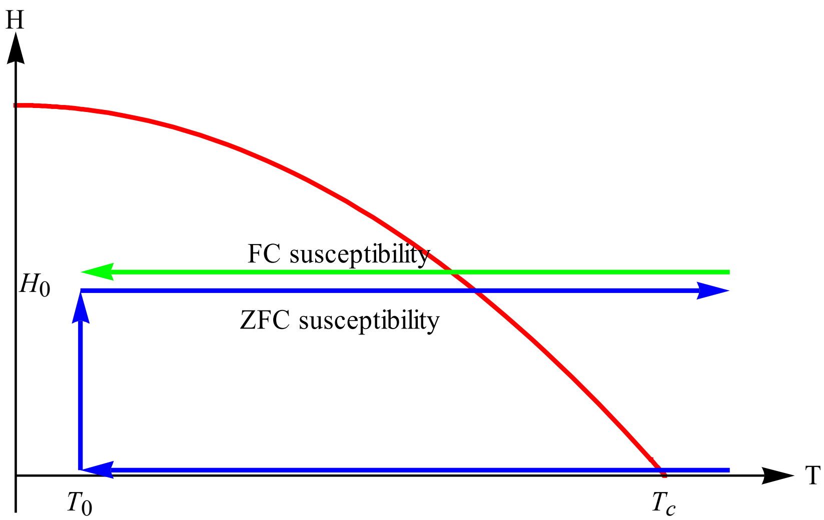


Fig. *H-T* phase diagram, where the procedure of the ZFC and FC susceptibility measurement for the type-I superconductor.

In the ZFC cooling process, the system undergoes a phase transition from the normal phase to the Meissner phase, since no external magnetic field is applied. Even if the external magnetic field ($H_0<H_c$ for a type-I superconductor) is applied at $T=T_0$ (at the lowest temperature), the Meissner phase with $B = 0$ remains maintained. In this sense, the ZFC susceptibility reflects the nature of the Meissner phase.

How about the FC susceptibility? Suppose that the system has crystalline defects inside it. The magnetic field is applied above T_c. Then the system is gradually cooled from above T_c. The magnetic flux penetrates into the systems. A part of magnetic flux is pinned to the defects. When the system is cooled down below T_c. Main part becomes in the Meissner phase with $B = 0$. However, a part of the magnetic flux is pinned in the defects and becomes frozen. In accompanying with this, the supercurrents flow around the defect, in order to maintain the frozen magnetic flux. Such defects contribute to the positive magnetization, leading to the FC susceptibility which is larger than the ZFC susceptibility at the same temperature.

8. Isotope effect

Lattice vibrations (phonons) are responsible for the isotope effect in superconductivity, which was discovered in 1950. The superconducting critical temperature of Hg varie with isotopic mass *M*, from T_c = 4.185 K to 4.146 K, as the isotopic mass *M* varies from 199.5 to 203.4. The critical temperature for superconductors depends on the isotopic mass of the crystal, according to

$$T_c M^{\alpha} = \text{constant} \qquad \text{(Isotope effect)}$$

with $\alpha = 0.5$. This indicates that lattice vibrations and hence electron-lattice interactions are deeply involved in superconductivity. In BCS theory, it is predicted that

$$T_c \propto \Theta_D \propto M^{-1/2}.$$

where M is the mass of isotopes.

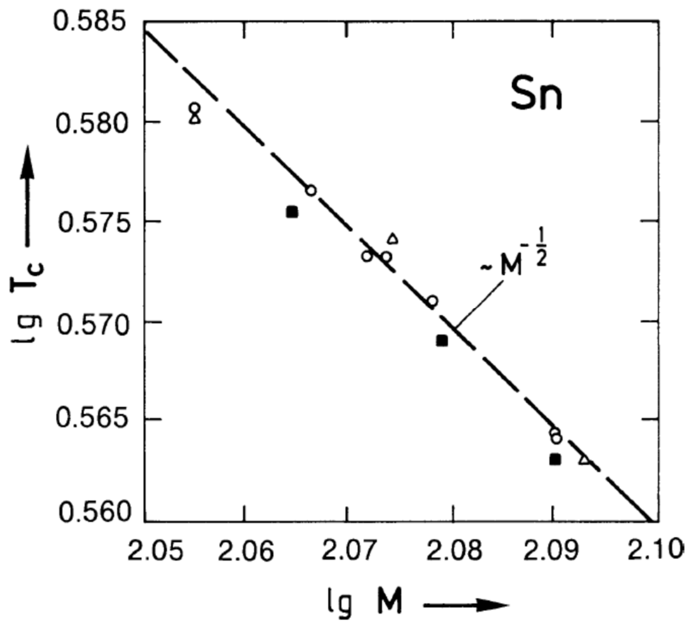


Fig. Isotope effect for Sn. The results of several authors are summarized. Maxwell (○); Lock, Pippard, and Shoenberg (■); Serin, Reynolds, and Lohman (Δ). $\alpha = 0.47 \pm 0.02$ (from ISSP).

8. Type-II superconductor

The difference between the type-I and type-II superconductors are characterized by the Ginzburg-Landau parameter k,

$$\kappa = \frac{\lambda}{\xi},$$

where λ is the penetration length of magnetic field and ξ is the coherence length.

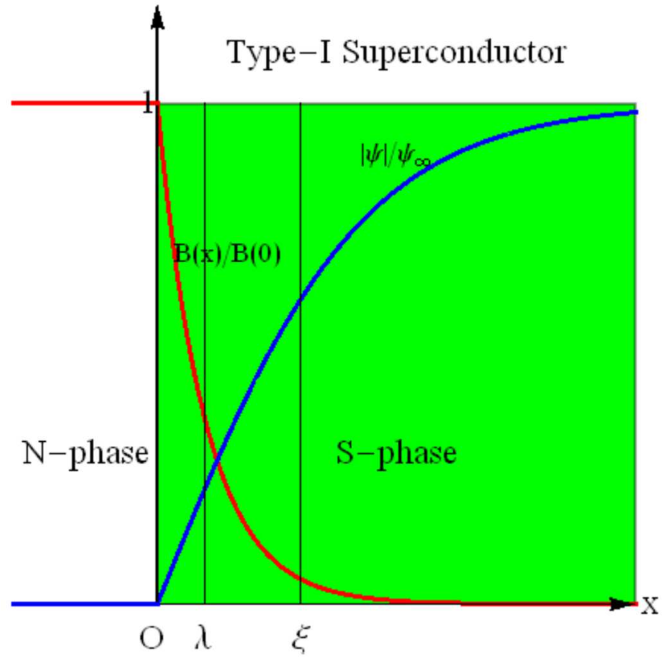


Fig. Boundary between the N-phase and S-phase for type-I superconductor. $\kappa = \frac{\lambda}{\xi} < \frac{1}{\sqrt{2}}$. κ is the Ginzburg-Landau parameter.

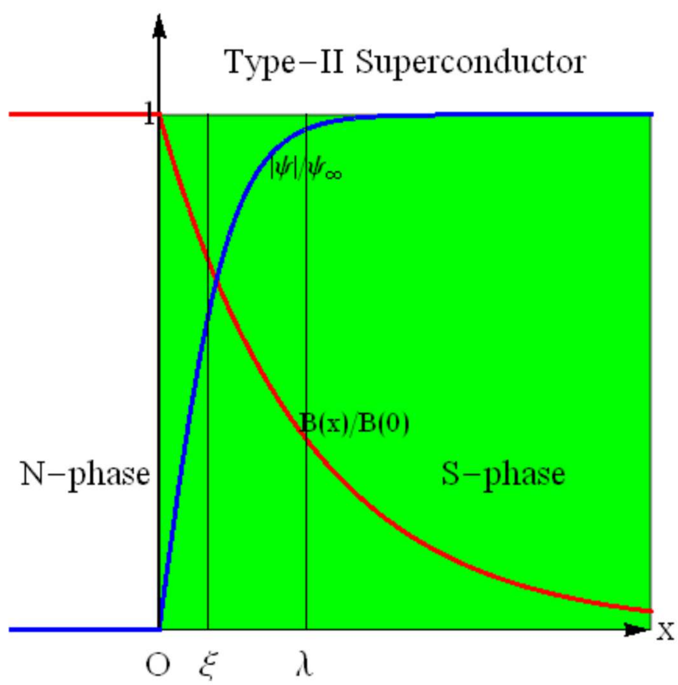
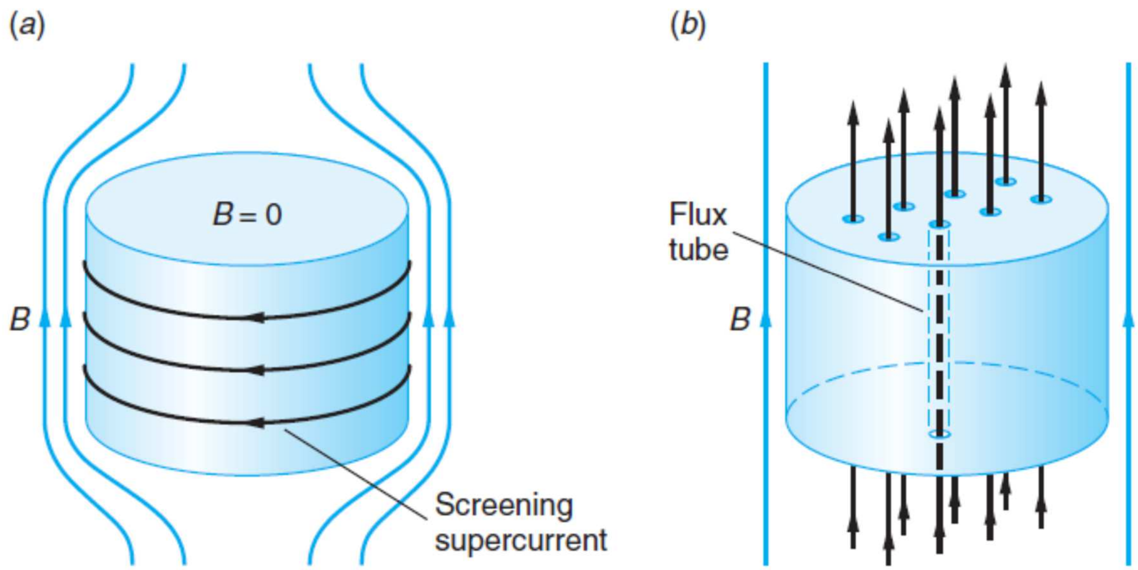


Fig. Boundary between the N-phase and S-phase for type-II superconductor. $\kappa = \frac{\lambda}{\xi} > \frac{1}{\sqrt{2}}$.

There are two phases in the phase diagram of H vs T below T_c for the type-II superconductor, the Meissner phase ($H<H_{c1}$) and the mixed phase (Shubnikov phase, or vortex state, mixed phase) for $H_{c1}<H<H_{c2}$.



(a) Meissner phase ($H<H_{c1}$). (b) Mixed phase ($H_{c1}<H<H_{c2}$)

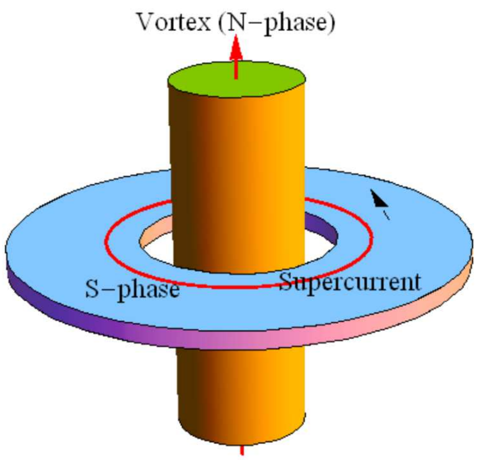


Fig. A single vortex which penetrates into the superconductor. The supercurrent flows in a counterclockwise direction in this figure.

9. Surface energy

Consider the interface between a region in the S-state and a region in the N-state. The interface has a surface energy that may be positive or negative .

Type II: The surface energy becomes negative.

Type I: The surface energy is positive.

10. Thermodynamics of superconductors

F_n: Free energy of superconducting phase

F_s: Free energy of normal phase

The Helmholtz free energy is

$$dF = d(E - ST) = TdS - PdV - SdT - TdS$$
$$= -PdV - SdT$$

Replacing

Intensive variable $P \to M$

Extensive variable $V \to H$

(see ISSP of Kittel),

we have the relation

$$dF = -MdH - SdT$$

For T = constant,

$$dF = -MdH - SdT \qquad \text{(thermodynamics identity).}$$

(a) The superconducting phase;

$$M = -\frac{1}{4\pi}H \qquad \text{(Meissner effect).}$$

Then we have

$$F_s(H) - F_s(H=0) = \int_0^H dF_S = \frac{1}{4\pi}\int_0^H dH = \frac{H^2}{8\pi},$$

or

$$F_s(H) = F_s(H=0) + \frac{H^2}{8\pi}.$$

(b) The normal phase

Since $M = 0$,

$$F_n(H) = F_n(H=0)$$

At the critical field $H_c(T)$ the energies are equal in the normal and superconducting states,

$$F_n(H_c) = F_s(H_c)$$

or

$$F_n(H=0) = F_s(H=0) + \frac{[H_c(T)]^2}{8\pi}$$

Then it follows that

$$\Delta F = F_n(0) - F_s(0) = \frac{[H_c(T)]^2}{8\pi}$$

which is the stabilization free energy of the superconducting state (or the condensation energy).

We define the entropy as

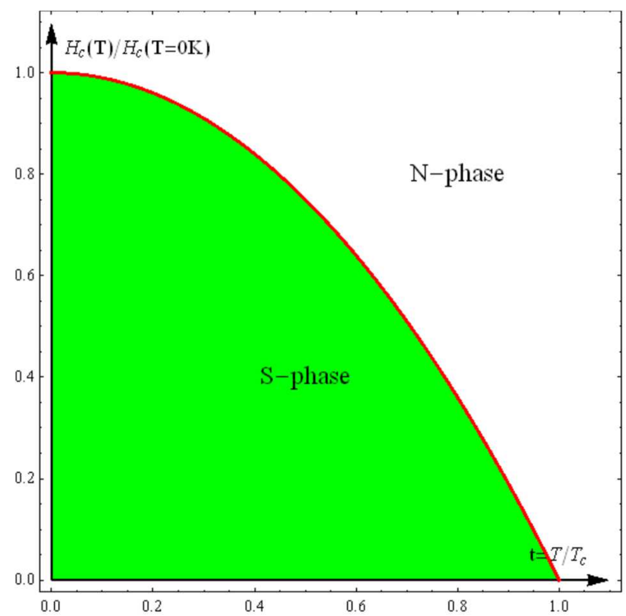
$$S_n = -\left(\frac{\partial F_n}{\partial T}\right)_V, \; S_s = -\left(\frac{\partial F_s}{\partial T}\right)_V .$$

The difference between these entropies is

$$S_n - S_s = -\left(\frac{\partial (F_n - F_s)}{\partial T}\right)_V = -\frac{1}{4\pi} H_c(T)\frac{dH_c(T)}{dT}$$

The latent heat of the transition is

$$L = T(S_n - S_s) = -\frac{T}{4\pi} H_c(T)\frac{dH_c(T)}{dT}$$



The parabolic law:

$$H_c(T) = H_c(0)(1 - \frac{T^2}{T_c^{\,2}})$$

$$\frac{dH_c(T)}{dT} = H_c(0)(-\frac{2T}{T_c^2}). \qquad \frac{d^2H_c(T)}{dT^2} = H_c(0)(-\frac{2}{T_c^2})$$

For $T = T_c$,

$$L = 0$$

Since the entropy is continuous at $T = T_c$, the phase transition at $T = T_c$ is the second order.

11. Second order phase transition

The specific heat is defined by

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V .$$

Then we get

$$(C_s - C_n)_V = T\left(\frac{\partial(S_s - S_n)}{\partial T}\right)_V$$
$$= \frac{T}{4\pi}\frac{d}{dT}[H_c(T)\frac{dH_c(T)}{dT}]$$
$$= \frac{T}{4\pi}\{H_c(T)\frac{d^2H_c(T)}{dT^2} + [\frac{dH_c(T)}{dT}]^2\}$$

At $T = T_c$,

$$(C_s - C_n)_V = \frac{T_c}{4\pi}[\frac{dH_c(T)}{dT}]^2 |_{T=T_c} = \frac{1}{\pi T_c}[H_c(0)]^2$$

Then the specific heat is discontinuous at $T = T_c$.

12. Thermodynamics for the type-II superconductor (de Gennes)

(a) Gibbs free energy per unit volume for the superconducting state G_s can be written as

$$G_s = F_s(B) - \frac{BH}{4\pi},$$

G_s has a minimum as a function of B for fixed H,

$$\left(\frac{\partial G_s}{\partial B}\right)_H = 0 .$$

(b) Gibbs free energy per unit volume for the normal state.
G_n can be written as

$$G_n = F_n(B) - \frac{BH}{4\pi} = (F_n{}^0 + \frac{B^2}{8\pi}) - \frac{BH}{4\pi} .$$

G_n has a minimum as a function of B for fixed H,

$$\left(\frac{\partial G_n}{\partial B}\right)_H = 0 .$$

Then we have

$$B = H ,$$

leading to the expression of G_n

$$G_n = F_n{}^0 - \frac{H^2}{8\pi} ,$$

in the normal state.

(iii)

Let the field vary from H to $H + \delta H$

$$\frac{\partial G_s}{\partial H} = -\frac{B}{4\pi} , \qquad \frac{\partial G_n}{\partial H} = -\frac{H}{4\pi} .$$

Thus we have

$$\frac{\partial}{\partial H}(G_n - G_s) = \frac{B - H}{4\pi} = M .$$

We now integrate the relation between $H = 0$ and $H = H_{c2}$. We note that

(i) At $H = H_{c2}$,

$$G_n = G_s$$

(ii) At $H = 0$,

$$G_n^0 = F_n^{\ 0}, \qquad G_s^0 = F_s^{\ 0},$$

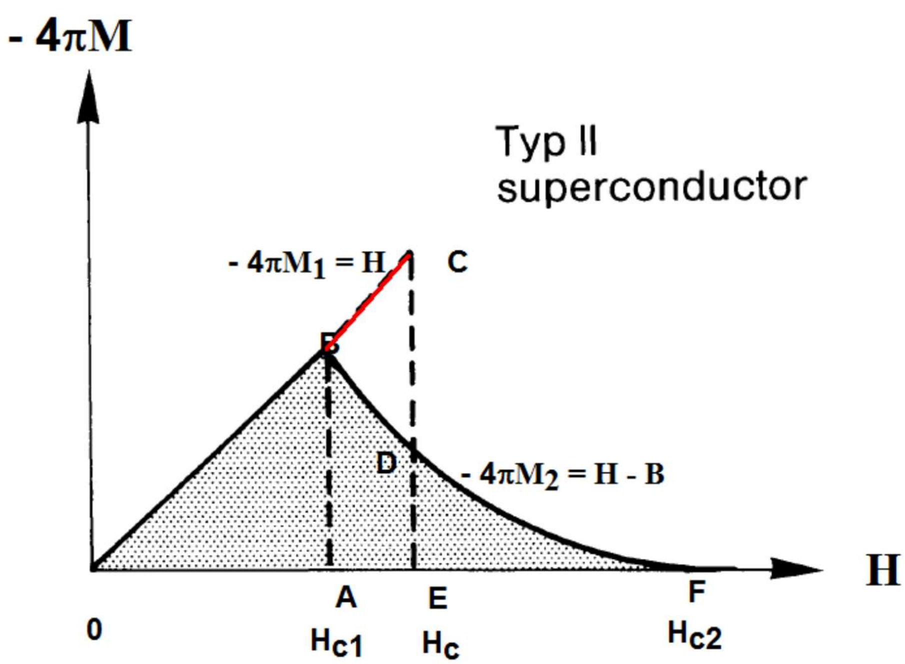
and by definition

$$F_n^0 - F_s^0 = \frac{H_c^{\ 2}}{8\pi}.$$

Then we get

$$\int_0^{H_{c2}} (-4\pi M) dH = -4\pi [G_n - G_s]|_0^{H_{c2}} = 4\pi (G_n - G_s)_{H=0} = \frac{1}{2} H_c^{\ 2}$$

13. The magnetization vs *H* for the type-II superconductor



(i)

$$\int_0^{H_c} HdH = \frac{1}{2}H_c^{\ 2}$$

(ii)

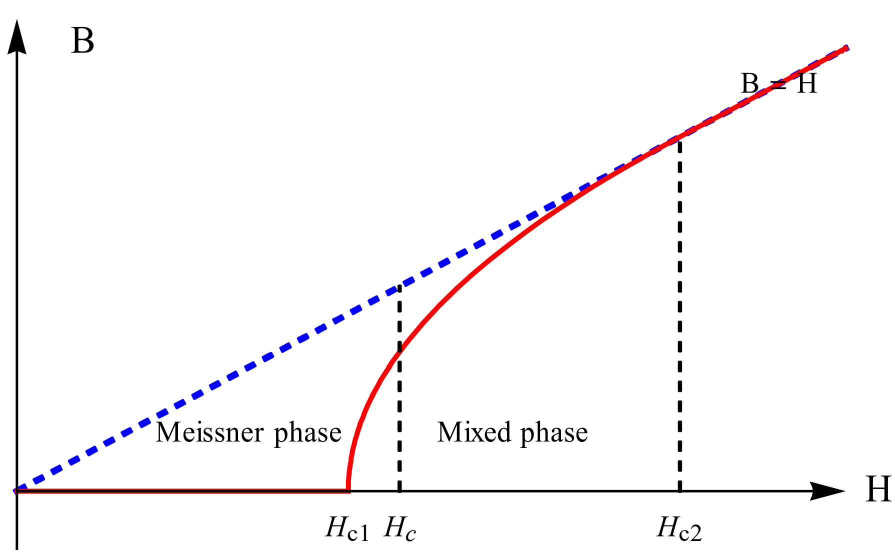
$$\int_0^{H_{c2}}(-4\pi M)dH = \int_0^{H_{c1}}(-4\pi M_1)dH + \int_{H_{c1}}^{H_{c2}}(-4\pi M_2)dH$$
$$= \int_0^{H_{c1}} HdH + \int_{H_{c1}}^{H_{c2}}(-4\pi M_2)dH$$
$$= \frac{1}{2}H_{c1}^{\ 2} + \int_{H_{c1}}^{H_{c2}}(-4\pi M_2)dH = \frac{1}{2}H_c^{\ 2}$$

or

$$\int_{H_{c1}}^{H_{c2}}(-4\pi M_2)dH = \frac{1}{2}(H_c^{\ 2} - H_{c1}^{\ 2}).$$

This means that the area (BCD) is equal to the area (DEF) in the magnetization vs H curve for the type-II superconductor.

14. *B* vs *H* for type-II superconductor



The magnetic induction B (the mean magnetic field in the interior of a type II superconductor is plotted as a function of an external magnetic field H. For $H<H_{c1}$, a type-II super conductor behaves exactly like a type-I superconductor, exhibiting a perfect diamagnetism (Meissner effect). At $H = H_{c1}$, normal cores with their associate vortices form and pass into the system. The magnetic flux threading the vortices is in the same direction as that due to H, so that the magnetic flux is no longer equal to zero. For $H_{c1}<H<H_{c2}$, the number of vortices which occupy the system is governed by the fact that vortices repel each other. The number of normal cores per unit area for a given strength of H is such that there is equilibrium between the reduction in free energy of the system due to the presence of each non-diamagnetic core and the existence of the mutual repulsion between vortices. As H is increased, the normal cores pack closer, so the average flux density in the system increases. At $H = H_{c2}$, there is a discontinuity change in the slope of the flux density.

A good type-II superconductor excludes the field completely for $H<H_{c1}$.Above H_{c1}, the field is partially excluded, but the specimen remains superconducting. For $H>H_{c2}$, the magnetic flux penetrates completely and superconducting vanishes.

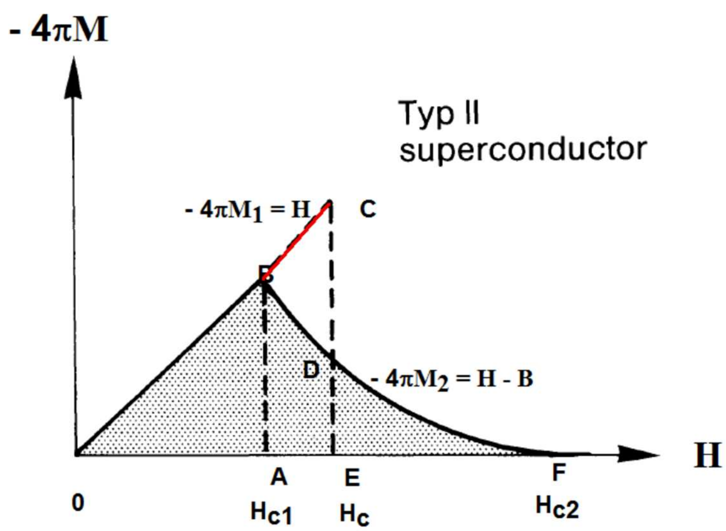


Figure shows the plot of -4πM vs H for typical type-II superconductors, where M is the magnetization and H is an external magnetic field. We note that the line BC for the typical type-I superconductor is described by

$$-4\pi M_1 = H \ .$$

The line BDE for the typical type-II superconductor is described by

$$-4\pi M_2 = H - B$$

where ***B*** is the magnetic induction (the mean field inside the superconductor). Note that The area (BCD) and area (DEF) are expressed by

$$\text{Area (BCD)} = \int_{H_{c1}}^{H_c} [(-4\pi M_1) - (-4\pi M_2)] dH = \int_{H_{c1}}^{H_c} [H - (H - B)] dH = \int_{H_{c1}}^{H_c} B dH$$

$$\text{Area (DEF)} = \int_{H_c}^{H_{c2}} (-4\pi M_2) dH = \int_{H_c}^{H_{c2}} (H - B) dH$$

From the condition that area (BCD) is equal to area (DEF), we get

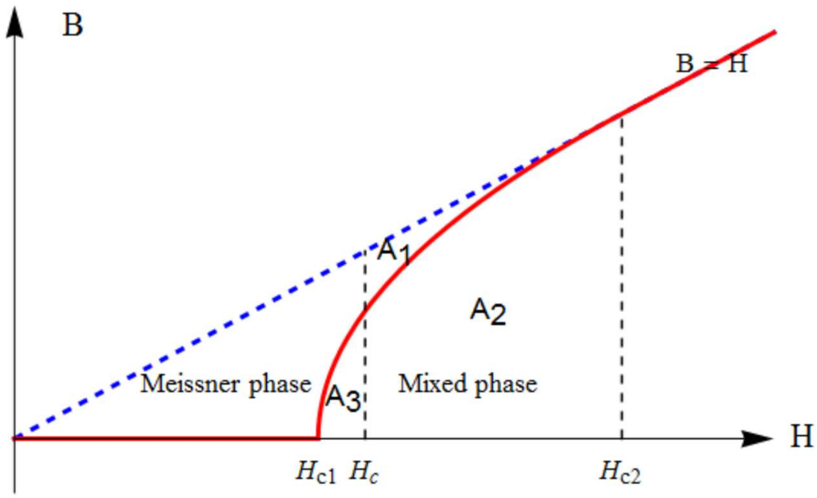
$$\int_{H_{c1}}^{H_c} B dH = \int_{H_c}^{H_{c2}} (H - B) dH$$

or

$$\int_{H_{c1}}^{H_c} B dH + \int_{H_c}^{H_{c2}} B dH = \int_{H_c}^{H_{c2}} H dH$$

or

$$\int_{H_{c1}}^{H_{c2}} B dH = \int_{H_c}^{H_{c2}} H dH$$



Since

$$\int_{H_{c1}}^{H_{c2}} BdH = A_2 + A_3, \qquad \int_{H_c}^{H_{c2}} HdH = A_1 + A_2,$$

we have

$$A_2 + A_3 = A_1 + A_2,$$

or

$$A_3 = A_1.$$

15. Specific heat of type-I superconductor Al

The specific heat as a function of T changes discontinuously at the critical temperature T_c. Figure shows the example of Al with T_c = 1.140 K and H_c = 100 Oe. The specific heat undergoes a sharp jump at T_c with decreasing temperature. Then it sink to below the value of the normal phase, at very low temperatures.

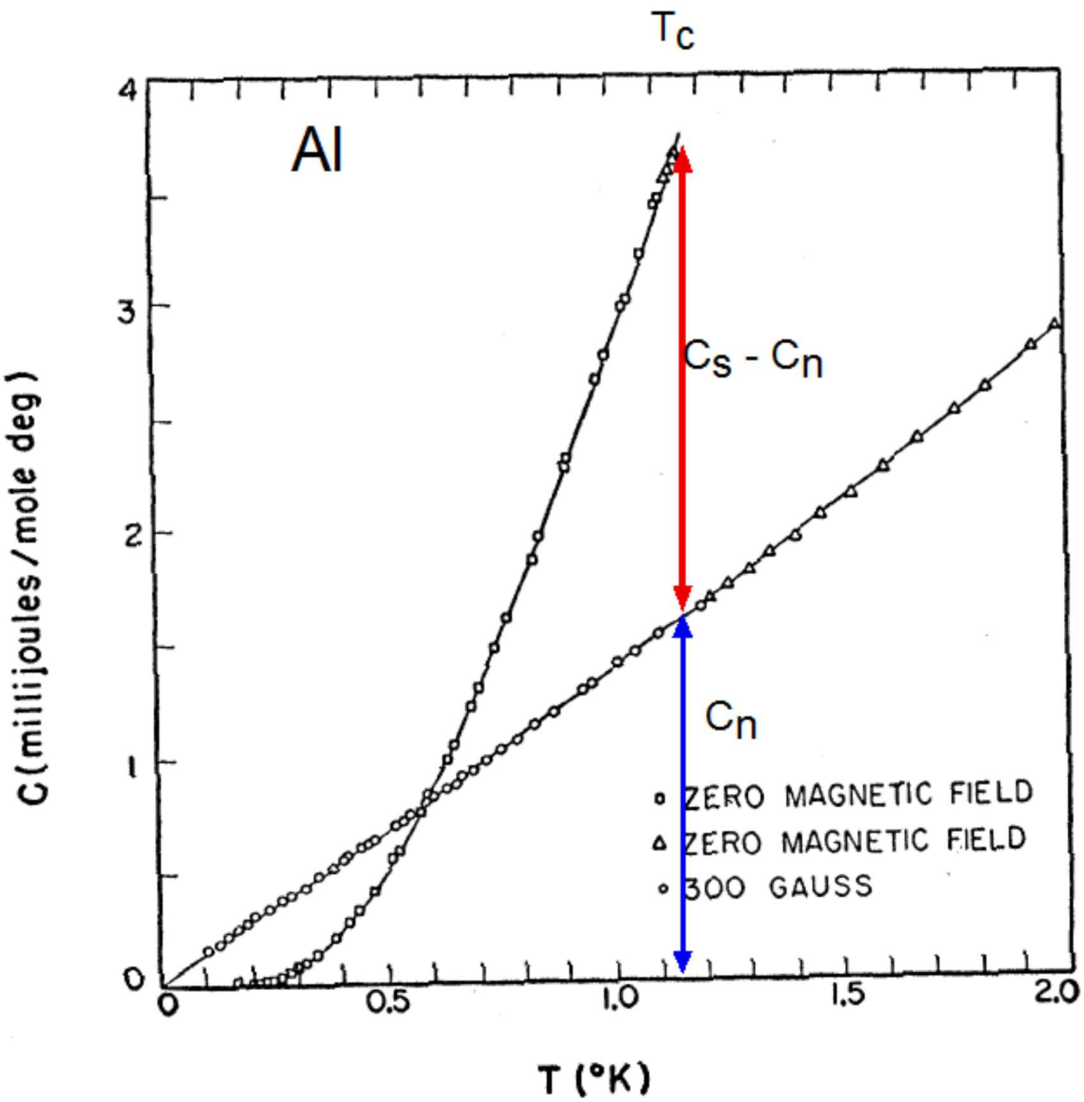


Fig. Specific heat of Al between 0.1 and 2.0 K. [N.E. Philips, Phys. Rev. **114**, 676 (1959)]. Al (T_c = 1.140 K, H_c = 100 Oe). This figure is made using AppleDraw based on the data obtained by Philips.

16. Specific heat in the normal state

The specific heat C_n of a normally conducting metal is composed of lattice part C_L and an electronic part C_{ne},

$$C_n = C_{ph} + C_{ne}$$

where C_e is the electronic specific heat

$$C_{ne} = \gamma T$$

with

$$\gamma = \frac{2\pi^2 k_B{}^2}{3} N(\varepsilon_F)$$

and C_{ph} is the contribution of phonon to the specific heat.

$$C_{ph} \approx \left(\frac{T}{\Theta}\right)^3, \qquad (\Theta = 427.7 \text{ K for Al})$$

which is negligibly small at low temperatures. Thus the normal specific heat in the vicinity of T_c ($T>T_c$) is well described by

$$C_n = \gamma T$$

with

$\gamma = 1.35$ mJ/(mol.K) for Al.

17. Specific heat in the superconducting state

The specific heat of Al below T_c is well described by

$$\frac{C_{es}}{\gamma T_c} = 7.1 \exp(-\frac{1.34 T_c}{T})$$

for Al. Such an exponential form of $1/T$ in specific heat near T_c suggests an energy band gap for the superconducting material. This band gap is one of the experimental evidence which supports the BCS theory of superconductivity. For $T \rightarrow 0$, the BCS theory predicts that

$$\frac{C_s}{\gamma T_c} = 9.17 \exp(-\frac{1.5 T_c}{T}),$$

for the specific heat of conventional superconductors.

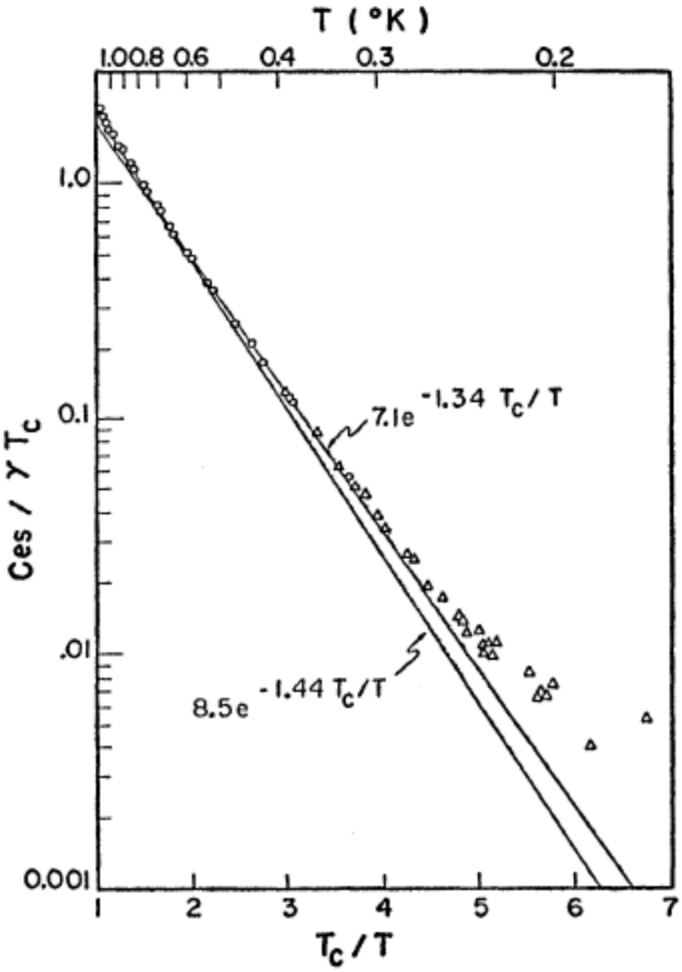


Fig. Electronic specific heat in the superconducting state, C_{es} for Al. The different symbols distinguish the results of two completely separate experiments [N.E. Philips, Phys. Rev. **114**, 676 (1959)]. Al (T_c = 1.140 K, H_c = 100 Oe).

18. Prediction from BCS theory on specific heat

The BCS theory (which will be discussed later) predicts that

$$\Delta C = (C_s - C_n)|_{T=T_c-0} = k_B{}^2 T_c N(\varepsilon_F)\frac{8\pi^2}{7\varsigma(3)}.$$

Then the ratio at $T = T_c$ is predicted as

$$\frac{C_s - C_n}{C_n}|_{T_c} = \frac{k_B{}^2 T_c N(\varepsilon_F)\frac{8\pi^2}{7\varsigma(3)}}{\frac{2\pi^2 k_B{}^2}{3}T_c N(\varepsilon_F)} = \frac{12}{7\varsigma(3)} = 1.43$$

For Al, this ratio is about 1.34, which is a little smaller that that predicted from the BCS theory. For $k_B T << \Delta$, it is predicted that

$$C_s \approx \left(\frac{\Delta}{k_B T}\right)^{3/2} \exp(-\frac{\Delta}{k_B T}).$$

The energy gap is given by

$$\Delta = \pi k B T_c \left(\frac{8}{7\varsigma(3)}\right)^{1/2} \left(\frac{T_c - T}{T_c}\right)^{1/2}.$$

((**Note**))
Comment by Richard Feynman, how to attack on the problem of superconductivity ["Richard Feynman and Condensed Matter Physics," D. Pines, Phys. Today 42, p.61 (1989)]

I decides it would be easiest to explain the specific heat rather than the electrical properties... But we do not have to explain any entire specific heat curve; we only have to explain any feature of it, like the existence of a transition, or that the specific heat near absolute zero is less than proportional to T. I chose the latter because being near absolute zero is a much simpler situation than being at any finite temperature. Thus the property we should study is this: Why does a superconductor have a specific heat less than T?

19. Quasi particle tunneling

Ivar Giaever (Giæver, born April 5, 1929, in Bergen, Norway) is a physicist who shared the Nobel Prize in Physics in 1973 with Leo Esaki and Brian Josephson "for their discoveries regarding tunnelling phenomena in solids". Giaever's share of the prize was specifically for his "experimental discoveries regarding tunneling phenomena in ... superconductors". Giaever is an institute professor emeritus at the Rensselaer Polytechnic Institute, a professor-at-large at the University of Oslo, and the president of Applied Biophysics.

http://en.wikipedia.org/wiki/Ivar_Giaever

Giaever (1960) discovered that for the superconductor-insulator-metal sandwich, the current-voltage characteristic changes from the straight line to the kinked curve. This result indicates that the superconductor has a energy gap centered at the Fermi energy. At $T = 0$ K, no current can flow until the applied voltage is $V = \Delta/e$. The energy gap corresponds to the break-up of a pair of electrons in the superconducting state.

20. Semiconductor model

(a) S-I-N junction

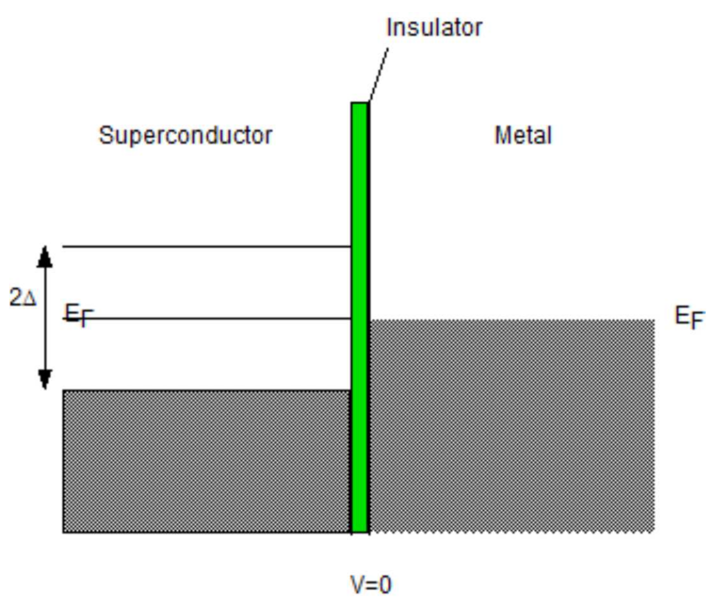


Fig. Semiconductor model for the S-I-N junction at $T = 0$ K. $V = 0$. The Fermi energy of the superconductor is at the same level as that of the metal.

