Superfluidity in Liquid ^{4}He

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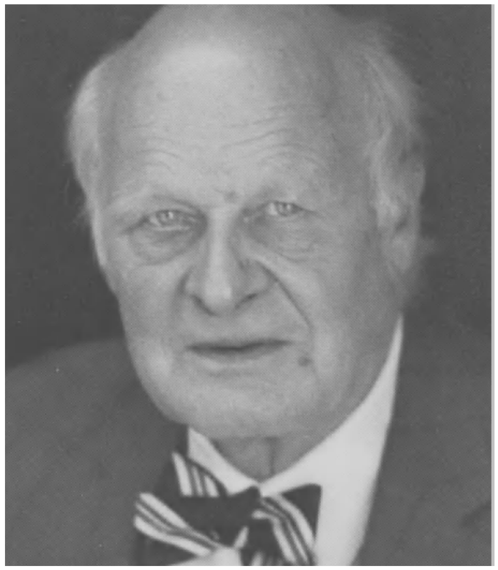
Fritz Wolfgang London (March 7, 1900 – March 30, 1954) was a Jewish-German physicist and professor at Duke University. His fundamental contributions to the theories of chemical bonding and of intermolecular forces (London dispersion forces) are today considered classic and are discussed in standard textbooks of physical chemistry. With his brother Heinz London, he made a significant contribution to understanding electromagnetic properties of superconductors with the London equations and was nominated for the Nobel Prize in Chemistry on five separate occasions.

London was the first theoretical physicist to make the fundamental, and at the time controversial, suggestion that superfluidity is intrinsically related to the Einstein condensation of bosons, a phenomenon now known as Bose–Einstein condensation. Bose recognized that the statistics of massless photons could also be applied to massive particles; he did not contribute to the theory of the condensation of bosons.

https://en.wikipedia.org/wiki/Fritz_London

Lars Onsager (November 27, 1903 – October 5, 1976) was a Norwegian-born American physical chemist and theoretical physicist. He held the Gibbs Professorship of Theoretical Chemistry at Yale University. He was awarded the Nobel Prize in Chemistry in 1968.



https://en.wikipedia.org/wiki/Lars_Onsager

1. Properties of liquid He

Helium exists in two stable isotropic forms, ^4He and ^3He. The phase diagram of ^4He is shown below. The normal boiling point is 4.2 K and the critical temperature is 5.19 K (1,718 Torr = 2.26 atm). Liquid He exists in two phases, He I and He II, separated by a phase boundary commonly called the lambda-line. At T_λ = 2.172 K, which is termed the λ-point. There is no latent heat associated with this transformation. The specific heat at saturated vapor pressure, becomes large as the λ-point is approached from either side. One of the most remarkable properties of He II is its ability to flow through very small capillaries or narrow channels without any friction at all.

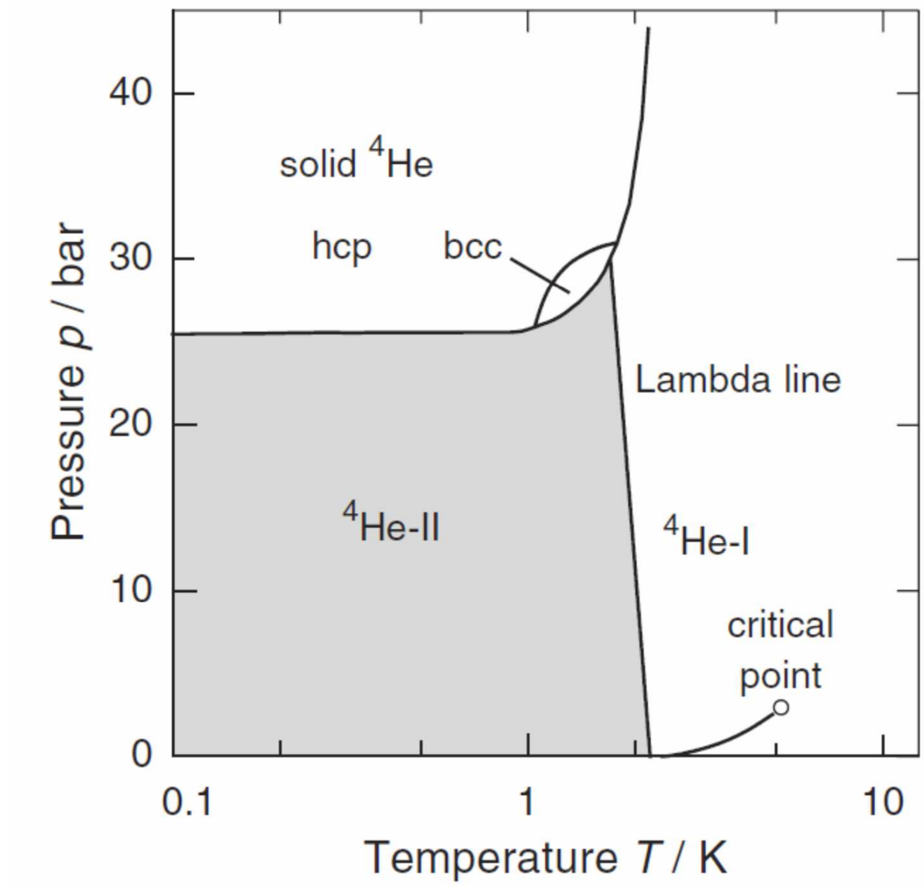


Fig. Phase diagram of liquid ^{4}He. He I (normal liquid) and He II (superfluid)

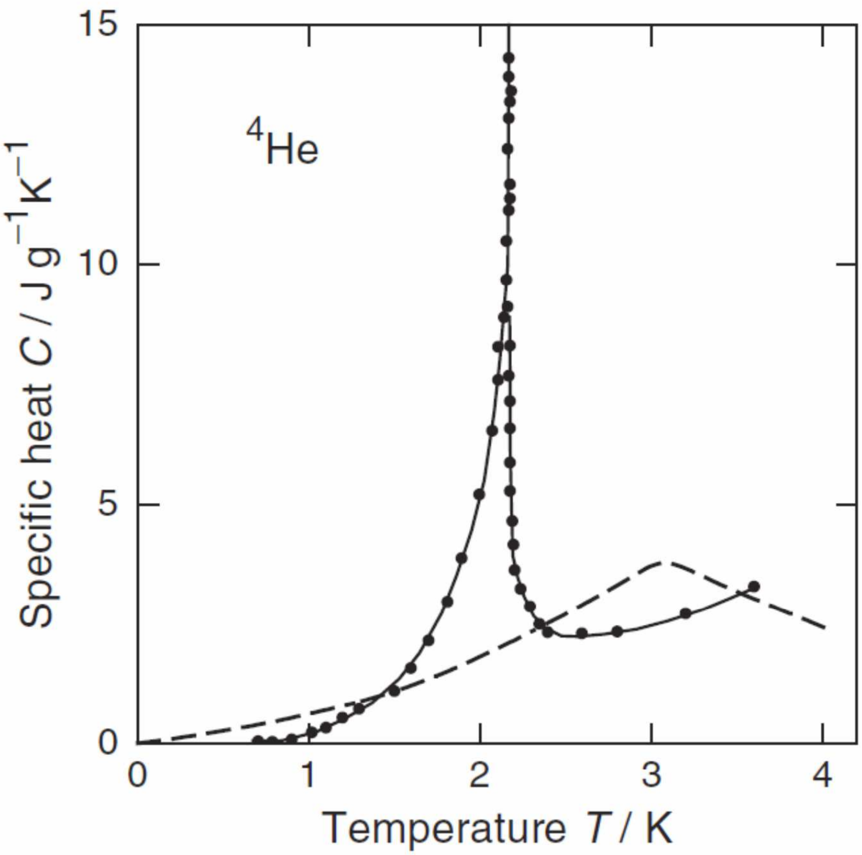
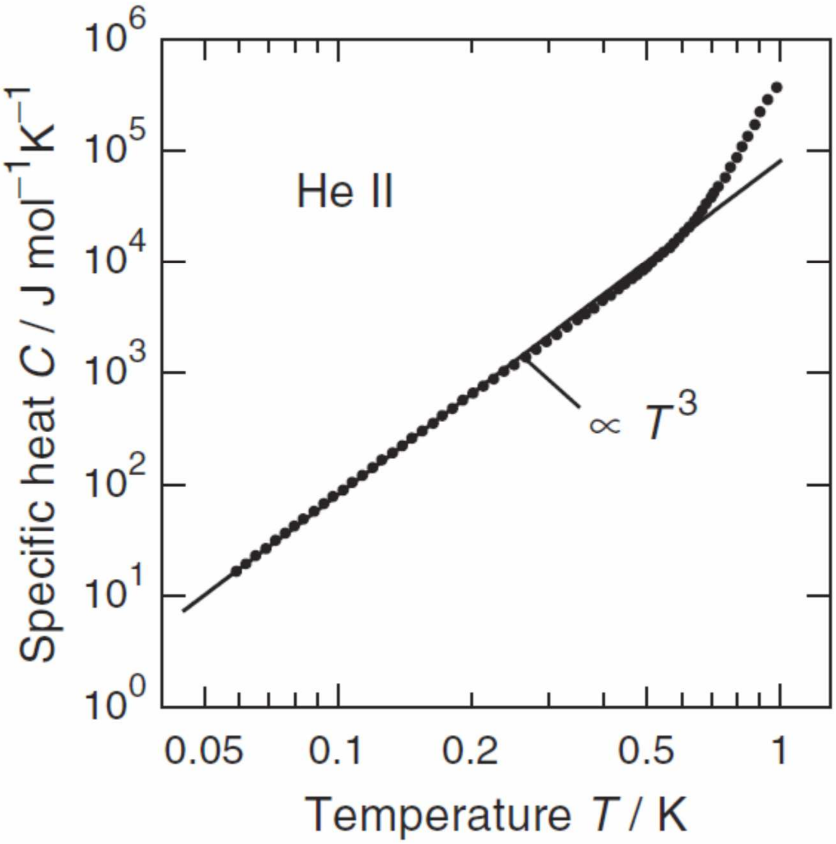
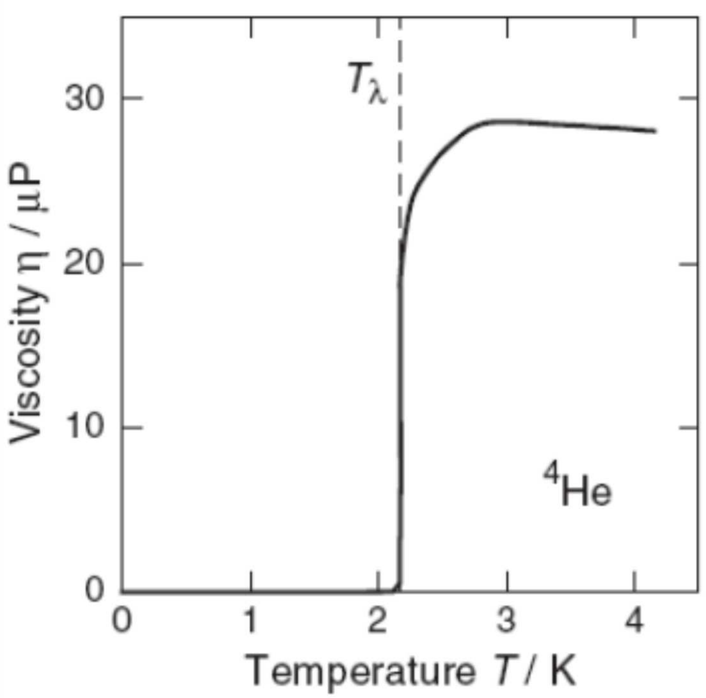


Fig. Specific heat of liquid 4He (point) in comparison with the theoretical curve for an ideal Bose gas with the parameters of liquid He (dashed line).

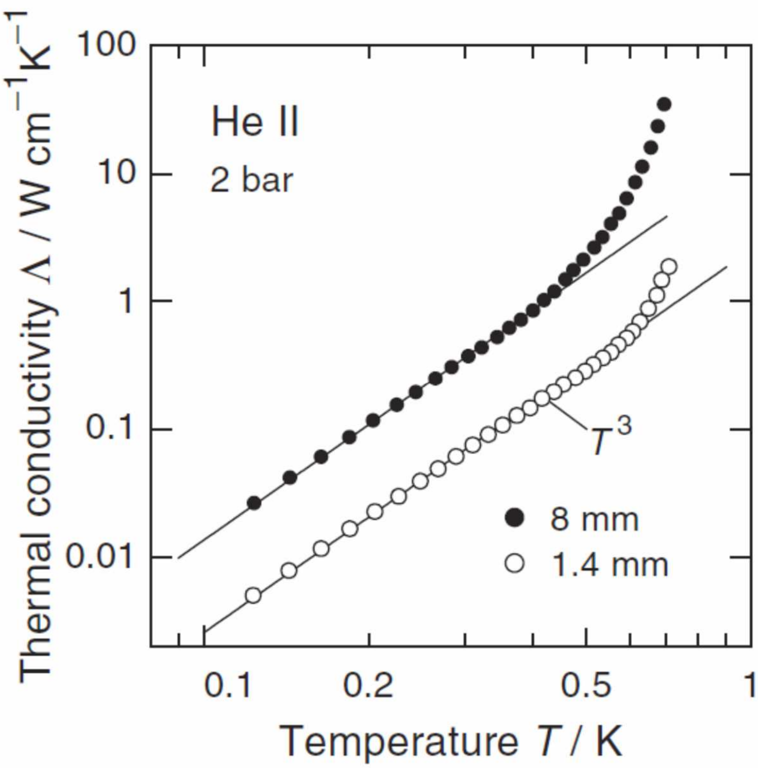


Specific heat C / J mol⁻¹K⁻¹
He II
∝ T³
Temperature T / K



Viscosity η / μP
Tλ
⁴He
Temperature T / K

Fig. Temperature dependence of the viscosity of He II as determined from flow experiments with thin capillaries.



2. Historical overview on the research on superfluid Helium 4.

In 1908, **Kammerlingh Onnes** succeeded for the first time in liquefying helium in Leiden. In 1938 **Allen and Misener**, and **Kapitza** independently discovered that it exhibited a superfluid behavior. **Fritz London** put forth his theory that superfluidity could be related to the Bose-Einstein condensation. **Tisza** suggested that the superfluid phase of the liquid could be described by a two-fluid model, the normal fluid and the superfluid. In 1941 **Landau** suggested that superfluidity can be understood in terms of the special nature of the thermally excited states of the liquid: the well-known phonons and rotons. This theory also led Landau to the two-fluid model. Experimentally the two-fluid model was supported by the experiment of the experiment of **Andronikashvili** and the discovery of second sound.

In 1946, **Onsager** put forth his idea of quantized circulation in superfluid helium. The well-known invariant called the hydrodynamic circulation is quantized. The quantum of circulation is h/m, where m is the exact mass of the bare helium atom. This is a surprising result in itself considering how strongly coupled atoms in a liquid really are. **Feynman** was working on the same problem and came to a somewhat different conclusion. He showed that the excitation spectrum postulated by **Landau** can be derived within a quantum-mechanical description. He considered that the vortices in the superfluid might take the form of a vortex filament with a core of atomic dimensions, truly a line vortex. In this picture, the multiple connectivity of a vortex arises because the superfluid is somehow excluded from the core and circulates about the core in quantized fashion. The quantization of circulation was experimentally confirmed by Hall and Vinen with the direct observation in a macroscopic scale. This work led to an appreciation for the first time of the full significance of London's "quantum mechanism on a macroscopic scale", and of the underlying importance of Bose-Einstein condensation in superfluidity.