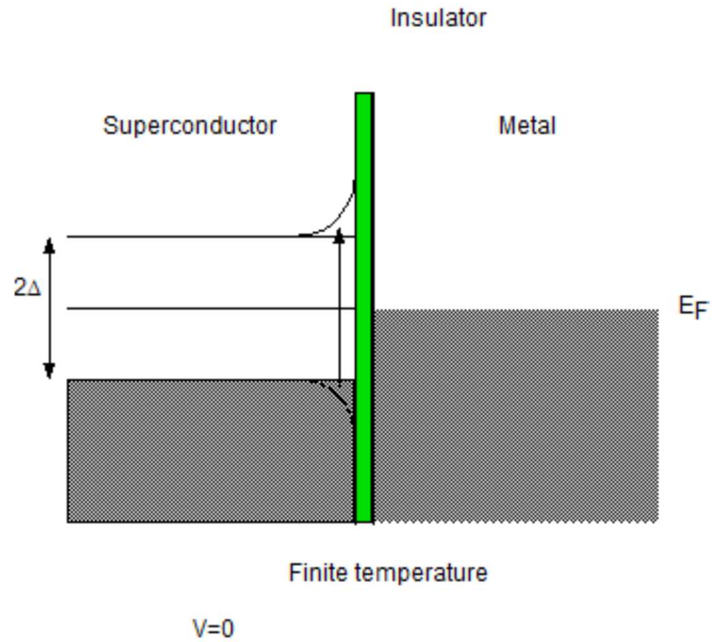


Fig. Semiconductor model for the S-I-N junction at finite temperature below T_c of the superconductor. $V = 0$. The Fermi energy of the superconductor is at the same level as that of the metal. As a result of the breaking up of Cooper pair, a part of electrons is excited to the upper level. There are holes in the lower level.

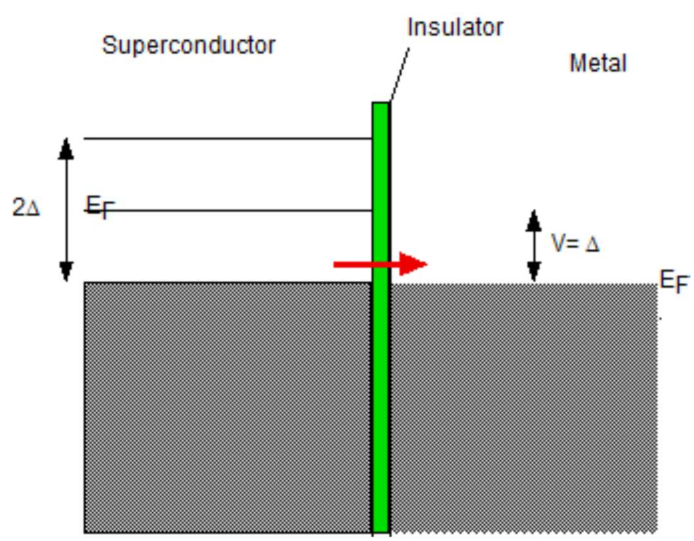


Fig. Semiconductor model for the S-I-N junction at finite temperature below T_c of the superconductor. $V = \Delta$. The Fermi energy of the superconductor is higher than that of the metal by V. The current flows between the superconductor and metal. The excitation of electrons is taken into account because of finite temperatures. However, for simplicity, this effect is not described in this figure.

(b) S_1-I-S_2 junction

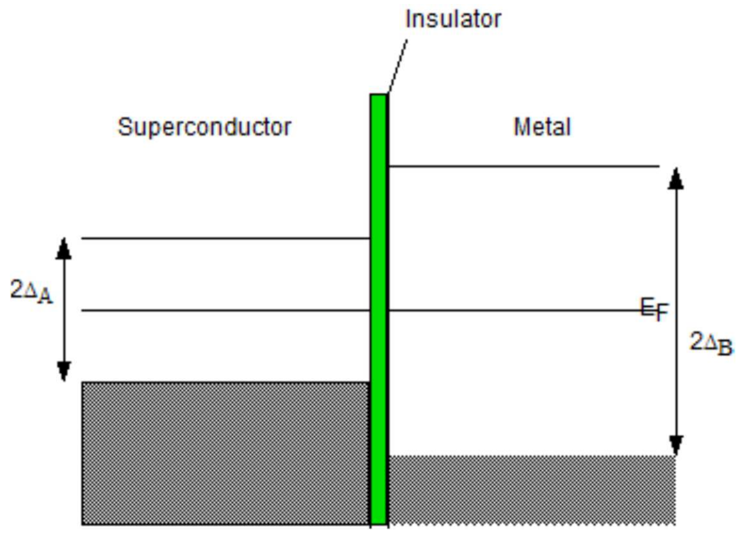


Fig. Semiconductor model for the S_1-I-S_2 junction at $T = 0$ K. $V = 0$. The energy gaps are $2\Delta_A$ for the superconductor S_1 and $2\Delta_B$ for the superconductor S_2. The Fermi energy of the superconductor S_1 is the same as that of the superconductor S_2.

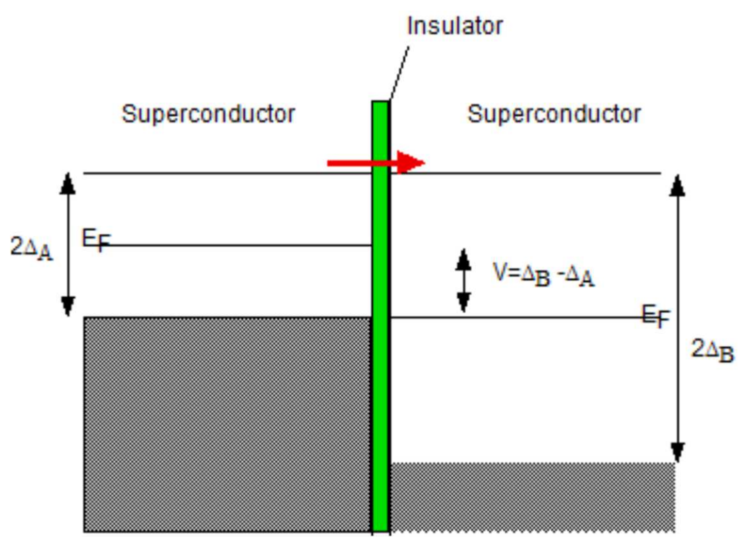


Fig. Semiconductor model for the S_1-I-S_2 junction at $T = 0$ K. $V = \Delta_B - \Delta_A$..

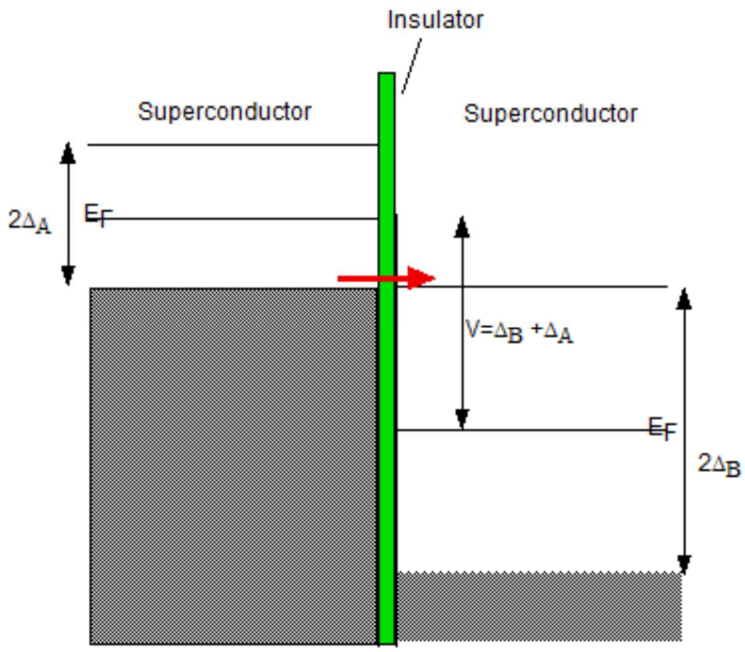


Fig. Semiconductor model for the S_1-I-S_2 junction at $T = 0$ K. $V = \Delta_B + \Delta_A$.

21. Result of Giaevr and Megerle

The experimental results were obtained by Giaever and Megerle. Reference: I. Giaever and K. Megerle, Phys. Rev. B **122**, 1101 (1961).

Al: $T_c = 1.140$ K, $H_c = 100$ Oe;
Pb: $T_c = 7.193$ K, $H_c = 803$ Oe
Sn: $T_c = 3.722$ K, $H_c = 309$ Oe

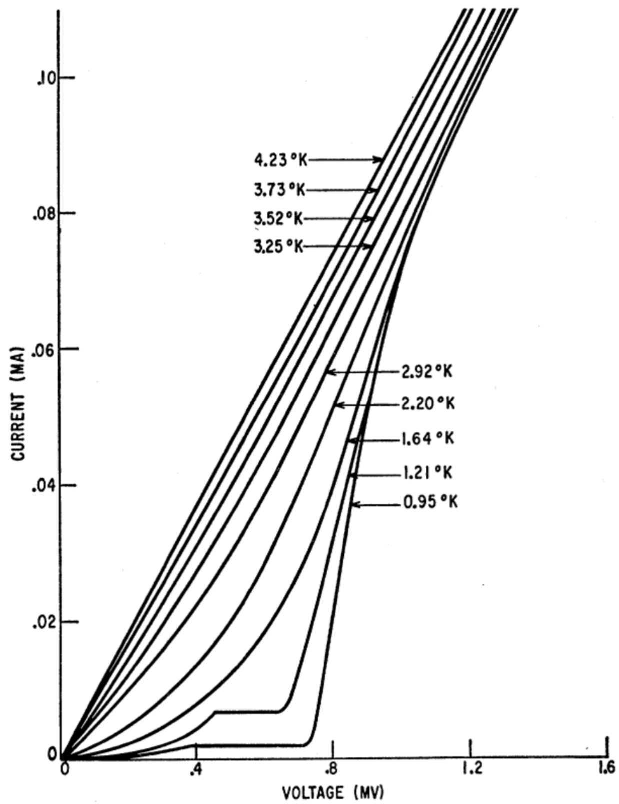


Fig. *I* vs *V* characteristics of an Al-Al_2O_3-Sn sandwich at various temperatures. Fig.6 [I. Giaever and K. Megerle, Phys. Rev. B 122, 1101 (1961)].

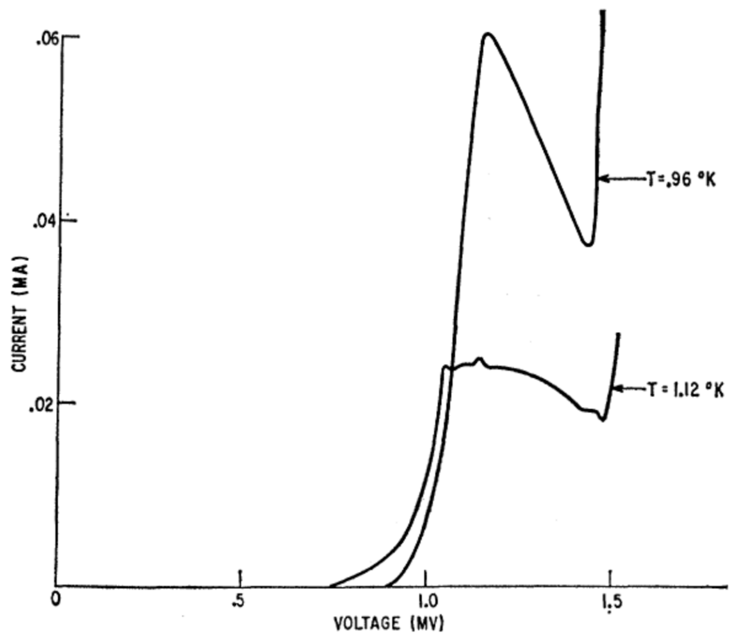
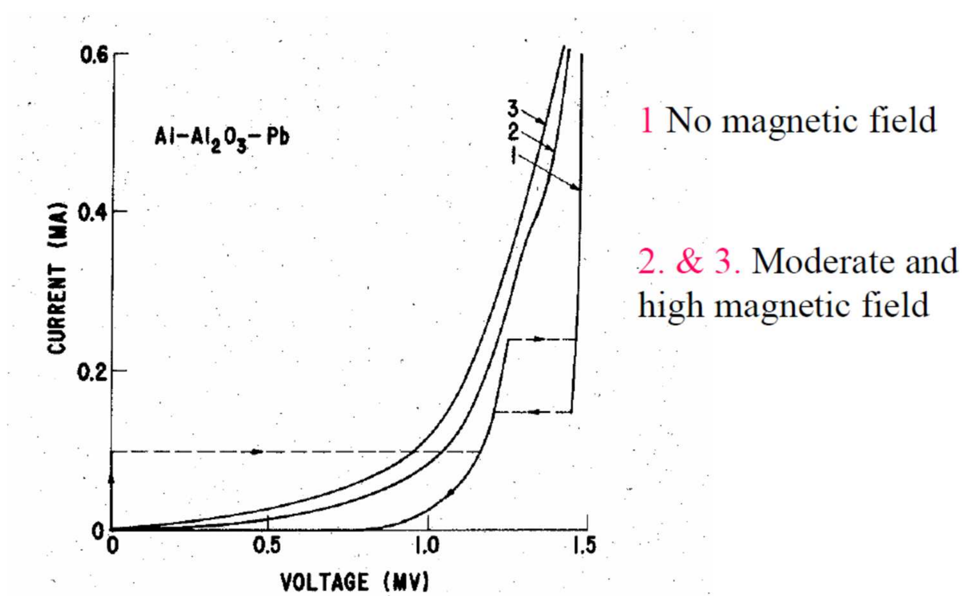


Fig. The negative-resistance region traced out for different Al-Al_2O_3-Pb sandwiches Fig.17 [I. Giaever and K. Megerle, Phys. Rev. B 122, 1101 (1961)].

22. Josephson junction which was missed by Giaever

In a series of his experiments, Giaever missed phenomena known as a DC Josephson effect at zero voltage.



The detail of the DC Josephson junction will be discussed later.

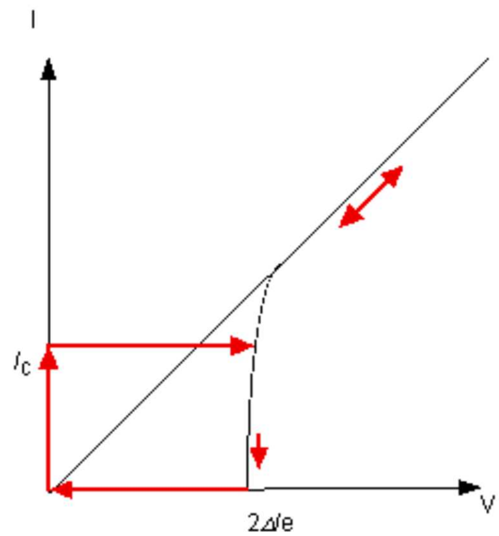


Fig. Schematic diagram of quasiparticle *I-V* characteristic (usually observed in a S-I-S Josephson *tunneling-type*). Josephson current (up to a maximum value I_c) flows at $V = 0$. Δ is an energy gap of the superconductor . The DC Josepson supercurrent flows under $V = 0$. For $V > 2\Delta/e$ the quasiparticle tunneling current is seen.

23. Energy gap

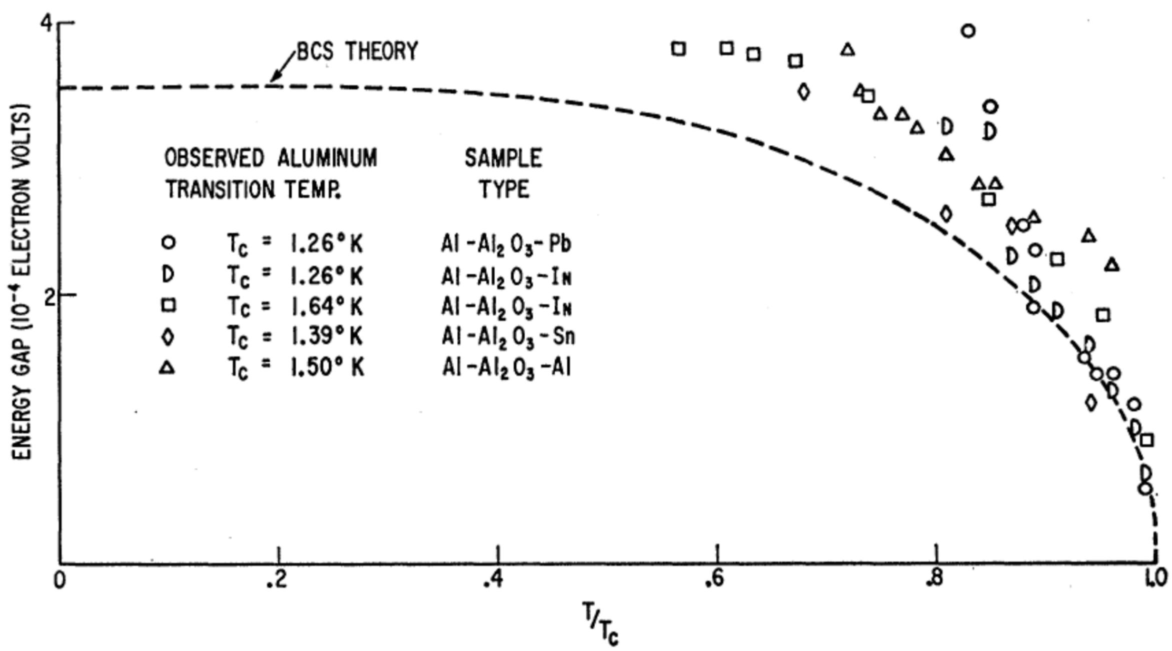


Fig. The energy gap of Pb, Sn, and In films as a function of reduced temperature, which is compared with the BCS theory.Fig.11 [I. Giaever and K. Megerle, Phys. Rev. B 122, 1101 (1961)].

24. Vortex state (Mixed state) $H_{c1}<H<H_{c2}$

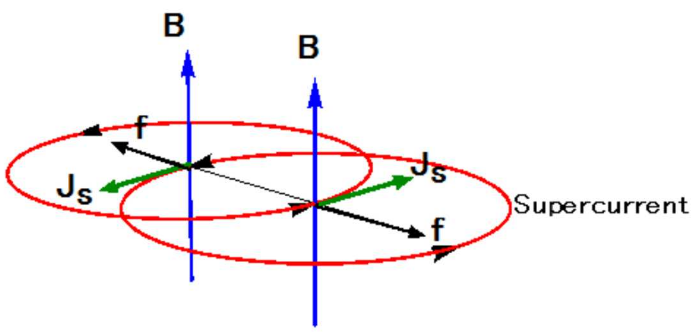


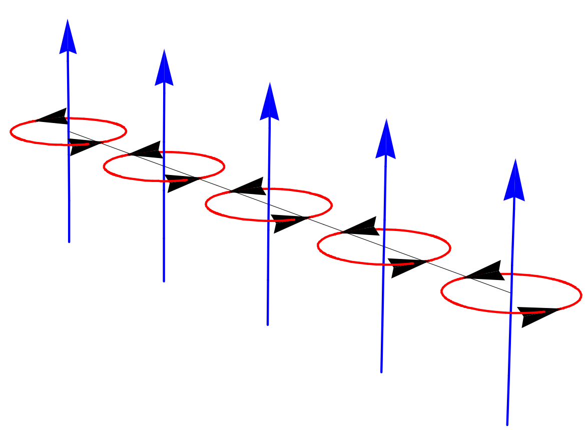
Fig. Repulsive interaction between vortices (fluxoids).

There is a repulsive force between two vortices. due to the Lorentz force. The repulsive force is given by

$$\boldsymbol{F} = \frac{l}{c}\boldsymbol{I}\times\boldsymbol{B} = \frac{lA}{c}\boldsymbol{J}\times\boldsymbol{B},$$

or the force density is given by

$$\boldsymbol{f} = \frac{\boldsymbol{F}}{V} = \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B}$$



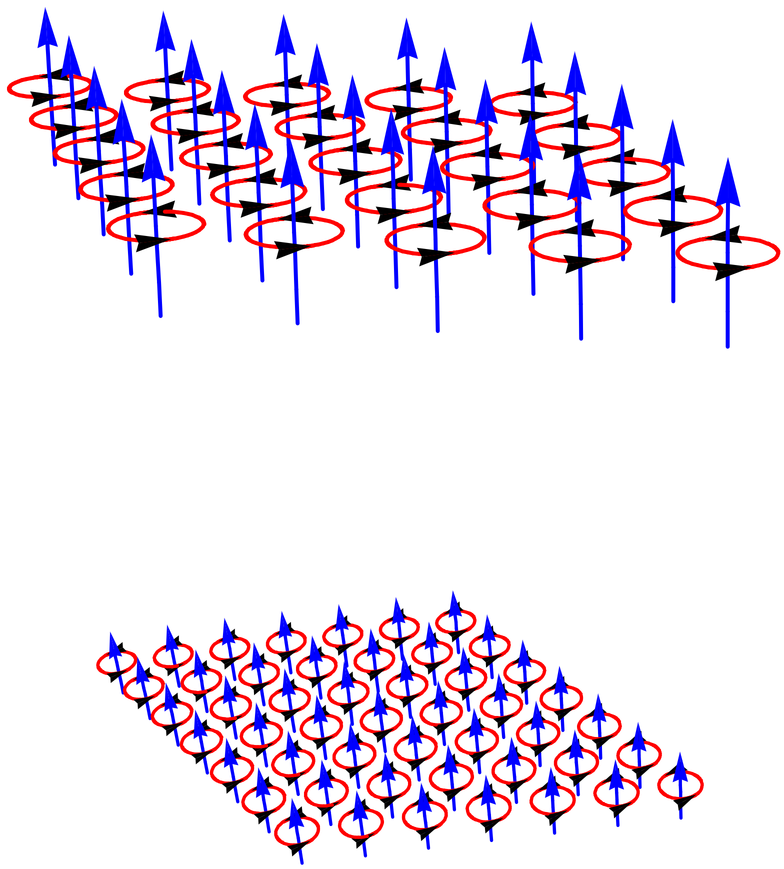


Fig. Abrikosov lattice for $H_{c1}<H<H_{c2}$. The vortices form a triangular lattice in the above case. The arrows show the direction of the supercurrents.

At $H = H_{c2}$, the spacing of the Abrikosov lattice is on the order of the coherence length. There is one vortex per lattice.

$$H_{c2} = \frac{\Phi_0}{2\pi\xi^2}$$

At $H = H_{c2}$, the fluxoids are packed together as tightly as possible, consistent with the preservation of the superconducting state.

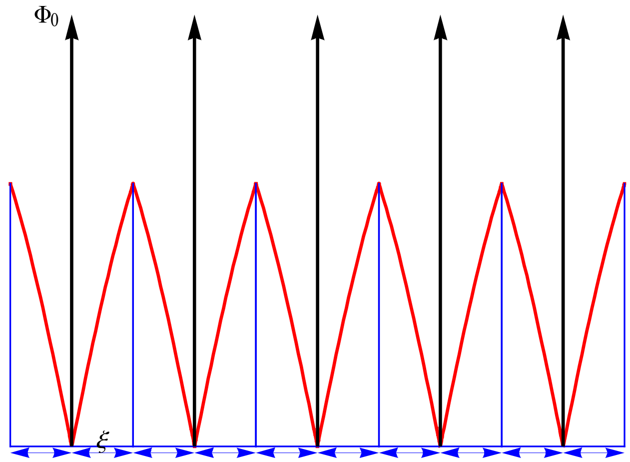


Fig. The spacing of the Abrikosov lattice is close to the coherence length ξ just below H_{c2}.

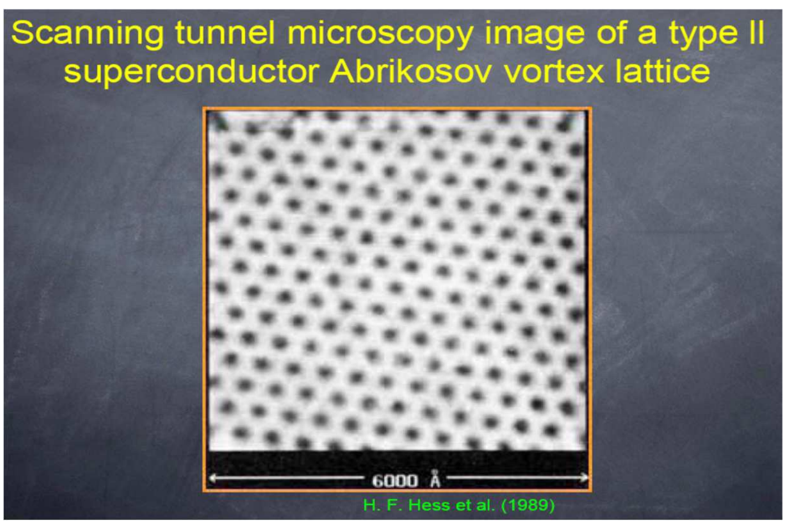


Fig. Scanning tunnel microscopy (STM) image of a type-II superconductor. Abrikosov vortex lattice (H.F. Hess et al. 1989). This figure is copied from PPT of Cooper.

25. The critical field H_{c1} and H_{c2} for type-II superconductor

The vortex state describes the circulation of superconducting currents in vortices throughout the bulk specimen. The vortex state is stable when the penetration of the applied field into the superconducting material causes the surface energy to become negative.

Estimation of H_{c1}:

At $H = H_{c1}$, there is one magnetic flux per the penetration depth.

$$\pi\lambda^2 H_{c1} = \Phi_0$$

H_{c1} is the field of nucleation of a single fluxoid

26. Movement of vortex leading to the disappearance of superconductivity

We consider Why the mixed state is still superconducting in spite of the penetration of magnetic flux.

$$\boldsymbol{f} = \frac{\boldsymbol{F}}{V} = \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B},$$

Because of the Lorentz force, the magnetic flux lines tend to move transverse to the current density ***J***.

If they do move, with velocity ***v***, they essentially induce an electric field of magnitude

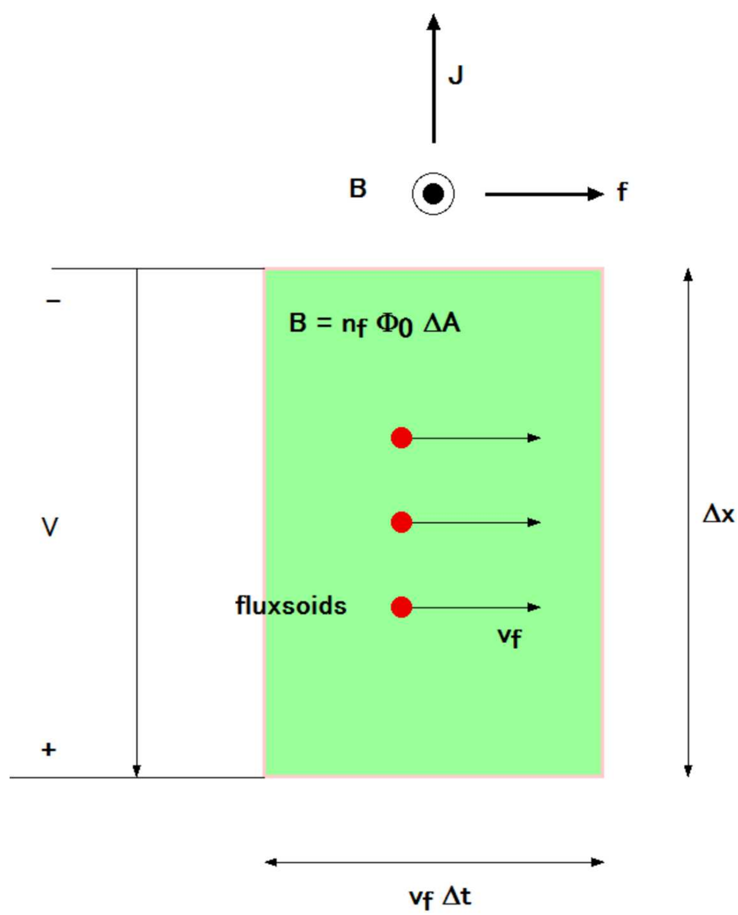


Fig. Movement of magnetic flux (fluxoid) when the magnetic field is applied to the system in the mixed phase.

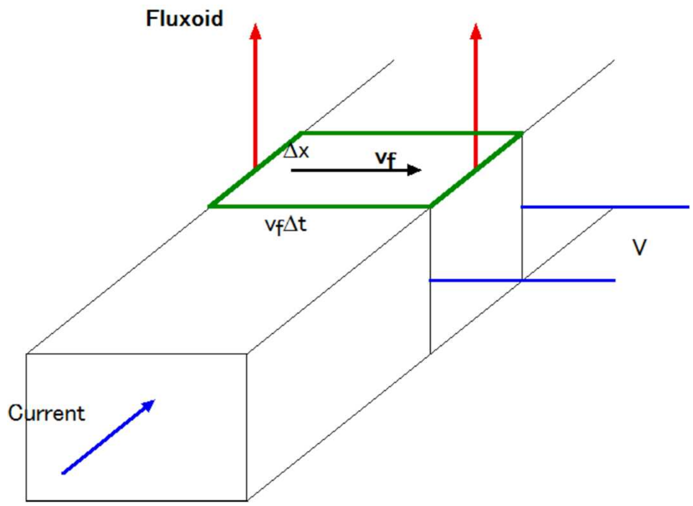


Fig. The generation of DC voltage through an AC Josephson effect, when the magnetic flux (fluxoid) moves at the velocity v_f.

The area is given by

$$\Delta A = (v_f \Delta t)\Delta x$$

The total change in the superconducting phase $\Delta\theta$ is given by

$$\Delta\theta = 2\pi n_f \Delta A = 2\pi n_f (v_f \Delta t)\Delta x$$

where n_f is the area density of vortices (fluxoids) and v_f is the velocity of vortices. The magnetic flux penetrating in the area ΔA is

$$\Delta\Phi = n_f \Phi_0 \Delta A = n_f \Phi_0 (v_f \Delta t)\Delta x$$

where

$$\Phi_0 = \frac{hc}{2e} = \frac{\pi\hbar c}{e}$$

27. Approach from the AC Josephson effect

We can evaluate the voltage ΔV produced across the distance Δx along the x direction, using the formula of the AC Josephson effect,

$$\frac{d\theta}{dt} = \frac{2e}{\hbar} V$$ **(AC Josephson effect)**

Since

$$\frac{\Delta\theta}{\Delta t} = \frac{2\pi n_f (v_f \Delta t)\Delta x}{\Delta t} = 2\pi n_f (v_f)\Delta x = \frac{2e}{\hbar}\Delta V$$

we get the electric field as

$$E = \frac{\Delta V}{\Delta x} = \pi n_f v_f \frac{\hbar}{e} = \pi n_f v_f \frac{\Phi_0}{\pi c} = n_f \frac{v_f}{c}\Phi_0 .$$

Note that the effective field B_0 is

$$B_0 = \frac{\Phi}{\Delta A} = n_f \Phi_0 .$$

Using this expression of B_0, we have the final form of $\boldsymbol{E}$ as

$$\boldsymbol{E} = \frac{\boldsymbol{v}_f \times \boldsymbol{B}}{c}.$$

In summary, in the mixed phase of the type-II superconductors, the resistivity is generated when the vortices which are pinned by impurities, dislocations, and so on, starts to move due to the Lorentz force. This means the destruction of the superconductivity.

28. Approach from the special relativity

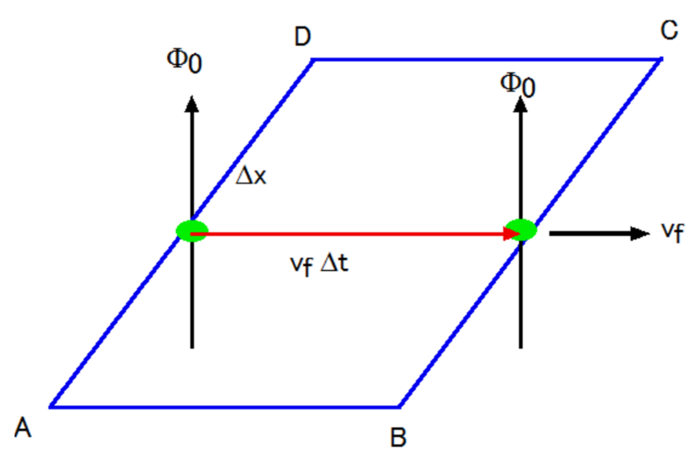


Fig. Application of the special relativity when the magnetic vortex moves at the velocity v_f.

Fig. Configuration where the quantum magnetic flux moves at the velocity v_f. The internal magnetic field (magnetic induction) B is $B = n_f\Phi_0$. As a result, the electric field is set up between the lines AB and

We consider the two frames K and K'. The frame K is the rest frame and the frame K' is moving to the right at a velocity v_f relative to the frame K. There is no electric field $\boldsymbol{E}$ in the K. The magnetic field B is

$$B_z = \frac{n_f \Phi_0 (\Delta x)(v_f \Delta t)}{(\Delta x)(v_f \Delta t)} = n_f \Phi_0$$

In the frame K', an electric field is newly established as

$$\boldsymbol{E}' = \frac{1}{c}\gamma \boldsymbol{v}_f \times \boldsymbol{B} = \frac{1}{c}\boldsymbol{v}_f \times \boldsymbol{B}'$$

where $\boldsymbol{B}'$ (= $\gamma\boldsymbol{B}$) and $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 1$

Since $\boldsymbol{B}'$ is nearly equal to $\boldsymbol{B}$, we have

$$\boldsymbol{E}' \approx \frac{1}{c}\boldsymbol{v}_f \times \boldsymbol{B}$$

---

29. Flux pinning preserving the superconductivity

Suppose that there is a pinning force which cancels out the Lorentz force. Then the velocity is equal to zero. Then we have $\Delta V = 0$, which means that the system is still superconducting state where the resistivity is equal to zero.

Flux pinning is the phenomenon that magnetic flux lines do not move (become trapped, or "pinned") in spite of the Lorentz force acting on them inside a current-carrying Type II superconductor. Flux pinning is only possible when there are defects in the crystalline structure of the superconductor (usually resulting from grain boundaries or impurities).

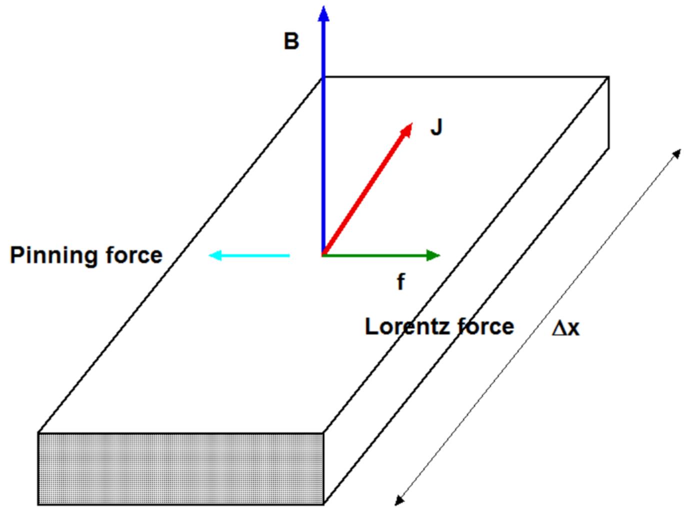


Fig. Balance of the Lorentz force and pinning force. The system is still in the superconducting phase.

---

30. Ginzburg-Landau (GL) Theory

It is surprising that the rich phenomenology of the superconducting state could be quantitatively described by the GL theory,[11] without knowledge of the underlying microscopic mechanism based on the BCS theory. It is based on the idea that the superconducting transition is one of the second order phase transition. In fact, the universality class of the critical behavior belongs to the three-dimensional *XY* system such as liquid ^{4}He. The general theory of the critical behavior can be applied to the superconducting phenomena. The order parameter is described by two components (complex number $\psi = |\psi| e^{i\theta}$). The amplitude $|\psi|$ is zero in the normal phase above a superconducting transition temperature T_c and is finite in the superconducting phase below T_c. In the presence of an external magnetic field, the order parameter has a spatial variation. When the spatial variation of the order parameter is taken into account, the free energy of the system can be expressed in terms of the order parameter ψ and its spatial derivative of ψ. In general, this is valid in the vicinity of T_c below T_c, where the amplitude $|\psi|$ is small and the length scale for spatial variation is long.

The order parameter ψ is considered as a kind of a wave function for a particle of charge q^* and mass m^*. The two approaches, the BCS theory and the GL theory, remained completely separate until Gorkov[13] showed that, in some limiting cases, the order parameter $\psi(\mathbf{r})$ of the GL theory is proportional to the pair potential $\Delta(\mathbf{r})$. At the same time this also shows that $q^* = 2e$ (<0) and $m^* = 2m$. Consequently, the Ginzburg-Landau theory acquired their definitive status.

The GL theory is a triumph of physical intuition, in which a wave function $\psi(\mathbf{r})$ is introduced as a complex order parameter. The parameter $|\psi(\mathbf{r})|^2$ represents the local density of superconducting electrons, $n_s(\boldsymbol{r})$. The macroscopic behavior of superconductors (in particular the type II superconductors) can be explained well by this GL theory. This theory also provides the qualitative framework for understanding the dramatic supercurrent behavior as a consequence of quantum properties on a macroscopic scale.

The superconductors are classed into two types of superconductor: type-I and type-II superconductors. The Ginzburg-Landau parameter κ is the ratio of λ to ξ, where λ is the magnetic-field penetration depth and ξ is the coherence length of the superconducting phase. The limiting value $\kappa = 1/\sqrt{2}$ separating superconductors with positive surface energy $(\kappa < 1/\sqrt{2})$ (type-I) from those with negative surface energy $(\kappa > 1/\sqrt{2})$ (type-II), is properly identified. For the type-II superconductor, the superconducting and normal regions coexist. The normal regions appear in the cores (of size ξ) of vortices binding individual magnetic flux quanta $\Phi_0 = 2\pi\hbar c/|q^*|$ on the scale λ, with the charge $|q^*| = 2|e|$ appearing in Φ_0 a consequence of the pairing mechanism. Since $\lambda > \xi$, the vortices repel and arrange in a so-called Abrikosov lattice. In his 1957 paper, Abrikosov[14] derived the periodic vortex structure near the upper critical field H_{c2}, where the superconductivity is totally suppressed, determined the magnetization $M(H)$, calculated the field H_{c1} of first penetration, analyzed the structure of individual vortex lines, found the structure of the vortex lattice at low fields.

31. Ginzburg-Landau theory-phenomenological approach

We introduce the order parameter $\psi(\boldsymbol{r})$ with the property that

$$\psi^*(\boldsymbol{r})\psi(\boldsymbol{r}) = n_s(\boldsymbol{r}),$$

which is the local concentration of superconducting electrons. We first set up a form of the free energy density $F_s(\boldsymbol{r})$,

$$F_s(\boldsymbol{r}) = F_N + \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{1}{2m^*}\left|(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\boldsymbol{A})\psi\right|^2 + \frac{\boldsymbol{B}^2}{8\pi},$$

where β is positive and the sign of α is dependent on temperature.

We must minimize the free energy with respect to the order parameter $\psi(\boldsymbol{r})$ and the vector potential $\boldsymbol{A}(\boldsymbol{r})$. We set

$$\Im = \int F_s(\boldsymbol{r})dr,$$

where the integral is extending over the volume of the system. If we vary

$$\psi(\boldsymbol{r}) \to \psi(\boldsymbol{r}) + \delta\psi(\boldsymbol{r}) \text{ and } \boldsymbol{A}(\boldsymbol{r}) \to \boldsymbol{A}(\boldsymbol{r}) + \delta\boldsymbol{A}(\boldsymbol{r}),$$

we obtain the variation in the free energy such that

$$\Im + \delta\Im .$$

By setting $\delta\Im = 0$, we obtain the GL equation

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\boldsymbol{A}\right)^2\psi = 0,$$

and the current density

$$\boldsymbol{J}_s = \frac{q^*\hbar}{2m^* i}[\psi^*\nabla\psi - \psi\nabla\psi^*] - \frac{q^{*2}|\psi|^2}{m^* c}\boldsymbol{A}$$

or

$$\boldsymbol{J}_s = \frac{q^*}{2m^*}[\psi^*(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\boldsymbol{A})\psi + \psi(-\frac{\hbar}{i}\nabla - \frac{q^*}{c}\boldsymbol{A})\psi^*].$$

At a free surface of the system we must choose the gauge to satisfy the boundary condition that no current flows out of the superconductor into the vacuum.

$$\boldsymbol{n}\cdot\boldsymbol{J}_s = 0 .$$

32. GL free energy and Thermodynamic critical field H_c

$\boldsymbol{A}$ = 0 and $\psi = \psi_\infty$ (real) has no space dependence. Why ψ is real? We have a gauge transformation;

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi \text{ and } \psi'(\boldsymbol{r}) = \exp(\frac{iq^*\chi}{c\hbar})\psi(\boldsymbol{r}).$$

We choose $\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi = \boldsymbol{A}$ with $\chi = \chi_0 =$ constant. Then we have $\psi'(\boldsymbol{r}) = \exp(\frac{iq^*\chi_0}{c\hbar})\psi(\boldsymbol{r})$, or $\psi(\boldsymbol{r}) = \exp(-\frac{iq^*\chi_0}{c\hbar})\psi'(\boldsymbol{r})$. Even if ψ' is complex number, ψ can be real number.

$$F_s = F_N + \alpha\psi_\infty^{\;2} + \frac{1}{2}\beta\psi_\infty^{\;4}.$$

When $\frac{\partial F}{\partial \psi_\infty} = 0$, F_s has a local minimum at

$$\psi_\infty = (-\alpha/\beta)^{1/2} = (|\alpha|/\beta)^{1/2}.$$

Then we have

$$F_s - F_N = -\frac{\alpha^2}{2\beta} = -\frac{H_c^{\;2}}{8\pi}$$

from the definition of the thermodynamic critical field:

$$H_c = \left(\frac{4\pi\alpha^2}{\beta}\right)^{1/2} = H_c(0)(1 - \frac{T^2}{T_c^{\;2}}).$$

Suppose that β is independent of T, then

$$\alpha = -\sqrt{\frac{\beta}{4\pi}}H_c(0)(1 - \frac{T^2}{T_c^{\;2}}) \approx 2\sqrt{\frac{\beta}{4\pi}}H_c(0)(\frac{T}{T_c} - 1) = \alpha_0(\frac{T}{T_c} - 1),$$

where

$$\alpha_0 = 2\sqrt{\frac{\beta}{4\pi}}H_c(0).$$

The parameter α is positive above T_c and is negative below T_c. Note that $\beta>0$. For $T<T_c$, the sign of α is negative:

$$F_s - F_N = \alpha_0 (t-1)\psi_\infty^{\ 2} + \frac{1}{2}\beta\psi_\infty^{\ 4},$$

where α_0>0 and $t = T/T_c$ is a reduced temperature.

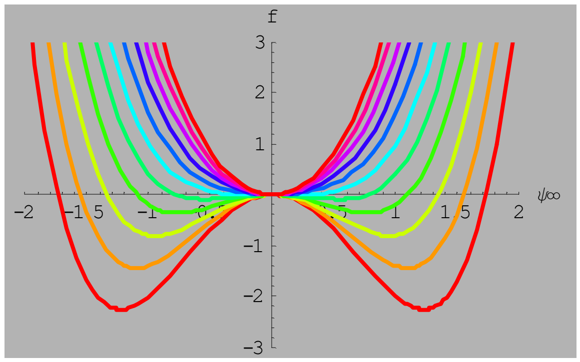


Fig. The GL free energy functions expressed by Eq.(5.7), as a function of ψ_∞. $\alpha_0 = 3$. $\beta = 1$. t is changed as a parameter. $t = T/T_c$. ($t = 0 – 2$) around $t = 1$.

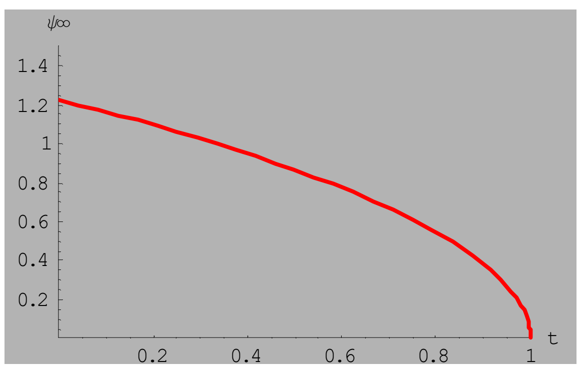


Fig. The order parameter ψ_∞ as a function of a reduced temperature $t = T/T_c$. $\alpha_0 = 3$. $\beta = 1$.

33. Coherence length ξ from GL theory

We assume that $\boldsymbol{A} = 0$. We choose the gauge in which ψ is real.

$$\alpha\psi + \beta\psi^3 - \frac{\hbar^2}{2m^*}\frac{d^2}{dx^2}\psi = 0.$$

We put $\psi = \psi_\infty f$.

$$-\frac{\hbar^2}{2m^*\alpha}\frac{d^2 f}{dx^2}+f-f^3=0\,.$$

We introduce the coherence length

$$\xi^2=-\frac{\hbar^2}{2m^*\alpha}=\frac{\hbar^2}{2m^*|\alpha|}=\frac{2\pi n_s^*\hbar^2}{m^*H_c^{\ 2}}\,,$$

where

$$\frac{\alpha^2}{2\beta}=\frac{H_c^{\ 2}}{8\pi}\,,\qquad\qquad \psi_\infty^{\ 2}=\frac{|\alpha|}{\beta}=n_s\,,$$

or

$$\xi=\sqrt{\frac{\hbar^2}{2m^*\alpha_0}}\,|1-\frac{T}{T_c}|^{-1/2}\,.$$

Note that the coherence length diverges as the temperature approaches T_c. Then we have

$$\xi^2\frac{d^2 f}{dx^2}+f-f^3=0\,,$$

with the boundary condition $f = 1$, $df/dx = 0$ at $x = \infty$ and $f = 0$ at $x = 0$.

$$\xi^2\frac{df}{dx}\frac{d^2 f}{dx^2}+(f-f^3)\frac{df}{dx}=0\,,$$

or

$$\frac{d}{dx}\frac{\xi^2}{2}\left(\frac{df}{dx}\right)^2=\frac{d}{dx}\left(\frac{f^4}{4}-\frac{f^2}{2}\right),$$

or

$$\frac{\xi^2}{2}\left(\frac{df}{dx}\right)^2=\frac{1}{4}(1-f^2)^2\,,$$

or

$$\frac{df}{dx} = \frac{1}{\sqrt{2}\xi}(1 - f^2).$$

The solution of this equation is given by

$$f = \tanh\left(\frac{x}{\sqrt{2}\xi}\right).$$

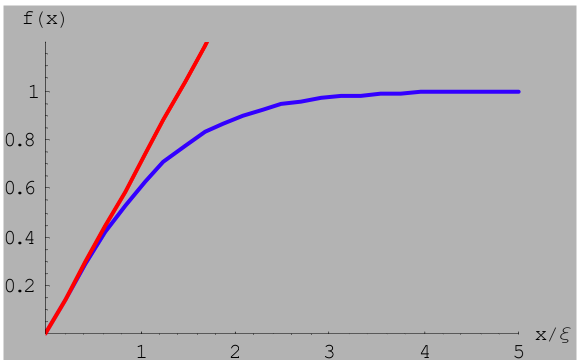


Fig. Normalized order parameter $f(x)$ as a function of x/ξ.

34. Physical meaning of coherence length ξ

The coherence length ξ is a measure of the distance within which the superconducting electron concentration cannot change drastically in a spatially-varying magnetic field. The coherence length ξ is a measure of the range over which we should average $\boldsymbol{A}$ to obtain the current density $\boldsymbol{J}$. It is also a measure of the minimum spatial extent of a transition layer between the normal metal and superconducting phases.

We consider two kinds of wavefunctions given by

$$|\psi\rangle = |k\rangle$$

$$|\varphi\rangle = \frac{1}{\sqrt{2}}[|k+q\rangle + |k\rangle]$$

where

$$\psi(x) = \langle x|\psi\rangle = \langle x|k\rangle = \frac{1}{\sqrt{2\pi}}e^{ikx}$$

and

$$\begin{aligned}\varphi(x) &= \langle x|\varphi\rangle \\ &= \frac{1}{\sqrt{2}}[\langle x|k\rangle + \langle x|k+q\rangle] \\ &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2\pi}}e^{ikx}(1+e^{iqx})\end{aligned}$$

where the probability amplitude is

$$|\psi(x)|^2 = \frac{1}{2\pi}$$

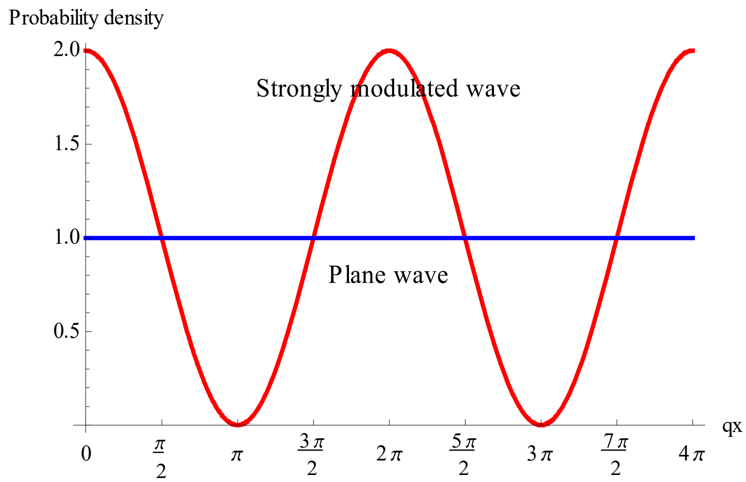
$$|\varphi(x)|^2 = \frac{1}{4\pi}(1+e^{iqx})(1+e^{-iqx}) = \frac{1}{2\pi}[1+\cos(qx)]$$

The kinetic energy;

$$\langle\psi|\hat{H}|\psi\rangle = \frac{\hbar^2 k^2}{2m}$$

$$\begin{aligned}\langle\varphi|\hat{H}|\varphi\rangle &= \frac{1}{2}(\langle k| + \langle k+q|)\hat{H}(|k\rangle + |k+q\rangle) \\ &= \frac{\hbar^2}{4m}(\langle k| + \langle k+q|)[k^2|k\rangle + (k+q)^2|k+q\rangle) \\ &= \frac{\hbar^2}{4m}[k^2 + (k+q)^2] \\ &= \frac{\hbar^2}{4m}(2k^2 + q^2 + 2kq) \approx \frac{\hbar^2}{2m}k^2 + \frac{\hbar^2}{2m}kq\end{aligned}$$

where we neglect q^2 for $q<<k$.



The increase of energy required to modulate is

$$\frac{\hbar^2}{2m}kq\ .$$

If this increase exceeds ε_g, the superconductivity will be destroyed. The critical value q0 of the modulation wave vector is given by

$$\frac{\hbar^2}{2m}k_F q_0 = \varepsilon_g$$

The intrinsic coherence length ξ_0 is defined as

$$\xi_0 \approx \frac{1}{q_0} = \frac{\hbar^2 k_F}{2m\varepsilon_g} = \frac{\hbar v_F}{2\varepsilon_g}$$

where v_F is the Fermi velocity.

((Note)) BCS theory

$$\xi_0 = \frac{2\hbar v_F}{\pi\varepsilon_g} = \frac{\hbar v_F}{\pi\Delta}$$

In impure materials and in alloys,

$$\xi < \xi_0$$

since in impure metals the electron eigenfunctions already have wiggles in them.

35. Lagrangian of particles with mass *m** and charge *q** in the presence of magnetic field

The Lagrangian L for the motion of a particle in the presence of magnetic field and electric field is given by

$$L = \frac{1}{2} m^* \boldsymbol{v}^2 - q^* (\varphi - \frac{1}{c} \boldsymbol{v} \cdot \boldsymbol{A}),$$

where m^* and q^* are the mass and charge of the particle. A is a vector potential and ϕ is a scalar potential.

The canonical momentum is defined as

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} = m\boldsymbol{v} + \frac{q}{c}\boldsymbol{A}.$$

The mechanical momentum (the measurable quantity) is given by

$$\boldsymbol{\pi} = m^* \boldsymbol{v} = \boldsymbol{p} - \frac{q^*}{c}\boldsymbol{A}.$$

The Hamiltonian H is given by

$$H = \boldsymbol{p} \cdot \boldsymbol{v} - L = (m^* \boldsymbol{v} + \frac{q^*}{c}\boldsymbol{A}) \cdot \boldsymbol{v} - L = \frac{1}{2} m^* \boldsymbol{v}^2 + q^* \varphi = \frac{1}{2m^*}(\boldsymbol{p} - \frac{q^*}{c}\boldsymbol{A})^2 + q^* \varphi.$$

The Hamiltonian formalism uses $\boldsymbol{A}$ and ϕ, and not $\boldsymbol{E}$ and $\boldsymbol{B}$, directly. The result is that the description of the particle depends on the gauge chosen.

36. Current density for the superconductors

We consider the current density for the superconductor. ψ is the order parameter of the superconductor and m^* and q^* are the mass and charge of the Cooper pairs. The current density is invariant under the gauge transformation.

$$\boldsymbol{J}_s = \frac{q^*}{m^*} \mathrm{Re}[\langle \psi | \hat{\boldsymbol{p}} - \frac{q^*}{c}\boldsymbol{A} | \psi \rangle],$$

This can be rewritten as

$$\begin{aligned}
\boldsymbol{J}_s &= \frac{q^*}{m^*}\mathrm{Re}[\psi^*(\frac{\hbar}{i}\nabla\psi - \frac{q^*}{c}\boldsymbol{A}\psi)] \\
&= \frac{q^*}{2m^*}[(\psi^*\frac{\hbar}{i}\nabla\psi - \frac{q^*}{c}\boldsymbol{A}\psi^*\psi) + (-\psi\frac{\hbar}{i}\nabla\psi^* - \frac{q^*}{c}\boldsymbol{A}\psi^*\psi)] \\
&= \frac{q^*\hbar}{2m^*i}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{q^{*2}|\psi|^2}{m^*c}\boldsymbol{A}
\end{aligned}$$

The density is also gauge independent.

$$\rho_s = |\langle\boldsymbol{r}|\psi\rangle|^2.$$

Now we assume that

$$\psi(\boldsymbol{r}) = |\psi(\boldsymbol{r})|e^{i\theta(\boldsymbol{r})}.$$

We note that

$$\begin{aligned}
\psi^*(\boldsymbol{p} - \frac{q^*}{c}\boldsymbol{A})\psi(\boldsymbol{r}) - &= \psi^*\frac{\hbar}{i}\nabla[e^{i\theta(\boldsymbol{r})}|\psi(\boldsymbol{r})|] - \frac{q^*}{c}\boldsymbol{A}|\psi(\boldsymbol{r})|^2 \\
&= |\psi(\boldsymbol{r})|e^{-i\theta(\boldsymbol{r})}\frac{\hbar}{i}[i|\psi(\boldsymbol{r})|e^{i\theta(\boldsymbol{r})}\nabla\theta(\boldsymbol{r}) \\
&\quad + e^{i\theta(\boldsymbol{r})}\nabla|\psi(\boldsymbol{r})|] - \frac{q^*}{c}\boldsymbol{A}|\psi(\boldsymbol{r})|^2 \\
&= \hbar|\psi(\boldsymbol{r})|^2[\nabla\theta(\boldsymbol{r}) - \frac{q^*}{c\hbar}\boldsymbol{A}] - i\hbar|\psi(\boldsymbol{r})|\nabla|\psi(\boldsymbol{r})|
\end{aligned}.$$

The last term is pure imaginary. Then the current density is obtained as

$$\boldsymbol{J}_s = \frac{q^*\hbar}{m^*}|\psi|^2(\nabla\theta - \frac{q^*}{c\hbar}\boldsymbol{A}) = q^*|\psi|^2\boldsymbol{v}_s,$$

or

$$\hbar\nabla\theta = \frac{q^*}{c}\boldsymbol{A} + m^*\boldsymbol{v}_s.$$

Since

$$\boldsymbol{\pi} = m^* \boldsymbol{v}_s = \boldsymbol{p} - \frac{q^*}{c}\boldsymbol{A},$$

we have

$$\boldsymbol{p} = \frac{q^*}{c}\boldsymbol{A} + m^* \boldsymbol{v}_s = \hbar \nabla \theta .$$

Note that $\boldsymbol{J}_s$ (or $\boldsymbol{v}_s$) is gauge-invariant. Under the gauge transformation, the wave function is transformed as

$$\psi'(\boldsymbol{r}) = \exp(\frac{iq^*\chi}{\hbar c})\psi(\boldsymbol{r}) .$$

This implies that

$$\theta \to \theta' = \theta + \frac{q^*\chi}{\hbar c},$$

Since $\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi$, we have

$$\begin{aligned}\boldsymbol{J}_s' &= \hbar(\nabla\theta' - \frac{q^*}{c\hbar}\boldsymbol{A}') \\ &= \hbar[\nabla(\theta + \frac{q^*\chi}{\hbar c}) - \frac{q^*}{c\hbar}(\boldsymbol{A} + \nabla\chi)] . \\ &= \hbar(\nabla\theta - \frac{q^*}{c\hbar}\boldsymbol{A})\end{aligned}$$

So the current density is invariant under the gauge transformation.

37. London's equation

We start with

$$\boldsymbol{p} = \hbar\nabla\theta = \frac{q^*}{c}\boldsymbol{A} + m^*\boldsymbol{v}_s ,$$

where $\boldsymbol{p}$ is the canonical momentum. We assume that $|\psi|^2 = n_s$ is independent of $\boldsymbol{r}$. Then we get

$$\boldsymbol{J}_s = q^* |\psi|^2 \boldsymbol{v}_s = q^* n_s \boldsymbol{v}_s .$$

Using these two equations, we get

$$\boldsymbol{p} = \hbar \nabla \theta = \frac{q^*}{c} \boldsymbol{A} + \frac{m^*}{q^* n_s} \boldsymbol{J}_s .$$

Suppose that $\boldsymbol{p} = \hbar \nabla \theta = 0$, which means that the phase θ is independent of $\boldsymbol{r}$. Then we have a London's equation,

$$\boldsymbol{J}_s = -\frac{q^{*2} n_s}{m^* c} \boldsymbol{A} .$$

From this equation, we get

$$\nabla \times \boldsymbol{J}_s = -\frac{q^{*2} n_s}{m^* c} \nabla \times \boldsymbol{A} = -\frac{q^{*2} n_s}{m^* c} \boldsymbol{B} .$$

Using the Maxwell's equation

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c} \boldsymbol{J}_s , \qquad \text{and} \qquad \nabla \cdot \boldsymbol{B} = 0 ,$$

we get

$$\nabla \times (\nabla \times \boldsymbol{B}) = \frac{4\pi}{c} \nabla \times \boldsymbol{J}_s = -\frac{4\pi n_s q^{*2}}{m^* c^2} \boldsymbol{B} ,$$

where

$$n_s = |\psi|^2 = \text{constant} \qquad \text{(independent of } \boldsymbol{r}\text{)}$$

$$\lambda_L{}^2 = \frac{m^* c^2}{4\pi n_s q^{*2}} , \qquad \text{(penetration depth).}$$

Then

$$\nabla\times(\nabla\times\boldsymbol{B}) = \nabla(\nabla\cdot\boldsymbol{B}) - \nabla^2\boldsymbol{B} = -\frac{1}{\lambda_L{}^2}\boldsymbol{B},$$

or

$$\nabla^2\boldsymbol{B} = \frac{1}{\lambda_L{}^2}\boldsymbol{B}.$$

Inside the system, ***B*** become s zero, corresponding to the Meissner effect.

38. Penetration depth and the surface current

We assume that the magnetic field is applied along the z direction.

$\boldsymbol{B}$=(0, 0, $B_z(x)$)

$$\frac{d^2B_z(x)}{dx^2} = \frac{1}{\lambda_L{}^2}B_z(x)$$

The solution of this equation is given by

$$B_z(x) = B_z(x=0)\exp(-\frac{x}{\lambda_L})$$

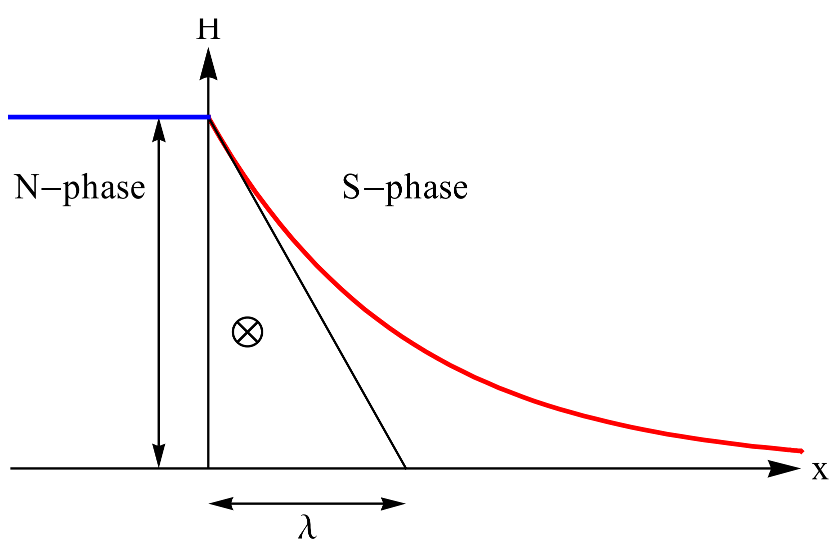


Fig. The distribution of the internal magnetic field B near the boundary between the normal phase and the superconducting phase. λ is the penetration depth. The direction of B is into the page.

where λ_L is the penetration depth. Then the surface super current $\boldsymbol{J}_s$ is given by

$$\boldsymbol{J}_s = \frac{c}{4\pi}\nabla\times\boldsymbol{B} = (-\frac{c}{4\pi})\boldsymbol{e}_y\frac{\partial B_z(x)}{\partial x} = \frac{c}{4\pi\lambda_L}\boldsymbol{e}_y\exp(-\frac{x}{\lambda_L})$$

Since $\frac{\partial B_z(x)}{\partial x}<0$ for $x>0$ (inside the S phase), the surface supercurrent flows along the positive y axis only in the region over the penetration depth λ from the

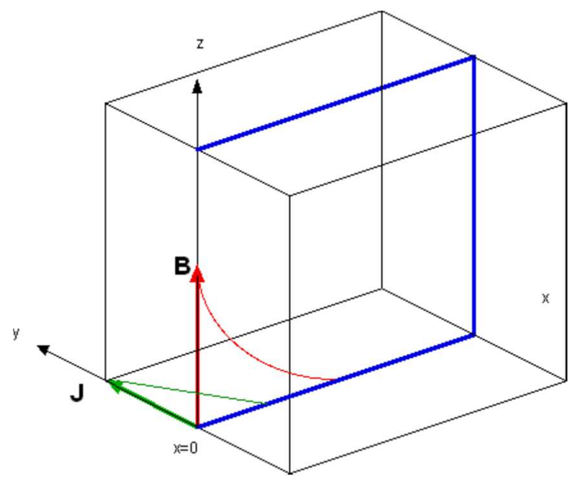


Fig. The distribution of the magnetic induction $B(x)$ (along the z axis) and the current density (along the y axis) near the boundary between the normal phase and the superconducting phase. The plane with $x = 0$ is the boundary.

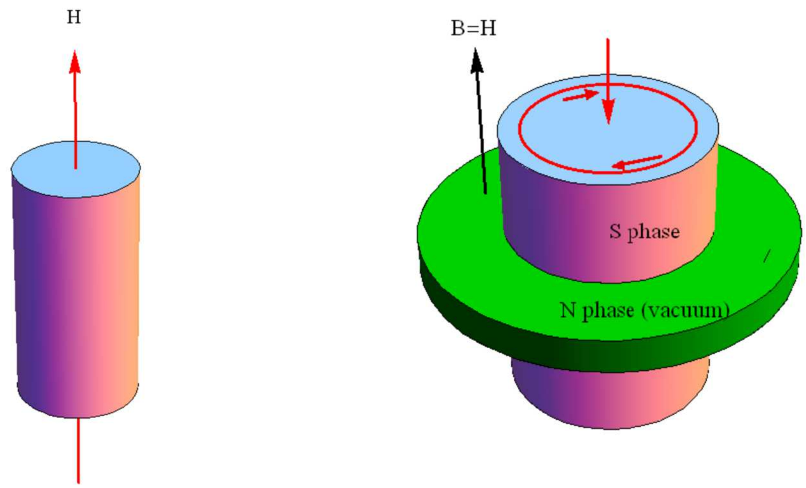


Fig. Suppose that a magnetic field is applied to the axis direction of the superconductor [vacuum or N phase (green) and *S* phase (pink)]. The supercurrent (red) flows around the surface of the S-phase (over the limited region of penetration depth from the surface) in the clock-wise direction. The external magnetic field is cancelled out by the magnetic field due to the supercurrent, leading to the perfect diamagnetism (Meissner effect).

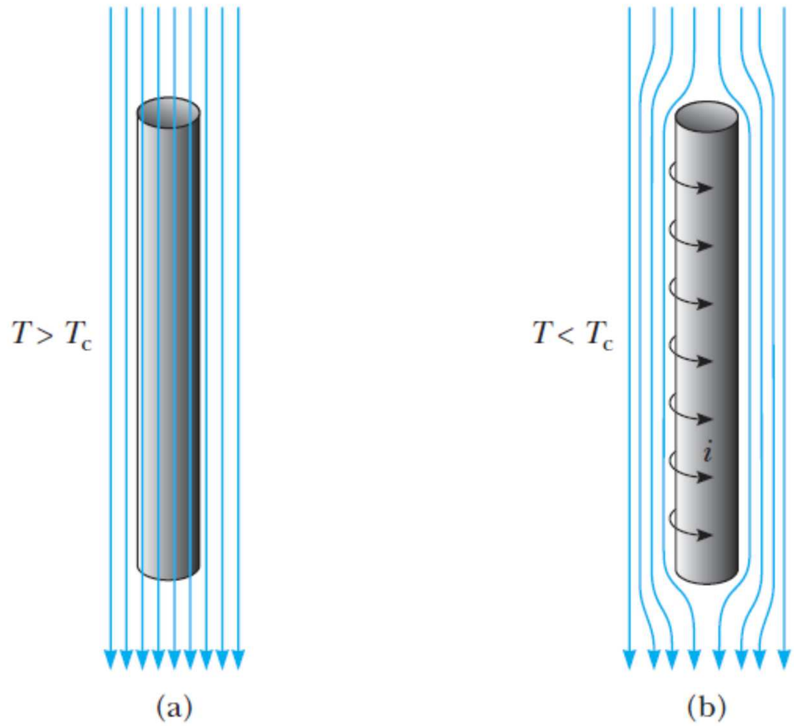


Fig. For $T<T_c$, the surface current flows near the surface, leading to the cancellation of the external magnetic field. Meissner effect.

39. Flux quantization

We start with the current density

$$\boldsymbol{J}_s = \frac{q^*\hbar}{m^*}|\psi|^2 (\nabla\theta - \frac{q^*}{c\hbar}\boldsymbol{A}) = q^*|\psi|^2 \boldsymbol{v}_s .$$

Suppose that $n_s = |\psi|^2$ =constant. Then we have

$$\nabla\theta = \frac{m^*}{q^*\hbar n_s}\boldsymbol{J}_s + \frac{q^*}{c\hbar}\boldsymbol{A},$$

or

$$\oint \nabla\theta\cdot d\boldsymbol{l} = \frac{m^*}{q^*\hbar n_s^*}\oint \boldsymbol{J}_s\cdot d\boldsymbol{l} + \frac{q^*}{c\hbar}\oint \boldsymbol{A}\cdot d\boldsymbol{l} .$$

The path of integration can be taken inside the penetration depth where $\boldsymbol{J}_s$ =0.

$$\oint \nabla\theta \cdot d\boldsymbol{l} = \frac{q^*}{c\hbar}\oint \boldsymbol{A}\cdot d\boldsymbol{l} = \frac{q^*}{c\hbar}\int (\nabla\times\boldsymbol{A})\cdot d\boldsymbol{a} = \frac{q^*}{c\hbar}\int \boldsymbol{B}\cdot d\boldsymbol{a} = \frac{q^*}{c\hbar}\Phi ,$$

where Φ is the magnetic flux. Then we find that

$$\Delta\theta = \theta_2 - \theta_1 = 2\pi n = \frac{q^*}{c\hbar}\Phi ,$$

where n is an integer. The phase θ of the wave function must be unique, or differ by a multiple of 2π at each point,

$$\Phi = \frac{2\pi c\hbar}{|q^*|} n .$$

The flux is quantized. When $|q^*| = 2|e|$, we have a magnetic quantum fluxoid;

$$\Phi_0 = \frac{2\pi c\hbar}{2|e|} = \frac{ch}{2|e|} = 2.06783372 \times 10^{-7} \text{ Gauss cm}^2$$

40. Parameters related to the superconductivity

Here the superconducting parameters are listed for convenience.

Thermodynamic field $\quad H_c = \sqrt{\frac{4\pi\alpha^2}{\beta}}, \qquad H_c^2 = 4\pi\psi_\infty^4\beta$

$$H_c = \frac{c\hbar}{\sqrt{2}|q^*|\lambda\xi}$$

Order parameter $\quad \psi_\infty^2 = n_s^* = |\alpha|/\beta .$

The parameters $\quad |\alpha| = \frac{H_c^2}{4\pi n_s^*}, \qquad \beta = \frac{H_c^2}{4\pi n_s^{*2}} .$

Quantum fluxoid $\quad \Phi_0 = \frac{2\pi\hbar c}{|q^*|} .$

Magnetic field penetration depth $\lambda = \sqrt{\frac{m^* c^2 \beta}{4\pi q^{*2} |\alpha|}}$, $\lambda = \sqrt{\frac{m^* c^2}{4\pi n_s^* q^{*2}}}$.

Coherence length $\xi = \frac{\hbar}{\sqrt{2m^* |\alpha|}}$, $\xi = \frac{\sqrt{2\pi n_s^*}\hbar}{H_c \sqrt{m^*}}$.

Ginzburg-Landau parameter $\kappa = \frac{\lambda}{\xi} = \frac{cm^* \sqrt{\beta}}{\sqrt{2\pi}\hbar |q^*|}$, $\kappa = \frac{cm^* H_c}{2\sqrt{2\pi}\hbar n_s^* |q^*|}$.

$\kappa < \frac{1}{\sqrt{2}}$ for type-I superconductor

$\kappa > \frac{1}{\sqrt{2}}$ for type-II superconductor

Note:

$|q^*| = 2e\ (>0) \quad m^* = 2m$, $\quad n_s^* = n_s/2$.

41. Critical fields H_{c1} and H_{c2} for the type II supercinductor

Lower critical field (type II) $H_{c1} = \frac{\Phi_0}{4\pi\lambda^2} \ln(\frac{\lambda}{\xi}) = \frac{H_c}{\sqrt{2}\kappa} \ln(\kappa)$.

Upper critical field (type II) $H_{c2} = \frac{\Phi_0}{2\pi\xi^2}$.

Surface-sheath field $H_{c3} = 1.695 H_{c2}$.

Relations between H_c, H_{c1}, and H_{c2}

$$\frac{\lambda\xi}{\Phi_0} = \frac{1}{2\sqrt{2}\pi H_c}, \qquad \frac{\kappa}{\lambda^2} = \frac{\sqrt{2}|q^*| H_c}{c\hbar}.$$

$$\frac{\kappa}{\lambda^2}\Phi_0 = 2\sqrt{2}\pi H_c, \qquad \kappa = \frac{2\sqrt{2}\pi\lambda^2 H_c}{\Phi_0}.$$

$$\sqrt{2}H_c = \frac{\Phi_0}{2\pi\lambda\xi}, \qquad \frac{H_c}{H_{c2}} = \frac{1}{\sqrt{2}\kappa}.$$

42. Vortex structure

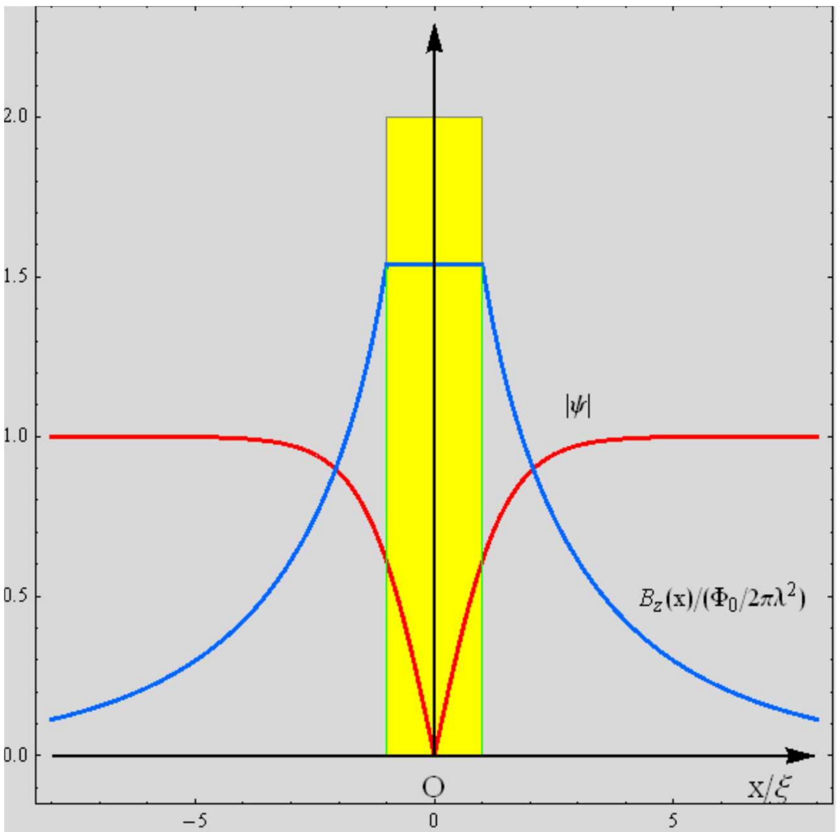


Fig. Plot of the magnetic induction $B(r) = \frac{\Phi_0}{2\pi\lambda^2} K_0(\frac{x}{\lambda})$ and the order parameter $|\psi(x)|/\psi_\infty$ as a function of x/ξ. We use the GL parameter as $\kappa = \lambda/\xi = 4$.

$$f = \frac{|\psi|}{\psi_\infty} = \tanh\left(\frac{x}{\sqrt{2}\xi}\right)$$

$$B(r) = \frac{\Phi_0}{2\pi\lambda^2} K_0(\frac{r}{\lambda})$$

where $K_n(x)$ is the modified Bessel function of the second-kind.

For $\xi < r << \lambda$,

$$B(r) = \frac{\Phi_0}{2\pi\lambda^2}[\ln(\frac{r}{\lambda}) + 0.115932].$$

For $r >> \lambda$,

$$B(r) = \frac{\Phi_0}{2\pi\lambda^2}\sqrt{\frac{\pi\lambda}{2r}}\exp(-\frac{r}{\lambda}).$$

The magnetic flux $\Phi(r)$ inside the circle with a radius r is

$$\begin{aligned}\Phi(r) &= \int_0^r 2\pi r' dr' B(r') = \frac{\Phi_0}{\lambda^2}\int_0^r r' dr' K_0(\frac{r'}{\lambda}) \\ &= \Phi_0 \int_0^{r/\lambda} x dx K_0(x) = \Phi_0[1 - \frac{r}{\lambda}K_1(\frac{r}{\lambda})]\end{aligned}.$$

When $r \to \infty$, $\Phi(r) = \Phi_0$.

In Fig, for simplicity we assume that

$$\frac{B(r)}{\frac{\Phi_0}{2\pi\lambda^2}} = K_0(\frac{r}{\lambda}) \qquad \text{for } r > \xi$$

and

$$\frac{B(r)}{\frac{\Phi_0}{2\pi\lambda^2}} = K_0(\frac{\xi}{\lambda}) = \text{const} \qquad \text{for } r < \xi.$$

The current density is calculated as

$$\boldsymbol{J} = \frac{c}{4\pi}\nabla \times \boldsymbol{B} = -\boldsymbol{e}_\varphi \frac{c}{4\pi}\frac{d}{dr}B(r).$$

The vortex line energy is given by

$$\varepsilon_1 = \left(\frac{\Phi_0}{4\pi\lambda}\right)^2 \frac{\xi}{\lambda} K_0(\frac{\lambda}{\xi}) K_1(\frac{\lambda}{\xi})$$

The interaction between vortex lines is

$$U_{12} = \frac{\Phi_0}{4\pi} B_1(\boldsymbol{r}_2) = \frac{{\Phi_0}^2}{8\pi^2\lambda^2} K_0(\frac{r_{12}}{\lambda}),$$

where $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$

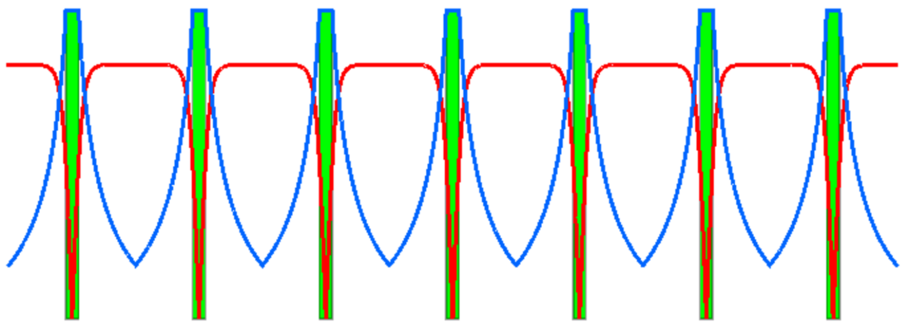
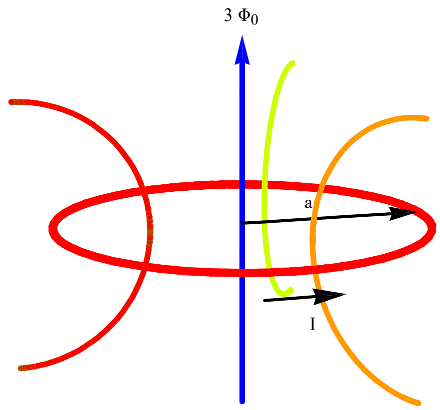


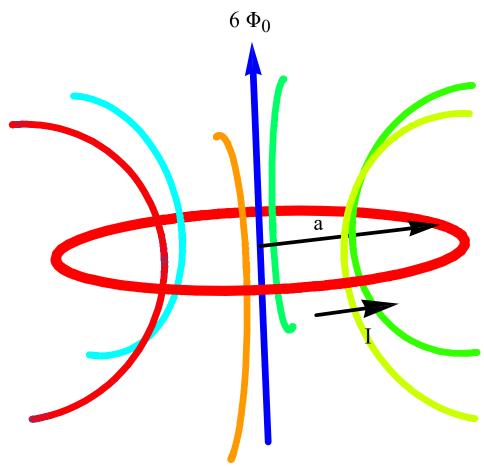
Fig. One-dimesional Abrikosov-type structure of vortices. The red line shows the distribution of the normalized order parameter and the blue line shows the distribution of the magnetic induction (the internal magnetic field). The center of each vortex is denoted by the green lines.