Here the superfluidity of He II is discussed in association with the statistical mechanics and quantum mechanics.

3 Two component fluid model (Tisza, Landau)

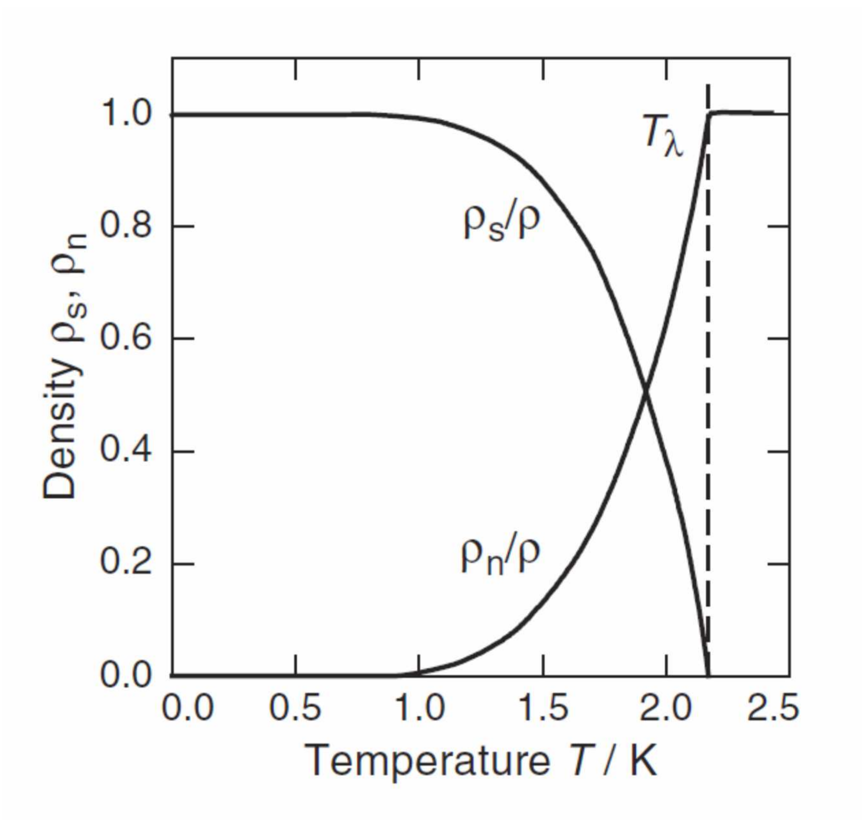


Fig. Density of the superfluid and normal-fluid component in He II as a function of temperature.

The two-fluid model of liquid helium (Landau) postulates that He II behaves as if it were a mixture of two fluids freely intermingling with each other without any viscous interaction. There two fluids are termed the normal fluid and have densities ρ_n and ρ_s such that

$$\rho = \rho_n + \rho_s$$

where ρ is the ordinary density of liquid He. The normal density ρ_n is a function of temperature, and increases from zero at $T = 0$ K, to the value ρ at the lambda point. Conversely, the superfluid density ρ_s is zero at the lambda point and increases to the value ρ at T = 0 K. The model postulates that the superfluid carries zero entropy, and experiences no resistance whaever to its flow, that is, it exhibits neither viscosity nor turbulence. This condition is specified by stipulating that the viscosity of the superfluid is zero, and that its velocity $\boldsymbol{v}_s$ satisfies the relation

$$\nabla \times \boldsymbol{v}_s = 0 \qquad \text{(irrotational).}$$

On the other hand, the normal fluid has a viscosity, the so-called normal viscosity η_n, and an entropy S_n equal to the entropy of liquid He.

We now derive equations of motion for the two fluids of the model. Let $\boldsymbol{j}$ denotes the momentum of unit volume of liquid He, and $\boldsymbol{v}_n$ and $\boldsymbol{v}_s$ the velocities of the two fluids. Then we have

$$\boldsymbol{j} = \rho_n \boldsymbol{v}_n + \rho_s \boldsymbol{v}_s$$

The flow $\boldsymbol{j}$ is also related to the density of liquid He by the equation of continuity

$$\nabla \cdot \boldsymbol{j} + \frac{\partial \rho}{\partial t} = 0$$

4. Probability current density (London) in quantum mechanics in Macroscopic scale

A superfluid has the special property of having phase, given by the wave function. The order parameter (wave function) for the BEC phase is given by

$$\psi(\boldsymbol{r}) = |\psi_0| e^{i\theta(\boldsymbol{r})}$$

where $|\psi_0|$ is independent of the position vector $\boldsymbol{r}$ and the phase $\theta(\boldsymbol{r})$ is a real-valued function of $\boldsymbol{r}$. The probability current density is

$$\begin{aligned}\boldsymbol{j} &= \mathrm{Re}[\psi^* \frac{\boldsymbol{p}}{m}\psi] \\ &= \mathrm{Re}[\psi^* \frac{\hbar}{mi}\nabla\psi] \\ &= \frac{\hbar}{2im}(\psi^*\nabla\psi - \psi\nabla\psi^*)\end{aligned}$$

Suppose that $\psi = |\psi_0| e^{i\theta(\boldsymbol{r})}$

$$\boldsymbol{j} = \frac{\hbar}{m}|\psi_0|^2 \nabla\theta = |\psi_0|^2 \boldsymbol{v}_s$$

with the velocity

$$\boldsymbol{v}_s = \frac{\hbar}{m}\nabla\theta$$

The velocity $\boldsymbol{v}_s$ of the superfluid is proportional to the gradient of the phase.

$$\boldsymbol{v}_s = \frac{\hbar}{m}\nabla\theta(\boldsymbol{r})$$

We not that

$$\boldsymbol{w} = \nabla\times\boldsymbol{v}_s = \nabla\times\frac{\hbar}{m}\nabla\theta(\boldsymbol{r}) = 0$$

In 1941, Landau suggested a test of this assumption in experiments with He II in a rotating vessel. Even before such experiments were conducted, Onsager speculated whether the assumption of $\nabla\times\mathbf{v}_s = 0$ is generally valid and suspected the occurrence of vortices in rotating He II.

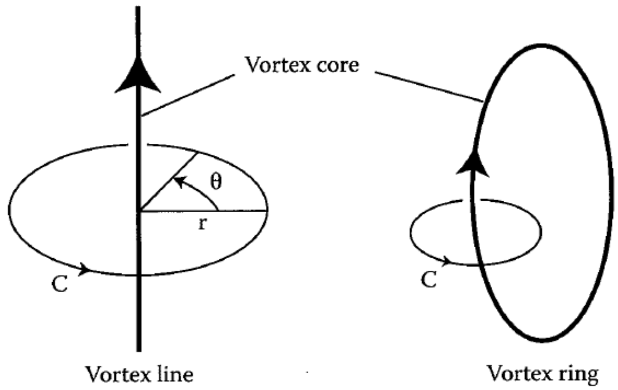
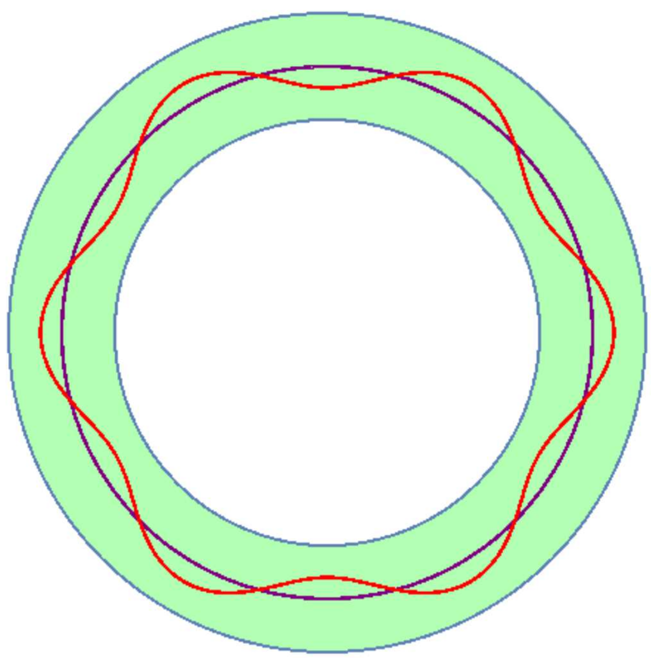


Fig. Vortex line and vortex ring.

6. Quantization of angular momentum



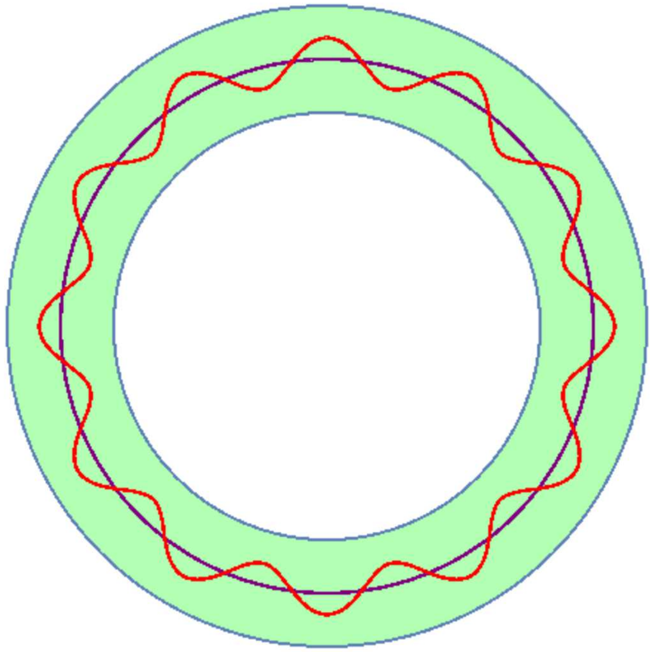


Fig. Quantum vortex with $\frac{2\pi r}{\lambda} = q$, where q = 6 and 12. λ is the wavelength.

The angular momentum is quantized. We note that

$$\oint \boldsymbol{p}\cdot d\boldsymbol{l} = \oint m\boldsymbol{v}_s \cdot d\boldsymbol{l}$$
$$= m\frac{\hbar}{m}\oint \nabla\theta\cdot d\boldsymbol{l}$$
$$= \hbar(2\pi q)$$

where q is integer. The momentum p is

$$p = \frac{\hbar(2\pi q)}{2\pi r} = \frac{\hbar q}{r}$$

The kinetic energy is

$$\varepsilon = \frac{p^2}{2m} = \frac{\hbar^2 q^2}{2mr^2}$$

The velocity is

$$v_s = \frac{p}{m} = \frac{\hbar q}{mr}.$$

The circulation:

$$\kappa = \oint \boldsymbol{v}_s \cdot d\boldsymbol{l} = \frac{\hbar}{m}\oint \nabla\theta \cdot d\boldsymbol{l} = \frac{\hbar}{m}(2\pi q) = \frac{hq}{m}$$

The angular momentum:

$$m\kappa = \oint \boldsymbol{p} \cdot d\boldsymbol{l} = 2\pi r p = 2\pi r(m v_s) = 2\pi(m v_s r) = 2\pi L_z$$

leading to

$$L_z = m v_s r = \frac{m}{2\pi}\kappa$$

7. Quantization of circulation

The circulation around any closed loop in the superfluid is zero, if the region enclosed is simply connected. The superfluid is thought to be irrotational (no rotation),

$$\oint_A (\nabla \times \boldsymbol{v}_s) \cdot d\boldsymbol{a} = \oint_L \boldsymbol{v}_s \cdot d\boldsymbol{l} = 0 \qquad \text{(Stoke's theorem).}$$

However, if the enclosed region actually contains a smaller region with an absence of superfluid (the singularity), the circulation around a doubly connected system is

$$\kappa = \oint_C \boldsymbol{v}_s \cdot d\boldsymbol{l} = \frac{\hbar}{m}\oint_C \nabla\theta(\boldsymbol{r}) \cdot d\boldsymbol{l} = \frac{\hbar}{m}\Delta\theta = \frac{\hbar}{m}2\pi q \qquad \text{(Stoke's theorem)}$$

where q is integer.

$$\frac{\hbar}{m} = \frac{\hbar}{4u} = 1.58769 \times 10^{-8}\ \text{m}^2/\text{s}$$

m is the mass of ^{4}He .

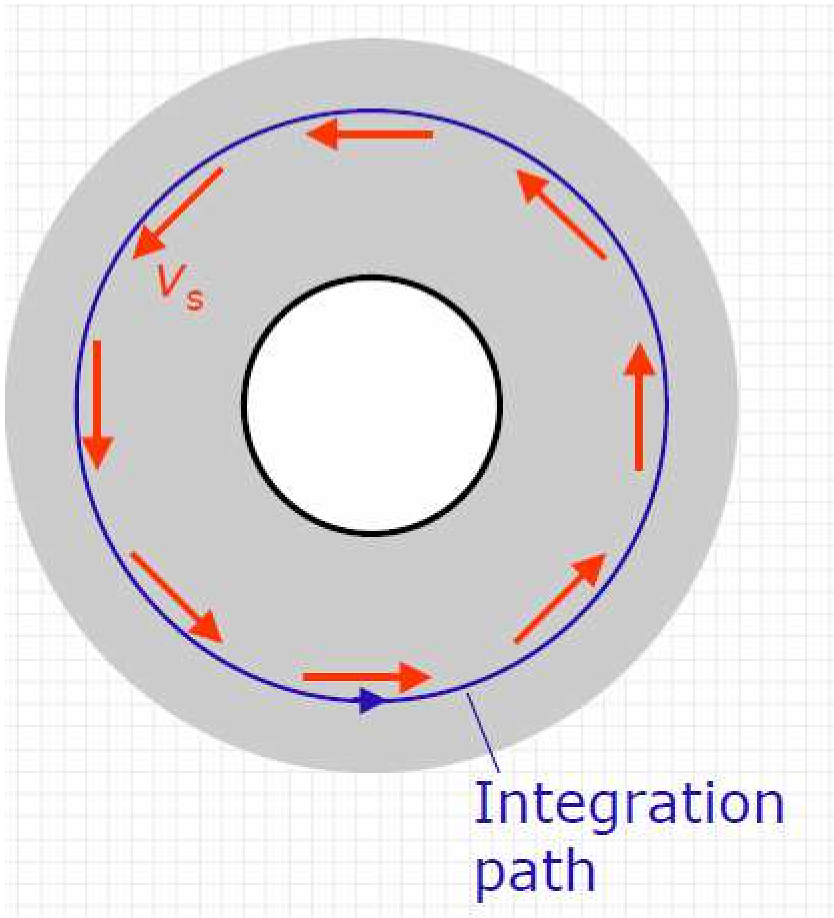


Fig. Circulating $\boldsymbol{v}_s$ in a doubly connected system. Circulation around a hole.

((**Note**))
Simply connected superfluid and multiply connected superfluid.

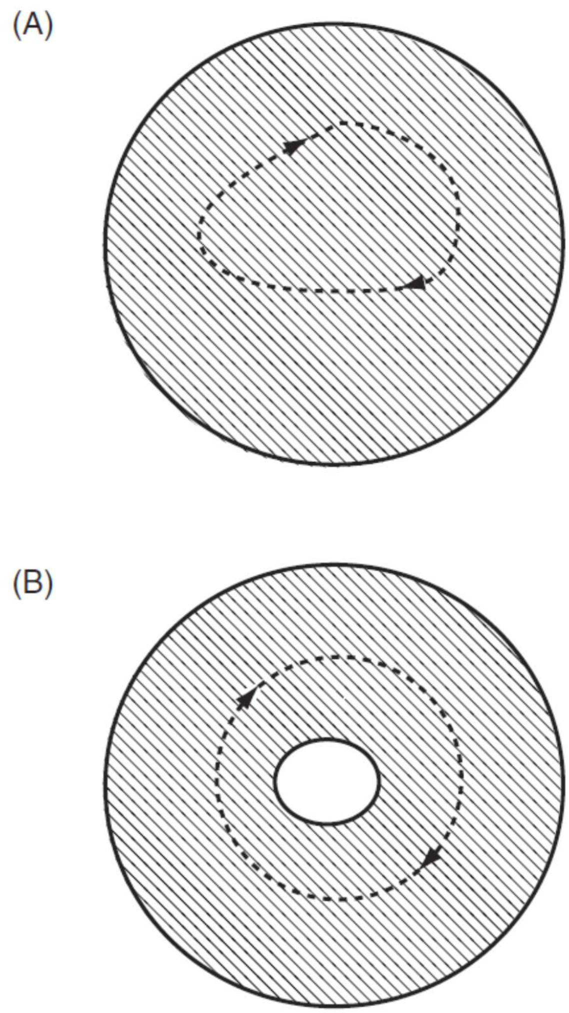


Fig. (a) Simply-connected superfluid. (b) Multiply-connected superfluid.

For the velocity $\boldsymbol{v}_s$, we have

$$v_s(2\pi r) = \frac{\hbar}{m}2\pi q = \kappa, \qquad v_s = \frac{\kappa}{2\pi r}$$

or

$$L_z = mv_s r = q\hbar$$

which means that the angular momentum is quantized.

The circulation around the path C is either zero or a multiple of the quantum of circulation κ,

$$\kappa = \frac{hq}{m} = \frac{2\pi\hbar}{m} q$$

For $q \neq 0$, the velocity around the singularity decreases to zero at infinity.

$$r \to \infty, \qquad v_\theta \to 0,$$

$$r \to 0, \qquad v_\theta \to \infty,$$

The sign of q determines the direction of the flow. q is called the charge of the vortex.

If n_s is the superfluid number density, the kinetic energy is (for q = 1),

$$\begin{aligned} E &= \frac{1}{2}\int_a^b n_s m \left(\frac{\hbar q}{mr}\right)^2 2\pi r dr l \\ &= \pi n_s \frac{q^2\hbar^2 l}{m} \ln\left(\frac{b}{a}\right) \\ &= \frac{\rho_s l}{4\pi} \kappa^2 \ln(\frac{b}{a}) \end{aligned}$$

where l is the depth of liquid, $\rho_s = mn_s$ is the superfluid density, a is the core radius of the vortex, b is the radius of the bucket, or the mean distance between vortices. The line energy per unit length is given by

$$\frac{E}{l} = \frac{\rho_s}{4\pi} l \kappa^2 \ln(\frac{b}{a})$$

Feynman conjectured how the vortex line might be arranged. First, note that since the circulation κ enters squared in the energy, doubly quantized vortex line would have four times the energy of a singly quantized line, and would likely be unstable to break up into four separate lines. The creation of many vortices with q = 1 is energetically more favorable than the creation of a smaller number of vortices with correspondingly higher circulation.

The angular momentum L_z per unit length associated with a single vortex is given by

$$\frac{L}{l} = \int_0^R r v_s \rho_s (2\pi r) dr$$

$$= \rho_s \int_0^R 2\pi r^2 \frac{\kappa}{2\pi r} dr$$

$$= \kappa\rho_s \int_0^R r dr$$

$$= \kappa\rho_s \frac{R^2}{2}$$

where

$$v_s = \frac{\kappa}{2\pi r}$$

We note that

$$\frac{E}{L} = \frac{\frac{\rho_s}{4\pi}\kappa^2 \ln(\frac{b}{a})}{\kappa\rho_s \frac{R^2}{2}} = \frac{\kappa \ln(\frac{b}{a})}{2\pi R^2}$$

We assume that

$$\frac{E}{L} \approx \frac{\frac{1}{2} I\omega_c^2}{I\omega_c} = \frac{1}{2}\omega_c$$

Thus we have

$$\omega_c = \frac{\kappa \ln(\frac{b}{a})}{\pi R^2} = \frac{hq}{m\pi R^2} \ln(\frac{b}{a})$$

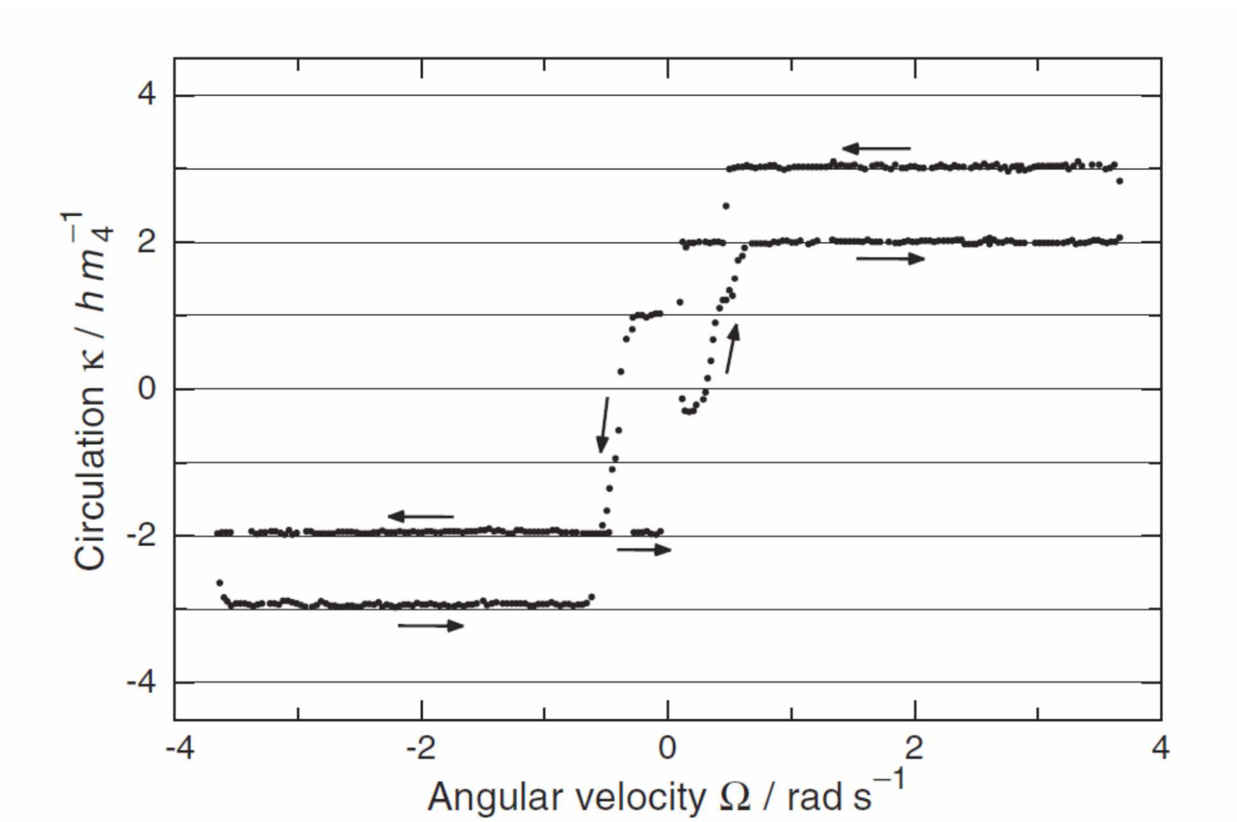


Fig. Circulation κ in units of h/m_4 as a function of the angular velocity of the rotating cylinder. The arrows indicate the sequence in which the angular velocity was changed.

8. Circulation of normal liquid

Normal fluid:

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$$

and

$$\omega_\theta = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)$$

Since

$$v_\theta = \Omega r$$

then we get

$$\omega_\theta = \frac{1}{r}\frac{\partial}{\partial r}(\Omega r^2) = \frac{1}{r}2\Omega r = 2\Omega : \qquad \text{solid body rotation}$$

The velocity v_θ is proportional to r in the normal phase.

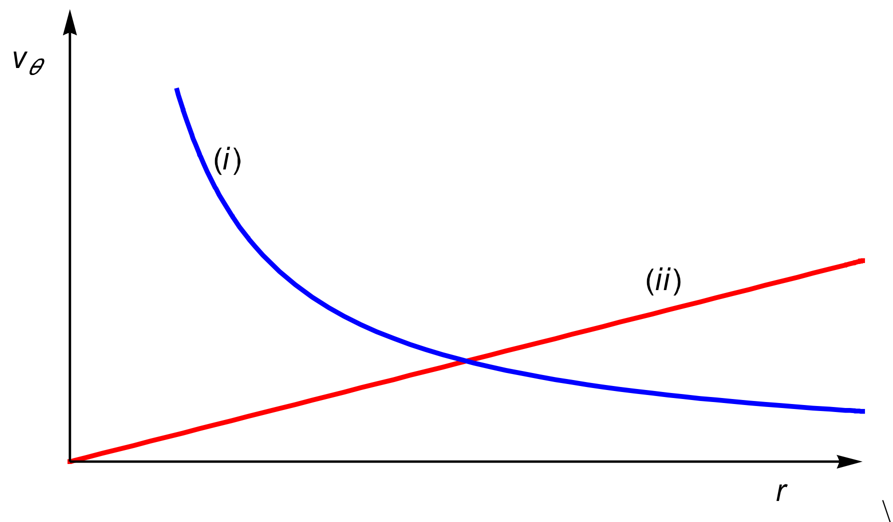


Fig. (i) $v_\theta = \frac{\hbar}{mr}q$ (q: integer). (ii) $v_\theta = \Omega r$ for the normal fluid.

9. Second London equation

Since

$$\boldsymbol{v}_s = \frac{\hbar}{m}\nabla\theta(\boldsymbol{r})$$

we have

$$\nabla\times\boldsymbol{v}_s = \nabla\times[\frac{\hbar}{m}\nabla\theta(\boldsymbol{r})] = 0 .$$

or

$$\nabla\times\boldsymbol{v}_s = 0 \qquad \text{(the second London equation)}$$

The superfluid is irrotational.

Here note that using the Stokes theorem we have

$$\kappa = \oint_C \boldsymbol{v}_s \cdot d\boldsymbol{l} = \oint (\nabla \times \boldsymbol{v}_s) \cdot d\boldsymbol{a} = \frac{\hbar}{m} 2\pi q$$

where d***a*** is the areal vector of the surface element (inside the closed loop C). This result leads to the expression of ω around the singularity as

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v}_s = \kappa \delta^2(\boldsymbol{r})\, \boldsymbol{e}_z$$

$\delta^2(\boldsymbol{r})$ is the two-dimensional Dirac delta function. So $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}_s$ is zero (the flow is irrotational) except at the origin (singularity).

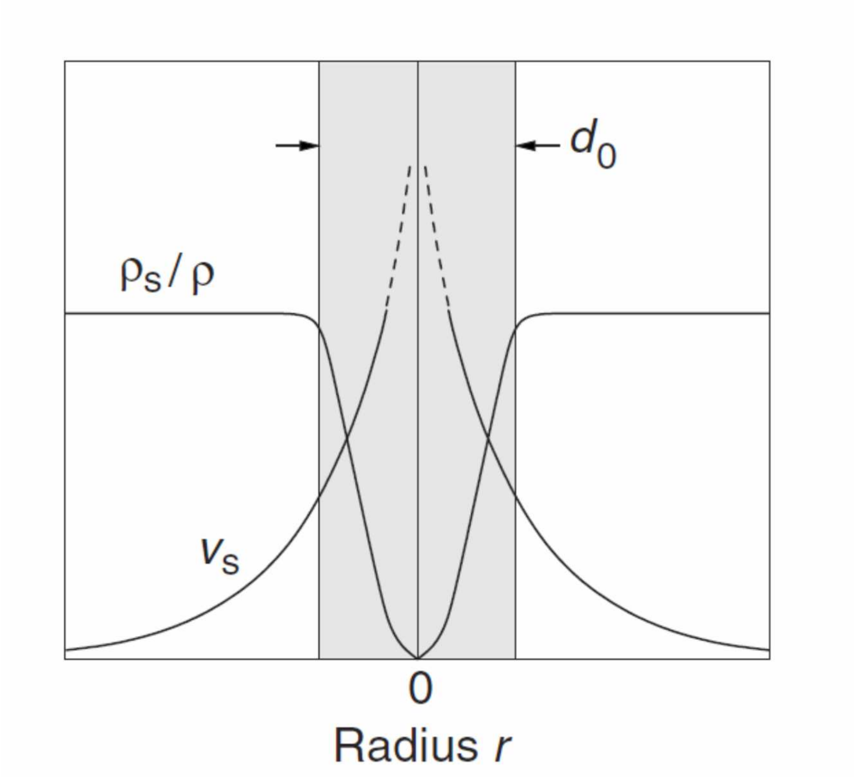


Fig. Radial (r) dependence of vs around the singularity (origin).

10. Anderson-Josephson evolution relation (time dependent Schrödinger equation)

We start with the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi \,.$$

We assume that the wave function ψ is given by

$$\psi = |\psi_0| e^{i\theta}$$

The phase angle θ is dependent on t and $\boldsymbol{r}$. Thus we have

$$i\hbar\frac{\partial\psi}{\partial t} = i\hbar\frac{\partial}{\partial t}|\psi_0|e^{i\theta} = -\hbar|\psi_0|e^{i\theta}\frac{\partial\theta}{\partial t} = H|\psi_0|e^{i\theta}$$

or

$$-\hbar\frac{\partial\theta}{\partial t} = H\,.$$

As regards the time derivative, we replace the Hamiltonian with the chemical potential μ.

$$\hbar\frac{d\theta}{dt} = -\mu \qquad \text{(Anderson-Josephson phase evolution relation).}$$

Since $v_s = \frac{\hbar}{m}\nabla\theta$, we have

$$m\frac{\partial v_s}{\partial t} = m\frac{\partial}{\partial t}(\frac{\hbar}{m}\nabla\theta) = \nabla(\hbar\frac{\partial\theta}{\partial t}) = -\nabla\mu$$

11. The Gibbs-Duhem relation (thermodynamics)

The Gibbs free energy is given by

$$G = \mu N = F + PV = E - ST + PV$$

leading to the differential form

$$dG = \mu dN + Nd\mu = dE - SdT - TdS + PdV + VdP$$

or

$$\mu dN + Nd\mu = TdS - PdV + \mu dN - SdT - TdS + PdV + VdP$$

or

$$Nd\mu = -SdT + VdP$$

So we get the relation

$$d\mu = \frac{V}{N}dP - \frac{S}{N}dT \qquad \text{(Gibbs-Duhem relation)}$$

Here N is the number of particles, V is the volume, S is the entropy, ρ is the pressure and T is the temperature. The chemical potential can be written as follows.

$$d\mu = \frac{V}{N}dP - \frac{S}{N}dT = \frac{m}{\rho}dP - \sigma m dT = m(\frac{dP}{\rho} - \sigma dT)$$

$$\frac{V}{N} = \frac{V}{M}\frac{M}{N} = \frac{m}{\rho}, \qquad \frac{S}{N} = \frac{S}{Nm}m = \sigma m$$

The Gibbs-Duhem can be rewritten as

$$\nabla\mu = m(\frac{1}{\rho}\nabla P - \sigma\nabla T),$$

where σ is the entropy per unit mass and ρ is the density.