43. Experimental verification of magnetic flux quantization (1961)

Two experimental measurements were published in 1961, by Deaver and Fairbank, and by Doll and Näbauer. In both these experiments the flux trapped by a superconducting hollow cylinder was found to be quantized, and the accuracy of the two experiments was comparable. Doll and Näbauer found that the measured flux quantum differed from London's work by as much as 60%, but Buyers and Yang were at Stanford, and the Stanford group found that they agreed with the theoretically expected value of $hc/2e$ to within 20 %. (D.J. Thouless, "Topological quantum number in norelativistic physics, " World Scientific, 1998).



3 Φ0
a
I



6 Φ0
a
I

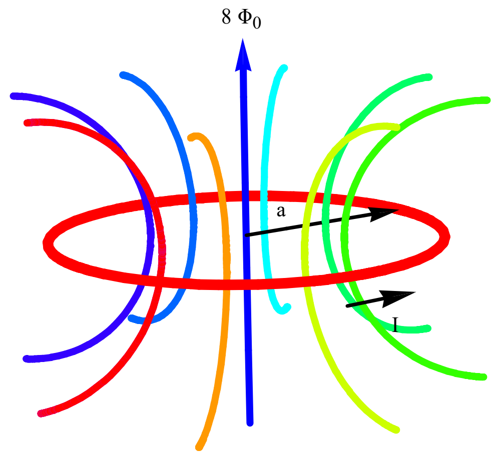


Fig. The magnetic flux lines (fluxoid) threading in the superconducting ring. Magnetic flux is quantized. n = 1, 2, 3,..... Note that n = 3, 6, and 8 in these figures. The direction of the supercurrent is denoted in the figure.

The magnetic field B can be measured inside the superconducting ring (with a radius r)

$$BA = n\Phi_0$$

or

$$B = \frac{n\Phi_0}{\pi r^2}$$

where $\Phi_0 =$ 2.06783366752 x 10^{-7} G.cm^2. Suppose that $r = 10^{-4}$ cm. Then we get

$B = 6.582\, n$ [Gauss] (n = 1, 2, 3,...).

which can be easily measured in the laboratories.

44. Cooper pair (1956)

44.1 Discussion by de Gennes

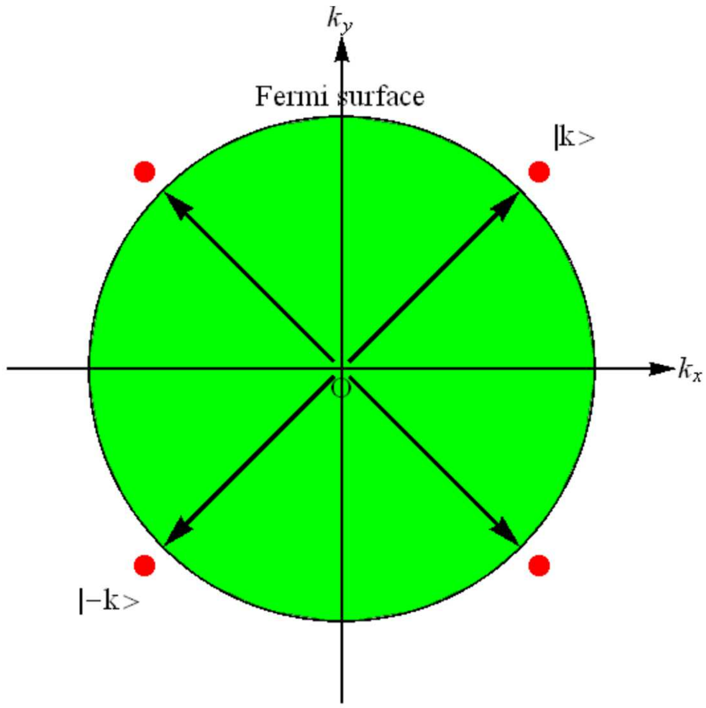
Even a weak attraction can bind pairs of electrons into a bound state. The Fermi sea of electrons is unstable against the formation of at least one bound pair,

(i) regardless of how weak the interaction is,

(ii) so long as it is attractive.

((**Assumption**))

The extra electrons are added to a Fermi sea at $T = 0$ K, with the stipulation that the extra electrons interact with each other but not with those in the sea except via the Pauli exclusion principle.



We expect the lowest energy state to have zero total momentum. Two electrons have equal and opposite momentum. The orbital wave function

$$\psi_0(r_1, r_2) = \sum_k g_k \exp(i\boldsymbol{k}\cdot\boldsymbol{r}_1)\exp(-i\boldsymbol{k}\cdot\boldsymbol{r}_2),$$

which needs to satisfy the condition of anti-symmetry of the total wavefunction with respect to the exchange of the two electrons.

$$\exp(i\boldsymbol{k}\cdot\boldsymbol{r}_1)\exp(-i\boldsymbol{k}\cdot\boldsymbol{r}_2) = \cos[\boldsymbol{k}\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)] + i\sin[\boldsymbol{k}\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)].$$

First term (even function): anti-symmetric single spin function

$$\chi(0,0) = \frac{1}{\sqrt{2}}[|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2 \cdot$$

Second term (odd function): symmetric triplet spin function

$$\chi(1,1) = |+\rangle_1 |+\rangle_2$$
$$\chi(1,0) = \frac{1}{\sqrt{2}}[|+\rangle_1 |-\rangle_2 + |-\rangle_1 |+\rangle_2$$
$$\chi(1,-1) = |-\rangle_1 |-\rangle_2$$

Note that

$$D_{1/2} \times D_{1/2} = D_1 + D_0. \qquad \text{(angular momentum addition rule)}$$

Anticipating an attractive interaction, we expect the singlet coupling to have lower energy, because the $\cos[\boldsymbol{k}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)]$ gives a larger probability amplitude of the electrons to be near each other. Thus, we consider a two electron singlet wavefunction,

$$\psi_0(\boldsymbol{r}_1,\boldsymbol{r}_2) = \sum_k g_k \cos[\boldsymbol{k}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)]\chi(0,0)$$
.

Here we follow the method which is used by de Gennes. He uses

$$\psi_0(\boldsymbol{r}_1,\boldsymbol{r}_2) = \sum_k g_{\boldsymbol{k}} \exp[i\boldsymbol{k}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)]$$
,

where $g_{\mathbf{k}}$ is the probability amplitude for finding one electron with ***k*** and the other electron in the state -***k***. Note that

$$g_{\mathbf{k}} = 0 \qquad \text{for } k<k_{\mathrm{F}}, \qquad \text{(Pauli principal)}$$

since the states with $k<k_F$ are already occupied.

Schrödinger equation:

$$-\frac{\hbar^2}{2m}(\nabla_1{}^2 + \nabla_2{}^2)\psi_0(\boldsymbol{r}_1,\boldsymbol{r}_2) + V(\boldsymbol{r}_1,\boldsymbol{r}_2)\psi_0(\boldsymbol{r}_1,\boldsymbol{r}_2) = (2\varepsilon_F - \Delta)\psi_0(\boldsymbol{r}_1,\boldsymbol{r}_2),$$

with

$$\varepsilon_F = \frac{\hbar^2}{2m}k_F{}^2.$$

Since

$$-\frac{\hbar^2}{2m}(\nabla_1{}^2+\nabla_2{}^2)\psi_0(\boldsymbol{r}_1,\boldsymbol{r}_2)=\sum_k g_{\boldsymbol{k}}(-\frac{\hbar^2}{2m})(\nabla_1{}^2+\nabla_2{}^2)\exp[i\boldsymbol{k}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)]$$
$$=\frac{\hbar^2}{m}\sum_k \boldsymbol{k}^2 g_{\boldsymbol{k}}\exp[i\boldsymbol{k}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)]$$

we have

$$\frac{\hbar^2}{m}\boldsymbol{k}^2 g_{\boldsymbol{k}}+\sum_{\boldsymbol{k}'} g_{\boldsymbol{k}'}V_{\boldsymbol{k},\boldsymbol{k}'}=(-\Delta+2\varepsilon_F)g_{\boldsymbol{k}}, \qquad \text{(Bethe-Goldstone equation)}$$

where

$$V_{\boldsymbol{k},\boldsymbol{k}'}=\frac{1}{\Omega}\int V(\boldsymbol{r})e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}},$$

where Ω is the volume of the system.

$$\sum_{\boldsymbol{k}'} g_{\boldsymbol{k}'}V_{\boldsymbol{k},\boldsymbol{k}'}=(-\Delta+2\varepsilon_F-\frac{\hbar^2}{m}\boldsymbol{k}^2)g_{\boldsymbol{k}}=(-\Delta-2\xi_{\boldsymbol{k}})g_{\boldsymbol{k}},$$

with

$$\xi_{\boldsymbol{k}}=\frac{\hbar^2}{2m}\boldsymbol{k}^2-\varepsilon_F.$$

((**Assumption**))

For simplicity, for

$$\varepsilon_F<\frac{\hbar^2}{2m}\boldsymbol{k}^2<\varepsilon_F+\hbar\omega_D. \qquad \varepsilon_F<\frac{\hbar^2}{2m}\boldsymbol{k}'^2<\varepsilon_F+\hbar\omega_D,$$

we assume that

$$V_{\boldsymbol{k},\boldsymbol{k}'}=-V,$$

otherwise, $V_{\boldsymbol{k},\boldsymbol{k}'} = 0$.

This interaction is attractive and constant in an energy band $\hbar\omega_D$ above the Fermi level. Then we get

$$(-\Delta - 2\xi_{\boldsymbol{k}})g_{\boldsymbol{k}} = \sum_{\boldsymbol{k}'} g_{\boldsymbol{k}'} V_{\boldsymbol{k},\boldsymbol{k}'} = -V\sum_{\boldsymbol{k}'} g_{\boldsymbol{k}'} .$$

From this we obtain

$$g_{\boldsymbol{k}} = \frac{V}{\Delta + 2\xi_{\boldsymbol{k}}} \sum_{\boldsymbol{k}'} g_{\boldsymbol{k}'} ,$$

or

$$\sum_{\boldsymbol{k}} g_{\boldsymbol{k}} = V \sum_{\boldsymbol{k}} \frac{1}{(\Delta + 2\xi_{\boldsymbol{k}})} \sum_{\boldsymbol{k}'} g_{\boldsymbol{k}'} ,$$

or

$$\frac{1}{V} = \sum_{\substack{\boldsymbol{k} \\ k > k_F}} \frac{1}{(\Delta + 2\xi_{\boldsymbol{k}})} .$$

((**Density of states**)) $N(\xi)$; the density of states per spin

$$N(\xi)d\xi = \frac{1}{(2\pi)^3} 4\pi k^2 dk .$$

Noting that

$$k = \sqrt{\frac{2m}{\hbar^2}} \sqrt{\xi + \varepsilon_F} ,$$

$$dk = \sqrt{\frac{2m}{\hbar^2}} \frac{d\xi}{2\sqrt{\xi + \varepsilon_F}} ,$$

we get

$$N(\xi)d\xi = \frac{1}{(2\pi)^3} 4\pi \left(\frac{2m}{\hbar^2}\right)^{3/2} (\xi + \varepsilon_F) \frac{d\xi}{2\sqrt{\xi + \varepsilon_F}}$$

$$= \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\xi + \varepsilon_F}\, d\xi$$

or

$$N(\xi) = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\xi + \varepsilon_F} .$$

$$\frac{1}{V} = \int_0^{\hbar\omega_D} 2N(\xi)d\xi\left(\frac{1}{2\xi + \Delta}\right).$$

If we assume $\hbar\omega_D << \varepsilon_F$, $N(\xi)$ can be considered as constant and replaced by its value $N(0)$ at the Fermi level.

$$1 = 2VN(0)\int_0^{\hbar\omega_D} d\xi\left(\frac{1}{2\xi + \Delta}\right) = VN(0)\ln\left(\frac{\Delta + 2\hbar\omega_D}{\Delta}\right).$$

In the limit of weak interaction ($N(0)V<<1$),

$$\exp[\frac{1}{VN(0)}] = 1 + \frac{2\hbar\omega_D}{\Delta} >> 1.$$

Then we get

$$\Delta = 2\hbar\omega_D \exp[-\frac{1}{VN(0)}].$$

The energy of the Cooper pair relative to the state where the two electrons are at the Fermi level.

44. Cooper pair (1956)

44.1 Discussion by de Gennes

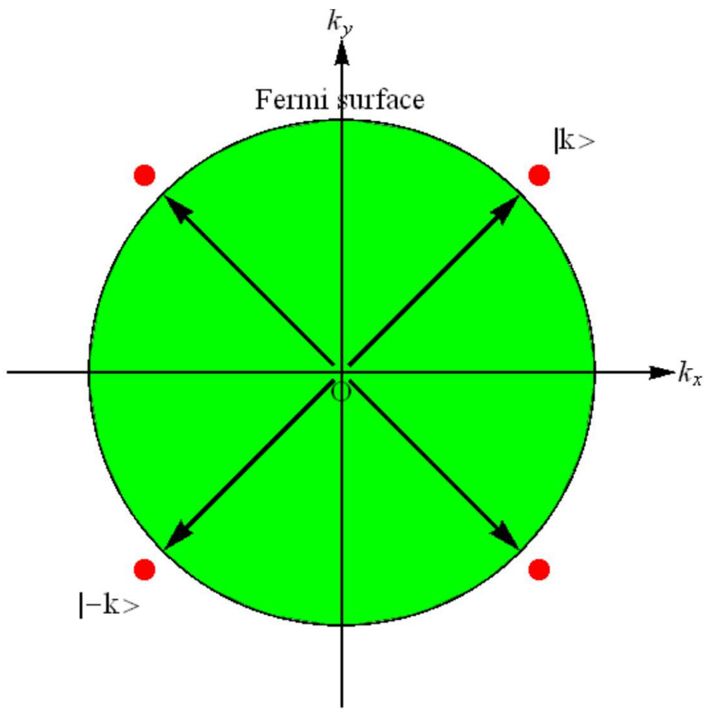
Even a weak attraction can bind pairs of electrons into a bound state. The Fermi sea of electrons is unstable against the formation of at least one bound pair,

(i) regardless of how weak the interaction is,

(ii) so long as it is attractive.

((**Assumption**))

The extra electrons are added to a Fermi sea at T = 0 K, with the stipulation that the extra electrons interact with each other but not with those in the sea except via the Pauli exclusion principle.



We expect the lowest energy state to have zero total momentum. Two electrons have equal and opposite momentum. The orbital wave function

$$\psi_0(r_1, r_2) = \sum_k g_k \exp(i\mathbf{k}\cdot\mathbf{r}_1)\exp(-i\mathbf{k}\cdot\mathbf{r}_2),$$

which needs to satisfy the condition of anti-symmetry of the total wavefunction with respect to the exchange of the two electrons.

$$\exp(i\mathbf{k}\cdot\mathbf{r}_1)\exp(-i\mathbf{k}\cdot\mathbf{r}_2) = \cos[\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)] + i\sin[\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)].$$

First term (even function): anti-symmetric single spin function

$$\chi(0,0) = \frac{1}{\sqrt{2}}[|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2.$$

Second term (odd function): symmetric triplet spin function

$$\chi(1,1) = |+\rangle_1 |+\rangle_2$$
$$\chi(1,0) = \frac{1}{\sqrt{2}}[|+\rangle_1 |-\rangle_2 + |-\rangle_1 |+\rangle_2$$
$$\chi(1,-1) = |-\rangle_1 |-\rangle_2$$

Note that

$$D_{1/2} \times D_{1/2} = D_1 + D_0 .$$ (angular momentum addition .s rule)

Anticipating an attractive interaction, we expect the singlet coupling to have lower energy, because the $\cos[\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)]$ gives a larger probability amplitude of the electrons to be near each other. Thus we consider a two electron singlet wavefunction,

$$\psi_0(\mathbf{r}_1, \mathbf{r}_2) = \sum_k g_k \cos[\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)]\chi(0,0) .$$

Here we follow the method which is used by de Gennes. He uses

$$\psi_0(\mathbf{r}_1, \mathbf{r}_2) = \sum_k g_{\mathbf{k}} \exp[i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)] ,$$

where $g_{\mathbf{k}}$ is the probability amplitude for finding one electron with ***k*** and the other electron in the state **-*k***. Note that

$g_{\mathbf{k}} = 0$ for $k<k_F$, (Pauli principal)

since the states with $k<k_F$ are already occupied.

Schrödinger equation:

$$-\frac{\hbar^2}{2m}(\nabla_1{}^2 + \nabla_2{}^2)\psi_0(\boldsymbol{r}_1, \boldsymbol{r}_2) + V(\boldsymbol{r}_1, \boldsymbol{r}_2)\psi_0(\boldsymbol{r}_1, \boldsymbol{r}_2) = E\psi_0(\boldsymbol{r}_1, \boldsymbol{r}_2) ,$$

with

$E = 2\varepsilon_F - \Delta$ (the energy eigenvalue of two systems)

$$\varepsilon_F = \frac{\hbar^2}{2m} k_F{}^2 .$$

Since

$$-\frac{\hbar^2}{2m}(\nabla_1{}^2 + \nabla_2{}^2)\psi_0(\mathbf{r}_1, \mathbf{r}_2) = \sum_k g_{\mathbf{k}} (-\frac{\hbar^2}{2m})(\nabla_1{}^2 + \nabla_2{}^2)\exp[i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)]$$

$$= \frac{\hbar^2}{m}\sum_k \mathbf{k}^2 g_{\mathbf{k}} \exp[i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)]$$

$$V(\boldsymbol{r}_1, \boldsymbol{r}_2)\psi_0(\boldsymbol{r}_1, \boldsymbol{r}_2) = \sum_{\boldsymbol{k}'} g_{\boldsymbol{k}'} V(\boldsymbol{r}_1, \boldsymbol{r}_2)\exp[i\boldsymbol{k}'\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)]$$

$$= \sum_{\boldsymbol{k}'} g_{\boldsymbol{k}'} \exp[i\boldsymbol{k}'\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)] \sum_{\boldsymbol{k}-\boldsymbol{k}'} V_{\boldsymbol{k}-\boldsymbol{k}'} \exp[i(\boldsymbol{k} - \boldsymbol{k}')\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)]$$

$$= \sum_{\boldsymbol{k},\boldsymbol{k}'} V_{\boldsymbol{k},\boldsymbol{k}'} g_{\boldsymbol{k}'} \exp[i\boldsymbol{k}\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)]$$

we have

$$\frac{\hbar^2}{m}\boldsymbol{k}^2 g_{\boldsymbol{k}} + \sum_{\boldsymbol{k}'} V_{\boldsymbol{k},\boldsymbol{k}'} g_{\boldsymbol{k}'} = (-\Delta + 2\varepsilon_F) g_{\boldsymbol{k}} ,$$ (Bethe-Goldstone equation)

where we assume that the potential energy can be described by the Fourier series

$$V(\boldsymbol{r}) = \sum_{\boldsymbol{k}-\boldsymbol{k}'} V_{\boldsymbol{k},\boldsymbol{k}'} e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}} = \sum_{\boldsymbol{k}-\boldsymbol{k}'} V_{\boldsymbol{k}-\boldsymbol{k}'} e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}} = \sum_{\boldsymbol{q}} V_{\boldsymbol{q}} e^{i\boldsymbol{q}\cdot\boldsymbol{r}}$$

Then we have

$$\sum_{\boldsymbol{k}'} V_{\boldsymbol{k},\boldsymbol{k}'} g_{\boldsymbol{k}'} = (-\Delta + 2\varepsilon_F - \frac{\hbar^2}{m}\boldsymbol{k}^2) g_{\boldsymbol{k}} = (-\Delta - 2\xi_{\boldsymbol{k}}) g_{\boldsymbol{k}} ,$$

with

$$\xi_{\mathbf{k}} = \frac{\hbar^2}{2m}\mathbf{k}^2 - \varepsilon_F .$$

((**Assumption**))
For simplicity, for

$$\varepsilon_F < \frac{\hbar^2}{2m}\mathbf{k}^2 < \varepsilon_F + \hbar\omega_D . \qquad \varepsilon_F < \frac{\hbar^2}{2m}\mathbf{k}'^2 < \varepsilon_F + \hbar\omega_D ,$$

we assume that

$$V_{\mathbf{k},\mathbf{k}'} = -V , \qquad (V>0 \text{ for attractive and } V<0 \text{ for repulsive interactions})$$

otherwise, $V_{\mathbf{k},\mathbf{k}'} = 0$.

This interaction is attractive and constant in an energy band $\hbar\omega_D$ above the Fermi level. Then we get

$$(-\Delta - 2\xi_k)g_k = \sum_{k'} V_{k,k'} g_{k'} = -V \sum_{k'} g_{k'} .$$

From this we obtain

$$g_{\mathbf{k}} = \frac{V}{\Delta + 2\xi_{\mathbf{k}}} \sum_{\mathbf{k}'} g_{\mathbf{k}'} ,$$

or

$$\sum_{\mathbf{k}} g_{\mathbf{k}} = V \sum_{\mathbf{k}} \frac{1}{(\Delta + 2\xi_{\mathbf{k}})} \sum_{\mathbf{k}'} g_{\mathbf{k}'} ,$$

or

$$\frac{1}{V} = \sum_{\substack{\mathbf{k} \\ k>k_F}} \frac{1}{(\Delta + 2\xi_{\mathbf{k}})} .$$

((**Density of states**)) $N(\xi)$; the density of states per spin

$$N(\xi)d\xi = \frac{1}{(2\pi)^3} 4\pi k^2 dk .$$

Noting that

$$k = \sqrt{\frac{2m}{\hbar^2}}\sqrt{\xi + \varepsilon_F} \,,$$

$$dk = \sqrt{\frac{2m}{\hbar^2}}\frac{d\xi}{2\sqrt{\xi + \varepsilon_F}} \,,$$

we get

$$\begin{aligned} N(\xi)d\xi &= \frac{1}{(2\pi)^3}4\pi\left(\frac{2m}{\hbar^2}\right)^{3/2}(\xi + \varepsilon_F)\frac{d\xi}{2\sqrt{\xi + \varepsilon_F}} \\ &= \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\xi + \varepsilon_F}\, d\xi \end{aligned}$$

or

$$N(\xi) = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\xi + \varepsilon_F} \,.$$

$$\frac{1}{V} = \int_0^{\hbar\omega_D} 2N(\xi)d\xi\left(\frac{1}{2\xi + \Delta}\right).$$

If we assume $\hbar\omega_D << \varepsilon_F$, $N(\xi)$ can be considered as constant and replaced by its value $N(0)$ at the Fermi level.

$$1 = 2VN(0)\int_0^{\hbar\omega_D} d\xi\left(\frac{1}{2\xi + \Delta}\right) = VN(0)\ln\left(\frac{\Delta + 2\hbar\omega_D}{\Delta}\right).$$

In the limit of weak interaction ($N(0)V<<1$),

$$\exp\left[\frac{1}{VN(0)}\right] = 1 + \frac{2\hbar\omega_D}{\Delta} >> 1.$$

Then we get

$$\Delta = 2\hbar\omega_D \exp\left[-\frac{1}{VN(0)}\right].$$

The energy of the Cooper pair relative to the state where the two electrons are at the Fermi level.

There exists a two-electron bound state of energy Δ>0. If we start from a free electron gas, and turn on the interaction V, we predict that electrons will group themselves in pairs giving up energy to the external world. The normal state is thus unstable. This means the existence of a bound state for the two added electrons.

Cooper showed that, where the interaction is attractive, the system energy is reduced by pairing. Therefore, the Fermi sea of single electrons is unstable, since any perturbation that moves two electrons above ε_F will lower the system energy. Note that the Cooper model is not a model for the superconducting ground state, and that

$$\Delta = 2\hbar\omega_D \exp[-\frac{1}{VN(0)}] = 2k_B\Theta_D \exp[-\frac{1}{VN(0)}].$$

44.2 Original discussion by Cooper

L.N. Cooper, Phys. Rev. 104, 1189 (1956).
J.R. Schrieffer, Theory of Superconductivity (W.A. Menjamin, New York, 1964).

Leon N. Cooper: Superconductivity and Beyond (25 th Army Science Conference, Orlando, Florida, November 28, 2006).

The method is more direct compared to the above theory. We start with

$$(-\Delta - 2\xi_{\boldsymbol{k}})g_{\boldsymbol{k}} = \sum_{\boldsymbol{k}'} V_{\boldsymbol{k},\boldsymbol{k}'} g_{\boldsymbol{k}'} = -V\sum_{\boldsymbol{k}'} g_{\boldsymbol{k}'},$$

where -V<0 for the attractive potential. Here we introduce E (the energy eigenvalue),

$$E = 2\varepsilon_F - \Delta, \qquad \xi_{\mathbf{k}} = \varepsilon_k - \varepsilon_F$$

Then we have

$$(E - 2\varepsilon_k)g_{\mathbf{k}} = \sum_{\mathbf{k}'} g_{\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} = -V\sum_{\mathbf{k}'} g_{\mathbf{k}'}$$

From this we get

$$g_{\mathbf{k}} = -\frac{V}{E - 2\varepsilon_{\mathbf{k}}}\sum_{\mathbf{k}'} g_{\mathbf{k}'},$$

or

$$\sum_{\mathbf{k}} g_{\mathbf{k}} = -V \sum_{\mathbf{k}} \frac{1}{E - 2\varepsilon_{\mathbf{k}}} \sum_{\mathbf{k'}} g_{\mathbf{k'}} ,$$

or

$$-\frac{1}{V} = \sum_{\substack{\mathbf{k} \\ k>k_F}} \frac{1}{(E - 2\varepsilon_{\mathbf{k}})} .$$

where E is the energy eigenvalue of the system

44.3 Example

First we consider the simple example. Suppose that

$N_1 = 10$, $\qquad 2\varepsilon_k = (2 + \frac{k}{N_1})$

where $k = 1, 2, ..., 10$. In other words,

$$2\varepsilon_k = 2, 2.1, 2.2, 2.3, \ldots, 2.8, 2.9, 3.$$

Then we get

$$-\frac{1}{V} = f(E) = \sum_{k=1}^{N_1} \frac{1}{E - 2\varepsilon_k} = \sum_{k=1}^{N_1} \frac{1}{E - (2 + \frac{k}{N_1})} = \frac{1}{E - 2.1} + \frac{1}{E - 2.2} + \ldots + \frac{1}{E - 2.9} + \frac{1}{E - 3}$$

We make a plot of $f(E)$ as a function of E.

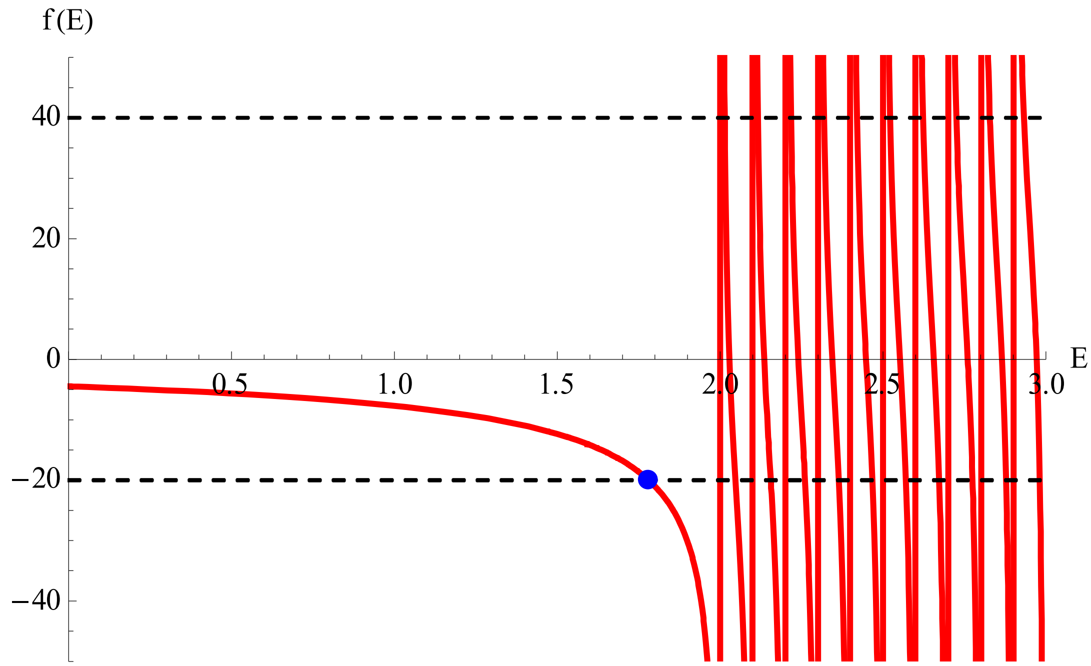


Fig. Plot of $f(E)$ as a function of E. The dashed line ($-1/V = 40$) for the repulsive interaction. The dashed line ($-1/V = -20$) for the attractive interaction. The last crossing on the left is the coherent state, split from the continuum by an energy gap, displaying the essential singularity in the coupling constant.

What is the solution of $-\dfrac{1}{V} = f(E)$?

(i) When $V<0$ (repulsive) the solution of E is always larger than 2.

(ii) When $V>0$ (attractive), we have a solution of E, which is lower than 2. This means that the two electrons forms a Cooper pair (bound state).

((**Note**))
Leon N. Cooper: Superconductivity and Beyond (25 th Army Science Conference, Orlando, Florida, November 28, 2006).

"Allow me (Cooper himself) to show you a page from my notes of that period with the pair solutions as they first appeared to me."

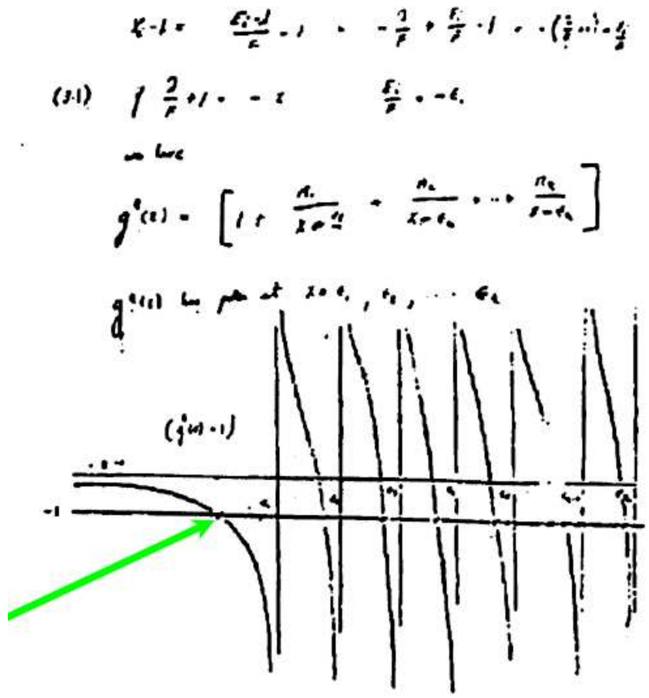


Fig. From the note of Cooper around 1956.

44.4 The derivation of energy gap

$$\frac{1}{V} = \sum_{k(>kF)} \frac{1}{2\varepsilon_k - E}$$
$$= 2\int_{\varepsilon_F}^{\varepsilon_F + \hbar\omega_D} \frac{1}{2\varepsilon - E} N(\varepsilon) d\varepsilon ,$$
$$\approx 2N(0)\int_{\varepsilon_F}^{\varepsilon_F + \hbar\omega_D} \frac{1}{2\varepsilon - E} d\varepsilon$$

or

$$\frac{1}{V} = N(0)\ln(2\varepsilon - E)\,|_{\varepsilon_F}^{\varepsilon_F + \hbar\omega_D} = N(0)\ln\left(\frac{2\varepsilon_F + 2\hbar\omega_D - E}{2\varepsilon_F - E}\right),$$

or

$$\exp[\frac{1}{N(0)V}] = \frac{2\varepsilon_F + 2\hbar\omega_D - E}{2\varepsilon_F - E} = 1 + \frac{2\hbar\omega_D}{2\varepsilon_F - E}$$

or

$$\exp[\frac{1}{N(0)V}]-1=\frac{2\hbar\omega_D}{2\varepsilon_F-E},$$

where $N(\varepsilon_F)$ is the density of states per spin (per unit volume). Then we have

$$2\varepsilon_F-E=\frac{2\hbar\omega_D}{\exp[\frac{1}{N(0)V}]-1}.$$

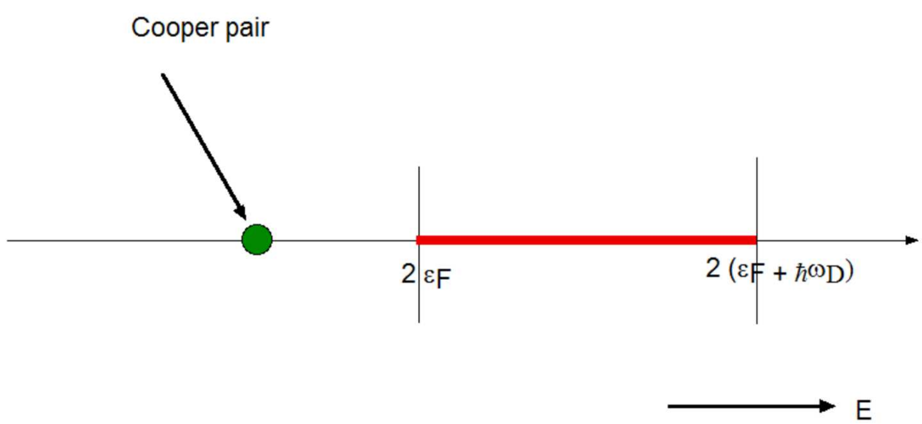
In the weak coupling ($N(0)V$ <<1), we have

$$E=2\varepsilon_F-2\hbar\omega_D\exp[-\frac{1}{N(0)V}].$$

Note that

$$\Delta_0=E-2\varepsilon_F=2\hbar\omega_D\exp[-\frac{1}{N(0)V}],$$

is the bound state energy for the Cooper pair (V>0; attractive interaction).



44.5 Eigenvalue problem

Degenerate matrices have some very surprising properties. We return to the original eigenvalue problem

$$(E-2\varepsilon_k)g_{\mathbf{k}}=\sum_{\mathbf{k}'}g_{\mathbf{k}'}V_{\mathbf{k},\mathbf{k}'}=-V\sum_{\mathbf{k}'}g_{\mathbf{k}'}$$

Here we assume that

$$E-2\varepsilon_{\boldsymbol{k}}+V=-\Delta+V\approx E_1$$

is independent of ***k*** since ε_k is nearly equal to ε_F (degenerate states).

For simplicity we consider the matrix (N x N) for the example. The eigenvalue problem is given by

$$\begin{pmatrix} 0 & -V & -V & . & . & . & . & . & -V & -V \\ -V & 0 & -V & . & . & . & . & . & . & -V \\ -V & -V & 0 & . & . & . & . & . & . & -V \\ . & . & . & 0 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & 0 & -V & . \\ -V & . & . & . & . & . & . & -V & 0 & -V \\ -V & -V & -V & . & . & . & . & . & -V & 0 \end{pmatrix} . \begin{pmatrix} g_1 \\ g_2 \\ . \\ . \\ . \\ . \\ . \\ . \\ g_{n-1} \\ g_n \end{pmatrix} = E_1 \begin{pmatrix} g_1 \\ g_2 \\ . \\ . \\ . \\ . \\ . \\ . \\ g_{n-1} \\ g_n \end{pmatrix}$$

where $-V<0$ (attractive interaction). When diagonalized, what happens to the energy eigenvalue? The schematic diagram is shown below.

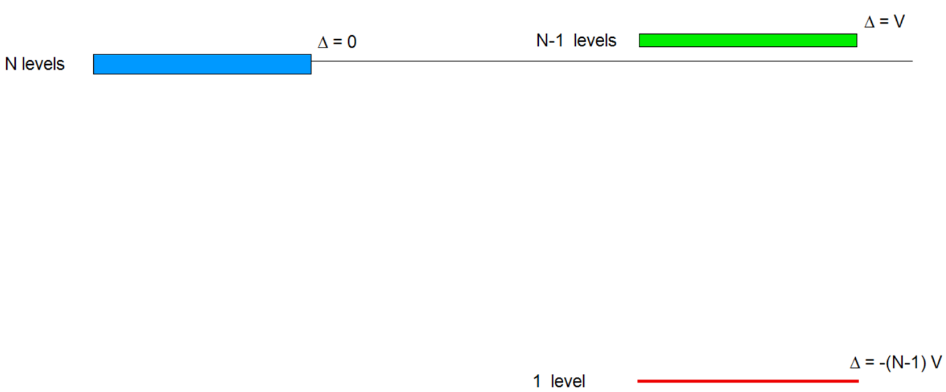


Fig. Schematic diagram of the energy levels.

The lower level is a coherent superposition of all the original states and is separated from them by an energy gap.

44.6 Numerical calculation using Mathematica

We solve the eigenvalue problems for the (24x24) matrix with $V = 0.1$ (-V<0 means attractive interaction). The results are as follows.

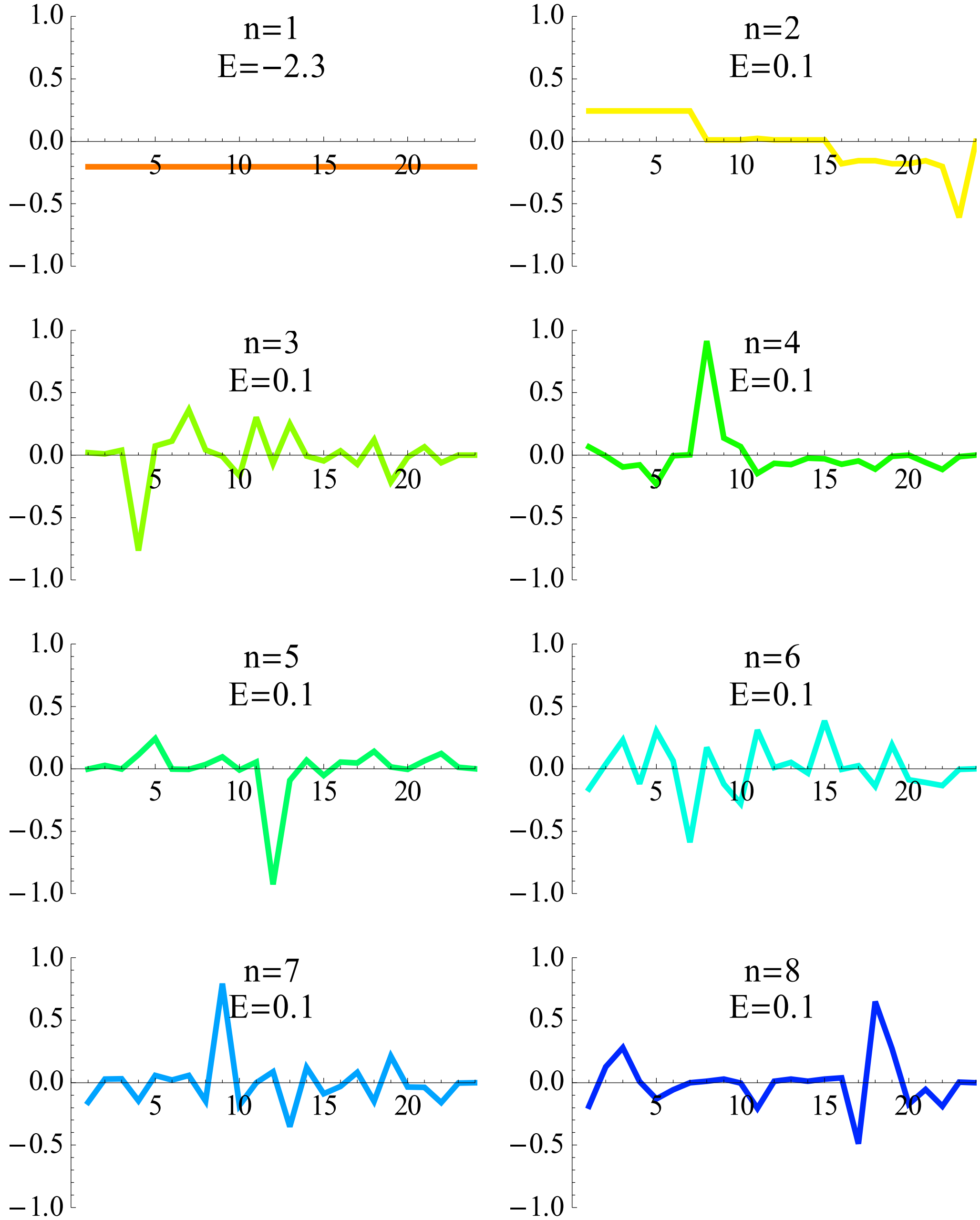
(i) The ground state (1 state) with the lowest energy; -(24-1) x 0.1 = -2.3

(ii) The degenerate states (23 states) with the energy 0.1.

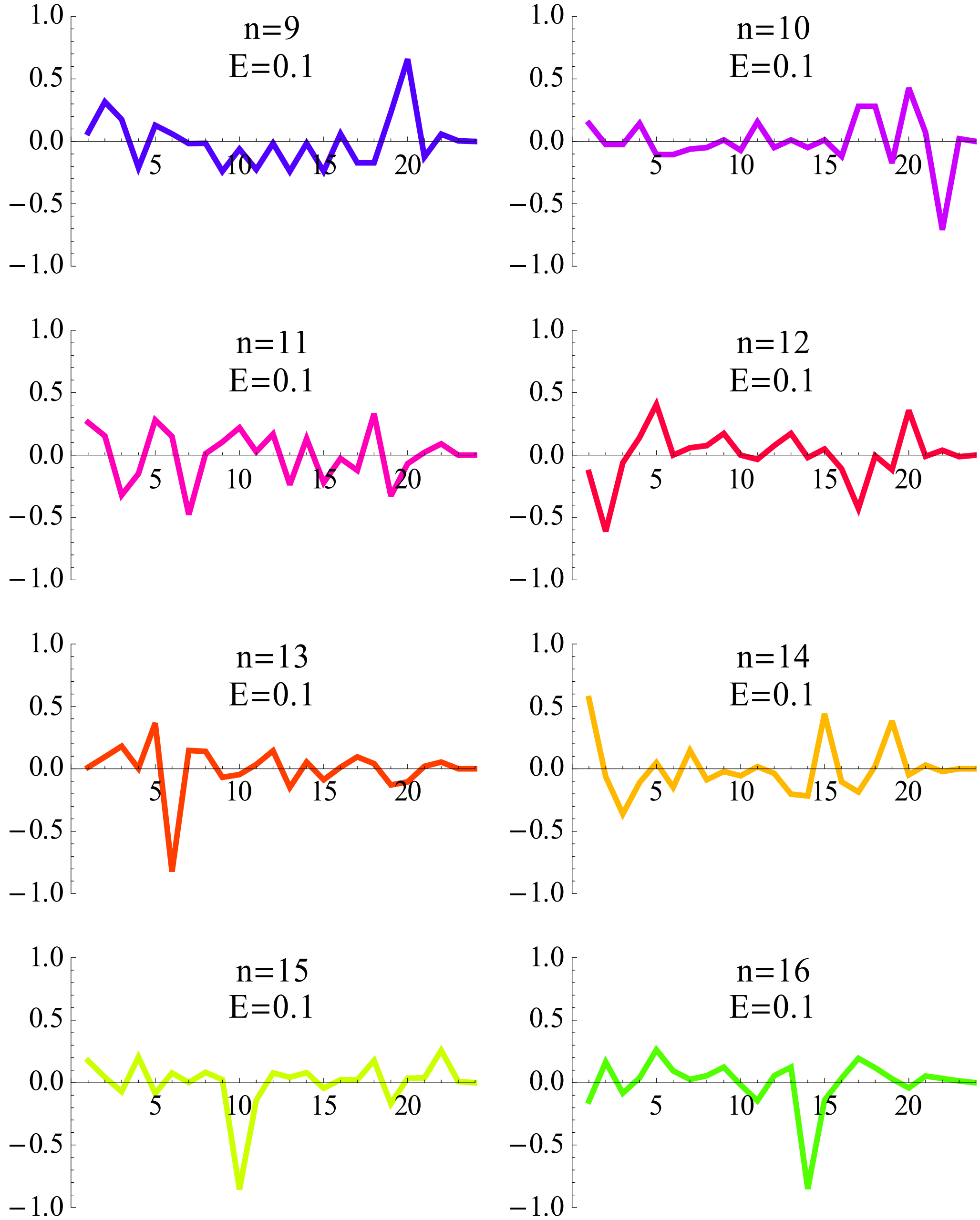
(iii) The ground state is a coherent states; linear combination of the state with the same amplitude.

((**Mathematica**))

```
Clear["Global`*"]; N1 = 24; a = 0.1; A[i_, j_] := -a;
B[i_, j_] = a KroneckerDelta[i, j]; M1 = Table[A[i, j], {i, 1, N1}, {j, 1, N1}];
M2 = Table[B[i, j], {i, 1, N1}, {j, 1, N1}]; M = M1 + M2; eq1 = Eigensystem[M];
eq1[[1]]; f[n_] := Table[{k, Normalize[eq1[[2, n]]][[k]]}, {k, 1, N1}];
g[n_] := ListPlot[f[n] , PlotStyle → {Thick, Hue[0.08 n]}, Joined → True,
  PlotRange → {{0, N1}, {-1, 1}},
  Epilog → {Text[Style["E=" <> ToString[ eq1[[1, n]]], Black, 12], {12, 0.6}],
    Text[Style["n=" <> ToString[ n], Black, 12], {12, 0.9}]}];
K1 = Table[g[m], {m, 1, 8}]; K2 = Table[g[m], {m, 9, 16}];
K3 = Table[g[m], {m, 17, 24}];
```



n=1
E=−2.3
n=2
E=0.1
n=3
E=0.1
n=4
E=0.1
n=5
E=0.1
n=6
E=0.1
n=7
E=0.1
n=8
E=0.1


n=9
E=0.1
n=10
E=0.1
n=11
E=0.1
n=12
E=0.1
n=13
E=0.1
n=14
E=0.1
n=15
E=0.1
n=16
E=0.1

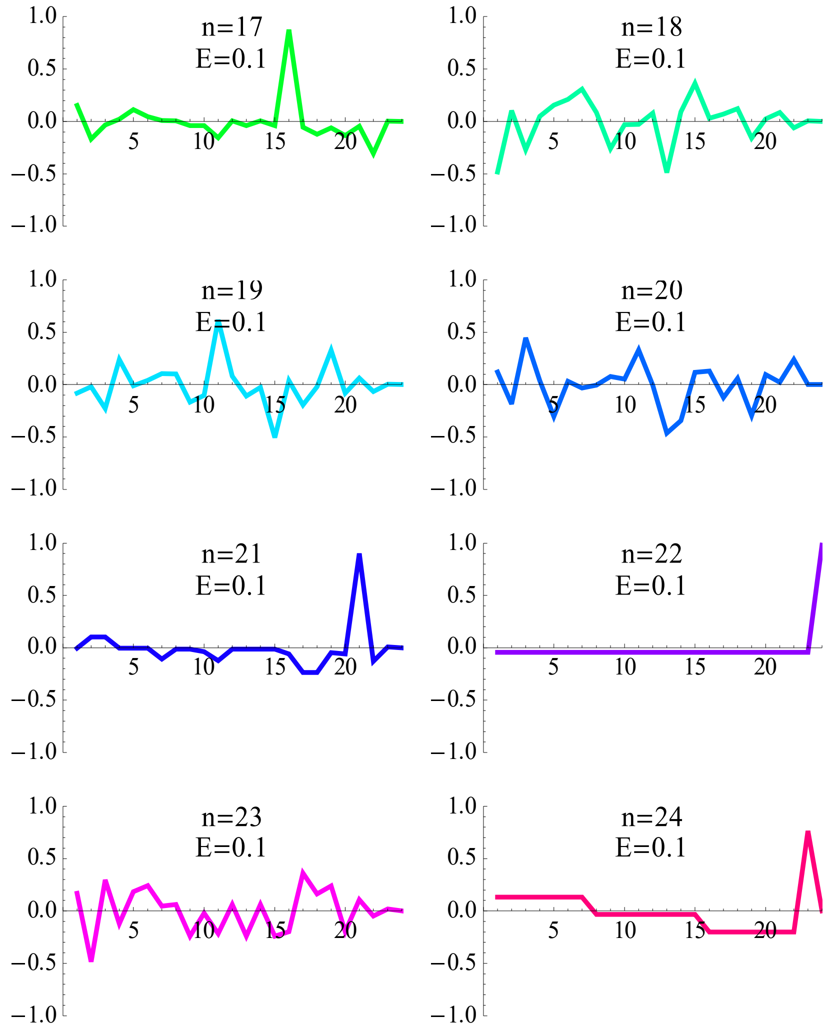


Fig. Pattern of superposition in the amplitudes of thenormalized wavefunction. E ($= E_1 = -\Delta + V$) is the eigenvalue. n is the mode number ($n = 1$ - 24). $n = 1$ corresponds to the ground state with $E = -2.3$. $n = 1$ is the coherent ground state. $-V = -0.1$ (attractive interaction). In this calculation we mean E by E_1.

44.7 Classification of the superconductivity : s-wave, p-wave, and d-wave

(a). Introduction

We consider two fermions which are subject to a central field. In this case the wave function can be described by

$$\psi(\boldsymbol{r}', \boldsymbol{r}'') = \langle \boldsymbol{r}', r'' | \psi \rangle = \psi(\boldsymbol{r} = \boldsymbol{r}' - \boldsymbol{r}'') .$$

This wave function can be decomposed into a radial part and a spherical-harmonics part, i.e.

$$\psi(\boldsymbol{r}) = \langle \boldsymbol{r} | n, l, m \rangle = R_{nl}(r) Y_l^m(\theta, \phi)$$

Note that exchanging the two particles is equivalent to inverting the vector $\boldsymbol{r}' - \boldsymbol{r}''$ (i.e. changing its sign). With such an inversion, the spherical harmonics undergo the transformation

$$Y_l^m(\theta, \phi) \to (-1)^l Y_l^m(\theta, \phi)$$

Suppose that the fermion is an electron with spin s = 1/2. In this case the total spin is S = 1 (triplet, symmetric state) and S = 0 (singlet, anti symmetric state). The wave function should be antisymmetric under the exchange of the position. This requires the conditions for l = even and S = 0, and l = odd and S = 1.

b. Parity operator and permutational operator

$$\hat{P}_{12} |\boldsymbol{r}', \boldsymbol{r}''\rangle = \hat{P}_{12} |\boldsymbol{r}'\rangle_1 |\boldsymbol{r}''\rangle_2 = |\boldsymbol{r}''\rangle_1 |\boldsymbol{r}'\rangle_2 ,$$

When

$$[\hat{P}_{12}, \hat{H}] = 0$$

we have the simultaneous eigenket $|\psi\rangle$;

$$\hat{P}_{12} |\psi\rangle = \pm |\psi\rangle , \qquad \hat{H} |\psi\rangle = E |\psi\rangle$$

Then we have

$$\langle \boldsymbol{r}', \boldsymbol{r}'' | \hat{P}_{12} | \psi\rangle = \pm \langle \boldsymbol{r}', \boldsymbol{r}'' | \psi\rangle$$

or

$$\langle \boldsymbol{r}'', \boldsymbol{r}' | \psi\rangle = \pm \langle \boldsymbol{r}', \boldsymbol{r}'' | \psi\rangle$$

We introduce the relative co-ordinate

$$\boldsymbol{r} = \boldsymbol{r}' - \boldsymbol{r}''$$

$$|\boldsymbol{r}\rangle = |\boldsymbol{r}', \boldsymbol{r}''\rangle , \quad |-\boldsymbol{r}\rangle = |\boldsymbol{r}'', \boldsymbol{r}'\rangle = \hat{\pi} |\boldsymbol{r}\rangle$$

Then we have

$$\langle -\boldsymbol{r} | \psi\rangle = \pm \langle \boldsymbol{r} | \psi\rangle$$

or

$$\langle \boldsymbol{r}|\hat{\pi}|\psi\rangle = \pm\langle \boldsymbol{r}|\psi\rangle$$

or

$$\hat{\pi}|\psi\rangle = \pm|\psi\rangle$$

This implies that

$$\hat{P}_{12} = \hat{\pi}$$

We consider the relative motion for the two electrons. The Hamiltonian for these systems, can be described by that for one electron with the reduced mass.

$$|\psi\rangle = |nlm\rangle$$

We note that

$$\hat{\pi}|l,m\rangle = (-1)^l|l,m\rangle$$

Then the orbital state with even integer of *l* has the even parity, while the orbital state with odd integer of *l* has the odd parity.

c. Classification of the symmetry for superconductivity

When we take into account of the spin states, the symmetry of the resultant eigenket should be anti-symmetric. In other words,

$S = 0$ (antisymmetric) with l = even.
$S = 1$ (symmetric) with l = odd.

S-state (BCS Cooper pair)

$l = 0$, $S = 0$ $\quad {}^1S \quad j = 1$ (3 states)

P-state (liquid ^{3}He superfluidity)

$l = 1$, $S = 1$ $\quad {}^3P \quad j = 2, 1, 0$ (9 states; 5+3+1=9)

D-state (high Tc superconductor)

$l = 2$, $S = 0$ $\quad {}^1D \quad j = 2$ (5 states)

The onset of superconductivity occurs with the condensation of electron pairs. These electron pairs, called the Cooper pair, can be in a state of either total spin $S = 0$ (spin singlet) or 1 (spin triplet). Being fermions, electrons anti-commute. Therefore the antisymmetric spin-singlet state is accompanied by a symmetric orbital wave function (even parity) and vice versa, in order to preserve the anti-symmetry of the total wave function.

45. BCS Hamiltonian

We start with a pairing Hamiltonian

$$H - \mu N = \sum_{k,\sigma} \xi_k a_{k\sigma}^{+} a_{k\sigma} + \sum_{k,q} V_{k,q} a_{k\uparrow}^{+} a_{-k\downarrow}^{+} a_{-q\downarrow} a_{q\uparrow} .$$

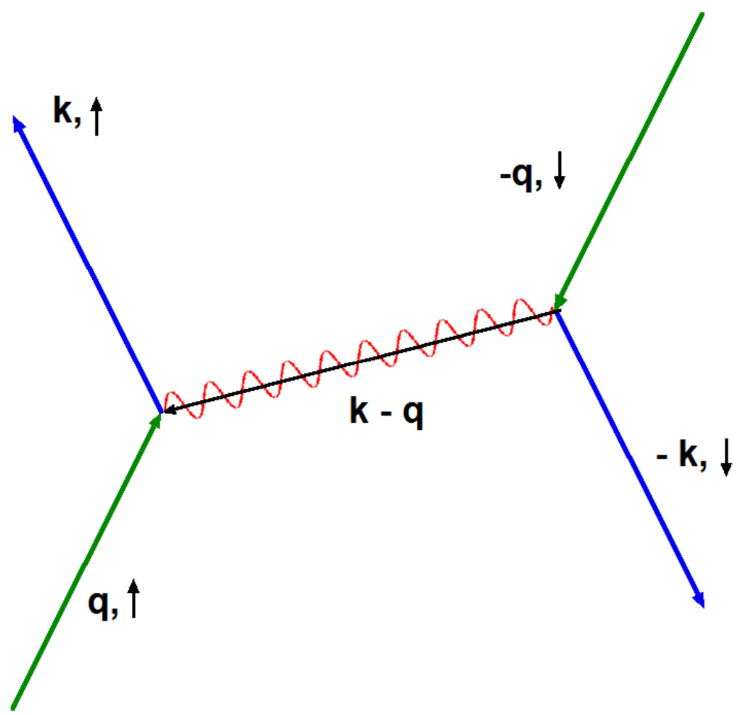


Fig. Two pairs of electrons (***q***, ↑; -***q***, ↓) and (***k***, ↑; -***k***, ↓), which are coupled with the electron phonon interaction ($V_{\boldsymbol{k},\boldsymbol{q}}$). The phonon has a momentum (***k*** - ***q***).

where μ is the chemical potential; $\mu = \varepsilon_F$ (the Fermi energy).

Here we use the mean field theory. The expectation value is all one needs to understand the nature of the ground state. This can be accomplished by setting

$$a_{k\uparrow}^{+} a_{-k\downarrow}^{+} \rightarrow (a_{k\uparrow}^{+} a_{-k\downarrow}^{+} - \langle a_{k\uparrow}^{+} a_{-k\downarrow}^{+} \rangle) + \langle a_{k\uparrow}^{+} a_{-k\downarrow}^{+} \rangle$$

$$a_{-k\downarrow} a_{k\uparrow} \rightarrow (a_{-k\downarrow} a_{k\uparrow} - \langle a_{-k\downarrow} a_{k\uparrow} \rangle) + \langle a_{-k\downarrow} a_{k\uparrow} \rangle$$

by treating the first term as small compared to the second term (its average), and expanding to first order in the small quantities. For simplicity we use

$$b_k = \left\langle a_{-k\downarrow} a_{k\uparrow} \right\rangle, \qquad b_k^* = \left\langle a_{k\uparrow}^+ a_{-k\downarrow}^+ \right\rangle$$

Using such an approiximation, we get

$$\begin{aligned}
a_{k\uparrow}^+ a_{-k\downarrow}^+ a_{-q\downarrow} a_{q\uparrow} &= [(a_{k\uparrow}^+ a_{-k\downarrow}^+ - b_k^*) + b_k^*][(a_{-q\downarrow} a_{q\uparrow} - b_q) + b_q] \\
&= b_k^* (a_{-q\downarrow} a_{q\uparrow} - b_q) + b_q (a_{k\uparrow}^+ a_{-k\downarrow}^+ - b_k^*) + b_k^* b_q \\
&= b_k^* a_{-q\downarrow} a_{q\uparrow} + b_q a_{k\uparrow}^+ a_{-k\downarrow}^+ - b_k^* b_q - b_k^* b_q + b_k^* b_q \\
&= b_k^* a_{-q\downarrow} a_{q\uparrow} + b_q a_{k\uparrow}^+ a_{-k\downarrow}^+ - b_k^* b_q
\end{aligned}$$

Then we have

$$\begin{aligned}
H - \varepsilon_F N &= \sum_{k,\sigma} \xi_k a_{k\sigma}^+ a_{k\sigma} \\
&\quad + \sum_{k,q} V_{k,q} [b_q^* a_{-k\downarrow} a_{k\uparrow} + b_q a_{k\uparrow}^+ a_{-k\downarrow}^+ - b_k^* b_q] \\
&= \sum_{k,\sigma} \xi_k a_{k\sigma}^+ a_{k\sigma} - \sum_k (\Delta_k a_{k\uparrow}^+ a_{-k\downarrow}^+ + \Delta_k^* a_{-k\downarrow} a_{k\uparrow} - \Delta_k b_k^*)
\end{aligned}$$

where

$$\xi_k = \varepsilon_k - \varepsilon_F .$$

The gap parameter is defined by

$$\Delta_k = -\sum_q V_{k,q} \left\langle a_{-q\downarrow} a_{q\uparrow} \right\rangle = -\sum_q V_{k,q} b_q ,$$

or

$$\Delta_k^* = -\sum_q V_{k,q} \left\langle a_{q\uparrow}^+ a_{-q\downarrow}^+ \right\rangle = -\sum_q V_{k,q} b_q^* .$$

The above Hamiltonian can be diagonalized using the Bogoliubov transformation,

$$\begin{pmatrix} \alpha_{k\uparrow} \\ \alpha_{-k\downarrow}^{+} \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k^{*} & u_k \end{pmatrix} \begin{pmatrix} a_{k\uparrow} \\ a_{-k\downarrow}^{+} \end{pmatrix},$$

where we assume that u_{k} is real and

$$|u_k|^2 + |v_k|^2 = u_k^{\;2} + |v_k|^2 = 1.$$

This condition can be derived from the commutation relation,

$$[\alpha_{k\uparrow}, \alpha_{k\uparrow}^{\;+}]_+ = \alpha_{k\uparrow}\alpha_{k\uparrow}^{\;+} + \alpha_{k\uparrow}^{\;+}\alpha_{k\uparrow} = 1.$$

Note that

$$\begin{aligned} [\alpha_{k\uparrow}, \alpha_{k\uparrow}^{\;+}]_+ &= [u_k a_{k\uparrow} - v_k a_{-k\downarrow}^{+}, u_k^{\;*} a_{k\uparrow}^{\;+} - v_k^{\;*} a_{-k\downarrow}] \\ &= |u_k|^2 [a_{k\uparrow}, a_{k\uparrow}^{\;+}]_+ + |v_k|^2 [a_{-k\downarrow}^{+}, a_{-k\downarrow}] \\ &\quad - u_k v_k^{\;*} [a_{k\uparrow}, a_{-k\downarrow}] - u_k^{\;*} v_k [a_{-k\downarrow}^{+}, a_{k\uparrow}^{\;+}] \\ &= |u_k|^2 + |v_k|^2 = u_k^{\;2} + |v_k|^2 = 1 \end{aligned}$$

The inverse Bogoliubov transformation is given by

$$\begin{pmatrix} a_{k\uparrow} \\ a_{-k\downarrow}^{+} \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k^{*} & u_k \end{pmatrix}^{-1} \begin{pmatrix} \alpha_{k\uparrow} \\ \alpha_{-k\downarrow}^{+} \end{pmatrix} = \begin{pmatrix} u_k & v_k \\ -v_k^{*} & u_k \end{pmatrix} \begin{pmatrix} \alpha_{k\uparrow} \\ \alpha_{-k\downarrow}^{+} \end{pmatrix}.$$

Using the Bogoliubov transformation, the above Hamiltonian can be rewritten as

$$\begin{aligned} H - \mu N &= \sum_k \xi_k (u_k \alpha_{k\uparrow}^{\;+} + v_k^{\;*} \alpha_{-k\downarrow})(u_k \alpha_{k\uparrow} + v_k \alpha_{-k\downarrow}^{+}) \\ &\quad + \sum_k \xi_k (-v_k^{\;*} \alpha_{k\uparrow} + u_k \alpha_{-k\downarrow}^{+})(u_k \alpha_{-k\downarrow} - v_k \alpha_{k\uparrow}^{\;+}) \\ &\quad - \sum_k [\Delta_k (u_k \alpha_{k\uparrow}^{\;+} + v_k^{\;*} \alpha_{-k\downarrow})(u_k \alpha_{-k\downarrow}^{+} - v_k^{\;*} \alpha_{k\uparrow}) + \\ &\quad + \Delta_k^{\;*} (u_k \alpha_{-k\downarrow} - v_k \alpha_{k\uparrow}^{+})(u_k \alpha_{k\uparrow} + v_k \alpha_{-k\downarrow}^{+}) - \Delta_k b_k^{\;*}] \end{aligned}$$

where $\xi_{-k} = \xi_k$. In order to diagonalize the Hamiltonian it is required that the coefficients of $\alpha_{-k\downarrow}\alpha_{k\uparrow}$ and $\alpha_{k\uparrow}^{\;+}\alpha_{-k\uparrow}^{\;+}$ should be equal to zero. Then we have

$$2\xi_k u_k v_k^* + \Delta_k v_k^{*2} - \Delta_k^* u_k^2 = 0,$$

and its conjugate

$$2\xi_k u_k v_k + \Delta_k^* v_k^2 - \Delta_k u_k^2 = 0.$$

We use the second quadratic equation.

$$\left(\frac{u_k}{v_k}\right)^2 - 2\frac{\xi_k}{\Delta_k}\left(\frac{u_k}{v_k}\right) - \left(\frac{\Delta_k^*}{\Delta_k}\right) = 0.$$

The solution of this equation is obtained as

$$\frac{u_k}{v_k} = \frac{\xi_k \pm E_k}{\Delta_k},$$

where

$$E_k = \sqrt{\xi_k^2 + |\Delta_k|^2}.$$

We will show later that the plus sign should be chosen to get the minimum energy. Then the Hamiltonian H can be rewritten as

$$H = E_g + \sum_k E_k(\alpha_{k\uparrow}^+ \alpha_{k\uparrow} + \alpha_{-k\uparrow}^+ \alpha_{-k_\uparrow}),$$

where

$$E_g = \sum_k [2\xi_k |v_k|^2 - \Delta_k u_k v_k^* - \Delta_k^* u_k v_k + \Delta_k b_k^*]$$

E_g is the ground state energy. The above Hamiltonian clearly indicates the energies of the excitations above the ground state. Since

$$\frac{u_k}{v_k} = \frac{\xi_k \pm E_k}{\Delta_k}$$

u_k and $\xi_k \pm E_k$ are real, which means that the phase of Δ_k is the same as that of v_k. We assume that

$$\Delta_k = |\Delta_k| e^{i\phi_k}$$

$$\frac{u_k^2 + |v_k|^2}{|v_k|^2} = \frac{1}{|v_k|^2} = \frac{(\xi_k \pm E_k)^2}{|\Delta_k|^2} + 1 = \frac{2E_k(E_k \pm \xi_k)}{|\Delta_k|^2}$$

Using the normalization condition ($u_k^2 + |v_k|^2 = 1$), we have

$$|v_k|^2 = \frac{(E_k \mp \xi_k)}{2E_k}$$

The energy of the ground state E_g can be rewritten as

$$\begin{aligned} E_g &= \sum_k [2\xi_k |v_k|^2 - \Delta_k u_k v_k^* - \Delta_k^* u_k v_k + \Delta_k b_k^*] \\ &= \sum_k [2\xi_k |v_k|^2 - 2|v_k|^2 (\xi_k \pm E_k) + \Delta_k b_k^*] \\ &= \sum_k [\mp 2|v_k|^2 E_k + \Delta_k b_k^*] \\ &= \sum_k [\mp (E_k \mp \xi_k) + \Delta_k b_k^*] \end{aligned}$$

where the double signs are in order. In order to get the minimum value of E_g, we need to choose

$$|v_k|^2 = \frac{(E_k - \xi_k)}{2E_k} = \frac{1}{2}(1 - \frac{\xi_k}{E_k}),$$

leading to

$$E_g = \sum_k [-E_k + \xi_k + \Delta_k b_k^*]$$

which is the ground state energy. Then we also have

$$u_k{}^2 = \frac{1}{2}(1 + \frac{\xi_k}{E_k}),$$

and

$$u_k{}^2 |v_k|^2 = \frac{1}{4}(1 - \frac{\xi_k{}^2}{E_k{}^2}) = \frac{|\Delta_k|^2}{4E_k{}^2}.$$

The condensation amplitude F is given by

$$F_k = u_k |v_k| = \frac{|\Delta_k|}{2E_k}.$$

We note that the quasi-particle occupation numbers follows the Fermi-Dirac function,

$$<\alpha_{k\downarrow}{}^+ \alpha_{k\uparrow}> = <\alpha_{-k\downarrow}{}^+ \alpha_{k\downarrow}> = f(E_k) = \frac{1}{\exp(\beta E_k) + 1},$$

where $f(E_k)$ is the Fermi-Dirac distribution function and $\beta = \frac{1}{k_B T}$. The parameter v_k is related to

$$\frac{u_k}{v_k} = \frac{\xi_k + E_k}{\Delta_k},$$

or

$$v_k = u_k \frac{\Delta_k}{\xi_k + E_k} = u_k \frac{|\Delta_k| e^{i\phi_k}}{\xi_k + E_k}.$$

The phase of Δ_k is the same as that of v_k.

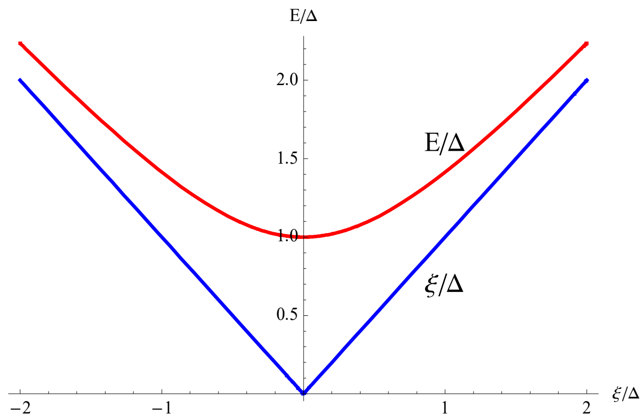


Fig. Excitation spectrum. $|\Delta_k| = \Delta$.

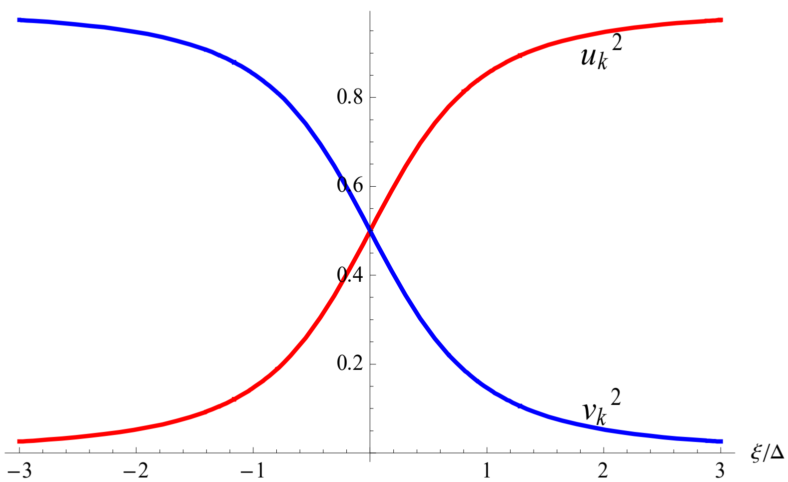


Fig. Plot of u_k^2 and $|v_k|^2$ as a function of ξ/Δ. $|\Delta_k| = \Delta$. In a normal phase, the momentum distribution ($|v_k|^2$) drops discontinuously at $\xi = 0$ ($k = k_F$). In the superconducting phase, this drop is smeared out on an interval $\delta k \approx 1/\xi_0$, where ξ_0 is the coherence length.

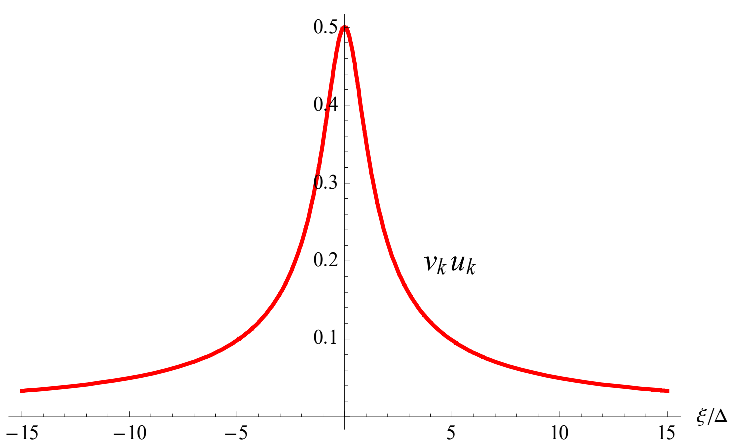


Fig. Plot of the condensation amplitude $F_k = u_k\,|v_k|$ as a function of ξ/Δ. $\left|\Delta_k\right| = \Delta$

46. The gap equation

We start with the energy gap equation at finite temperature given by

$$\Delta_k = -\sum_q V_{k,q}\left\langle a_{-q\downarrow}a_{q\uparrow}\right\rangle$$

,

where

$$\left\langle a_{-q\downarrow}a_{q\uparrow}\right\rangle = \frac{\Delta_q}{2E_q}\tanh(\frac{\beta E_q}{2})$$

Note that

$$\begin{aligned}
\left\langle a_{-q\downarrow}a_{q\uparrow}\right\rangle &= \left\langle(-v_q\alpha_{q\uparrow}^{+} + u_q\alpha_{-q\downarrow})(u_q\alpha_{q\uparrow} + v_q\alpha_{-q\downarrow}^{+})\right\rangle \\
&= u_q v_q\left\langle\alpha_{-q\downarrow}\alpha_{-q\downarrow}^{+}\right\rangle - u_q v_q\left\langle\alpha_{q\uparrow}^{+}\alpha_{q\uparrow}\right\rangle \\
&= u_q v_q[1 - \left\langle\alpha_{-q\uparrow}^{+}\alpha_{-q\uparrow}\right\rangle] - u_q v_q\left\langle\alpha_{q\uparrow}^{+}\alpha_{q\uparrow}\right\rangle \\
&= u_q v_q[1 - 2\left\langle\alpha_{q\uparrow}^{+}\alpha_{q\uparrow}\right\rangle] \\
&= u_q v_q[1 - 2f(E_q)] \\
&= \frac{u_q^{\;2}\Delta_q}{\xi_q + E_q}\tanh(\frac{\beta E_q}{2}) = \frac{\Delta_q}{2E_q}\tanh(\frac{\beta E_q}{2})
\end{aligned}$$

where we use

$$<\alpha_{q\uparrow}\alpha_{-q\downarrow}> = <\alpha_{-q\downarrow}^{+}\alpha_{q\uparrow}^{+}> = 0,$$

$$[\alpha_{q\uparrow}, \alpha_{q\uparrow}^{\;+}]_{+} = \alpha_{q\uparrow}\alpha_{q\uparrow}^{\;+} + \alpha_{q\uparrow}^{\;+}\alpha_{q\uparrow} = 1.$$

Then we get

$$\Delta_k = -\sum_q V_{k,q} \frac{\Delta_q}{2E_q} \tanh(\frac{\beta E_q}{2}) .$$

In order to simplify this equation, we assume that

$$V_{k,q} = -V \qquad \text{if } \left|\xi_k\right| \le \hbar\omega_D \text{ and } \left|\xi_q\right| \le \hbar\omega_D .$$

$$V_{k,q} = 0 \qquad \text{otherwise.}$$

where $\hbar\omega_D = k_B\Theta$ and Θ is the Debye temperature. Then Δ_k becomes independent of k and the above equation reduces

$$\Delta = V\sum_q \frac{\Delta}{2E_q} \tanh(\frac{\beta E_q}{2}) .$$

The final form of the gap equation is

$$\frac{1}{V} = \sum_q \frac{1}{2E_q} \tanh(\frac{\beta E_q}{2}) = \sum_q \frac{1}{2\sqrt{\xi_q{}^2 + \left|\Delta_q\right|^2}} \tanh(\frac{\beta}{2}\sqrt{\xi_q{}^2 + \left|\Delta_q\right|^2}) ,$$

We assume that $\left|\Delta_q\right| = \Delta$.

(1) The energy gap at *T* = 0 K

$$\begin{aligned}
\frac{1}{V} &= N(0)\int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{d\xi}{2\sqrt{\xi^2 + \Delta^2}} \\
&= N(0)\int_0^{\hbar\omega_D} \frac{d\xi}{\sqrt{\xi^2 + \Delta^2}} \\
&= N(0)\ln[\frac{\hbar\omega_D + \sqrt{\Delta^2 + (\hbar\omega_D)^2}}{\Delta}] \\
&\approx N(0)\ln[\frac{2\hbar\omega_D}{\Delta}]
\end{aligned}$$

where $N(0)$ is the density of states per spin at the Fermi surface.

(2) Critical temperature T_c

The energy gap is equal to zero at the critical temperature Tc.

$$\Delta = 0 \qquad \text{at } T = T_c.$$

This equation can be rewritten as

$$\begin{aligned}
\frac{1}{N(0)V} &= \int_0^{\hbar\omega_D} \frac{d\xi}{\xi}\tanh(\frac{\xi}{2k_BT_c}) \\
&= \int_0^{\hbar\omega_D/(2k_BT_c)} \frac{d\xi}{\xi}\tanh(\xi) \\
&= [\ln\xi\tanh(\xi)]_0^{\hbar\omega_D/(2k_BT_c)} - \int_0^{\hbar\omega_D/(2k_BT_c)} \frac{\ln\xi d\xi}{\cosh^2(\xi)}, \\
&= [\ln\xi\tanh(\xi)]_0^{\hbar\omega_D/(2k_BT_c)} - \int_0^{\infty} \frac{\ln\xi d\xi}{\cosh^2(\xi)} \\
&= \ln(\frac{\hbar\omega_D}{2k_BT_c})\tanh(\frac{\hbar\omega_D}{2k_BT_c}) - [\ln(\frac{\pi}{4}) - \gamma] \\
&= \ln(\frac{\hbar\omega_D}{k_BT_c}\frac{2e^{-\gamma}}{\pi})
\end{aligned}$$

where γ = 0.577216 is the Euler's constant. Here we use the fact that

$$\frac{\hbar\omega_D}{k_BT_c} = \frac{\Theta}{T_c}$$

is very large. For Al (type-I superconductor), in fact, T_c = 1.14 K and Θ = 428 K, The n we have

$$\frac{\Theta}{T_c} = \frac{428}{1.14} = 375.439 >> 1, \qquad \tanh(\frac{\hbar\omega_D}{2k_BT_c}) = \tanh(\frac{\Theta}{2T_c}) = 1$$

Then we get

$$\frac{1}{N(0)V} \approx \ln(\frac{\hbar\omega_D}{2k_BT_c}) + 0.81878$$

since $-[\ln(\frac{\pi}{4})-\gamma]=0.81878$. This can be rewritten as

$$\frac{\hbar\omega_D}{k_B T_c}=2\exp[\frac{1}{N(0)V}]\exp(-0.81878)$$
$$\approx 0.881939\exp[\frac{1}{N(0)V}]$$

or

$$k_B T_c = 1.13387\hbar\omega_D \exp[-\frac{1}{N(0)V}].$$

Since

$$\Delta_0 \approx 2\hbar\omega_D \exp[-\frac{1}{VN(0)}]$$

we have

$$\frac{2\Delta_0}{k_B T_c}=\frac{4}{1.13387}=3.52774$$

or

$$2\Delta_0 = 3.52774 k_B T_c = 0.303997\ T_c[\mathrm{K}]\ (\mathrm{meV})$$

$2\Delta_0$ is the energy gap for the breaking up of 1 Cooper pairs (two electrons).