12. The first London equation

We consider the Bernoulli effect for a system in dynamic flow. The pressure P is replaced by

$$P \rightarrow P + \frac{1}{2}\rho v_s^{\ 2}$$

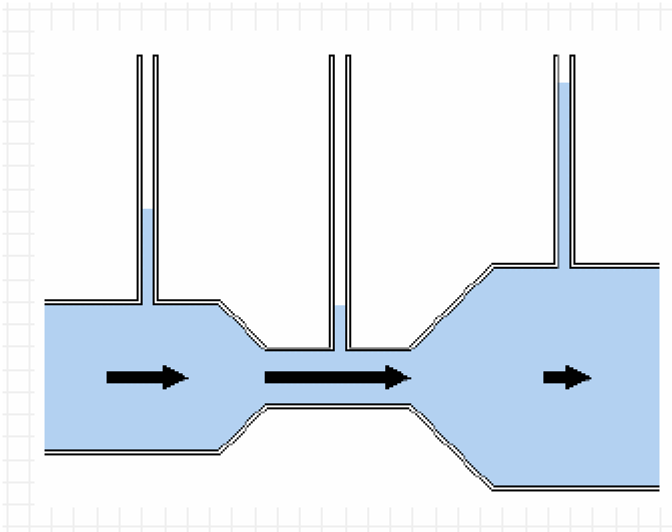


Fig. The principle based on the Bernoulli equation (derived from the work energy theorem for fluid). $P + \rho g h + \frac{1}{2}\rho v^2 = \text{const}$.

Then we have

$$\frac{1}{m}\nabla\mu = \frac{1}{\rho}\nabla(P + \frac{1}{2}\rho v_s^{\,2}) - \sigma\nabla T$$

The time evolution of v_s is

$$\begin{aligned}\frac{\partial \boldsymbol{v}_s}{\partial t} &= -\frac{1}{m}\nabla\mu \\ &= -\frac{1}{\rho}\nabla(P + \frac{1}{2}\rho v_s^{\,2}) + \sigma\nabla T \\ &= -\frac{1}{2}\nabla v_s^{\,2} - \frac{1}{\rho}\nabla P + \sigma\nabla T\end{aligned}$$

or

$$\frac{\partial \boldsymbol{v}_s}{\partial t} = -\frac{1}{2}\nabla v_s^{\,2} - \frac{1}{\rho}\nabla P + \sigma\nabla T. \qquad \textbf{(The first London equation)}$$

13. Heat flux density

The heat flux density (erg/cm^2 s) is expressed by

$$q = \rho T \sigma v_n$$

The amount of heat per unit mass

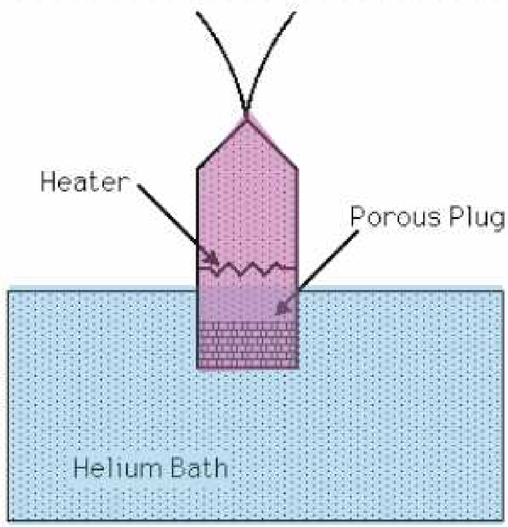
$$Q = T\sigma$$

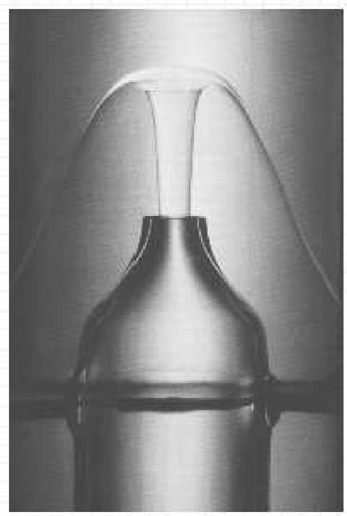
where σ is the entropy per unit mass. Once equilibrium has been established, the heat flow may be expressed in terms of the entropy, average temperature *T*, and flow rate as

$$\dot{q} = \rho T \sigma \dot{v}_n$$

14. Fountain effect (mechanocaloric effect)

In a nearly blocked porous plug only the superfluid can flow while the heat flow is blocked. This is enough to lift the superfluid against gravity and force it out at the top of vessel. It is an unusual example of osmosis. Both the superfluid and normal fluid components carry particle current, but only the normal component carries heat current.





We assume a steady state.

$$\frac{\partial v_s}{\partial t} = 0 = -\frac{1}{\rho}\nabla P + \sigma \nabla T$$

or

$$\nabla P = \rho\sigma\nabla T, \qquad \text{(London's equation)}$$

or

$$(P_1 - P_2) = \rho\sigma(T_1 - T_2)$$

A temperature difference implies a pressure difference.

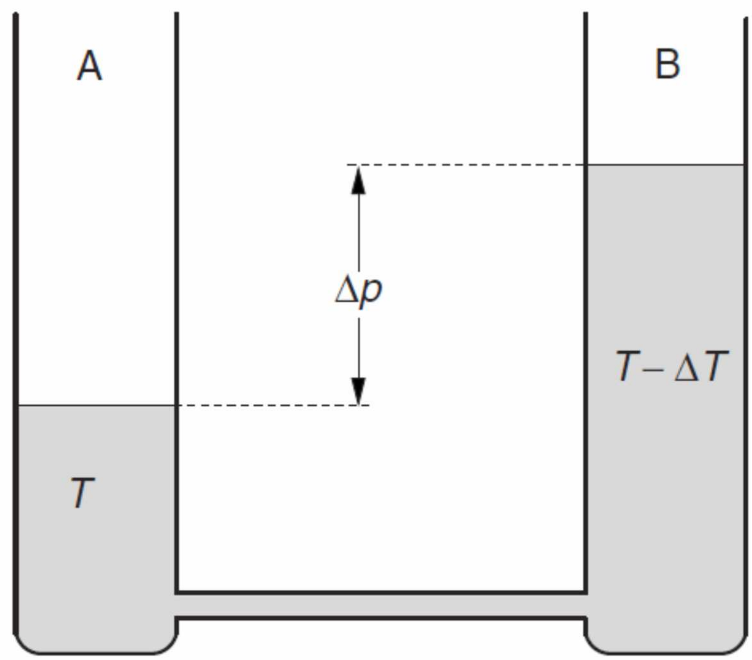


Fig. Schematic illustration of the principle of the thermomechanical effect.

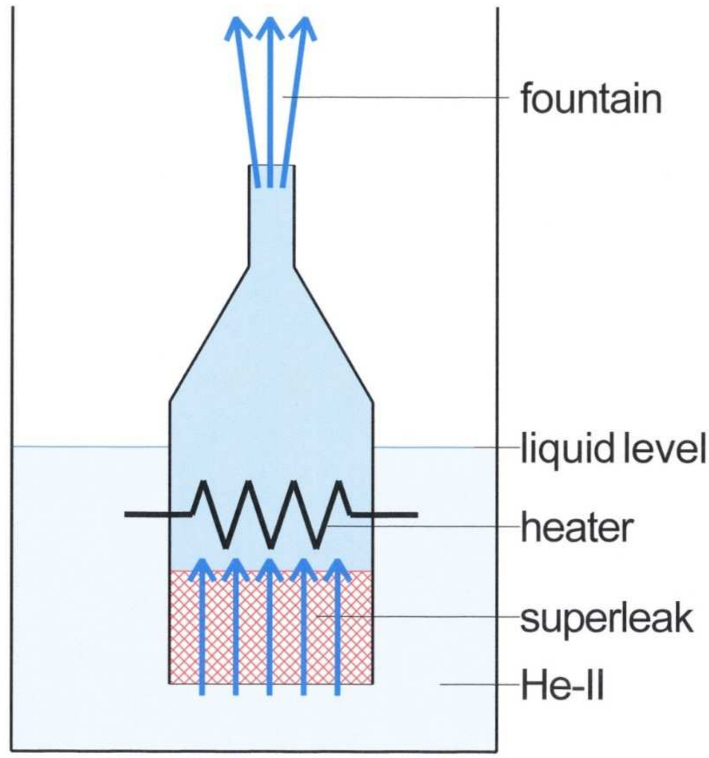
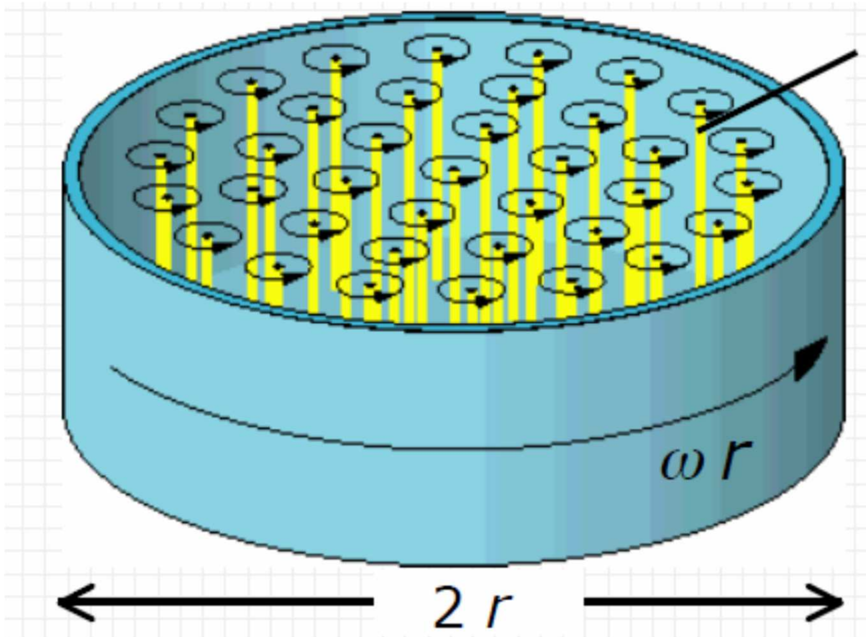
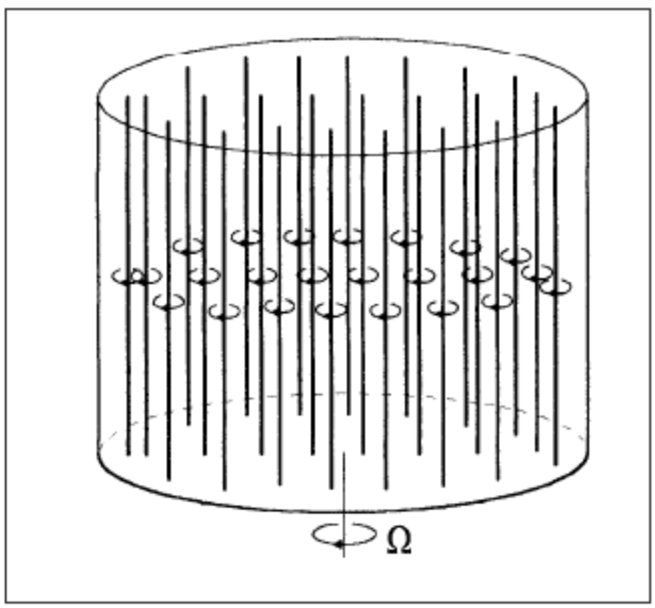


Fig Demonstration of the fountain effect. A capillary tube is "closed" at one end by a superleak and is placed into a bath of superfluid helium and then heated. The helium flows up through the tube and squirts like a fountain.

15. Quantum vortices in the rotating He II



ω r
2 r



Ω

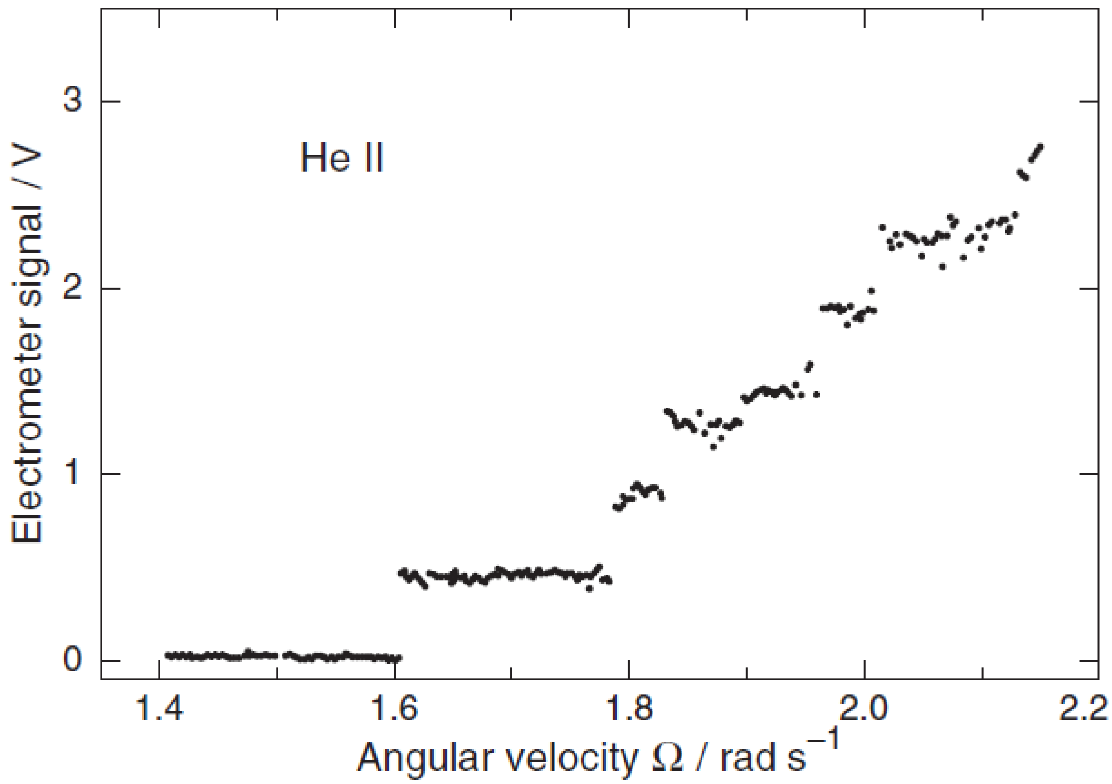


Fig. Electrometer signal as a function of angular velocity. The velocity of rotation of He II was increased steadily in this experiment