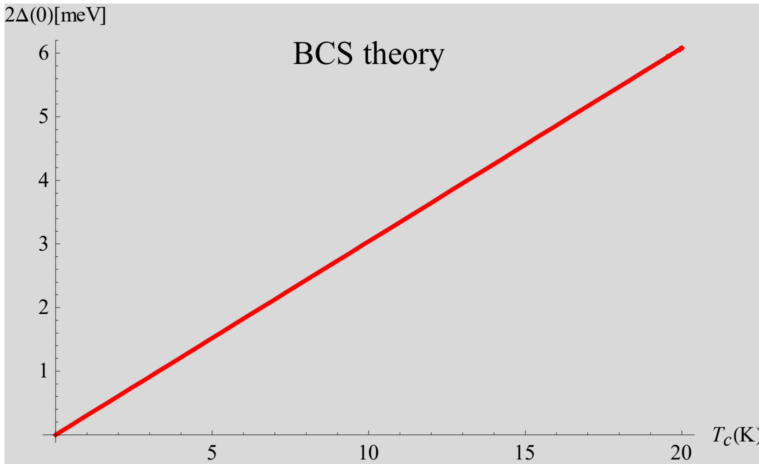


Fig. $2\Delta_0$ [meV] vs T_c [K], predicted from the BCS theory.

((Note))

The ratio $3.52774\ (=\frac{2\Delta_0}{k_B T_c})$ is a 'universal' value independent of T_c and other variables. It should be noted , however, that this value is obtained within the so-called weak coupling approximation in the BCS theory. The actual value of the ratio for ordinary superconducting metals with low T_c is close to this weak coupling value, but for metals with higher T_c, this ratio deviates from the universal constant.

	T_c	$2\Delta_0/k_B T_c$	$\Delta C/C_n$
BCS		3.52774	1.43
Al	1.18	3.53	1.43
Cd	0.52	3.44	1.32
Sn	3.72	3.61	1.60
Hg	4.15	3.65	2.37
Pb	7.20	3.95	2.71
Nb	9.25	3.65	1.87

47. Temperature dependence of energy gap

We start with the energy gap equation

$$
\begin{aligned}
\frac{1}{N(\varepsilon_F)V} &= \int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{d\xi}{2\sqrt{\xi^2+\Delta^2}} \tanh(\frac{\sqrt{\xi^2+\Delta^2}}{2k_BT}) \\
&= \int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{d\xi}{2\xi}\tanh(\frac{\xi}{2k_BT}) + \int_{-\hbar\omega_D}^{\hbar\omega_D} [\frac{1}{2\sqrt{\xi^2+\Delta^2}}\tanh(\frac{\sqrt{\xi^2+\Delta^2}}{2k_BT}) - \frac{1}{2\xi}\tanh(\frac{\xi}{2k_BT})] \\
&= \ln(\frac{\hbar\omega_D}{k_BT}\frac{2e^{-\gamma}}{\pi}) + \int_{-\hbar\omega_D}^{\hbar\omega_D} [\frac{1}{2\sqrt{\xi^2+\Delta^2}}\tanh(\frac{\sqrt{\xi^2+\Delta^2}}{2k_BT}) - \frac{1}{2\xi}\tanh(\frac{\xi}{2k_BT})] \\
&= \ln(\frac{\hbar\omega_D}{k_BT}\frac{2e^{-\gamma}}{\pi}) + \int_{0}^{\hbar\omega_D} [\frac{1}{\sqrt{\xi^2+\Delta^2}}\tanh(\frac{\sqrt{\xi^2+\Delta^2}}{2k_BT}) - \frac{1}{\xi}\tanh(\frac{\xi}{2k_BT})]
\end{aligned}
$$

Using the expansion formula

$$
\tanh(x) = \sum_{n=0}^{\infty} \frac{8x}{4x^2+(2n+1)^2\pi^2}
$$

we get

$$
\begin{aligned}
\frac{1}{N(0)V} &= \ln(\frac{\hbar\omega_D}{k_BT}\frac{2e^{-\gamma}}{\pi}) + \left(\frac{4}{k_BT}\right)\int_0^{\hbar\omega_D} \sum_{n=0}^{\infty} [\frac{1}{4\left(\frac{\sqrt{\xi^2+\Delta^2}}{2k_BT}\right)^2+(2n+1)^2\pi^2} \\
&\quad - \frac{1}{4\left(\frac{\xi}{2k_BT}\right)^2+(2n+1)^2\pi^2}] \\
&= \ln(\frac{\hbar\omega_D}{k_BT}\frac{2e^{-\gamma}}{\pi}) - 4\Delta^2 k_BT \sum_{n=0}^{\infty}\int_0^{\hbar\omega_D} \frac{d\xi}{[k_B{}^2T^2(2n+1)^2\pi^2+\xi^2]^2}
\end{aligned}
$$

Here

$$
\begin{aligned}
\int_0^{\hbar\omega_D} \frac{d\xi}{[k_B{}^2T^2(2n+1)^2\pi^2+\xi^2]^2} &= \frac{1}{k_B{}^3T^3}\int_0^{\hbar\omega_D/k_BT} \frac{dx}{[(2n+1)^2\pi^2+x^2]^2} \\
&= \frac{1}{k_B{}^3T^3}\int_0^{\infty} \frac{dx}{[(2n+1)^2\pi^2+x^2]^2} \\
&= \frac{1}{k_B{}^3T^3}\frac{1}{4\pi^2(2n+1)^3}
\end{aligned}
$$

Then we have

$$\frac{1}{N(0)V} = \ln(\frac{\hbar\omega_D}{k_B T}\frac{2e^{-\gamma}}{\pi}) - \frac{4\Delta^2}{4\pi^2 k_B{}^2 T^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}$$
$$= \ln(\frac{\hbar\omega_D}{k_B T}\frac{2e^{-\gamma}}{\pi}) - \frac{\Delta^2}{\pi^2 k_B{}^2 T^2}\frac{7}{8}\varsigma(3)$$

where

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3} = \frac{7}{8}\varsigma(3).$$

and $\zeta(3) = 1.2206$. Using the relation

$$\frac{1}{N(0)V} = \ln(\frac{\hbar\omega_D}{k_B T_c}\frac{2e^{-\gamma}}{\pi}),$$

we have

$$\ln(\frac{\hbar\omega_D}{k_B T_c}\frac{2e^{-\gamma}}{\pi}) = \ln(\frac{\hbar\omega_D}{k_B T}\frac{2e^{-\gamma}}{\pi}) - \frac{\Delta^2}{\pi^2 k_B{}^2 T^2}\frac{7}{8}\varsigma(3),$$

or

$$\ln(\frac{T}{T_c}) = -\frac{\Delta^2}{\pi^2 k_B{}^2 T^2}\frac{7}{8}\varsigma(3).$$

The energy gap is obtained as

$$\Delta(T) = \pi k_B T_c\left[\frac{8}{7\varsigma(3)}\right]^{1/2}\frac{T}{T_c}\sqrt{\ln(\frac{T_c}{T})}$$
$$= \pi k_B T_c\left[\frac{8}{7\varsigma(3)}\right]^{1/2}(1-\frac{T}{T_c})^{1/2}$$
$$= 3.06326 k_B T_c (1-\frac{T}{T_c})^{1/2}$$

in the vicinity of $T = T_c$. We make a plot of two functions (a) $(1-\frac{T}{T_c})^{1/2}$ and (b) $\frac{T}{T_c}\sqrt{\ln(\frac{T_c}{T})}$ as a function of T below T_c. It is found that these curves agree very well.

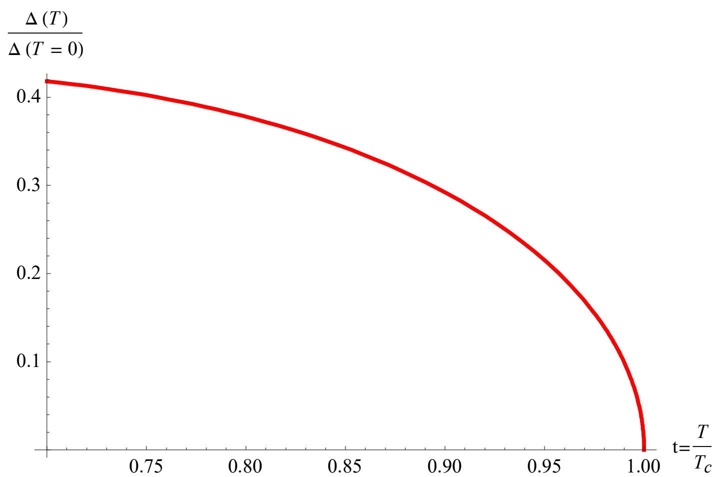


Fig. Plot of $f(t) = \Delta(T)/(3.06326 k_B T_c)$ as a function of reduced temperature $t = T/T_c$. We show two curves, $f(t) = (1-t)^{1/2}$ (red line) and $t\sqrt{\ln(1/t)}$ (blue line).

((**Mathematical note**))

Suppose that

$$t = \frac{T}{T_c} = 1 - x$$

where x is close to $x \approx 1$ (but $x>0$)

$$\left(\frac{T}{T_c}\sqrt{\ln(\frac{T_c}{T})}\right)^2 = t^2 \ln(\frac{1}{t})$$
$$= -t^2 \ln(t)$$
$$= -(1-x)^2 \ln(1-x)$$
$$= x - \frac{3}{2}x^2 + \frac{1}{3}x^3 + 0(x^4) \approx x$$

Then in the vicinity of $T = T_c$ (but $T<T_c$), we have

$$\left(\frac{T}{T_c}\sqrt{\ln(\frac{T_c}{T})}\right)^2 \approx (1-t), \qquad \text{or} \qquad \left(\frac{T}{T_c}\sqrt{\ln(\frac{T_c}{T})}\right)^2 \approx (1-t)^{1/2}$$

The energy gap is an order parameter for the superconducting transition. The critical exponent β for the order parameter is $\beta = 1/2$ as is predicted from the mean field theory. We note that

$$\Delta_0 = \frac{3.5277}{2} k_B T_c .$$

Using this, we have

$$\frac{\Delta(T)}{\Delta_0} = 2\times\frac{3.06326 k_B T_c}{3.5277 k_B T_c}(1-\frac{T}{T_c})^{1/2} = 1.73669(1-\frac{T}{T_c})^{1/2} .$$

48. Density of states

The BCS spectrum is easily seen to have a minimum $\Delta_{\mathbf{k}}$ for a given direction $\boldsymbol{k}$ on the Fermi surface; $\Delta_{\mathbf{k}}$ therefore, in addition to playing the role of order parameter for the superconducting transition, is also the energy gap in the quasi-particle spectrum. To see this explicitly, we can simply do a change of variables in all energy integrals from the normal metal eigenenergies ξ_{k} to the quasiparticle energies E_{k}:

$$\xi_k = \xi = \varepsilon - \varepsilon_F ,$$

$$E = \sqrt{\xi^2 + \Delta^2} , \qquad \text{or} \qquad \xi = \pm\sqrt{E^2 - \Delta^2}$$

$$N(E)dE = N_n(\xi)d\xi = N_n(0)d\xi ,$$

since $\xi \approx 0$. Then we get

$$N(E) = N_n(0)\frac{d\xi}{dE} = N_n(0)\frac{E}{\sqrt{E^2 - \Delta^2}} ,$$

or

$$\frac{N(E)}{N_n(0)} = \frac{E}{\sqrt{E^2 - \Delta^2}} = \frac{\frac{E}{\Delta}}{\sqrt{\left(\frac{E}{\Delta}\right)^2 - 1}}.$$

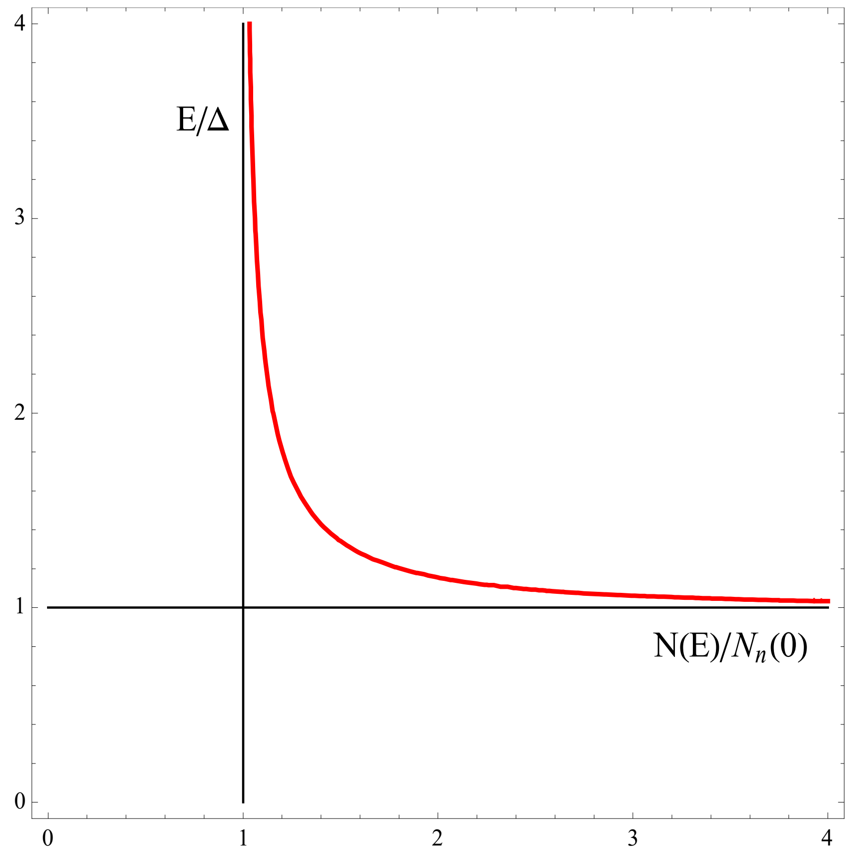


Fig. Plot of $\frac{E}{\Delta}$ vs the normalized density of states $\frac{N(E)}{N_n(0)}$. $N(E)$ is the density of state per spin.

49. Evaluation of the energy of ground state, E_g

$$b_k^* = u_k v_k^*, \qquad b_k = u_k^* v_k = u_k v_k$$

Then we have

$$\sum_k (-E_k + \xi_k + \Delta_k b_k^* + \Delta_k^* b_k) = \sum_k (-E_k + \xi_k + \Delta_k u_k v_k^* + \Delta_k^* u_k^* v_k)$$

$$= \sum_k (-E_k + \xi_k + 2\frac{u_k^2 |\Delta_k|^2}{\xi_k + E_k})$$

$$= \sum_k (-E_k + \xi_k + \frac{|\Delta_k|^2}{E_k})$$

$$= \sum_k (\xi_k + \frac{|\Delta_k|^2 - E_k}{E_k})$$

$$= \sum_k \xi_k (1 - \frac{\xi_k}{E_k})$$

Then we have

$$E_g = \sum_k (-E_k + \xi_k + \Delta_k b_k^*)$$

$$= \sum_k \xi_k (1 - \frac{\xi_k}{E_k}) - \sum_k \Delta_k^* b_k$$

We assume that Δ_k is independent of k. Then the energy gap equation can be expressed by

$$\sum_k b_k = \frac{\Delta}{V}, \qquad \sum_k b_k^* = \frac{\Delta^*}{V}$$

Then the ground state energy E_g can be rewritten as

$$E_g = \sum_k \xi_k (1 - \frac{\xi_k}{E_k}) - \frac{\Delta^* \Delta}{V} = \sum_k \xi_k (1 - \frac{\xi_k}{E_k}) - \frac{\Delta^2}{V}$$

Here we assume that Δ is a positive real value.

$$E_g(\Delta) + \frac{\Delta^2}{V} = -\sum_k \frac{\xi_k^2}{E_k} + \sum_k \xi_k$$

$$= -\sum_k \frac{\xi_k^2}{E_k} + \sum_k |\xi_k| + \sum_k (\xi_k - |\xi_k|)$$

$$= -\sum_k \frac{\xi_k^2}{E_k} + \sum_k |\xi_k| + E_n$$

where E_n is the energy of the normal state. Then the above equation can be rewritten as

$$\begin{aligned}
E_g(\Delta) + \frac{\Delta^2}{V} - E_n &= -N(0)\int_{-\hbar\omega_D}^{\hbar\omega_D} d\xi \frac{\xi^2}{\sqrt{\xi^2+\Delta^2}} + N(0)\int_{-\hbar\omega_D}^{\hbar\omega_D} |\xi| d\xi \\
&= -2N(0)\int_0^{\hbar\omega_D} d\xi \frac{\xi^2}{\sqrt{\xi^2+\Delta^2}} + 2N(0)\int_0^{\hbar\omega_D} \xi d\xi \\
&= -2N(0)\Delta^2 \int_0^{\hbar\omega_D/\Delta} dx \frac{x^2}{\sqrt{x^2+\Delta^2}} + 2N(0)\Delta^2 \int_0^{\hbar\omega_D/\Delta} x dx \\
&= -N(0)\Delta^2 \frac{\hbar\omega_D}{\Delta}\sqrt{1+\left(\frac{\hbar\omega_D}{\Delta}\right)^2} + N(0)\Delta^2 \sinh^{-1}\left(\frac{\hbar\omega_D}{\Delta}\right) + N(0)\Delta^2\left(\frac{\hbar\omega_D}{\Delta}\right)^2 \\
&= N(0)\Delta^2[-\frac{\hbar\omega_D}{\Delta}\sqrt{1+\left(\frac{\hbar\omega_D}{\Delta}\right)^2} + \left(\frac{\hbar\omega_D}{\Delta}\right)^2] + N(0)\Delta^2 \sinh^{-1}\left(\frac{\hbar\omega_D}{\Delta}\right)] \\
&\approx -N(0)\Delta^2\left(\frac{\hbar\omega_D}{\Delta}\right) + N(0)\Delta^2 \sinh^{-1}\left(\frac{\hbar\omega_D}{\Delta}\right)]
\end{aligned}$$

Thus we have

$$\begin{aligned}
E_g(\Delta) + \frac{\Delta^2}{V} - E_n &\approx -N(0)\Delta^2 \frac{\hbar\omega_D}{\Delta} + \frac{\Delta^2}{V} \\
&\approx -N(0)\frac{\Delta^2}{2} + \frac{\Delta^2}{V}
\end{aligned}$$

or

$$E_n - E_g(\Delta) = N(0)\frac{\Delta^2}{2}$$

We note that the energy gap Δ is related to the thermodynamic critical field $H_c(0)$ as

$$N(0)\frac{\Delta^2}{2} = \frac{1}{8\pi}[H_c(0)]^2$$

((**Note**))
Here we use the formula

$$\int \frac{x^2}{\sqrt{1+x^2}}dx = \frac{1}{2}(x\sqrt{1+x^2} - \sinh^{-1} x),$$

$$\int_0^{\hbar\omega_D} \frac{d\xi}{\sqrt{\xi^2+\Delta^2}} = \sinh^{-1}(\frac{\hbar\omega_D}{\Delta}) = \ln[\frac{\hbar\omega_D}{\Delta} + \sqrt{\left(\frac{\hbar\omega_D}{\Delta}\right)^2 + 1},$$

The energy gap equation id

$$1 = N(0)V \int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{d\xi}{2\sqrt{\xi^2+\Delta^2}} = N(0)V \int_0^{\hbar\omega_D} \frac{d\xi}{\sqrt{\xi^2+\Delta^2}} = N(0)V \sinh^{-1}(\frac{\hbar\omega_D}{\Delta}),$$

or

$$\sinh[\frac{1}{N(0)V}] = \frac{\hbar\omega_D}{\Delta},$$

or

$$\Delta = \frac{\hbar\omega_D}{\sinh[\frac{1}{VN(0)}]} \approx 2\hbar\omega_D \exp[-\frac{1}{VN(0)}] \approx 2\hbar\omega_D.$$

50. Physical meaning

The energy gap at $T = 0$ K is given by

$$\Delta_0 = \frac{3.5277}{2} k_B T_c.$$

Then we get

$$\frac{[H_c(0)]^2}{8\pi} = \frac{N(0)}{2}\Delta_0{}^2 = \frac{N(0)}{2}(\frac{3.5277}{2}k_B T_c)^2 = 1.55558 N(0)(k_B T_c)^2.$$

Note that the unit of left-hand side is Oe^2 = erg/cm^3. Here we use the following relation for the electronic specific heat in the normal state

$$C_{en} = \gamma_e T,$$

with

$$\gamma_e = \frac{2\pi^2}{3} N(0) k_B{}^2 .$$

Then we have

$$\frac{[H_c(0)]^2}{8\pi} = 1.55558 \frac{3\gamma_e}{2\pi^2} T_c{}^2 .$$

Suppose that the atom (showing the superconductivity) has a molar mass M (g) and a density ρ (g/cm^3).

$$\frac{[H_c(0)]^2}{8\pi} = 1.55558 \frac{3}{2\pi^2} \times (10^{-3} \times 10^7) \times \gamma(mJ/(molK^2)) \frac{\rho}{M} T_c{}^2 ,$$

where

$\frac{M}{\rho}$ is the volume (cm^3) per mol; [mol/cm^3].

$$\begin{aligned} H_c(0) &= \sqrt{8\pi \times 1.55558 \times \frac{3}{2\pi^2} \times (10^7 \times 10^{-3}) \times \gamma(mJ/(molK)^2) \frac{\rho}{M}} T_c \\ &= \sqrt{8\pi \times 1.55558 \times \frac{3}{2\pi^2} \times (10^7 \times 10^{-3}) \times \gamma(mJ/(molK)^2) \frac{\rho}{M}} T_c \\ &= 243.76 \sqrt{\gamma(mJ/(molK)^2) \frac{\rho}{M}} T_c \end{aligned}$$

in the units of Oe.

(i) $H_c(0)$ is proportional to $\sqrt{\gamma \frac{\rho}{M}} T_c$.

(ii) If T_c is very high, then $H_c(0)$ is also very high.

((**Example**))

(i) Pb,

ρ = 11.34 g/cm^3, M = 207.2 g/mol.

γ = 2.98 mJ/(mol K^2). T_c = 7.193 K

$H_c(0)$=708.1 Oe (calculation)

$H_c(0)$=803 Oe (experiment)

(ii) Sn

ρ = 7.365 g/cm^3, M = 118.692 g/mol.

γ = 1.78 mJ/(mol K^2). T_c = 3.722 K

$H_c(0)$=301.5 Oe (calculation)

$H_c(0)$=309 Oe (experiment)

(iii) Al

ρ = 2.70 g/cm^3, M = 26.9815 g/mol.

γ = 1.35 mJ/(mol K^2). T_c = 1.140 K

$H_c(0)$=102.1 Oe (calculation)

$H_c(0)$=105 Oe (experiment)

All the experimental data of g, Hc and Tc are obtained from the ISSP Kittel (8th-edition).

---

51. BCS Ground state

According to BCS, the ground state can be expressed by

$$|BCS\rangle = \prod_k (u_k + v_k B_k^+)|\Phi_{vac}\rangle,$$

where B_k is a pairing operator, and is defined by

$$B_k = a_{-k\downarrow} a_{k\uparrow},$$

and an operator Hermitian conjugate to the operator B_k is given by

$$B_k^+ = a_{k\uparrow}^+ a_{-k\downarrow}^+ = -a_{-k\downarrow}^+ a_{k\uparrow}^+ .$$

The parameters u_k and v_k can be determined such that the BCS ground state is to be minimized over the space of u_k and v_k. Another method (as is already discussed above) is the diagonalization of the BCS Hamiltonian by using the Bogoliubov transformation. We will show that these two methods are equivalent.

Since there is no quasi particle in the ground state, we have

$$\alpha_{k\uparrow}\left|BCS\right\rangle = 0, \ \alpha_{-k\downarrow}\left|BCS\right\rangle = 0.$$

$\left|\Phi_{vac}\right\rangle$ is the vacuum state. We use the Bogoliubov transformation;

$$\begin{pmatrix}\alpha_{k\uparrow}\\ \alpha_{-k\downarrow}^{+}\end{pmatrix} = \begin{pmatrix}u_k & -v_k\\ v_k^{*} & u_k\end{pmatrix}\begin{pmatrix}a_{k\uparrow}\\ a_{-k\downarrow}^{+}\end{pmatrix}, \qquad \begin{pmatrix}\alpha_{k\uparrow}^{+}\\ \alpha_{-k\downarrow}\end{pmatrix} = \begin{pmatrix}u_k & -v_k^{*}\\ v_k & u_k\end{pmatrix}\begin{pmatrix}a_{k\uparrow}^{+}\\ a_{-k\downarrow}\end{pmatrix},$$

or

$$\begin{pmatrix}a_{k\uparrow}\\ a_{-k\downarrow}^{+}\end{pmatrix} = \begin{pmatrix}u_k & v_k\\ -v_k^{*} & u_k\end{pmatrix}\begin{pmatrix}\alpha_{k\uparrow}\\ \alpha_{-k\downarrow}^{+}\end{pmatrix}, \qquad \begin{pmatrix}a_{k\uparrow}^{+}\\ a_{-k\downarrow}\end{pmatrix} = \begin{pmatrix}u_k & v_k^{*}\\ -v_k & u_k\end{pmatrix}\begin{pmatrix}\alpha_{k\uparrow}^{+}\\ \alpha_{-k\downarrow}\end{pmatrix}.$$

Here we assume that $u_k^{*} = u_k$ (real number). v_k is in general a complex number.

Then the above relations can be rewritten as

$$\alpha_{k\uparrow}\left|BCS\right\rangle = (u_k a_{k\uparrow} - v_k a_{-k\downarrow}^{+})\left|BCS\right\rangle = 0,$$

and

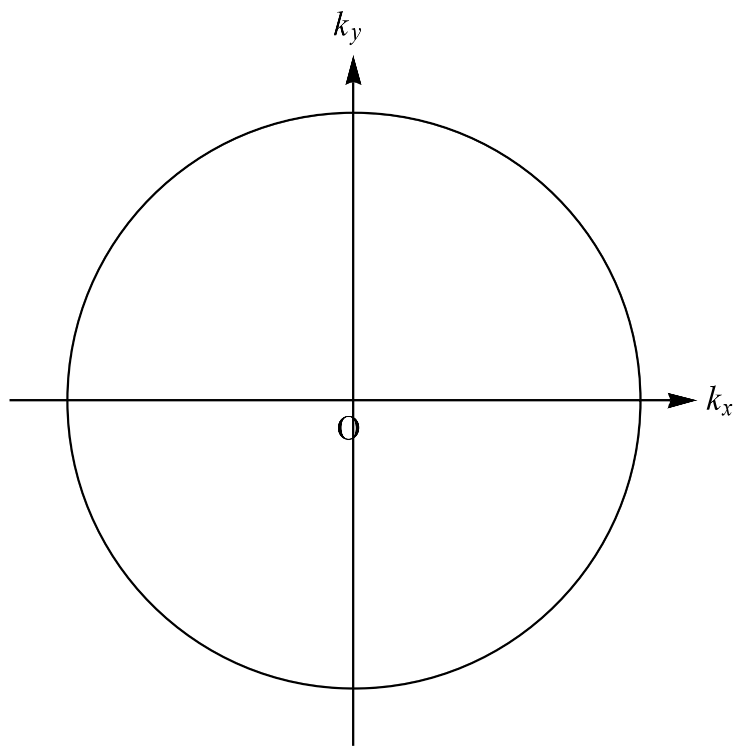
$$\alpha_{-k\downarrow}\left|BCS\right\rangle = (u_k a_{-k\downarrow} + v_k a_{k\uparrow}^{+})\left|BCS\right\rangle = 0.$$

This means that $\left|BCS\right\rangle$ is a vacuum state for quasi-particles.

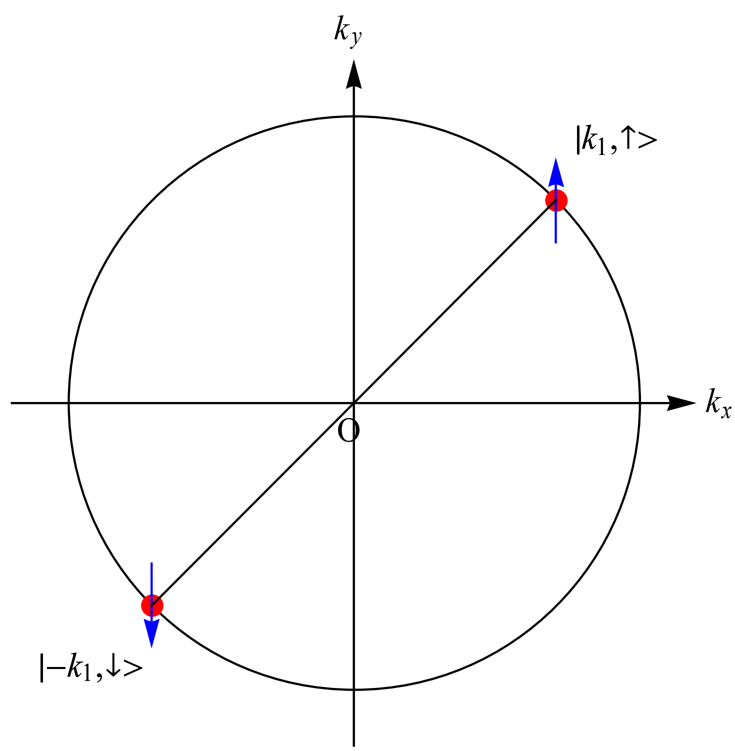
52. The formation of the BCS ground state by the successive addition of Cooper pairs

The BCS state can be formulated from the successive addition of Cooper pairs to the vacuum state.

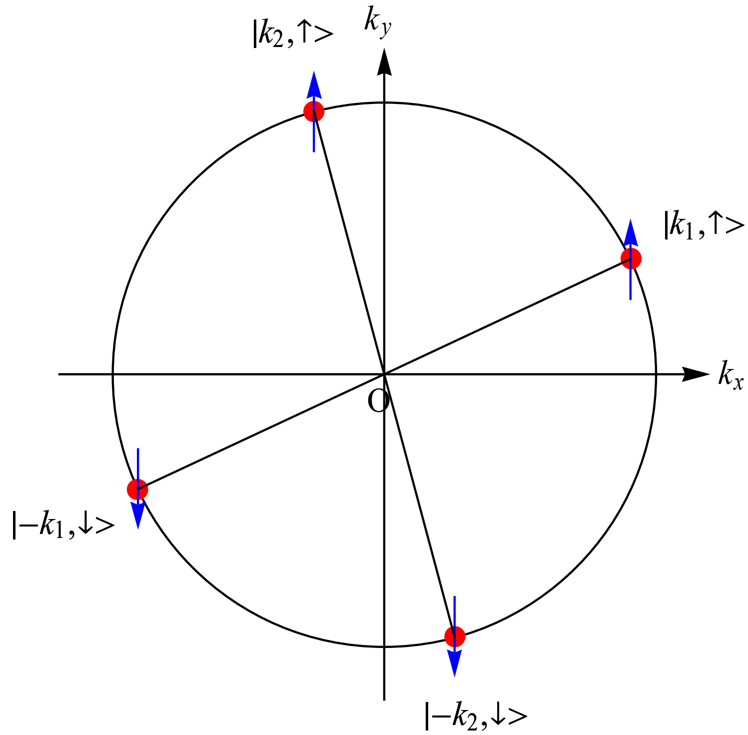
$\left|\Phi_{vac}\right\rangle$;



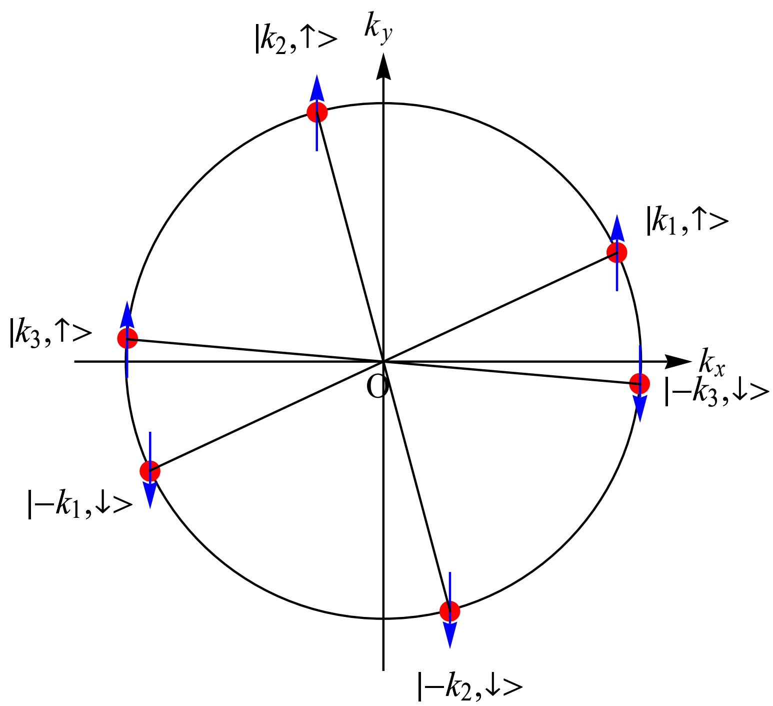
$(u_{k_1} + v_{k_1} a^{+}_{k_1\uparrow} a^{+}_{-k_1\downarrow}) |\Phi_{vac}\rangle$;



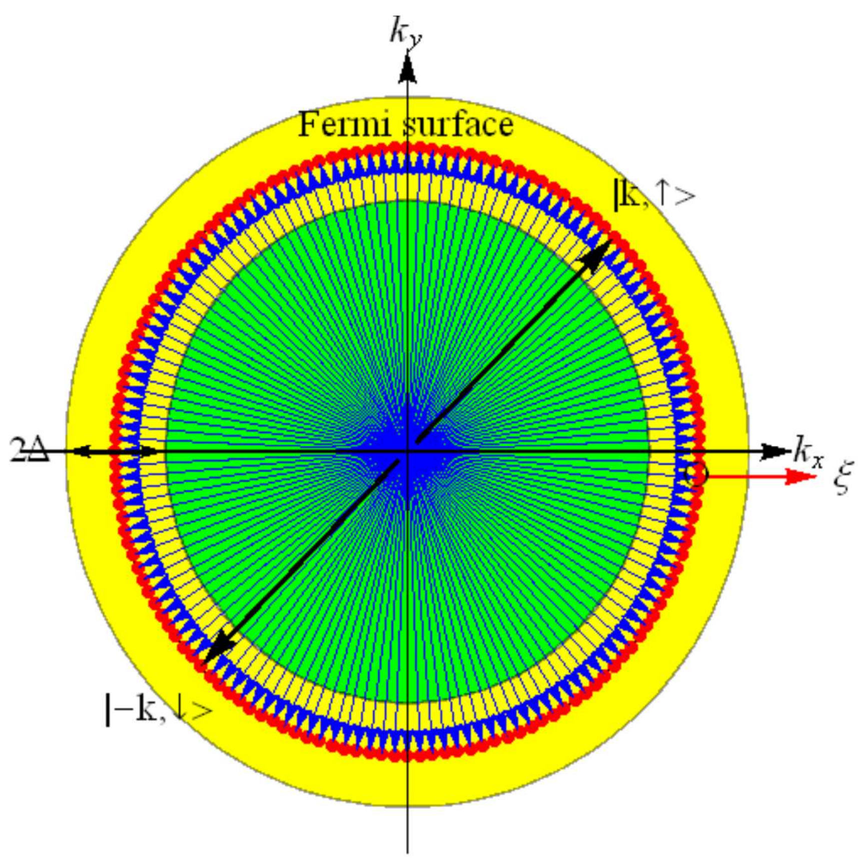
$$(u_{k_2} + v_{k_2} a^+_{k_2\uparrow} a^+_{-k_2\downarrow})(u_{k_1} + v_{k_1} a^+_{k_1\uparrow} a^+_{-k_1\downarrow})\left|\Phi_{vac}\right\rangle;$$



$$(u_{k_1} + v_{k_1} a^+_{k_1\uparrow} a^+_{-k_1\downarrow})(u_{k_2} + v_{k_2} a^+_{k_2\uparrow} a^+_{-k_2\downarrow})(u_{k_1} + v_{k_1} a^+_{k_1\uparrow} a^+_{-k_1\downarrow})\left|\Phi_{vac}\right\rangle;$$



$$\left|BCS\right\rangle = \prod_k (u_k + v_k B_k^+)\left|\Phi_{vac}\right\rangle$$



53. The commutation relation

53.1. Fermion operators

The Fermion operators satisfies the anticommutation relation.

$$[a_{k\sigma}, a_{k'\sigma'}^{+}]_{+} = a_{k\sigma}a_{k'\sigma'}^{+} + a_{k'\sigma'}^{+}a_{k\sigma} = \delta_{k,k'}\delta_{\sigma,\sigma'}.$$

$$[a_{k\sigma}a_{k'\sigma'}]_{+} = [a_{k\sigma}^{+}, a_{k'\sigma'}^{+}]_{+} = 0.$$

Using the vacuum state $|\Phi_{vac}\rangle$, we have

$$[a_{k\sigma}, a_{k\sigma}^{+}]_{+}|\Phi_{vac}\rangle = |\Phi_{vac}\rangle,$$

or

$$(a_{k\sigma}a_{k\sigma}^{+} + a_{k\sigma}^{+}a_{k\sigma})|\Phi_{vac}\rangle = |\Phi_{vac}\rangle,$$

or

$$a_{k\sigma}a_{k\sigma}^{\ +}|\Phi_{vac}\rangle = (1 - a_{k\sigma}^{\ +}a_{k\sigma})|\Phi_{vac}\rangle = |\Phi_{vac}\rangle.$$

53.2 Paring operators B_k and B_k^+

The application of B_k to $|\Phi_{vac}\rangle$ yields

$$B_k|\Phi_{vac}\rangle = a_{-k\downarrow}a_{k\uparrow}|\Phi_{vac}\rangle = 0,$$

and

$$\langle\Phi_{vac}|B_k^{\ +} = 0.$$

Similarly, we get

$$B_k^{\ +}|\Phi_{vac}\rangle = a_{k\uparrow}^{+}a_{-k\downarrow}^{+}|\Phi_{vac}\rangle = -a_{-k\downarrow}^{+}a_{k\uparrow}^{+}|\Phi_{vac}\rangle,$$

The application of $B_kB_k^{\ +}$ to $|\Phi_{vac}\rangle$ yields

$$
\begin{aligned}
B_k B_k{}^{+}|\Phi_{vac}\rangle &= a_{-k\downarrow}a_{k\uparrow}a_{k\uparrow}^{+}a_{-k\downarrow}^{+}|\Phi_{vac}\rangle \\
&= -a_{k\uparrow}a_{-k\downarrow}a_{k\uparrow}^{+}a_{-k\downarrow}^{+}|\Phi_{vac}\rangle \\
&= a_{k\uparrow}a_{-k\downarrow}a_{-k\downarrow}^{+}a_{k\uparrow}^{+}|\Phi_{vac}\rangle \\
&= a_{k\uparrow}(1-a_{-k\downarrow}^{+}a_{-k\downarrow})a_{k\uparrow}^{+}|\Phi_{vac}\rangle \\
&= a_{k\uparrow}a_{k\uparrow}^{+}|\Phi_{vac}\rangle + a_{k\uparrow}a_{-k\downarrow}^{+}a_{k\uparrow}^{+}a_{-k\downarrow}|\Phi_{vac}\rangle \\
&= a_{k\uparrow}a_{k\uparrow}^{+}|\Phi_{vac}\rangle \\
&= |\Phi_{vac}\rangle
\end{aligned}
$$

54. Fermion operators and boson operators

(1)

$$
\begin{aligned}
[a_{k\uparrow}, B_k^{+}]_{+} &= a_{k\uparrow}a_{k\uparrow}^{+}a_{-k\downarrow}^{+} + a_{k\uparrow}^{+}a_{-k\downarrow}^{+}a_{k\uparrow} \\
&= a_{-k\downarrow}^{+}a_{k\uparrow}a_{k\uparrow}^{+} - a_{-k\downarrow}^{+}a_{k\uparrow}^{+}a_{k\uparrow} \\
&= a_{-k\downarrow}^{+}(1-2a_{k\uparrow}^{+}a_{k\uparrow})
\end{aligned}
$$

(2)

$$
\begin{aligned}
[a_{-k\uparrow}, B_k^{+}]_{+} &= a_{-k\uparrow}a_{k\uparrow}^{+}a_{-k\downarrow}^{+} + a_{k\uparrow}^{+}a_{-k\downarrow}^{+}a_{-k\uparrow} \\
&= -a_{k\uparrow}^{+}a_{-k\uparrow}a_{-k\downarrow}^{+} + a_{k\uparrow}^{+}a_{-k\downarrow}^{+}a_{-k\uparrow} \\
&= -a_{k\uparrow}^{+}(1-2a_{-k\downarrow}^{+}a_{-k\uparrow})
\end{aligned}
$$

(3)

$$
\begin{aligned}
[a_{-k\downarrow}^{+}, B_k^{+}]_{+} &= a_{-k\downarrow}^{+}a_{k\uparrow}^{+}a_{-k\downarrow}^{+} + a_{k\uparrow}^{+}a_{-k\downarrow}^{+}a_{-k\downarrow}^{+} \\
&= -a_{k\uparrow}^{+}a_{-k\downarrow}^{+}a_{-k\downarrow}^{+} + a_{k\uparrow}^{+}a_{-k\downarrow}^{+}a_{-k\downarrow}^{+} \\
&= 0
\end{aligned}
$$

(4)

$$
\begin{aligned}
[a_{k\uparrow}^{+}, B_k^{+}]_{+} &= a_{k\uparrow}^{+}a_{k\uparrow}^{+}a_{-k\downarrow}^{+} + a_{k\uparrow}^{+}a_{-k\downarrow}^{+}a_{k\uparrow}^{+} \\
&= a_{k\uparrow}^{+}a_{k\uparrow}^{+}a_{-k\downarrow}^{+} - a_{k\uparrow}^{+}a_{k\uparrow}^{+}a_{-k\downarrow}^{+} \\
&= 0
\end{aligned}
$$

(5)

$$
\begin{aligned}
[a_{-k\downarrow}, B_k^+]_+ &= a_{-k\downarrow} a_{k\uparrow}^+ a_{-k\downarrow}^+ + a_{k\uparrow}^+ a_{-k\downarrow}^+ a_{-k\downarrow} \\
&= -a_{k\uparrow}^+ a_{-k\downarrow} a_{-k\downarrow}^+ + a_{k\uparrow}^+ a_{-k\downarrow}^+ a_{-k\downarrow} \\
&= -a_{k\uparrow}^+ (1 - 2a_{-k\downarrow}^+ a_{-k\downarrow})
\end{aligned}
$$

(6)

$$
\begin{aligned}
[a_{k\downarrow}, B_k^+]_+ &= a_{k\downarrow} a_{k\uparrow}^+ a_{-k\downarrow}^+ + a_{k\uparrow}^+ a_{-k\downarrow}^+ a_{k\downarrow} \\
&= a_{k\downarrow} a_{k\uparrow}^+ a_{-k\downarrow}^+ - a_{k\uparrow}^+ a_{k\downarrow} a_{-k\downarrow}^+ \\
&= (a_{k\downarrow} a_{k\uparrow}^+ + a_{k\uparrow}^+ a_{k\downarrow}) a_{-k\downarrow}^+ \\
&= a_{-k\downarrow}^+
\end{aligned}
$$

When these operators are applied to $\left|\Phi_{vac}\right\rangle$, we have

(a)

$$[a_{k\uparrow}, B_k^+]_+ \left|\Phi_{vac}\right\rangle = a_{k\uparrow} B_k^+ \left|\Phi_{vac}\right\rangle = a_{-k\downarrow}^+ (1 - 2a_{k\uparrow}^+ a_{k\uparrow}) \left|\Phi_{vac}\right\rangle = a_{-k\downarrow}^+ \left|\Phi_{vac}\right\rangle,$$

or

$$a_{k\uparrow} B_k^+ \left|\Phi_{vac}\right\rangle = a_{-k\downarrow}^+ \left|\Phi_{vac}\right\rangle.$$

(b)

$$[a_{-k\uparrow}, B_k^+]_+ \left|\Phi_{vac}\right\rangle = a_{-k\uparrow} B_k^+ \left|\Phi_{vac}\right\rangle = -a_{k\uparrow}^+ (1 - 2a_{-k\downarrow}^+ a_{-k\uparrow}) \left|\Phi_{vac}\right\rangle = -a_{k\uparrow}^+ \left|\Phi_{vac}\right\rangle.$$

or

$$a_{-k\uparrow} B_k^+ \left|\Phi_{vac}\right\rangle = -a_{k\uparrow}^+ \left|\Phi_{vac}\right\rangle.$$

(c)

$$[a_{-k\downarrow}, B_k^+]_+ \left|\Phi_{vac}\right\rangle = -a_{k\uparrow}^+ (1 - 2a_{-k\downarrow}^+ a_{-k\downarrow}) \left|\Phi_{vac}\right\rangle = -a_{k\uparrow}^+ \left|\Phi_{vac}\right\rangle.$$

or

$$a_{-k\downarrow} B_k^+ \left|\Phi_{vac}\right\rangle = -a_{k\uparrow}^+ \left|\Phi_{vac}\right\rangle.$$

(d)

$$[a_{k\downarrow}, B_k^+]_+ \left|\Phi_{vac}\right\rangle = (a_{k\downarrow} B_k^+ + B_k^+ a_{k\downarrow}) \left|\Phi_{vac}\right\rangle = a_{-k\downarrow}^+ \left|\Phi_{vac}\right\rangle,$$

or

$$a_{k\downarrow}B_k^+\left|\Phi_{vac}\right\rangle = a_{-k\downarrow}^+\left|\Phi_{vac}\right\rangle.$$

55. The nature of the BCS ground state

Here we show that,

$$\alpha_{k\uparrow}\left|BCS\right\rangle = 0. \qquad \alpha_{-k\downarrow}\left|BCS\right\rangle = 0.$$

((**Proof**))

$$\begin{aligned}
\alpha_{k\uparrow}\left|BCS\right\rangle &= \prod_{l\neq k}(u_l + v_l B_l^+)(u_k a_{k\uparrow} - v_k a_{-k\downarrow}^+)(u_k + v_k B_k^+)\left|\Phi_{vac}\right\rangle \\
&= \prod_{l\neq k}(u_l + v_l B_l^+)(u_k{}^2 a_{k\uparrow} + u_k v_k a_{k\uparrow} B_k^+ - u_k v_k a_{-k\downarrow}^+ - v_k{}^2 a_{-k\downarrow}^+ B_k^+)\left|\Phi_{vac}\right\rangle \\
&= \prod_{l\neq k}(u_l + v_l B_l^+)(u_k v_k a_{k\uparrow} B_k^+ - u_k v_k a_{-k\downarrow}^+ - v_k{}^2 a_{-k\downarrow}^+ B_k^+)\left|\Phi_{vac}\right\rangle \\
&= \prod_{l\neq k}(u_l + v_l B_l^+)(u_k v_k a_{-k\uparrow} - u_k v_k a_{-k\downarrow}^+\left|\Phi_{vac}\right\rangle = 0
\end{aligned}$$

and

$$\begin{aligned}
\alpha_{-k\downarrow}\left|BCS\right\rangle &= \prod_{l\neq k}(u_l + v_l B_l^+)(u_k a_{-k\downarrow} + v_k a_{k\uparrow}^+)(u_k + v_k B_k^+)\left|\Phi_{vac}\right\rangle \\
&= \prod_{l\neq k}(u_l + v_l B_l^+)(u_k{}^2 a_{-k\downarrow} + u_k v_k a_{-k\downarrow} B_k^+ + u_k v_k a_{k\uparrow}^+ + v_k{}^2 a_{k\uparrow}^+ B_k^+)\left|\Phi_{vac}\right\rangle \\
&= \prod_{l\neq k}(u_l + v_l B_l^+)(-u_k v_k a_{k\uparrow}^+ + u_k v_k a_{k\uparrow}^+)\left|\Phi_{vac}\right\rangle = 0
\end{aligned}$$

Here we note that

$$a_{-k\downarrow}^+ B_k^+\left|\Phi_{vac}\right\rangle = a_{-k\downarrow}^+ a_{k\uparrow}^+ a_{-k\downarrow}^+\left|\Phi_{vac}\right\rangle = 0,$$

$$a_{k\uparrow}^+ B_k^+\left|\Phi_{vac}\right\rangle = a_{k\uparrow}^+ a_{k\uparrow}^+ a_{-k\downarrow}^+\left|\Phi_{vac}\right\rangle = 0,$$

from the Pauli's exclusion principle; No two electrons can occupy the one state denoted by $\left|k,\sigma\right\rangle$, where k is the wavenumber and σ is the spin up or down state.

56. Paring operators

$$B_k = a_{-k\downarrow}a_{k\uparrow},$$

$$B_k^+ = a_{k\uparrow}^+ a_{-k\downarrow}^+$$

The pairing operators obey commutation relations given by

$$[B_k, B_{k'}^+]_- = (1 - n_{k\uparrow} - n_{-k\downarrow})\delta_{k,k'}$$

with

$$n_{k\uparrow} = a_{k\uparrow}^+ a_{k\uparrow}, \qquad n_{-k\downarrow} = a_{-k\downarrow}^+ a_{-k\downarrow}.$$

and

$$[B_k, B_{k'}]_- = 0.$$

$$[B_k, B_{k'}]_+ = 2B_k B_{k'}(1 - \delta_{k,k'}).$$

$$\begin{aligned} B_k B_k^+ \left|\Phi_{vac}\right\rangle &= a_{-k\downarrow} a_{k\uparrow} a_{k\uparrow}^+ a_{-k\downarrow}^+ \left|\Phi_{vac}\right\rangle \\ &= a_{-k\downarrow} a_{-k\downarrow}^+ a_{k\uparrow} a_{k\uparrow}^+ \left|\Phi_{vac}\right\rangle \\ &= a_{-k\downarrow} a_{-k\downarrow}^+ \left|\Phi_{vac}\right\rangle \\ &= \left|\Phi_{vac}\right\rangle \end{aligned}$$

since

$$a_{k\sigma} a_{k\sigma}^+ \left|\Phi_{vac}\right\rangle = \left|\Phi_{vac}\right\rangle,$$

we have

$$B_k \left|\Phi_{vac}\right\rangle = 0, \qquad \left\langle\Phi_{vac}\right| B_k^{\ +} = 0.$$

57. Normalization of the ground state:

$$\begin{aligned} \left\langle BCS \middle| BCS \right\rangle &= \left\langle \Phi_{vac} \right| \prod_k (u_k + v_k^{\ *} B_k) \prod_k (u_k + v_k B_k^+) \left| \Phi_{vac} \right\rangle \\ &= \left\langle \Phi_{vac} \right| \prod_k (u_k + v_k^{\ *} B_k)(u_k + v_k B_k^+) \left| \Phi_{vac} \right\rangle \\ &= \left\langle \Phi_{vac} \right| \prod_k (u_k^{\ 2} + u_k v_k^{\ *} B_k + u_k v_k B_k^+ + v_k v_k^{\ *} B_k B_k^+) \left| \Phi_{vac} \right\rangle \\ &= \left\langle \Phi_{vac} \right| \prod_k (u_k^{\ 2} + v_k v_k^{\ *}) \left| \Phi_{vac} \right\rangle \\ &= \left\langle \Phi_{vac} \middle| \Phi_{vac} \right\rangle = 1 \end{aligned}$$

since

$$u_k{}^2 + v_k v_k{}^* = 1 .$$

58. Excited state: quasi particles

$$\begin{pmatrix} \alpha_{k\uparrow} \\ \alpha^+_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k{}^* & u_k \end{pmatrix} \begin{pmatrix} a_{k\uparrow} \\ a^+_{-k\downarrow} \end{pmatrix}, \qquad \begin{pmatrix} \alpha^+_{k\uparrow} \\ \alpha_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & -v_k{}^* \\ v_k & u_k \end{pmatrix} \begin{pmatrix} a^+_{k\uparrow} \\ a_{-k\downarrow} \end{pmatrix}.$$

(i)

$$\begin{aligned}
\alpha^+_{k\uparrow}\left|BCS\right\rangle &= \prod_{l\neq k}(u_l + v_l B_l^+)(u_k a^+_{k\uparrow} - v_k{}^* a_{-k\downarrow})(u_k + v_k B_k^+)\left|\Phi_{vac}\right\rangle \\
&= \prod_{l\neq k}(u_l + v_l B_l^+)(u_k{}^2 a^+_{k\uparrow} + u_k v_k a^+_{k\uparrow} B_k^+ - u_k v_k{}^* a_{-k\downarrow} - v_k{}^* v_k a_{-k\downarrow} B_k^+)\left|\Phi_{vac}\right\rangle \\
&= \prod_{l\neq k}(u_l + v_l B_l^+)(u_k{}^2 a^+_{k\uparrow} + v_k{}^* v_k a^+_{k\uparrow})\left|\Phi_{vac}\right\rangle \\
&= a^+_{k\uparrow}\prod_{l\neq k}(u_l + v_l B_l^+)\left|\Phi_{vac}\right\rangle
\end{aligned}$$

since

$$a_{-k\downarrow} B_k^+ \left|\Phi_{vac}\right\rangle = -a^+_{k\uparrow}\left|\Phi_{vac}\right\rangle ,$$

and

$$u_k{}^2 + v_k{}^* v_k = 1 .$$

(ii)

$$\begin{aligned}
\alpha^+_{-k\downarrow}\left|BCS\right\rangle &= \prod_{l\neq k}(u_l + v_l B_l^+)(v_k{}^* a_{k\uparrow} + u_k a^+_{-k\downarrow})(u_k + v_k B_k^+)\left|\Phi_{vac}\right\rangle \\
&= \prod_{l\neq k}(u_l + v_l B_l^+)(u_k v_k{}^* a_{k\uparrow} + v_k v_k{}^* a_{k\uparrow} B_k^+ + u_k{}^2 a^+_{-k\downarrow} + u_k v_k a^+_{-k\downarrow} B_k^+\left|\Phi_{vac}\right\rangle \\
&= \prod_{l\neq k}(u_l + v_l B_l^+)(v_k v_k{}^* a^+_{-k\downarrow} + u_k{}^2 a^+_{-k\downarrow})\left|\Phi_{vac}\right\rangle \\
&= a^+_{-k\downarrow}\prod_{l\neq k}(u_l + v_l B_l^+)\left|\Phi_{vac}\right\rangle
\end{aligned}$$

(iii)

$$\alpha_{k\uparrow}^{+}\alpha_{-k\downarrow}^{+}\left|BCS\right\rangle = \prod_{l\neq k}(u_l + v_l B_l^{+})(u_k a_{k\uparrow}^{+} - v_k^{*} a_{-k\downarrow})(v_k^{*} a_{k\uparrow} + u_k a_{-k\downarrow}^{+})(u_k + v_k B_k^{+})\left|\Phi_{vac}\right\rangle$$
$$= \prod_{l\neq k}(u_l + v_l B_l^{+})(u_k a_{k\uparrow}^{+} - v_k^{*} a_{-k\downarrow}) a_{-k\downarrow}^{+}\left|\Phi_{vac}\right\rangle$$
$$= \prod_{l\neq k}(u_l + v_l B_l^{+})(u_k a_{k\uparrow}^{+} a_{-k\downarrow}^{+} - v_k^{*} a_{-k\downarrow} a_{-k\downarrow}^{+})\left|\Phi_{vac}\right\rangle$$
$$= (u_k B_k - v_k^{*})\prod_{l\neq k}(u_l + v_l B_l^{+})\left|\Phi_{vac}\right\rangle$$

From these, it is found that

ground state Cooper pair (***k***↑, -***k***↓): $(u_k + v_k B_k^{+})$.
the first excited state (***k***↑): $a_{k\uparrow}^{+}$.
the first excited state (-***k***↓): $a_{-k\downarrow}^{+}$.
the excited state (***k***↑) and (-***k***↓): $(u_k B_k - v_k^{*})$.

59. Condensation amplitude

$$\left\langle BCS\right| n_{k\uparrow}\left|BCS\right\rangle = \left\langle BCS\right| a_{k\uparrow}^{+} a_{k\uparrow}\left|BCS\right\rangle = v_k v_k^{*}.$$

((**Proof**))

$$\left\langle BCS\right| a_{k\uparrow}^{+} a_{k\uparrow}\left|BCS\right\rangle = \left\langle\Phi_{vac}\right| [\prod_{l\neq k}(u_l + v_l^{*} B_l)\prod_{l\neq k}(u_l + v_k B_l^{+})](u_k + v_k^{*} B_k) a_{k\uparrow}^{+} a_{k\uparrow}(u_k + v_k B_k^{+})\left|\Phi_{vac}\right\rangle$$
$$= \left\langle\Phi_{vac}\right| [\prod_{l\neq k}(u_l + v_l^{*} B_l)\prod_{l\neq k}(u_l + v_k B_l^{+})](u_k + v_k^{*} B_k) v_k a_{k\uparrow}^{+} a_{k\uparrow} B_k^{+})\left|\Phi_{vac}\right\rangle$$
$$= \left\langle\Phi_{vac}\right| [\prod_{l\neq k}(u_l + v_l^{*} B_l)\prod_{l\neq k}(u_l + v_k B_l^{+})](u_k + v_k^{*} B_k) v_k a_{k\uparrow}^{+} a_{-k\downarrow}^{+}\left|\Phi_{vac}\right\rangle$$
$$= \left\langle\Phi_{vac}\right| [\prod_{l\neq k}(u_l + v_l^{*} B_l)\prod_{l\neq k}(u_l + v_k B_l^{+})](u_k + v_k^{*} B_k) v_k B_k^{+}\left|\Phi_{vac}\right\rangle$$
$$= \left\langle\Phi_{vac}\right| [\prod_{l\neq k}(u_l + v_l^{*} B_l)\prod_{l\neq k}(u_l + v_k B_l^{+})](u_k v_k B_k^{+} + v_k^{*} v_k)\left|\Phi_{vac}\right\rangle$$
$$= v_k^{*} v_k\left\langle\Phi_{vac}\right| \prod_{l\neq k}(u_l + v_l^{*} B_l)(u_l + v_k B_l^{+}))\left|\Phi_{vac}\right\rangle$$
$$= v_k^{*} v_k = \left|v_k\right|^2$$

using

$$a_{k\uparrow} B_k^{+}\left|\Phi_{vac}\right\rangle = a_{-k\downarrow}^{+}\left|\Phi_{vac}\right\rangle, \qquad B_k B_k^{+}\left|\Phi_{vac}\right\rangle = \left|\Phi_{vac}\right\rangle.$$

Similarly, we have

$$
\begin{aligned}
\langle BCS|a_{k\downarrow}^{+}a_{k\downarrow}|BCS\rangle &= \langle \Phi_{vac}|[\prod_{l\neq k}(u_l + v_l{}^{*}B_l)\prod_{l\neq k}(u_l + v_k B_l^{+})](u_k + v_k{}^{*}B_k)a_{k\downarrow}^{+}a_{k\downarrow}(u_k + v_k B_k^{+})|\Phi_{vac}\rangle \\
&= \langle \Phi_{vac}|[\prod_{l\neq k}(u_l + v_l{}^{*}B_l)\prod_{l\neq k}(u_l + v_k B_l^{+})](u_k + v_k{}^{*}B_k)v_k a_{k\downarrow}^{+}a_{k\downarrow}B_k^{+})|\Phi_{vac}\rangle \\
&= \langle \Phi_{vac}|[\prod_{l\neq k}(u_l + v_l{}^{*}B_l)\prod_{l\neq k}(u_l + v_k B_l^{+})](u_k + v_k{}^{*}B_k)v_k a_{k\uparrow}^{+}a_{-k\downarrow}^{+}|\Phi_{vac}\rangle \\
&= \langle \Phi_{vac}|[\prod_{l\neq k}(u_l + v_l{}^{*}B_l)\prod_{l\neq k}(u_l + v_k B_l^{+})](u_k + v_k{}^{*}B_k)v_k B_k^{+}|\Phi_{vac}\rangle \\
&= \langle \Phi_{vac}|[\prod_{l\neq k}(u_l + v_l{}^{*}B_l)\prod_{l\neq k}(u_l + v_k B_l^{+})](u_k v_k B_k^{+} + v_k{}^{*}v_k)|\Phi_{vac}\rangle \\
&= v_k{}^{*}v_k\langle \Phi_{vac}|\prod_{l\neq k}(u_l + v_l{}^{*}B_l)(u_l + v_k B_l^{+})|\Phi_{vac}\rangle \\
&= v_k{}^{*}v_k = |v_k|^2
\end{aligned}
$$

using

$$a_{k\downarrow}B_k^{+}|\Phi_{vac}\rangle = a_{-k\downarrow}^{+}|\Phi_{vac}\rangle .$$

F_k is the condensation amplitude in the state ***k***.

$$F_k = \langle BCS|B_k^{+}|BCS\rangle .$$

$$
\begin{aligned}
F_k &= \langle BCS|a_{k\uparrow}^{+}a_{-k\downarrow}^{+}|BCS\rangle \\
&= \langle \Phi_{vac}|[\prod_{l\neq k}(u_l + v_l{}^{*}B_l)\prod_{l\neq k}(u_l + v_k B_l^{+})](u_k + v_k{}^{*}B_k)B_k^{+}(u_k + v_k B_k^{+})|\Phi_{vac}\rangle \\
&= \langle \Phi_{vac}|[\prod_{l\neq k}(u_l + v_l{}^{*}B_l)\prod_{l\neq k}(u_l + v_k B_l^{+})](u_k + v_k{}^{*}B_k)(u_k B_k^{+} + v_k B_k^{+}B_k^{+})|\Phi_{vac}\rangle \\
&= \langle \Phi_{vac}|[\prod_{l\neq k}(u_l + v_l{}^{*}B_l)\prod_{l\neq k}(u_l + v_k B_l^{+})](u_k{}^{2}B_k^{+} + u_k v_k{}^{*}B_k B_k^{+} + u_k v_k B_k^{+}B_k^{+} \\
&\quad + v_k{}^{*}v_k B_k B_k^{+}B_k^{+})|\Phi_{vac}\rangle \\
&= u_k v_k{}^{*}
\end{aligned}
$$

since

$$[B_k, B_{k'}]_{-} = 0 .$$

$$[B_k, B_{k'}]_{+} = 2B_k B_{k'}(1-\delta_{k,k'}) .$$

$$B_k B_k^{+}|\Phi_{vac}\rangle = |\Phi_{vac}\rangle .$$

$$[B_k, B_{k'}^{+}]_{-} = (1 - n_{k\uparrow} - n_{-k\downarrow})\delta_{k,k'} .$$

$$
\begin{aligned}
\langle BCS|B_k|BCS\rangle &= \langle BCS|a_{-k\downarrow}a_{k\uparrow}|BCS\rangle \\
&= \langle \Phi_{vac}|[\prod_{l\neq k}(u_l + v_l^* B_l)\prod_{l\neq k}(u_l + v_k B_l^+)](u_k + v_k^* B_k)B_k(u_k + v_k B_k^+)|\Phi_{vac}\rangle \\
&= \langle \Phi_{vac}|[\prod_{l\neq k}(u_l + v_l^* B_l)\prod_{l\neq k}(u_l + v_k B_l^+)](u_k + v_k^* B_k)(u_k B_k + v_k B_k B_k^+)|\Phi_{vac}\rangle \\
&= \langle \Phi_{vac}|[\prod_{l\neq k}(u_l + v_l^* B_l)\prod_{l\neq k}(u_l + v_k B_l^+)](u_k^2 B_k + u_k v_k^* B_k^2 + u_k v_k B_k B_k^+ \\
&\quad + v_k^* v_k B_k B_k B_k^+)|\Phi_{vac}\rangle \\
&= u_k v_k
\end{aligned}
$$

---

60. The number operator:

The number operator is define by

$$
N = \sum_{k,\sigma} a_{k\sigma}^+ a_{k\sigma} = \sum_k (a_{k\uparrow}^+ a_{k\uparrow} + a_{k\downarrow}^+ a_{k\downarrow}) = \sum_k (n_{k\uparrow} + n_{k\downarrow}) .
$$

The average of N for the BCS state is

$$
\langle N\rangle = \langle BCS|N|BCS\rangle = 2\sum_k |v_k|^2 .
$$

$$
\langle N^2\rangle = \langle BCS|N^2|BCS\rangle = (2\sum_k |v_k|^2)^2 + \sum_k 4u_k^2 |v_k|^2 .
$$

The fluctuation is given by

$$
(\Delta N)^2 = \langle N^2\rangle - \langle N\rangle^2 = \sum_k 4u_k^2 |v_k|^2 .
$$

Note that

$$
\langle BCS|a_{k\uparrow}^+ a_{k\uparrow} a_{k'\downarrow}^+ a_{k'\downarrow}|BCS\rangle = |v_k|^2 \delta_{k,k'} + |v_k|^2 |v_{k'}|^2 (1 - \delta_{k,k'}) .
$$

The quantities u_k and $|v_k|$ will typically be numbers of order 1, so since the numbers of allowed ***k***-states appearing in the k sums scale with the volume Ω of the system, we have

$$
\langle N\rangle \approx \Omega, \qquad \text{and} \qquad (\Delta N)^2 \approx \Omega .
$$

Therefore the width of the distribution of numbers in the BCS state is

$$
\frac{\Delta N}{< N >} \approx \frac{\sqrt{\Omega}}{\Omega} = \frac{1}{\sqrt{\Omega}} = \frac{1}{\sqrt{\Omega}} .
$$

As $N \approx 10^{23}$ particles, this relative error implied by the number non-conservation in the BCS state becomes negligible.

---

61. Relation between $|N\rangle$ state and $|BCS\rangle$ state.

Here we assume that

$$h_k = \frac{|v_k|}{u_k} \qquad v_k = |v_k| e^{i\theta},$$

Then we get

$$|BCS(\theta)\rangle = \prod_k u_k \prod_k (1 + h_k e^{i\theta} B_k^+)|\Phi_{vac}\rangle,$$

or

$$|BCS(\theta)\rangle \approx \{1 + \frac{1}{1!} e^{i\theta} \sum_k h_k B_k^+ + \frac{1}{2} e^{i2\theta} \left(\sum_k h_k B_k^+ \right)^2 \}|\Phi_{vac}\rangle,$$

where θ is the phase of the order parameter. This indicates that the BCS state can be expressed as the superposition of $|N\rangle$ as

$$|BCS(\theta)\rangle = \sum_{N=0}^{\infty} \sqrt{A_N} \exp(\frac{i\theta}{2} N)|N\rangle,$$

where $|N\rangle$ is the state having the N particles and $\langle N|N'\rangle = \delta_{N,N'}$.

$$\begin{aligned}
\langle N|BCS(\theta)\rangle &= \langle N|\sum_{N'=0}^{\infty} \sqrt{A_{N'}} \exp(\frac{i\theta}{2} N')|N'\rangle \\
&= \sum_{N'=0}^{\infty} \sqrt{A_{N'}} \exp(\frac{i\theta}{2} N')\langle N|N'\rangle \\
&= \sqrt{A_N} \exp(\frac{i\theta}{2} N)
\end{aligned}$$

or

$$\sqrt{A_N} = \exp(-\frac{i\theta}{2} N)\langle N|BCS(\theta)\rangle.$$

We also have the expression

$$\left|N\right\rangle = \frac{1}{\sqrt{A_N}} \int_0^{2\pi} \frac{d\theta}{2\pi} \exp(-\frac{i\theta}{2} N) \left|BCS(\theta)\right\rangle .$$

A_N is the probability for finding the system in the $\left|N\right\rangle$. A_N may be expressed by a Gaussian distribution given by

$$A_N = \frac{1}{\sqrt{2\pi}\Delta N} \exp[-\frac{(N-<N>)^2}{2(\Delta N)^2},$$

where

$$\left\langle N\right\rangle = N^* = 2\sum_k \left|v_k\right|^2, \quad (\Delta N)^2 = \sum_k 4u_k{}^2 \left|v_k\right|^2 .$$

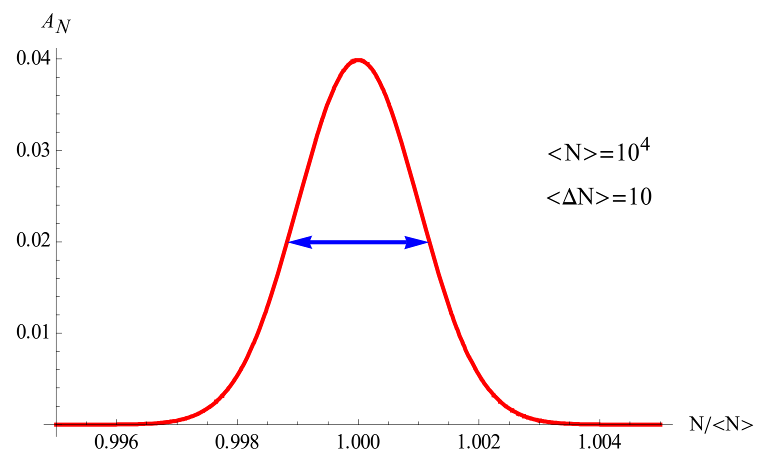


Fig. A_N which has a Gaussian distribution with $<N> = N^* = 10^4$ and $\Delta N = 10$. The blue arrow is the full-width at half maximum. In this figure, $\delta(N/<N>) = 2.355 \times 10^{-3}$.

For an arbitrary operator F, the matrix element is given by

$$\begin{aligned}\left\langle BCS(\theta)\right|F\left|BCS(\theta)\right\rangle &= \sum_{N,N'} \left\langle BCS(\theta)\middle|N\right\rangle\left\langle N\right|F\left|N'\right\rangle\left\langle N'\middle|BCS(\theta)\right\rangle \\ &= \sum_{N,N'} \sqrt{A_N} \exp(-\frac{i\theta}{2}N)\left\langle N\right|F\left|N'\right\rangle\sqrt{A_N'} \exp(-\frac{i\theta}{2}N') \\ &= \sum_{N,N'} \sqrt{A_N}\sqrt{A_N'} \exp[-\frac{i\theta}{2}(N-N')]\left\langle N\right|F\left|N'\right\rangle\end{aligned}$$

where

$$\langle N | BCS(\theta)\rangle = \sqrt{A_N}\exp(\frac{i\theta}{2}N), \qquad \langle BCS(\theta)| N\rangle = \sqrt{A_N}\exp(-\frac{i\theta}{2}N).$$

(a) Suppose that F conserved the number of particles'

$$\begin{aligned}\langle BCS(\theta)|F|BCS(\theta)\rangle &= \sum_{N,N'}\sqrt{A_N}\sqrt{A_N}'\exp[-\frac{i\theta}{2}(N'-N)]\langle N'|F|N\rangle \\ &= \sum_N A_N\langle N|F|N\rangle \\ &= \langle N^*|F|N^*\rangle\sum_N A_N \\ &= \langle N^*|F|N^*\rangle\end{aligned}$$

(b) If F acting on $|N\rangle$ gives a state of $N+p$ particles.

$$\begin{aligned}\langle BCS(\theta)|F|BCS(\theta)\rangle &= \sum_{N,N'}\sqrt{A_N}\sqrt{A_N}'\exp[-\frac{i\theta}{2}(N'-N)]\langle N'|F|N\rangle \\ &= \sum_N A_N\exp(-\frac{ip\theta}{2})\langle N+p|F|N\rangle \\ &= \exp(-\frac{ip\theta}{2})\langle N^*+p|F|N^*\rangle\sum_N A_N \\ &= \exp(-\frac{ip\theta}{2})\langle N^*+p|F|N^*\rangle\end{aligned}$$

These expectation values do not vanish in the state $|BCS(\theta)\rangle$. It is clear that this state is much simpler than $|N\rangle$, and, moreover, manifestly shows essential features of superconductivity.

62. BCS Theory

We start the so-called the BCS Hamiltonian

$$H-\mu N = \sum_{k,\sigma}\xi_k a_{k,\sigma}^+ a_{k,\sigma} + \sum_{k,l}V_{kl}a_{k\uparrow}^+ a_{-k\downarrow}^+ a_{-l\downarrow}a_{l\uparrow}.$$

We assume that u_k and v_k are real.

$$W_{BCS} = \langle BCS|\hat{H}-\mu\hat{N}|BCS\rangle = 2\sum_k \xi_k v_k^{\ 2} + \sum_{k,l}V_{kl}u_k v_k u_l v_l,$$

which is to be minimized subject to the constant that

$$u_k{}^2 + v_k{}^2 = 1 .$$

It is useful to put

$$u_k = \sin\theta_k , \qquad v_k = \cos\theta_k ,$$

$$W_{BCS} = \langle BCS|\hat{H} - \mu\hat{N}|BCS\rangle = 2\sum_k \xi_k \cos\theta_k{}^2 + \frac{1}{4}\sum_{k,l} V_{kl} \sin(2\theta_k)\sin(2\theta_l) .$$

Then we get

$$\frac{\partial}{\partial\theta_k}\langle BCS|\hat{H} - \mu\hat{N}|BCS\rangle = -2\sum_k \xi_k \sin(2\theta_k) + \frac{1}{2}\sum_{k,l} V_{kl} \cos(2\theta_k)\sin(2\theta_l) = 0 ,$$

or

$$\xi_k \tan(2\theta_k) = \frac{1}{2}\sum_l V_{kl} \sin(2\theta_l) .$$

We define the energy gap equation by

$$\Delta_k = -\sum_l V_{kl} \sin\theta_l \cos\theta_l = -\sum_l V_{kl} v_l u_l = -\frac{1}{2}\sum_l V_{kl} \sin(2\theta_l) .$$

Thus we have

$$\xi_k \tan(2\theta_k) = -\Delta_k .$$

We find

$$\cos(2\theta_k) = -\frac{\xi_k}{E_k} ,$$

$$\sin(2\theta_k) = \frac{\Delta_k}{E_k} ,$$

where

$$E_k = \sqrt{\xi_k{}^2 + \Delta_k{}^2} \ .$$

The gap energy is determined from the gap equation

$$\Delta_k = -\sum_l V_{kl} \frac{\Delta_l}{2E_l} = -\sum_l V_{kl} \frac{\Delta_l}{2\sqrt{\xi_l{}^2 + \Delta_l{}^2}} \ .$$

We choose the simplified interaction (BCS interaction),

$V_{kl} = -V$ if $|\xi_k| \le \hbar\omega_D$ and $|\xi_l| \le \hbar\omega_D$,

$V_{kl} = 0$ otherwise.

and

$\Delta_k = \Delta$ (independent of k) if $|\xi_k| \le \hbar\omega_D$,

$\Delta_k = 0$ otherwise.

Then we have

$$\Delta = VN(0) \int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{\Delta d\xi}{2\sqrt{\xi^2 + \Delta^2}} ,$$

or

$$\frac{1}{VN(0)} = \int_0^{\hbar\omega_D} \frac{d\xi}{\sqrt{\xi^2 + \Delta^2}} = \sinh^{-1}(\frac{\hbar\omega_D}{\Delta}) .$$

This equation only has a solution for positive V,

$$\Delta = \frac{\hbar\omega_D}{\sinh[\frac{1}{VN(0)}]} \approx 2\hbar\omega_D \exp[-\frac{1}{VN(0)}] .$$

where $N(0)$ is the density of states per spin at the Fermi level.

---

63. Summary

v_k^2: probability that the Cooper pair ($\boldsymbol{k}\uparrow$, $-\boldsymbol{k}\downarrow$) is occupied.
u_k^2: probability that the Cooper pair ($\boldsymbol{k}\uparrow$, $-k\downarrow$) is unoccupied.

where

$$u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 = 1,$$

$$u_{\mathbf{k}}^2 = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right), \qquad v_{\mathbf{k}}^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right),$$

where E_k is the excitation energy and is defined by

$$E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}$$

$F_{\mathbf{k}}$ is the condensation amplitude and is defined by

$$F_{\mathbf{k}} = u_{\mathbf{k}} v_{\mathbf{k}} = \frac{\Delta}{2E_{\mathbf{k}}}.$$

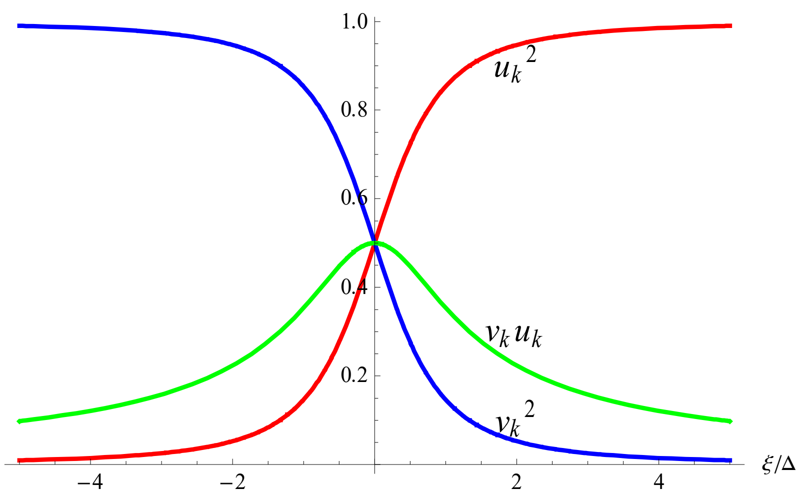


Fig. Plot of u_k^2, v_k^2, and $F_k = u_k v_k$ as a function of ξ/Δ. We assume that v_k is real for convenience.

There is an energy gap between the ground state and the excited state. The excitations consists of the breaking of pairs. The excitation energy is the sum of E_k for each excited electrons in each pair. The superconducting state is a condensed state in the sense that a finite energy Δ is required to produce an excited state of the whole system.