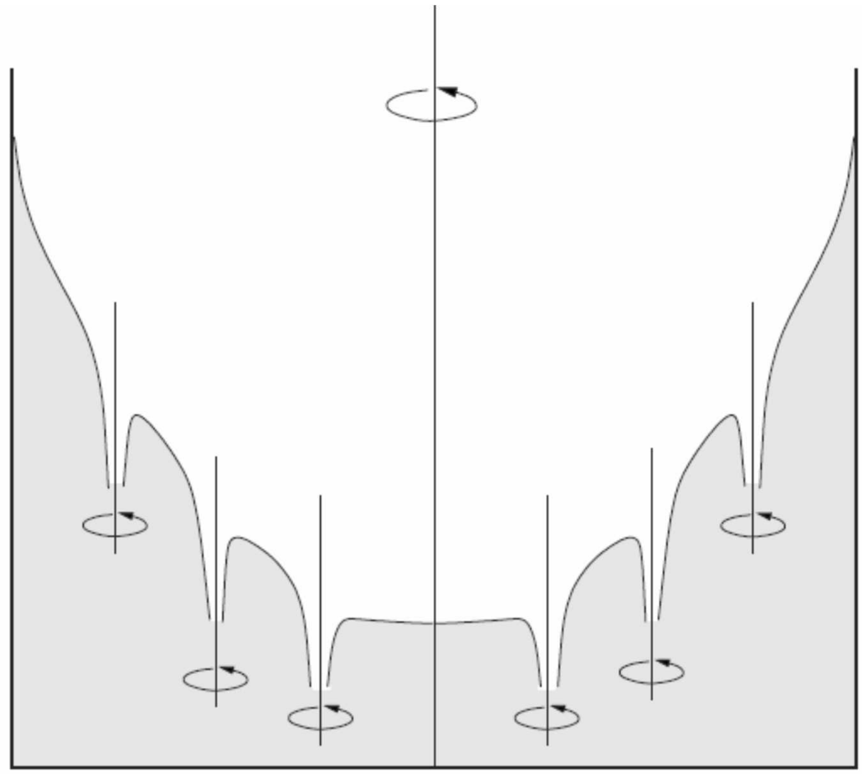
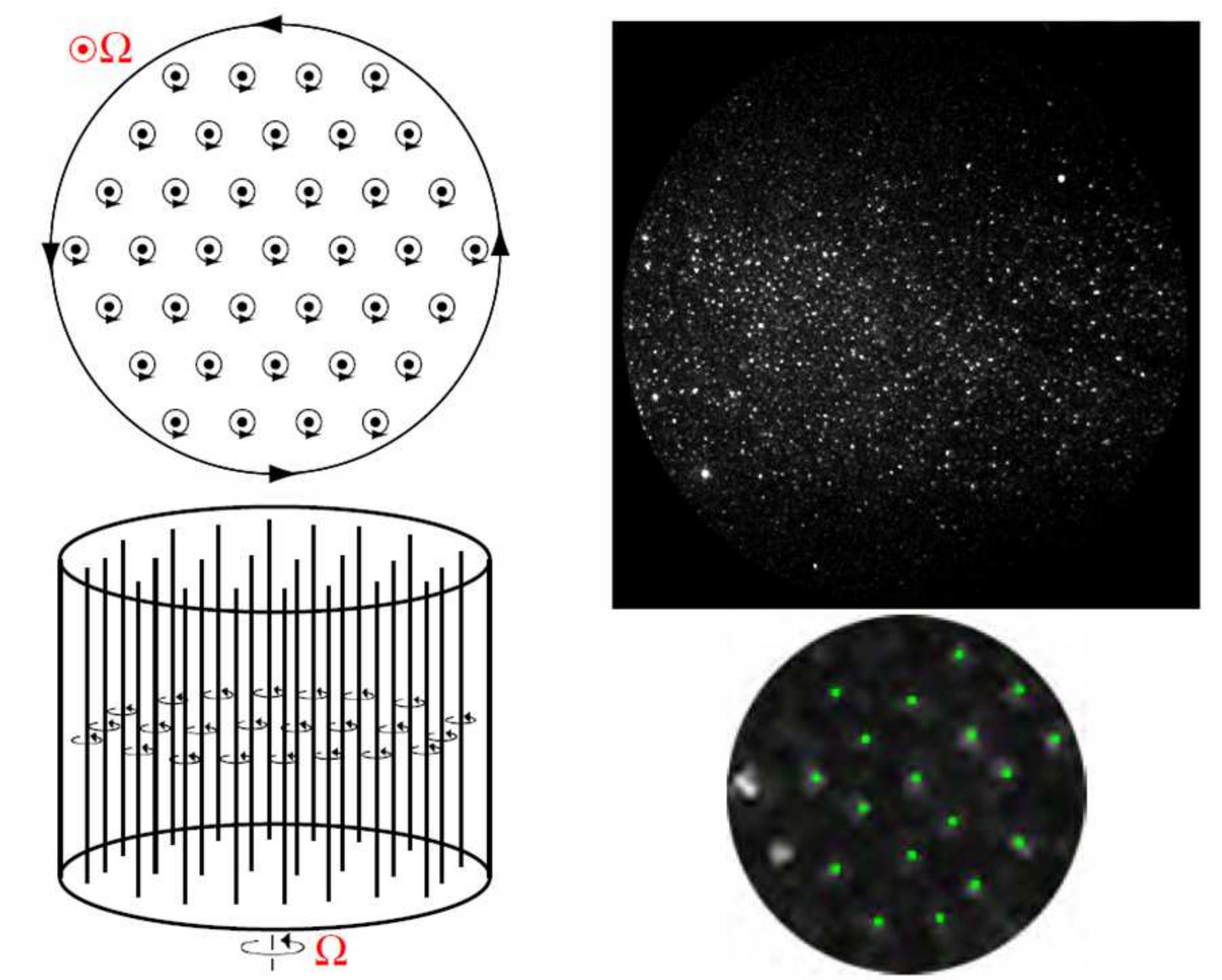


Fig. Schematic illustration of vortices in a rotating vessel containing He II.


⊙Ω
Ω

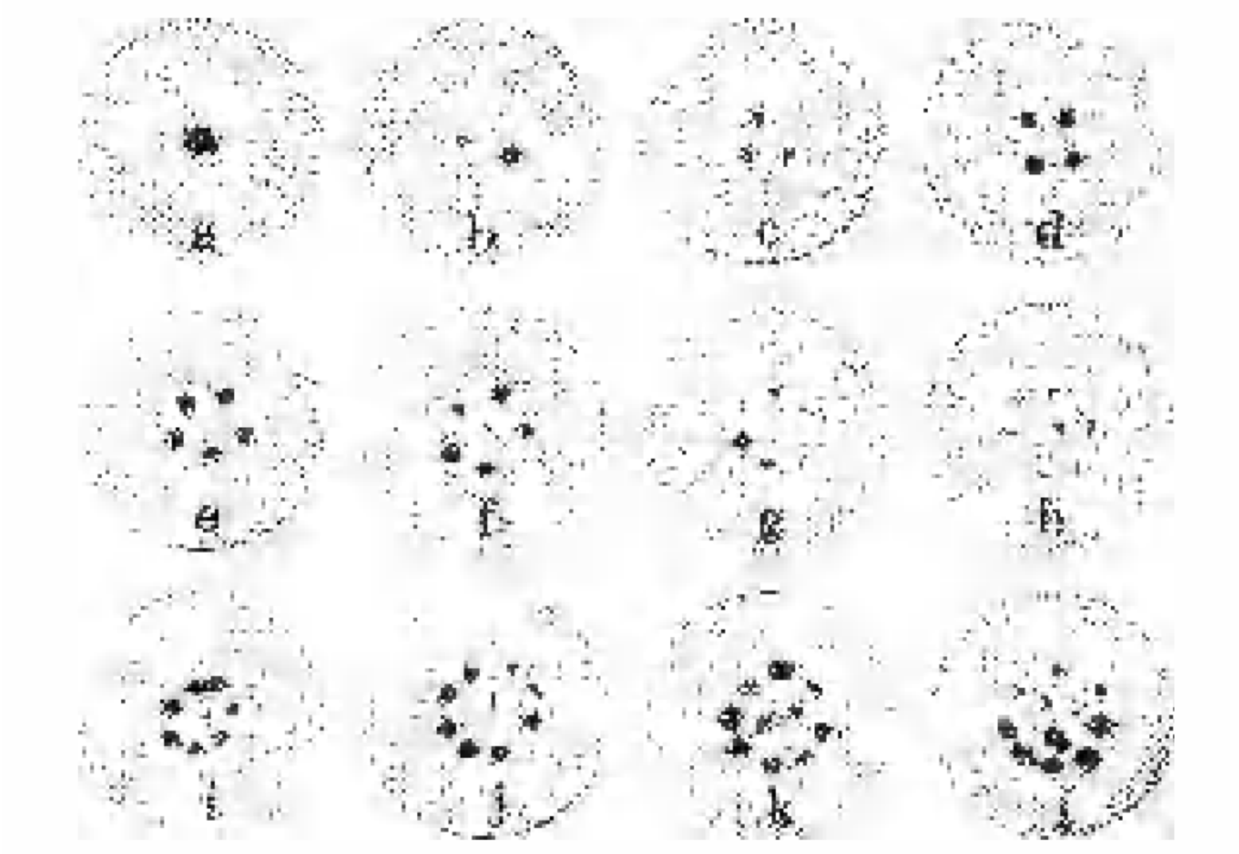


Fig. Visualization of quantized vortices in rotating He II at $T = 0.1$ K.

16. Beaker experiment

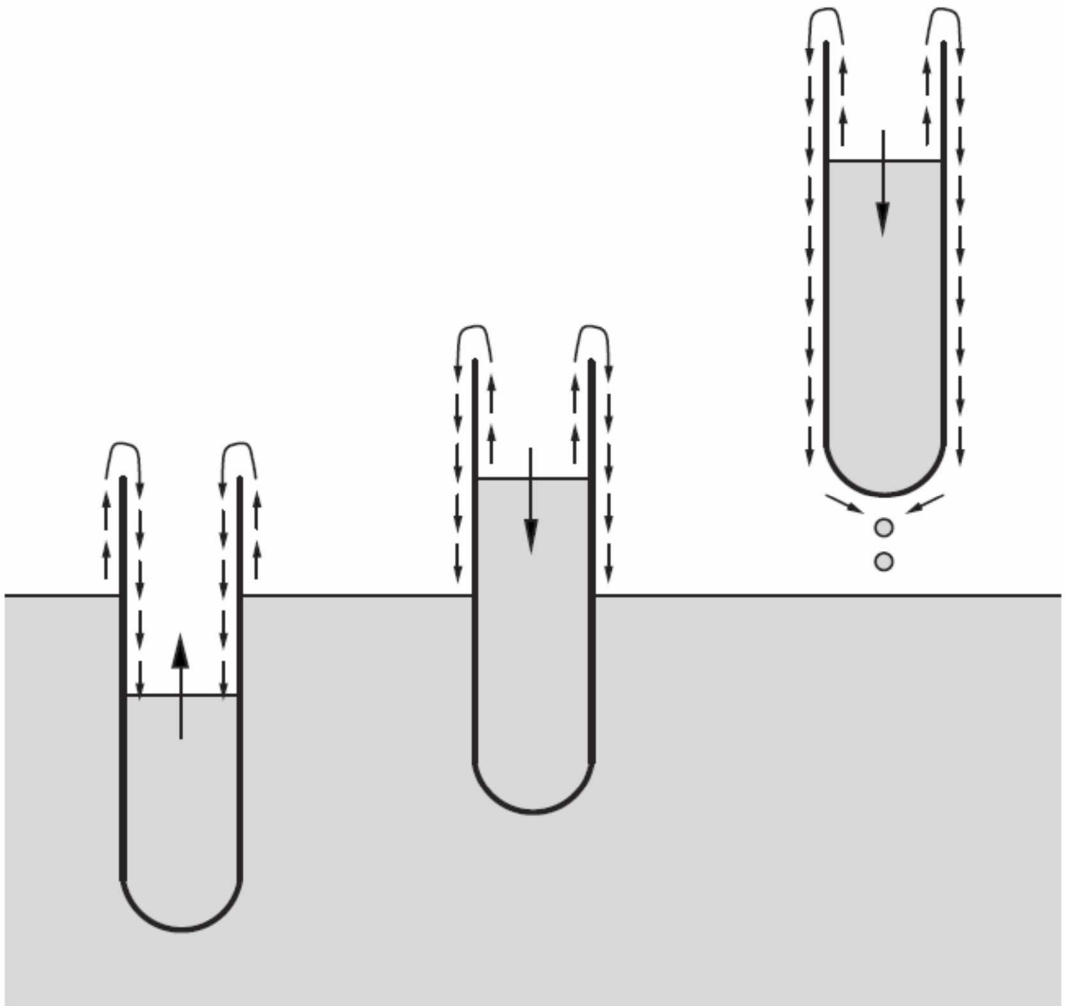


Fig. Schematic illustration of beaker experiments. Pioneering film flow experiments of Daunt and Mendelssohn. (a) Beaker filling through the film. (b) Beaker emptying through the film. (c) Drops forming on the bottom of the beaker.

17. Andronikashivili experiment

A schematic drawing of the specially designed torsion oscillator that Andronikashvili used in 1948 to determine the normal-fluid density ρ_n is shown in Fig. The complete normal-fluid component ρ_n, but not the superfluid component ρ_s, was dragged with the discs above and below the lambda point.