64. Thermodynamic critical field

If we make the substitutions

$$2u_k v_k = \frac{\Delta}{E_k}, \qquad v_k^{\ 2} = \frac{1}{2}(1 - \frac{\xi_k}{E_k})$$

into

$$W_{BCS} = 2\sum_k \xi_k v_k^{\ 2} + \sum_{k,l} V_{kl} u_k v_k u_l v_l ,$$

then we get

$$W_{BCS} = \sum_k \xi_k (1 - \frac{\xi_k}{E_k}) + \frac{1}{4}\sum_{k,k'} V_{k,k'} \frac{\Delta_k}{E_k}\frac{\Delta_{k'}}{E_{k'}},$$

or

$$W_{BCS} = (E_0 + \frac{\Delta^2}{V} - \frac{1}{2}N(0)\Delta^2) - \frac{\Delta^2}{V} = E_0 - \frac{1}{2}N(0)\Delta^2 .$$

Comparing this to the normal state energy ($\Delta = 0$), measured relative to ε_F,

$$E_0 = \sum_{k<k_F} 2\xi_k .$$

we get

$$W_{BCS} - E_0 = -\frac{1}{2}N(0)\Delta^2 < 0.$$

This indicates that

$$\frac{1}{2}N(0)\Delta^2 = \frac{[H_c(0)]^2}{8\pi}.$$

65. Mean field approximation at finite temperatures

At finite temperatures, quasi-particles are thermally excited. We need to take into account of the effect in the mean field theory. To this end, the average over the BCS ground state is replaced by the average in the thermal equilibrium, such that

$$\langle BCS | a_{k\uparrow}^{+} a_{-k'\downarrow} | BCS \rangle \to \qquad \langle a_{k\uparrow}^{+} a_{-k'\downarrow} \rangle .$$

$$\langle BCS | a_{k\uparrow}^{+} a_{k\uparrow} | BCS \rangle \to \qquad \langle a_{k\uparrow}^{+} a_{k\uparrow} \rangle .$$

Here we use the Bogoliubov transformation given by

$$\begin{pmatrix} a_{k\uparrow} \\ a_{-k\downarrow}^{+} \end{pmatrix} = \begin{pmatrix} u_k & v_k \\ -v_k^{*} & u_k \end{pmatrix} \begin{pmatrix} \alpha_{k\uparrow} \\ \alpha_{-k\downarrow}^{+} \end{pmatrix},$$

The Fermi-Dirac distribution function is given by

$$<\alpha_{k\downarrow}{}^{+} \alpha_{k\uparrow} >=< \alpha_{-k\downarrow}{}^{+} \alpha_{k\downarrow} >= f(E_k) = \frac{1}{\exp(\beta E_k) + 1},$$

where < > is the average in the thermal equilibrium at the finite temperature. We also note that

$$<\alpha_{k\downarrow} \alpha_{k\uparrow} >=< \alpha_{-k\downarrow} \alpha_{k\downarrow} >=< \alpha_{k\downarrow}^{+} \alpha_{k\uparrow}^{+} >=< \alpha_{-k\downarrow}^{+} \alpha_{k\uparrow}^{+} >= ... = 0 ,$$

since the number of particles are not conserved. Based on these rules, we can calculate the following quantities.

(1)

$$\begin{aligned} \langle a_{k\uparrow} a_{-k'\downarrow} \rangle &=< (u_k \alpha_{k\uparrow} + v_k \alpha_{-k\downarrow}^{+})(-v_{k'} \alpha_{k'\uparrow}^{+} + u_{k'} \alpha_{-k'\downarrow}) > \\ &= -u_k v_{k'} < \alpha_{k\uparrow} \alpha_{k'\uparrow}^{+} > + u_{k'} v_k < \alpha_{-k\downarrow}^{+} \alpha_{-k'\downarrow} > \\ &= u_k v_k [- < \alpha_{k\uparrow} \alpha_{k\uparrow}^{+} > + < \alpha_{-k\downarrow}^{+} \alpha_{-k\downarrow} >] \delta_{k,k'} \\ &= -[1 - 2 f(E_k)] u_k v_k \delta_{k,k'} \end{aligned}$$

and

$$\left\langle a_{-k\downarrow} a_{k'\uparrow} \right\rangle = [1 - 2f(E_k)] u_k v_k \delta_{k,k'}$$

(2)

$$\begin{aligned}
\left\langle a_{k\uparrow}^{+} a_{k\uparrow} \right\rangle &= < (u_k \alpha_{k\uparrow}^{+} + v_k^{*} \alpha_{-k\downarrow})(u_k \alpha_{k\uparrow} + v_k \alpha_{-k\downarrow}^{+}) > \\
&= u_k^{2} < \alpha_{k\uparrow}^{+} \alpha_{k\uparrow} > + u_k v_k^{*} < \alpha_{-k\downarrow} \alpha_{k\uparrow} > + u_k v_k \left\langle \alpha_{k\downarrow}^{+} \alpha_{-k\uparrow}^{+} \right\rangle + v_k v_k^{*} \left\langle \alpha_{-k\downarrow} \alpha_{-k\downarrow}^{+} \right\rangle \\
&= u_k^{2} < \alpha_{k\uparrow}^{+} \alpha_{k\uparrow} > + |v_k|^2 [1 - \left\langle \alpha_{-k\downarrow}^{+} \alpha_{-k\downarrow} \right\rangle] \\
&= u_k^{2} f(E_k) + |v_k|^2 (1 - f(E_k) \\
&= |v_k|^2 + (u_k^{2} - |v_k|^2) f(E_k)
\end{aligned}$$

(3)

$$\begin{aligned}
\left\langle a_{k\uparrow}^{+} a_{-k\downarrow}^{+} \right\rangle &= < (u_k \alpha_{k\uparrow}^{+} + v_k^{*} \alpha_{-k\downarrow})(-v_k^{*} \alpha_{k\uparrow} + u_k \alpha_{-k\downarrow}^{+}) > \\
&= -u_k v_k^{*} < \alpha_{k\uparrow}^{+} \alpha_{k\uparrow} > + u_k v_k^{*} < \alpha_{-k\downarrow} \alpha_{-k\downarrow}^{+} > \\
&= -u_k v_k^{*} < \alpha_{k\uparrow}^{+} \alpha_{k\uparrow} > + u_k v_k^{*} [1 - \left\langle \alpha_{-k\downarrow}^{+} \alpha_{-k\downarrow} \right\rangle] \\
&= u_k v_k^{*} (1 - 2f(E_k)
\end{aligned}$$

66. Fermi-Dirac function

The Fermi-Dirac distribution function is defined by

$$f(E) = \frac{1}{\exp(\beta E) + 1} = \frac{1}{\exp(\beta\sqrt{\xi^2 + \Delta^2}) + 1} = \frac{1}{\exp(\beta\Delta\sqrt{\frac{\xi^2}{\Delta^2} + 1}) + 1},$$

where

$$\beta\Delta = \frac{3.5277 k_B T_c}{2 k_B T} = 1.76385 \frac{T_c}{T},$$

and

$$2\Delta = 3.5277 k_B T_c. \qquad \text{(BCS prediction)}$$

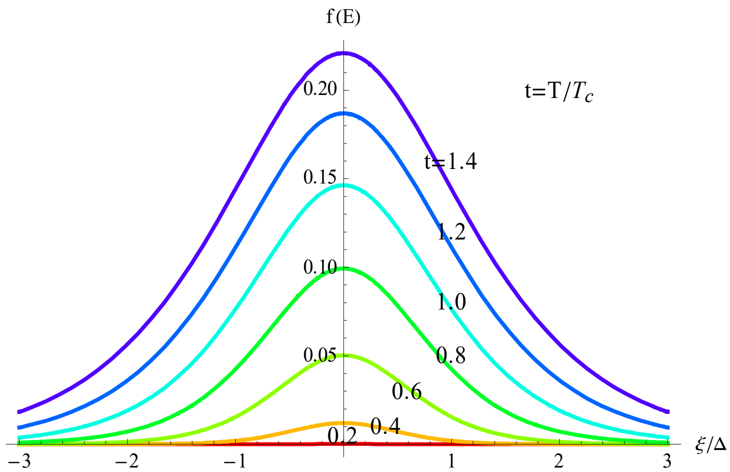


Fig. Plot of $f(E)$ as a function of ξ/Δ. t is the reduced temperature; $t = T/T_c$. t = 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, and 1.4.

This function is an unusual Fermi function. The denominator features $\exp(\beta E)$, rather than the conventional form $\exp[\beta(\varepsilon-\varepsilon_F)]$. Because of $E>0$, $f(E)$ reaches its maximum at the Fermi level, and it decays to zero when k lies either above or below. We also have the relation

$$1-f(E)=1-\frac{1}{\exp(\beta E)+1}=\frac{\exp(\beta E)}{\exp(\beta E)+1}=\frac{1}{\exp(-\beta E)+1},$$

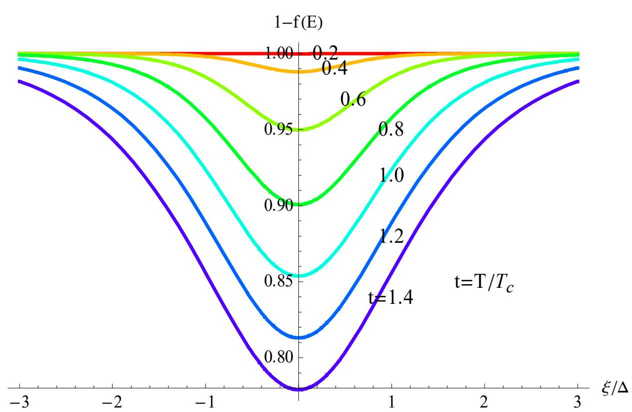


Fig. Plot of 1-$f(E)$ as a function of ξ/Δ. t is the reduced temperature; $t = T/T_c$. t = 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, and 1.4.

67. Order parameter: coherence length and phase

The wavefunction is defined as

$$\varphi_\uparrow(\mathbf{r}) = \frac{1}{\sqrt{\Omega}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} a_{\mathbf{k}\uparrow}\ , \qquad \varphi_\downarrow(\mathbf{r}) = \frac{1}{\sqrt{\Omega}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} a_{\mathbf{k}\downarrow}$$

where Ω is the volume of the system. Then the order parameter of the superconducting phase is defined by the form of correlation of the wavefunctions at ***r*** and ***r***'.

$$\begin{aligned}\psi(\mathbf{r}-\mathbf{r}') &= \langle \varphi_\downarrow(\mathbf{r})\varphi_\uparrow(\mathbf{r}')\rangle \\ &= \frac{1}{\Omega}\sum_{\mathbf{k},\mathbf{k}'} e^{i(\mathbf{k}\cdot\mathbf{r}-\mathbf{k}'\cdot\mathbf{r}')}\langle a_{\mathbf{k}\downarrow}a_{-\mathbf{k}'\uparrow}\rangle \\ &= \frac{1}{2\Omega}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\frac{\Delta_k}{E_k}\tanh(\frac{\beta E_k}{2})\end{aligned}$$

where we use the above formula

$$\begin{aligned}\langle a_{\mathbf{k}\downarrow}a_{-\mathbf{k}'\uparrow}\rangle &= u_{-\mathbf{k}}v_{-\mathbf{k}}[1-2f(E_{-\mathbf{k}})]\delta_{\mathbf{k},\mathbf{k}'} \\ &= u_{\mathbf{k}}v_{\mathbf{k}}[1-2f(E_{\mathbf{k}})]\delta_{\mathbf{k},\mathbf{k}'} \\ &= \frac{\Delta_{\mathbf{k}}}{2E_{\mathbf{k}}}\tanh(\frac{\beta E_{\mathbf{k}}}{2})\delta_{\mathbf{k},\mathbf{k}'}\end{aligned}$$

When $T\to 0$, $f(E_{\mathbf{k}})=0$, and

$$\langle a_{\mathbf{k}\downarrow}a_{-\mathbf{k}\uparrow}\rangle = \langle BCS|a_{\mathbf{k}\downarrow}a_{-\mathbf{k}\uparrow}|BCS\rangle = u_{-\mathbf{k}}v_{-\mathbf{k}} = u_{\mathbf{k}}v_{\mathbf{k}}$$

since $u_{-\mathbf{k}} = u_{\mathbf{k}}$. Then the order parameter $\psi(\mathbf{r})$ at $T = 0$ can be evaluated as

$$\psi(\mathbf{r}) = \frac{1}{\Omega}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}u_{\mathbf{k}}v_{\mathbf{k}} = \frac{1}{2\Omega}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}\frac{\Delta_{\mathbf{k}}}{E_{\mathbf{k}}} \approx \frac{\sin(k_F r)}{k_F r}K_0(\frac{r}{\pi\xi_0})\quad , \qquad (1)$$

where $K_0(x)$ is the modified Bessel function of the second kind. ξ_0 is the coherence length and is defined by

$$\xi_0 = \frac{\hbar v_F}{\pi\Delta_0} = \frac{\hbar v_F}{\pi(1.763 k_B T_c)}$$

Note that in the limit of $x\to\infty$,

$$K_0(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x}.$$

The form of $\psi(\mathbf{r})$ shows that the coherence length ξ_0 represents the spatial extension of the Cooper pair.

68. Phase of the order parameter

The order parameter at the finite temperature can be expressed by

$$\begin{aligned}\psi(\mathbf{r}) &= \langle \psi_\downarrow(\mathbf{r})\psi_\uparrow(0)\rangle \\ &= \frac{1}{\Omega}\sum_{\mathbf{k}} e^{\mathbf{k}\cdot\mathbf{r}} \langle a_{-\mathbf{k}\downarrow} a_{\mathbf{k}\uparrow}\rangle\end{aligned}$$

The phase of the order parameter comes from the phase of $v_{\mathbf{k}}$ ($v_k = |v_k| e^{i\theta}$) as

$$\begin{aligned}\langle a_{-\mathbf{k}\downarrow} a_{\mathbf{k}\uparrow}\rangle &= u_{\mathbf{k}} v_{\mathbf{k}} [1 - 2f(E_{\mathbf{k}})] \\ &= \frac{|\Delta_{\mathbf{k}}|}{2E_{\mathbf{k}}} e^{i\theta} \tanh(\frac{\beta E_{\mathbf{k}}}{2})\end{aligned}$$

Then the order parameter can be rewritten as

$$\psi(\mathbf{r}) = e^{i\theta}|\psi(\mathbf{r})| = \frac{1}{\Omega} e^{i\theta} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{|\Delta_{\mathbf{k}}|}{2E_{\mathbf{k}}} \tanh(\frac{\beta E_{\mathbf{k}}}{2}).$$

The parameter θ is the phase of the order parameter. The order parameter has a wave-like nature. Such character is found to appear in the Josephson effect.

---

69 Coherence length of Al and Sn

(1) Al (type-I superconductor)

k_F = 1.75 x 10^8/cm.
v_F = 2.02 x 10^8 cm/s.
ε_F = 11.63 eV.
H_c = 105 Oe.
$2\Delta_0$= 0.340 meV = 3.94553 K.
Δ_0= 1.973 K.

T_c = 1.140 K.

2 $\Delta_0/k_B T_c$ = 3.461.

$$\xi_0 = \frac{\hbar v_F}{\pi \Delta_0} = 2489.5 \text{ nm.}$$

((**Experiment**)) Al

Coherence length: ξ_0 = 1600 nm.

London penetration depth λ_L = 16 nm.

(2) Sn (type-I superconductor)

$k_F = 1.62 \times 10^8$/cm.

$v_F = 1.88 \times 10^8$ cm/s.

ε_F = 10.03 eV.

H_c = 309 Oe.

2 Δ_0 = 1.15 meV = 13.345 K.

Δ_0 = 6.6726 K.

T_c = 3.722 K.

2 $\Delta_0/k_B T_c$ = 3.585.

$$\xi_0 = \frac{\hbar v_F}{\pi \Delta_0} = 685.0 \text{ nm} = 6850 \text{ Å.}$$

((**Experimental values**)) Sn

Coherence length: ξ_0 = 230 nm.

London penetration depth λ_L = 34 nm.

---

70. Approach from the Heisenberg's principle of uncertainty

In the BCS theory, the Cooper pairs are formed from electrons having the energy close to the Fermi energy. Using the Heisenberg's principle uncertainty, we have

$$\Delta p \Delta x \approx \hbar \text{ .}$$

Since

$$\Delta\varepsilon = v_F \Delta p = \Delta_0 \qquad \text{or} \qquad \Delta p = \frac{\Delta_0}{v_F},$$

Δx can be evaluated as

$$\Delta x \approx \xi \approx \frac{\hbar}{\Delta p} = \frac{\hbar v_F}{\Delta_0},$$

where v_F is the Fermi velocity. Note that the average distance between electrons d is approximated by

$$d \approx \frac{\hbar v_F}{\varepsilon_F}.$$

since

$$k_F \approx \frac{1}{d}, \qquad \text{and} \qquad \varepsilon_F = \frac{\hbar^2 k_F{}^2}{2m} = \hbar \frac{\hbar k_F}{m} k_F = \hbar v_F k_F .$$

When $\varepsilon_F >> \Delta$, we have

$$\xi >> d .$$

71. Character in charge of quasi-particles

The charge density is defined by

$$\begin{aligned} Q &= -e \sum_{k,\sigma} \left\langle a_{k\sigma}^{+} a_{k\sigma} \right\rangle \\ &= -2e \sum_{k} \{ |v_k|^2 + (u_k{}^2 - |v_k|^2) f(E_k)] \} \\ &= Q_s + Q_{qp} \end{aligned}$$

Here Q_s is the contribution of the condensate of the Cooper pairs, and Q_{qp} is the contribution of the quasi-particle excitations,

$$Q_s = -2e \sum_k |v_k|^2 ,$$

and

$$Q_{qp} = -2e\sum_k (u_k{}^2 - |v_k|^2) f(E_k)].$$

Here we use the formula

$$\langle a_{k\uparrow}^+ a_{k\uparrow} \rangle = |v_k|^2 + (u_k{}^2 - |v_k|^2) f(E_k).$$

We make a plot of

$$h_k = -(u_k{}^2 - |v_k|^2) f(E_k),$$

It s found that the sign of h_k changes from positive (hole-like for $\xi<0$) to negative (electron-like for $\xi>0$). This means that the quasi-particles behave like an electron for $\xi>0$ and like a hole for $\xi<0$.

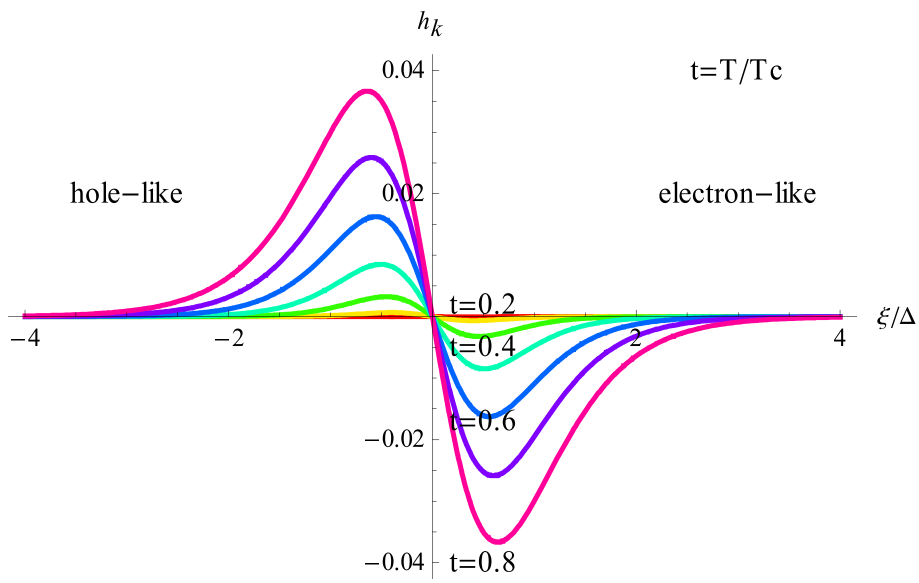


Fig. Plot of $h_k = -(u_k{}^2 - |v_k|^2) f(E_k)$, corresponding to the charge of quasiparticle, as a function of ξ/Δ. $t = T/T_c$. t is changed as a parameter.

72. Condensation amplitude at finite temperature

The condensation amplitude at finite temperature is given by

$$F = \langle a_{k\uparrow}^{+} a_{-k\downarrow}^{+} \rangle = u_k v_k^{*}[1 - 2f(E_k)].$$

Here we assume that v_k is real. Then we have

$$F_k = u_k v_k [1 - 2f(E_k)] = \frac{\Delta_k}{2E_k}[1 - 2f(E_k)].$$

We make a plot of F_k as a function of $\xi_k/\Delta_k = \xi/\Delta$, where $t = T/T_c$ is changed as a parameter.

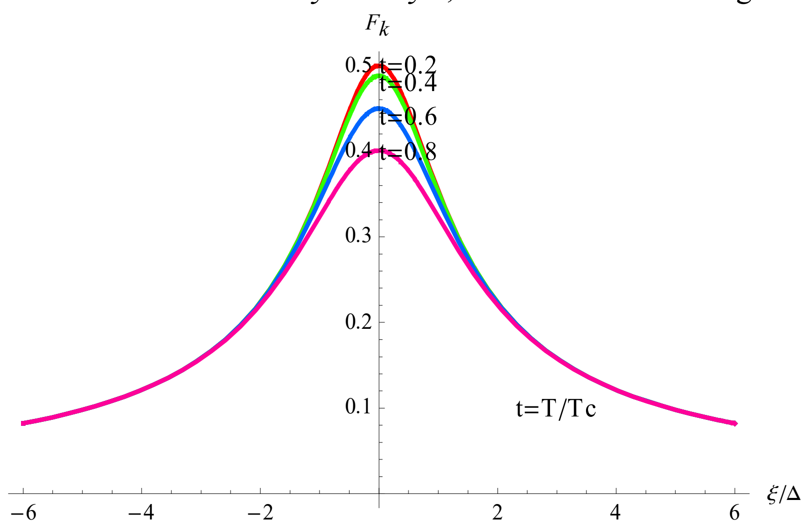


Fig. Plot of the condensation amplitude F as a function of ξ/Δ. ξ/Δ. t is the reduced temperature; $t = T/T_c$. t = 0.2, 0.4, 0.6, and 0.8. We assume that the energy gap is independent of t.

73. Specific heat

The electronic entropy in the superconducting phase is given by

$$S_{es} = -2k_B \int_0^{\infty} dEN(E)\{f(E)\ln f(E) + [1 - f(E)]\ln[1 - f(E)]\},$$

where $f(E)$ is the Fermi Dirac function and the factor 2 comes from the degree of freedom in spin.

$$f(E) = \frac{1}{\exp(\beta E) + 1}.$$

Then the specific heat is calculated as

$$C_{es} = T\frac{\partial S}{\partial T} = \frac{2}{T}\int dEN(E)(E^2 - \frac{T}{2}\frac{d\Delta^2}{dT})[-\frac{\partial f(E)}{\partial E}].$$

(i) At low temperatures ($T<<T_c$), where Δ is independent of T.

$$f(E) \approx \exp(-\beta E).$$

The density of states is defined as

$$N(E) = N(0)\frac{E}{\sqrt{E^2 - \Delta_0{}^2}}.$$

Then we have

$$\begin{aligned}
C_{es} &= \frac{2}{T}\int_{\Delta_0}^{\infty} dEN(E)E^2[-\frac{\partial f(E)}{\partial E}] \\
&= \frac{2N(0)}{T}\frac{1}{k_B T}\int_{\Delta_0}^{\infty}\frac{dEE^3 e^{\beta E}}{(1+e^{\beta E})^2\sqrt{E^2-\Delta_0{}^2}} \\
&\approx \frac{2N(0)}{T}\frac{1}{k_B T}\int_{\Delta_0}^{\infty}\frac{dEE^3 e^{-\beta E}}{\sqrt{E^2-\Delta_0{}^2}} \\
&= \frac{2N(0)\Delta_0{}^3}{k_B T^2}[K_1(\beta\Delta_0) + \frac{1}{\beta\Delta_0}K_2(\beta\Delta_0)] \\
&\approx \frac{2N(0)\Delta_0{}^3}{k_B T^2}K_1(\beta\Delta_0)
\end{aligned}$$

in the limit of $\beta\Delta_0\to\infty$, where $K_n(x)$ is the modified Bessel function of the second kind and has a symptotic form of

$$K_n(x) \approx \frac{e^{-x}}{\sqrt{\frac{2x}{\pi}}}$$

which is independent if n. Then we have

$$C_{es} = \sqrt{2\pi}N(0)k_B\Delta_0\left(\frac{\Delta_0}{k_B T}\right)^{3/2}\exp(-\frac{\Delta_0}{k_B T})$$

and

$$\frac{C_{es}}{\gamma T_c} = \frac{\sqrt{2\pi} N(0) k_B \Delta_0}{\frac{2}{3}\pi^2 k_B{}^2 N(0) T_c} \left(\frac{\Delta_0}{k_B T}\right)^{3/2} \exp(-\frac{\Delta_0}{k_B T})$$

$$= \frac{3\sqrt{2}\Delta_0}{2\pi^{3/2} k_B T_c} \left(\frac{\Delta_0}{k_B T}\right)^{3/2} \exp(-\frac{\Delta_0}{k_B T})$$

$$= \frac{3\sqrt{2} \times 3.5277}{4\pi^{3/2}} \left(\frac{\Delta_0}{k_B T}\right)^{3/2} \exp(-\frac{\Delta_0}{k_B T})$$

$$= 0.671959 \left(\frac{1.764 T_c}{T}\right)^{3/2} \exp(-\frac{1.764 T_c}{T})$$

where

$$2\Delta_0 = 3.5277 k_B T_c .$$

We make a scaling plot of $C_{es} / [\sqrt{2\pi} N(0) k_B \Delta_0]$ as a function of $\frac{\Delta_0}{k_B T}$.

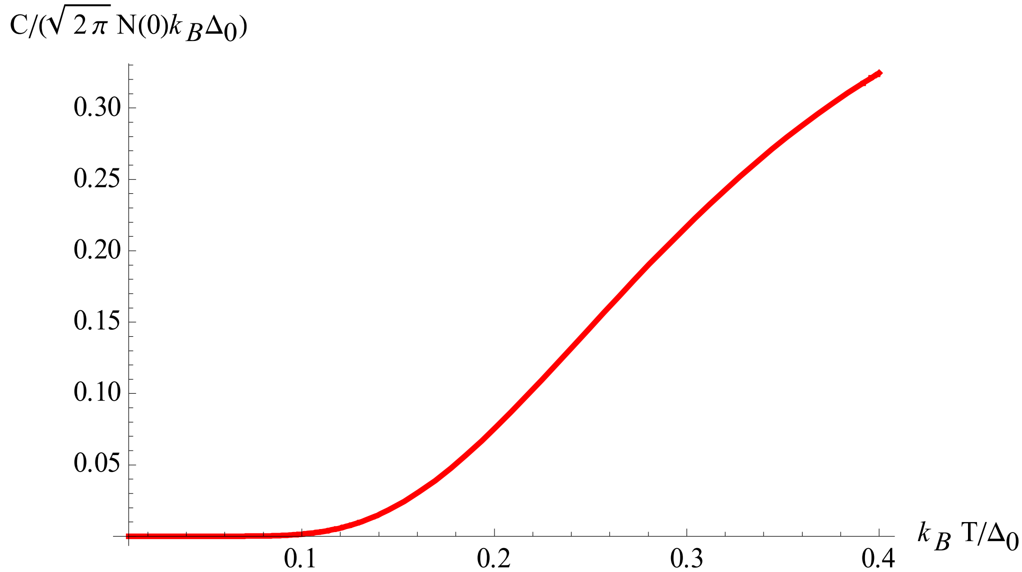


Fig. Plot of $C_{es} / [\sqrt{2\pi} N(0) k_B \Delta_0]$ as a function of $k_B T/\Delta_0$ for $T \ll \Delta_0/k_B$.

(ii) At $T = T_c$, where $\Delta = 0$. The first term is continuous through the transition ($\Delta \to 0$)

$$C_n = T\frac{\partial S}{\partial T} = \frac{2}{T_c}\int dEN(E)E^2[-\frac{\partial f(E)}{\partial E}]$$
$$= \frac{2}{T_c}N(0)2\int_0^\infty d\xi[-\xi^2\frac{\partial f(\xi)}{\partial \xi}]$$
$$= \frac{2}{T_c}2\frac{\pi^2}{6\beta^2}N(0)$$
$$= \frac{2\pi^2}{3}k_B{}^2T_cN(0) = \gamma T_c$$

Note that $E = |\xi|$ in the limit of $\Delta \rightarrow 0$. The specific heat jump at T_c is given by

$$\Delta C = (C_s - C_n)_{T=T_c-0}$$
$$= -\left(\frac{d\Delta^2}{dT}\right)_{T=T_c-0}\int d\xi N(\xi)(-\frac{\partial f(\xi)}{\partial \xi})$$
$$= -N(0)\left(\frac{d\Delta^2}{dT}\right)_{T=T_c-0}$$

where we use

$$-\frac{\partial f(\xi)}{\partial \xi} = \delta(\xi).$$

$\delta(\xi)$ is the Dirac delta function. Using the relation,

$$\Delta = \pi k_B T_c\left[\frac{8}{7\varsigma(3)}\right]^{1/2}(1-\frac{T}{T_c})^{1/2} = 3.0626\,k_B T_c(1-\frac{T}{T_c})^{1/2},$$

we get

$$\Delta C = \pi^2 k_B{}^2T_cN(0)\frac{8}{7\zeta(3)} = 0.950751\pi^2 k_B{}^2T_cN(0)$$
.

Then the ratio is evaluated as

$$\frac{\Delta C}{C_n} = \frac{\pi^2k_B{}^2T_cN(0)\frac{8}{7\zeta(3)}}{\frac{2\pi^2}{3}k_B{}^2T_cN(0)} = \frac{12}{7\zeta(3)} = 1.42613. \qquad \text{(BCS prediction).}$$

((Mathematica))

```
Integrate[e^(-x β) x^3 / Sqrt[x^2 - Δ^2], {x, Δ, ∞}] //
 Simplify[#, {β > 0, Δ > 0}] &
```

$$\frac{\Delta^2 (\beta \Delta \,\text{BesselK}[1, \beta \Delta] + \text{BesselK}[2, \beta \Delta])}{\beta}$$

```
Integrate[2 x / (Exp[β x] + 1), {x, 0, ∞}] //
 Simplify[#, {β > 0}] &
```

$$\frac{\pi^2}{6 \beta^2}$$

REFERENCES

1. J. Bardeen, L.N. Cooper, and J.R. Schrieffer, Phys. Rev. **108**, 1175 (1957).
2. P.G. de Gennes, *Superconductivity of Metals and Alloys* (W.A. Benjamin, New York, 1966).
3. M. Tinkham, *Introduction to Superconductivity*, Reprint edition (Robert E. Krieger Publishing Company, INC, Malabar, Florida, 1980).
4. J.B. Ketterson and S.N. Song, *Superconductivity* (Cambridge University Press, 1999).
5. T. Tsuneto, *Superconductivity and superfluidity* (Cambridge University Press, 1998).
6. J.R. Schrieffer, *Theory of Superconductivity*, revised edition (Addison-Wesley, Reading, 1983).
7. W. Buckel and R. Leiner, *Superconductivity, Fundamentals and Applications*, Wiley-Vch Verlag GmbH & Co. KGaA, Weinheim, 2004).
8. C.G. Kuper, *An Introduction to the Theory of Superconductivity* (Clarendon Press, Oxford, 1968).
9. G. Rickayzen, *Theory of Superconductivity* (Interscience, 1964).
10. D. Saint James, G. Sarma, and E. Thomas, *Type II Superconductors* (Clarendon Press, Oxford, 1971).
11. S. Nakajima, *Introduction to Superconductivity* (in Japanese) (Baifukan, Tokyo, 1971).
12. C. Kittel, *Quantum Theory of Solids*, second revised printing (John Wiley & Sons, New York, 1987).
13. H. Ibach and H. Lűth, *Solid-State Physics, 4*-th edition (Springer, Berlin, 2009).
14. R.P. Feynman*, Statistical Mechanics* (Benjamin, Reading, MA, 1972).
15. M.P. Marder, *Condensed Matter Physics* (John Wiley & Sons, New York, 2010).

16. L. Hoddeson and V. Daitch, *True Ginus. The life and science of John Bardeen* (Joseph Henry Press, Washington DC, 2001).
17. D.J. Thouless, Topological quantum numbers in nonrelativistic physics (World Scientific, Singapore, 1998).
18. M. Suzuki and I.S. Suzuki, Ginzburg-Landau theory for superconductivity, http://www2.binghamton.edu/physics/docs/ginzburg-landau.pdf

---

APPENDIX
Formula related to the BCS theory

The magnetic quantum flux (fluxoid, or fluxon) is

$$\Phi_0 = \frac{hc}{2e} = \frac{2\pi\hbar c}{2e} = 2.06783372 \times 10^{-7} \text{ (Gauss.cm}^2\text{)}.$$

The coherence length at $T = 0$ K

$$\xi_0 = \frac{\hbar v_F}{\pi \Delta_0}.$$

The energy gap at $T = 0$ K:

$$\Delta_0 = 2\hbar\omega_D \exp[-\frac{1}{VN(0)}] = 2k_B\Theta_D \exp[-\frac{1}{VN(0)}].$$

The energy gap

$$2\Delta_0 = 3.5277 k_B T_c.$$

The energy gap

$$\frac{\Delta(T)}{\Delta_0} = 1.73669(1 - \frac{T}{T_c})^{1/2}.$$

The specific heat discontinuity:

$$\frac{C_s - C_n}{C_n}|_{T_c} = \frac{12}{7\varsigma(3)} = 1.43.$$

Bogoliubov transformation:

$$\begin{pmatrix} \alpha_{k\uparrow} \\ \alpha^{+}_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k^{*} & u_k \end{pmatrix} \begin{pmatrix} a_{k\uparrow} \\ a^{+}_{-k\downarrow} \end{pmatrix}, \qquad \begin{pmatrix} \alpha^{+}_{k\uparrow} \\ \alpha_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & -v_k^{*} \\ v_k & u_k \end{pmatrix} \begin{pmatrix} a^{+}_{k\uparrow} \\ a_{-k\downarrow} \end{pmatrix}.$$

or

$$\begin{pmatrix} a_{k\uparrow} \\ a^{+}_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & v_k \\ -v_k^{*} & u_k \end{pmatrix} \begin{pmatrix} \alpha_{k\uparrow} \\ \alpha^{+}_{-k\downarrow} \end{pmatrix}, \qquad \begin{pmatrix} a^{+}_{k\uparrow} \\ a_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & v_k^{*} \\ -v_k & u_k \end{pmatrix} \begin{pmatrix} \alpha^{+}_{k\uparrow} \\ \alpha_{-k\downarrow} \end{pmatrix}.$$

The excitation energy of quasiparticle

$$E_k = \sqrt{\xi_k{}^2 + |\Delta_k|^2}\,.$$

$$u_k{}^2 = \frac{1}{2}\left(1 + \frac{\xi_k}{E_k}\right).$$

$$|v_k|^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right).$$

$$\xi_k = \varepsilon_k - \varepsilon_F\,.$$

$$v_k = u_k \frac{\Delta_k}{\xi_k + E_k}\,.$$

$$u_k |v_k| = \frac{\Delta_k}{2E_k}\,.$$

$$\Delta_k = |\Delta_k| e^{i\phi},$$

u_k is real. $\quad v_k = |v_k| e^{i\phi}$.

Density of states for the quasi-particles

$$\frac{N(E)}{N_n(0)} = \frac{E}{\sqrt{E^2 - \Delta^2}} = \frac{\frac{E}{\Delta}}{\sqrt{\left(\frac{E}{\Delta}\right)^2 - 1}}\,.$$

$$\frac{1}{2}N(0)\Delta^2 = \frac{[H_c(0)]^2}{8\pi}.$$

The BCS ground state:

$$\left|BCS\right\rangle = \prod_k (u_k + v_k B_k^+)\left|\Phi_{vac}\right\rangle.$$

$$\left\langle BCS\right|a_{k\uparrow}^+ a_{-k\downarrow}^+\left|BCS\right\rangle = u_k v_k^*.$$

$$\left\langle BCS\right|a_{-k\downarrow} a_{k\uparrow}\left|BCS\right\rangle = u_k v_k = \frac{\Delta_k}{2E_{\mathbf{k}}}.$$

$$\left\langle BCS\right|a_{k\uparrow}^+ a_{k\uparrow}\left|BCS\right\rangle = \left|v_k\right|^2.$$

$$\alpha_{k\uparrow}\left|BCS\right\rangle = (u_k a_{k\uparrow} - v_k a_{-k\downarrow}^+)\left|BCS\right\rangle = 0.$$

$$\alpha_{-k\downarrow}\left|BCS\right\rangle = (u_k a_{-k\downarrow} + v_k a_{k\uparrow}^+)\left|BCS\right\rangle = 0.$$

$$(\Delta N)^2 = \left\langle N^2\right\rangle - \left\langle N\right\rangle^2 = \sum_k 4u_k^2\left|v_k\right|^2.$$

$$\left\langle N\right\rangle = \left\langle BCS\right|N\left|BCS\right\rangle = 2\sum_k \left|v_k\right|^2.$$

$$[a_{k\sigma}, a_{k'\sigma'}^+]_+ = a_{k\sigma}a_{k'\sigma'}^+ + a_{k'\sigma'}^+ a_{k\sigma} = \delta_{k,k'}\delta_{\sigma,\sigma'}.$$

$$[a_{k\sigma}a_{k'\sigma'}]_+ = [a_{k\sigma}^+, a_{k'\sigma'}^+]_+ = 0.$$

$$a_{k\sigma}a_{k\sigma}^+\left|\Phi_{vac}\right\rangle = (1 - a_{k\sigma}^+ a_{k\sigma})\left|\Phi_{vac}\right\rangle = \left|\Phi_{vac}\right\rangle.$$

$$[B_k, B_{k'}]_- = 0.$$

$$[B_k, B_{k'}]_+ = 2B_k B_{k'}(1 - \delta_{k,k'}).$$

$$B_k B_k^+\left|\Phi_{vac}\right\rangle = \left|\Phi_{vac}\right\rangle.$$

$$[B_k, B_{k'}^+]_- = (1 - n_{k\uparrow} - n_{-k\downarrow})\delta_{k,k'}.$$

$$[a_{k\uparrow}, B_k^+]_+ = a_{-k\downarrow}^+(1-2a_{k\uparrow}^+ a_{k\uparrow}) .$$

$$[a_{-k\uparrow}, B_k^+]_+ = -a_{k\uparrow}^+(1-2a_{-k\downarrow}^+ a_{-k\uparrow}) .$$

$$B_k \left|\Phi_{vac}\right\rangle = a_{-k\downarrow} a_{k\uparrow} \left|\Phi_{vac}\right\rangle = 0 .$$

$$B_k{}^+ \left|\Phi_{vac}\right\rangle = a_{k\uparrow}^+ a_{-k\downarrow}^+ \left|\Phi_{vac}\right\rangle = -a_{-k\downarrow}^+ a_{k\uparrow}^+ \left|\Phi_{vac}\right\rangle .$$

$$B_k B_k{}^+ \left|\Phi_{vac}\right\rangle = \left|\Phi_{vac}\right\rangle .$$

$$\left\langle a_{k\uparrow}^+ a_{-k'\downarrow} \right\rangle = -u_k v_k [1-2f(E_k)]\delta_{k,k'} .$$

$$\left\langle a_{-k\downarrow} a_{k'\uparrow}^+ \right\rangle = u_k v_k [1-2f(E_k)]\delta_{k,k'}$$

$$\left\langle a_{k\uparrow}^+ a_{k\uparrow} \right\rangle = \left|v_k\right|^2 + (u_k{}^2 - \left|v_k\right|^2) f(E_k) .$$

$$H = E_g + \sum_k E_k (\alpha_{k\uparrow}{}^+ \alpha_{k\uparrow} + \alpha_{-k\uparrow}{}^+ \alpha_{-k_\uparrow})$$

$$E_g = \sum_k [-E_k + \xi_k + \Delta_k b_k^*]$$