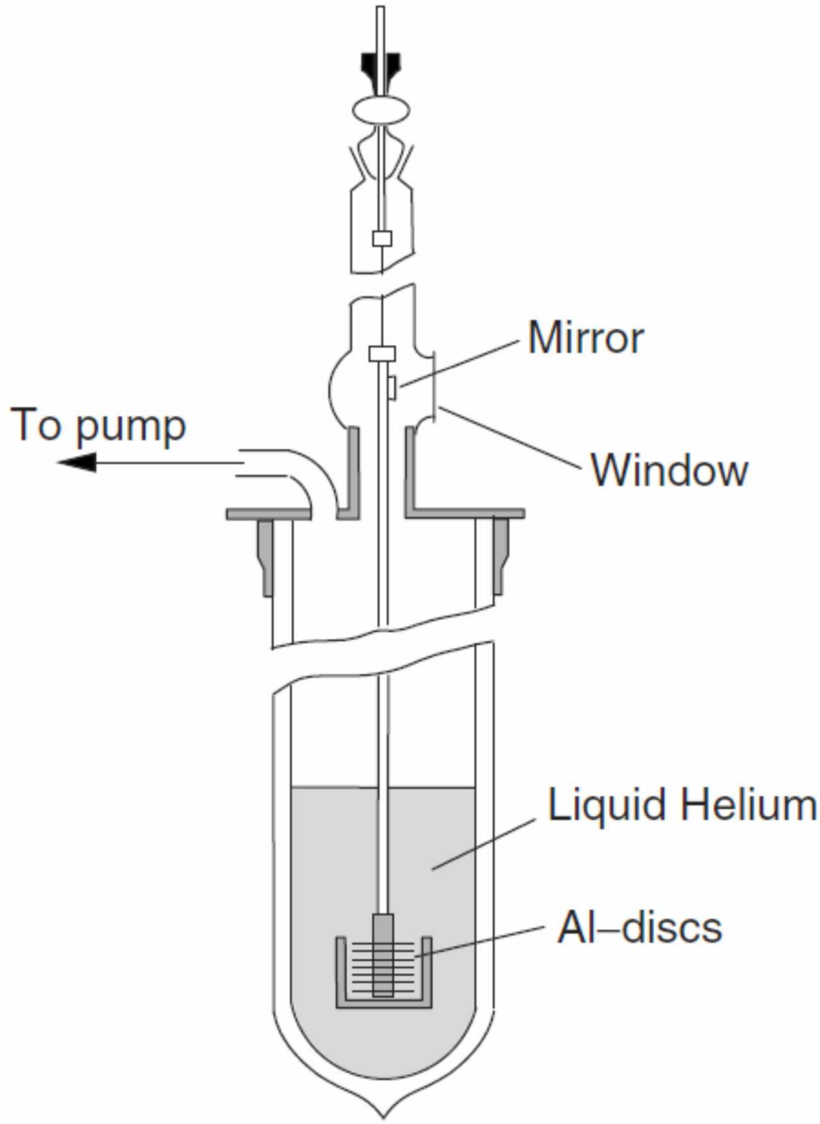
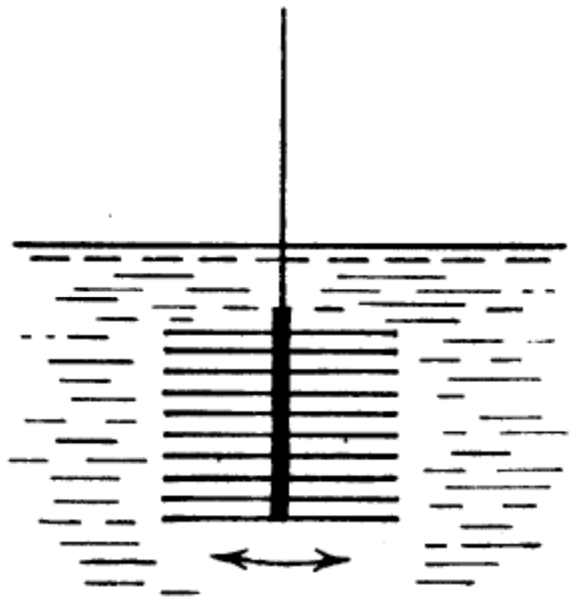


Fig. Schematic drawing of the apparatus used by Andronikashivili to determine the normal-fluid density ρ_n of He II.



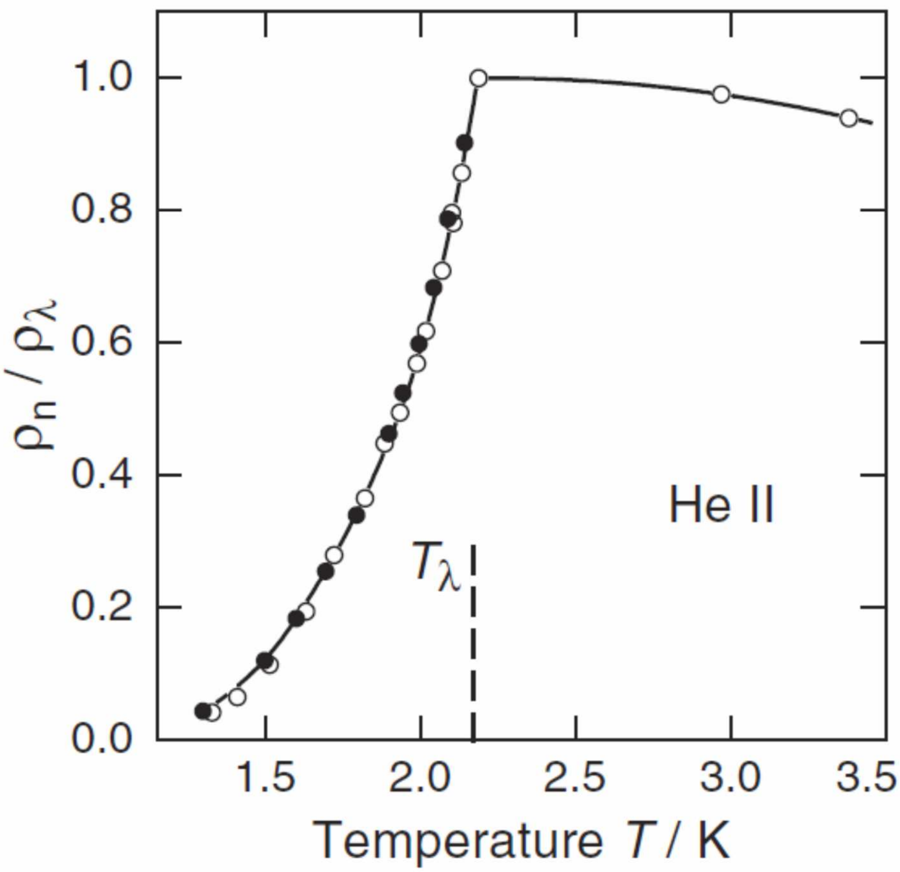
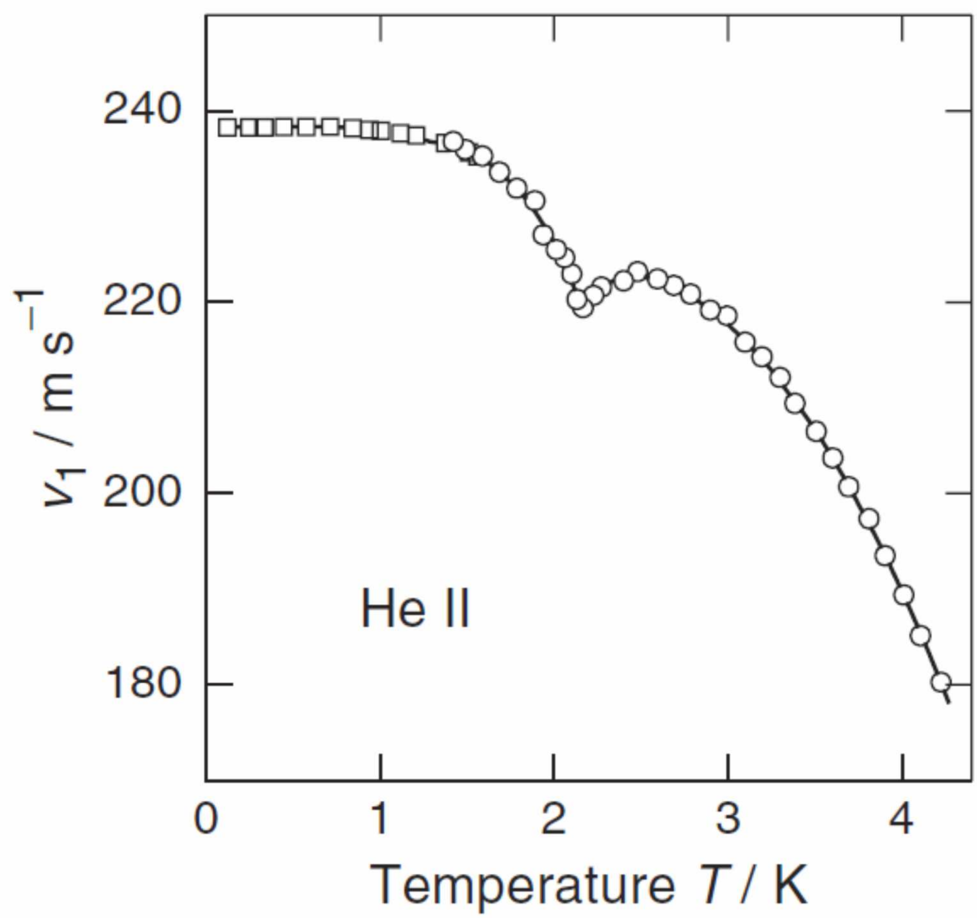


Fig. Temperature dependence of the normal-fluid density ρ_n normalized to the density ρ_λ at T_λ. The data were obtained with two different methods: ○ Andronikashvili viscometer, and ● second-sound measurements.

18. First sound, second sound, and third sound



240
220
200
180
v_1 / m s^{-1}
He II
0
1
2
3
4
Temperature T / K

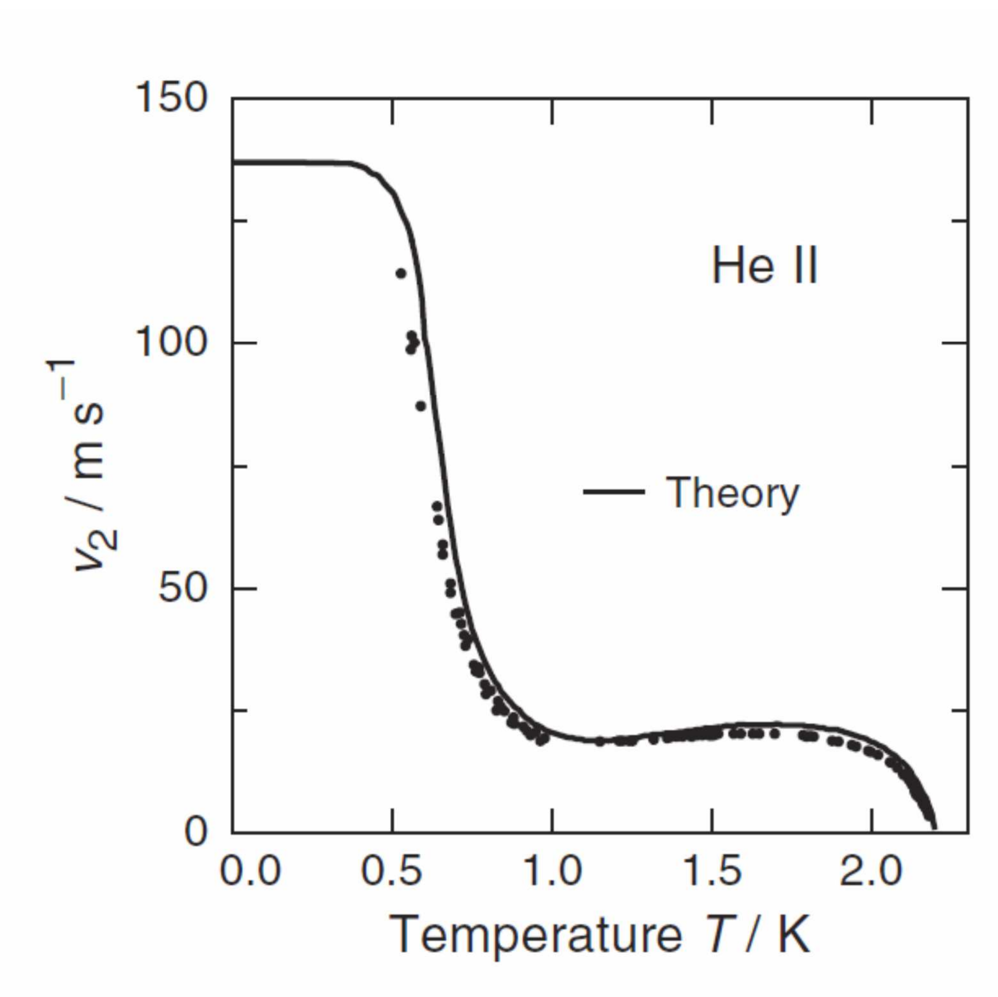
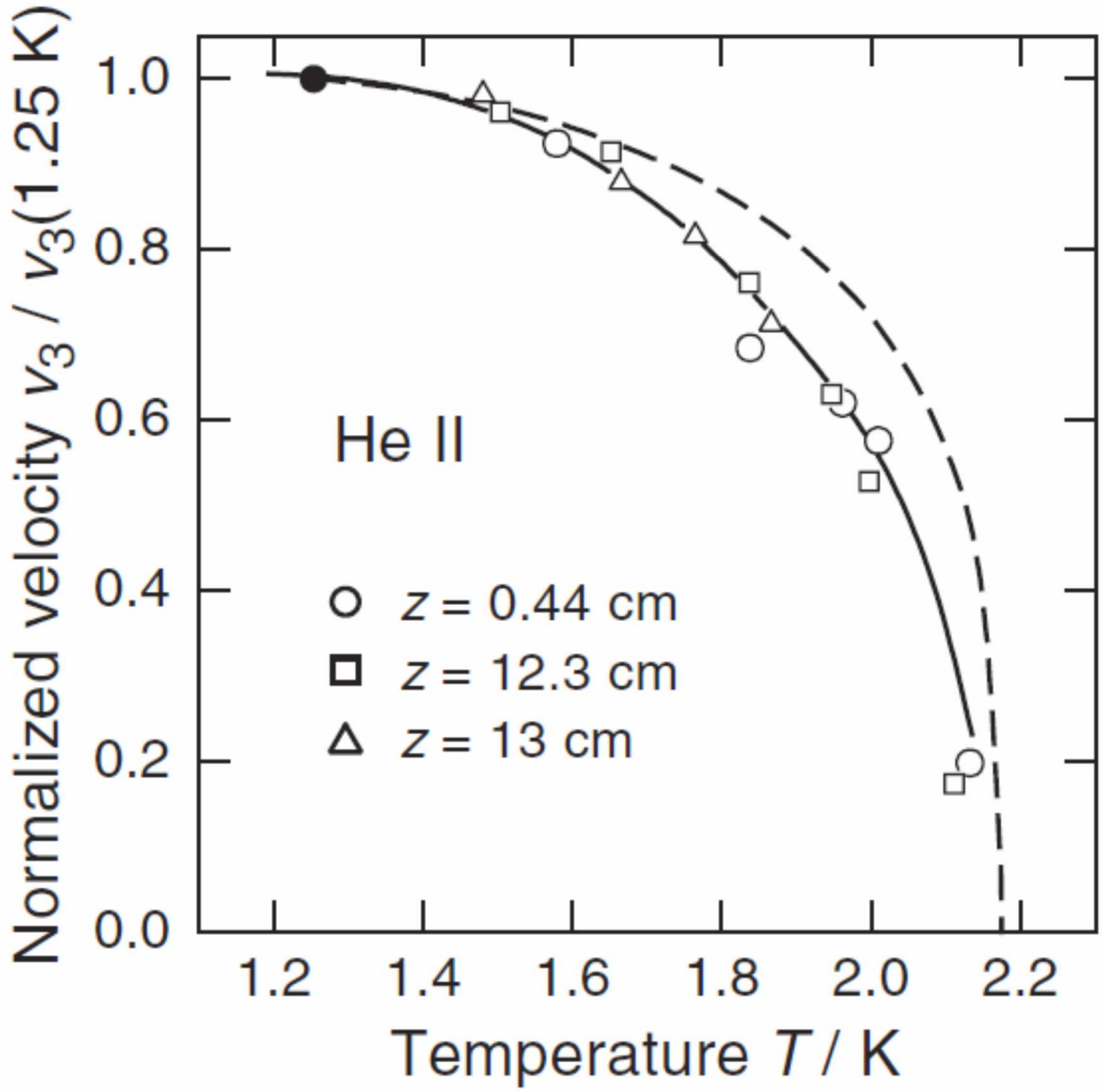
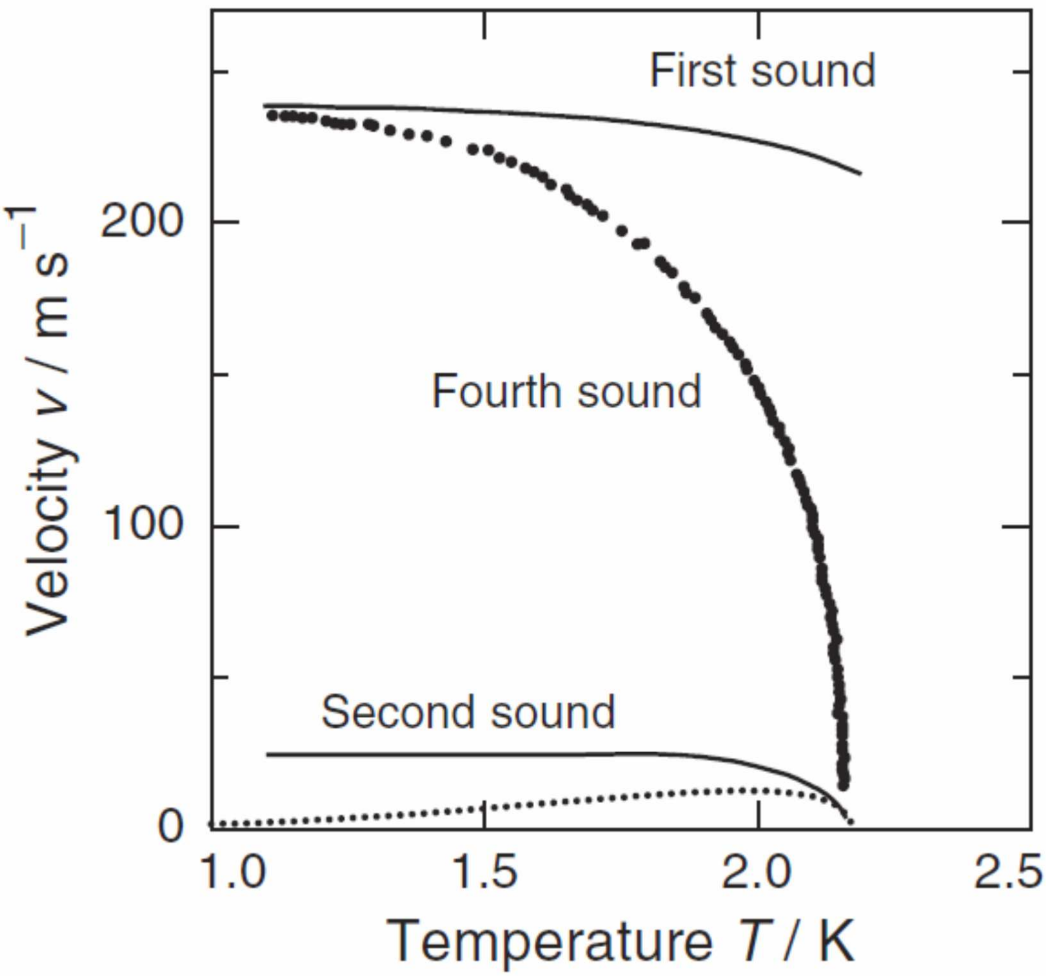


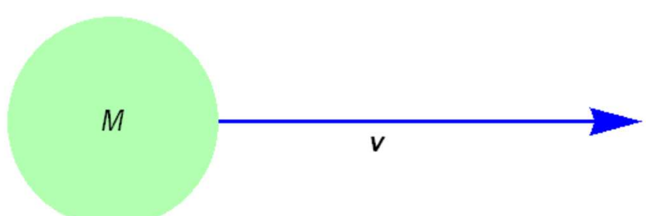
Fig. Velocity of second sound in He II as a function of temperature. The solid line shows the theoretical prediction.

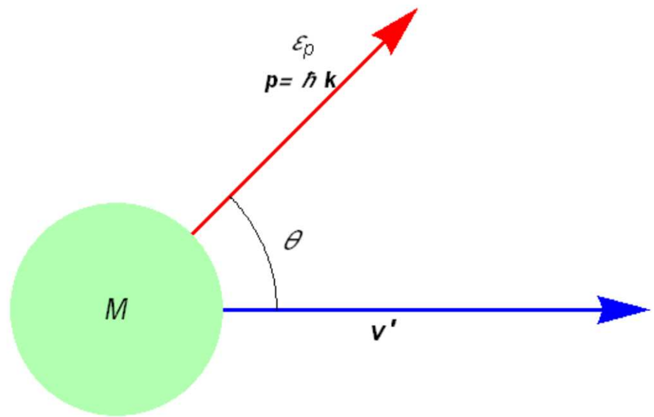


Normalized velocity v_3 / v_3(1.25 K)
1.0
0.8
0.6
0.4
0.2
0.0
1.2
1.4
1.6
1.8
2.0
2.2
Temperature T / K
He II
z = 0.44 cm
z = 12.3 cm
z = 13 cm



19. Critical velocity





Concept of the critical velocity

$$Mv = Mv' + \boldsymbol{p} \qquad \text{(Momentum conservation)}$$

$$\frac{1}{2}M\boldsymbol{v}^2 = \frac{1}{2}M\boldsymbol{v}'^2 + \varepsilon_{\boldsymbol{p}} \qquad \text{(Energy conservation)}$$

Using the relation

$$v' = v - \frac{\boldsymbol{p}}{M}$$

we have

$$\frac{1}{2}M\boldsymbol{v}^2 = \frac{1}{2}M(\boldsymbol{v} - \frac{\boldsymbol{p}}{M})^2 + \varepsilon_{\boldsymbol{p}} = \frac{1}{2}M\boldsymbol{v}^2 + \frac{\boldsymbol{p}^2}{2M} - \boldsymbol{v}\cdot\boldsymbol{p} + \varepsilon_{\boldsymbol{p}}$$

or

$$\frac{\boldsymbol{p}^2}{2M} = \boldsymbol{v}\cdot\boldsymbol{p} - \varepsilon_{\boldsymbol{p}}$$

We assume that M is so large that the $\frac{\boldsymbol{p}^2}{2M}$ can be neglected. If ϕ is the angle between p and v, we then have

$$\varepsilon_{\boldsymbol{p}} = \boldsymbol{v}\cdot\boldsymbol{p} = vp\cos\phi \le vp$$

Thus the condition

$$v \geq \frac{\varepsilon_p}{p}$$

must be satisfied for excitation to be created. Thus the critical velocity is given by

$$v_c = \left[\frac{\varepsilon_p}{p}\right]_{\min}$$

The superfluidity can therefore occur if

$$v_c > 0$$

a condition which is known as the Landau criterion for superfluidity.

The energy dispersion relation was determined using inelastic neutron scattering. The energy spectrum of the rotons is described by

$$\varepsilon = \Delta + \frac{1}{2m^*}(p - p_0)^2$$

where m^* denotes the effective mass of a helium atom.

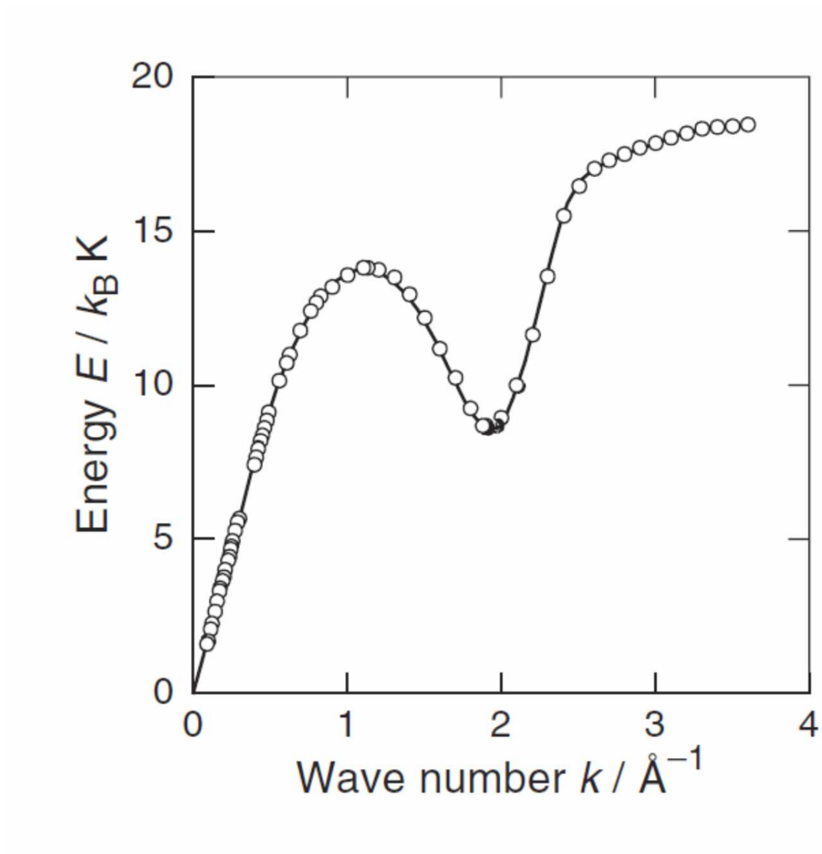


Fig. Dispersion curve of He II as determined experimentally (inelastic neutron scattering)

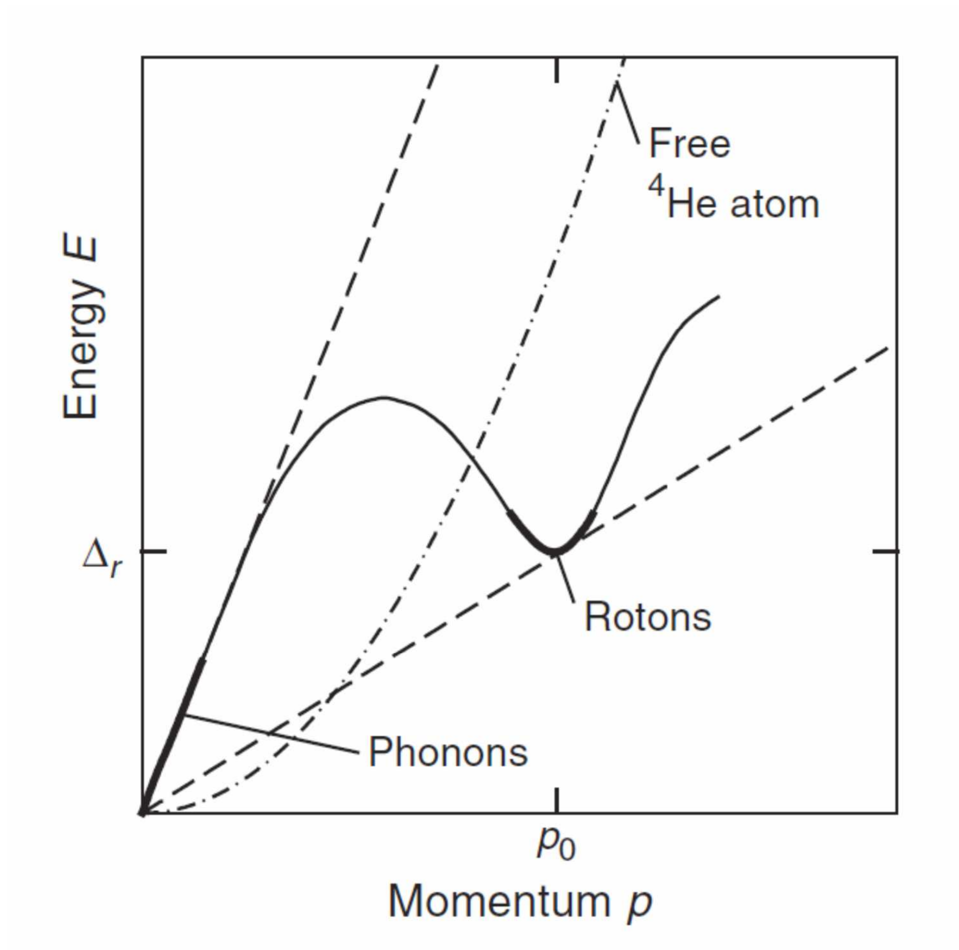


Fig. Phonon-roton spectrum in comparison with free ^{4}He atoms. The two dashed lines are tangents to the dispersion curve and reflect the critical velocities of phonons and rotons.

At $T = 1$ K, one finds that

$$\frac{\Delta}{k_B} = 8.67 \ \text{K}, \qquad \frac{p_0}{\hbar} = 1.94 \ \text{Å}^{-1}, \qquad m^* = 0.15 m_4$$

We note that the critical velocity is evaluated as

$$8\, v_c = \frac{\Delta}{p_0} = 58.5 \text{ m/s} \qquad \text{(roton).}$$

((**Mathematica**))

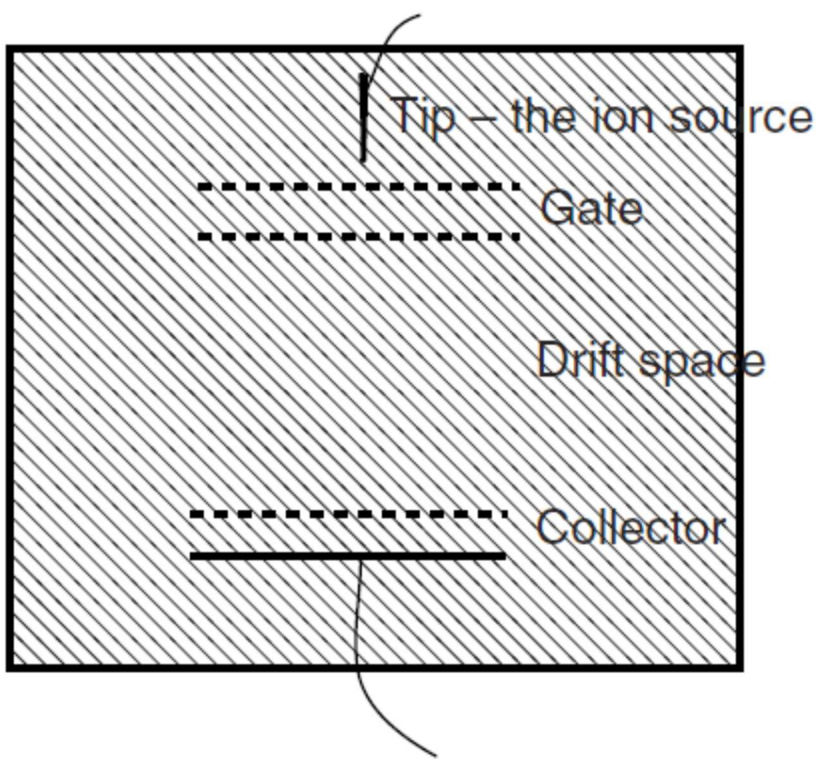
```
Clear["Global`*"];
rule1 = {kB → 1.3806504 × 10^-16,
   ℏ → 1.054571628 10^-27, Å → 10^-8, THz → 10^12,
   α → 7.2973525376 × 10^-3, m → 10^2};

vc = (8.67 kB)/(1.94 ℏ Å^-1 m) //. rule1

58.5093
```

((Ion mobility and the Landau critical velocity))

An direct measurement of significance is the study of ions in liquid helium. In this type of experiment, ions are injected into the liquid by applying a high voltage to a fine metal tip. The ion motion to a collector electrode can be controlled and studied with suitably arranged grids set at specific potential.



The measurement of negative ion mobility provides an experimental verification of the critical velocity. There is virtually no dissipation seen until the Landau critical velocity is reached. This a direct measurement of the critical velocity, and the magnitude agrees remarkably well with that calculated from the dispersion relation.

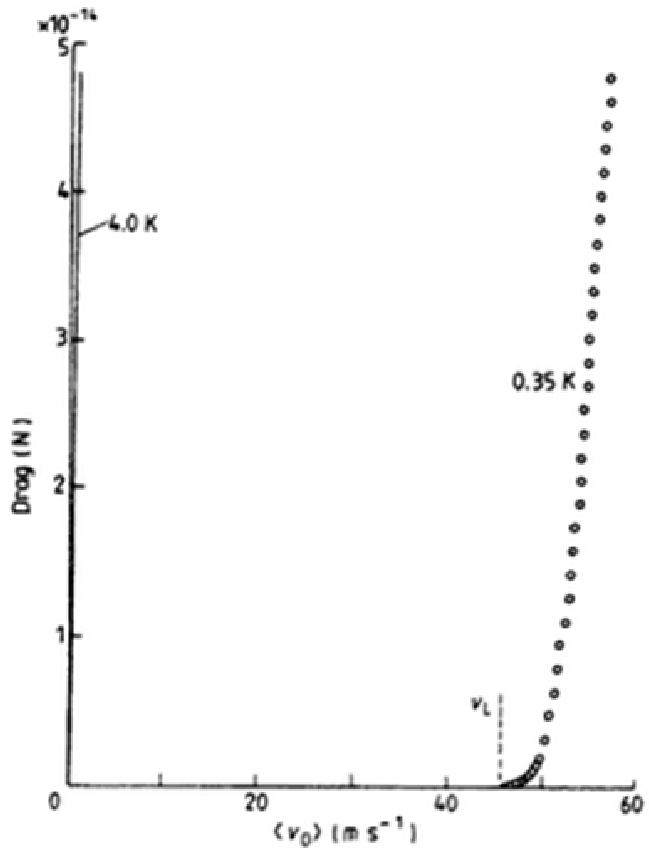


Fig. The critical velocity determined experimentally. $T = 0.35$ K.

20. DC and AC Josephson effect

Here we consider two vessels containing He II that are connected by a weak link. Suppoe that there is a difference in the chemical potential such that

$$\Delta\mu = \mu_2 - \mu_1.$$

Let ψ_1 and ψ_2 be the probability amplitude of macroscopic wave function on either side of the aperture. We can write for the Schrödinger equation for the two vessels,

$$i\hbar\frac{\partial\psi_1}{\partial t} = \mu_1\psi_1 + \kappa\psi_2, \qquad i\hbar\frac{\partial\psi_2}{\partial t} = \mu_2\psi_2 + \kappa\psi_1$$

or

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\mu_1 & \kappa\\ \kappa & \mu_2\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

where κ represents the coupling across the weak link.

Let

$$\psi_1 = \sqrt{\rho}e^{i\phi_1}, \qquad \psi_2 = \sqrt{\rho}e^{i\phi_2}$$

where

$$|\psi_1|^2 = \rho, \qquad |\psi_2|^2 = \rho$$

where ϕ_1 and ϕ_2 are the phases of the two wave functions. We solve the problem using Mathematica.

((Mathematica))

```
Clear["Global`*"];
eq1 = ⅈ ℏ D[ψ1[t], t] == μ1 ψ1[t] + κ ψ2[t];
eq2 = ⅈ ℏ D[ψ2[t], t] == μ2 ψ2[t] + κ ψ1[t];
rule1 = {ψ1 → (Sqrt[ρ[#]] Exp[ⅈ ϕ1[#]] &)};
rule2 = {ψ2 → (Sqrt[ρ[#]] Exp[ⅈ ϕ2[#]] &)};
eq11 = eq1 /. rule1 /. rule2 // FullSimplify
```

$$\frac{1}{\sqrt{\rho[t]}}\left(2\, e^{i\,\phi2[t]}\,\kappa\,\rho[t] + e^{i\,\phi1[t]}\,(-i\,\hbar\,\rho'[t] + 2\,\rho[t]\,(\mu1 + \hbar\,\phi1'[t]))\right) == 0$$

```
eq21 = eq2 /. rule1 /. rule2 // FullSimplify
```

$$\frac{1}{\sqrt{\rho[t]}}\left(2\, e^{i\,\phi1[t]}\,\kappa\,\rho[t] + e^{i\,\phi2[t]}\,(-i\,\hbar\,\rho'[t] + 2\,\rho[t]\,(\mu2 + \hbar\,\phi2'[t]))\right) == 0$$

Then we have

$$-i\hbar\dot{\rho} + 2\hbar\rho\dot{\phi}_1 + 2\rho\mu_1 + 2\kappa\rho e^{i\delta} = 0 \qquad (1)$$

$$-i\hbar\dot{\rho} + 2\hbar\rho\dot{\phi}_2 + 2\rho\mu_2 + 2\kappa\rho e^{-i\delta} = 0 \qquad (2)$$

where the phase difference is defined by

$$\delta = \phi_2 - \phi_1 .$$

Now equate the real and imaginary parts of Eqs.(1) and (2),

$$\dot{\rho} = \frac{2\kappa\rho}{\hbar}\sin\delta \qquad (3)$$

and

$$\hbar\dot{\phi}_1 + \mu_1 + \kappa\cos\delta = 0$$

$$\hbar\dot{\phi}_2 + \mu_2 + \kappa\cos\delta = 0$$

leading to

$$\dot{\delta} = -\frac{1}{\hbar}(\mu_2 - \mu_1) = -\frac{1}{\hbar}\Delta\mu \tag{4}$$

For $\Delta\mu$ =0 we have a constant phase difference $\delta = \phi_2 - \phi_1$ that results in a stationary mass flow without any pressure applied. This phenomenon is called the *dc Josephson effect*. For $\Delta\mu \neq 0$ we find an oscillating mass flow with frequency

$$\omega_J = \frac{1}{\hbar}\Delta\mu\, t$$

This phenomenon is called the *ac Josephson effect*.

21. Experiment on Josephson effect in liquid He II

Phys. Rev. Letts. <u>106</u>, 055302 (2011)
Quantum Coherence in a Superfluid Josephson Junction
by S. Narayana and Y. Sato

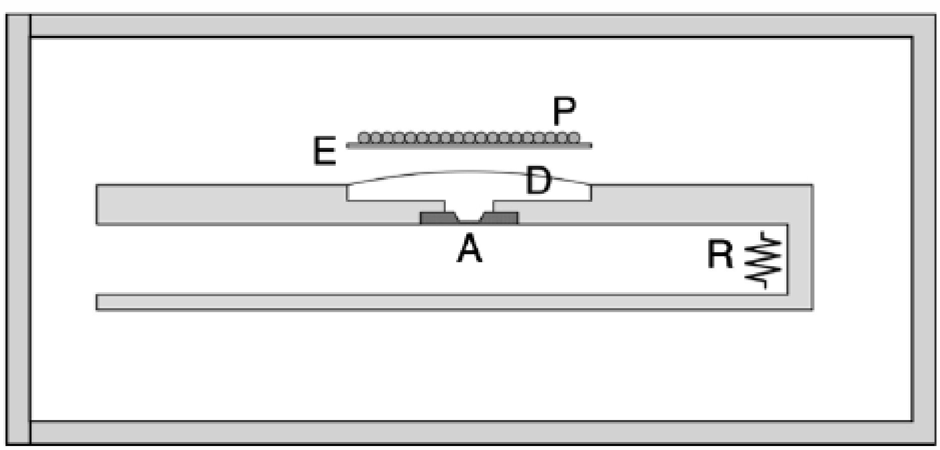


Fig.1 Experimental apparatus. A flexible diaphragm (D) and a rigid electrode (E) form an electrostatic pressure pump. The diaphragm also forms the input element of a sensitive

displacement sensor through a nearby pickup coil (P) connected to a dc- SQUID (not shown). A heater (R) is used to induce quantum phase gradients across an aperture array (A).

The experimental apparatus is schematically shown in Fig.1. Unshaded regions are filled with superfluid ^{4}He, and the entire apparatus is immersed in a temperature regulated ^{4}He bath. Two reservoirs of superfluid ^{4}He are coupled through an array of apertures (A). In an ideal weak coupling limit, the mass current $I(t)$ across the junction driven by a chemical potential difference $\Delta\mu$ is governed by the Josephson current-phase relation

$$I = I_0 \sin\theta \,,$$

where I_0 is the junction critical current, and θ is the quantum phase difference across the junction. The phase difference θ evolves in time according to the Anderson phase evolution equation

$$\dot{\theta} = -\frac{\Delta\mu}{\hbar} \,.$$

A constant chemical potential difference $\Delta\mu$ counterintuitively leads to oscillatory mass current at the junction

$$I(t) = I_0 \sin\left(\frac{\Delta\mu}{\hbar} t\right) \qquad \text{(AC Josephson effect).}$$

To observe such Josephson phenomena, a ‘‘weak’’ coupling must be established between two quantum fluids. This is achieved by using an aperture whose size matches the superfluid healing length ξ_4. Near the superfluid transition temperature $T_\lambda = 2.17K$, the correlation length ξ_4 diverges as

$$\xi_4 = 0.4\left(1 - \frac{T}{T_\lambda}\right)^{-0.67}, \qquad \text{(nm).}$$

Because of this property, various Josephson phenomena emerge at temperatures roughly 1 – 10 mK away from T_λ if the aperture size is 50 nm. A Josephson mass current signal from a single ≈50 nm aperture is on the order of 10^{-16} kg/sec, too small to detect. Furthermore, large thermally induced fluctuations in superfluid order parameter phase at 2 K are expected to destroy the Josephson effect even if one had the capability to detect such a small signal.

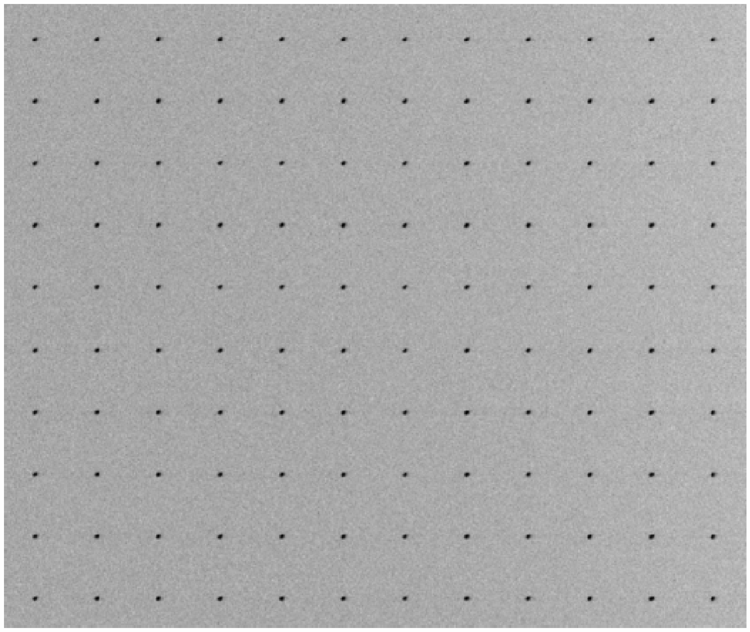


Fig.2 SEM image of aperture array. Each black dot is a 60 nm diameter aperture spaced 2 μm apart from each other.

Typically, 5000 to 10 000 apertures (each one ≈50 nm in size) have been utilized to raise the overall signal to a detectable level. In such experiments, phase fluctuations expected to wipe out the Josephson effect in a single aperture have shown no adverse effect. This has led to a model where thermal fluctuations are shared among N junctions with their phases rigidly locked together. In a weakly coupled regime, individual natures of various apertures have never been observed in any dynamical behaviors. All experiments done up to this point have always shown that thousands of apertures are amazingly locked together and act as a ‘‘single junction’’ in the Josephson regime.

An array used in this experiment consists of 75 x 75 60 nm apertures spaced on a 2 μm square lattice e-beam lithographed in a 60 nm-thick silicon nitride window. An SEM image of the array is shown in Fig. 2. As can be seen in the apparatus schematic (Fig. 1), an aperture array is configured as a part of a wall surface of a horizontal channel. The fluid volume within the channel is significantly smaller than the volume outside, and the outer can of the apparatus is well heat sunk to an even larger helium bath. Therefore, a resistor (R) placed at the end of a channel works as a local heat source while the fluid outside the channel behaves as a heat sink. When power $\dot{Q}$ is applied to the heater, superfluid fraction of the fluid (with density ρ_s) flows towards the heat source while the normal component (with density ρ_n) flows away carrying all the entropy. Landau's two-fluid model predicts superfluid velocity to be v_s

$$\dot{v}_s = \frac{\rho_n}{m_4 \rho \rho_s T \sigma} \dot{\boldsymbol{Q}}$$

where ρ is the total fluid density, σ is the specific entropy, and T is the temperature. London's wave function view of the condensate leads to superfluid velocity related to quantum phase gradient by

$$v_s = \frac{\hbar}{m_4} \nabla \theta \,.$$

A phase gradient is then induced along the channel (and along the aperture array):

$$\nabla \dot{\theta} = \frac{1}{\hbar} \frac{\rho_n}{\rho \rho_s T \sigma} \dot{\boldsymbol{Q}} \,. \tag{1}$$

This apparatus configuration allows us to directly apply finite phase gradient along the array of apertures, giving us an unique opportunity to probe their collective dynamics. In operation, we apply a pressure difference (and hence a chemical potential difference) across the aperture array by pulling a diaphragm [labeled (D) in Fig. 1] towards a nearby electrode (E). In response, the array exhibits Josephson mass current oscillation. The diaphragm motion (indicating fluid flow through the array) is detected using a dc-SQUID based displacement transducer. We record the overall mass current oscillation amplitude while applying finite external phase gradient [Eq. (1)] along the array in an attempt to unlock their phases and reveal their individuality.

In Fig. 3, we plot the overall mass current oscillation amplitude as a function of heat input in the channel for three different temperatures. If all the apertures are indeed phase locked due to strong coupling or interactions, the oscillation amplitude from the array should remain constant. However, as the heater power is increased, we find that the oscillation amplitude varies. The surprisingly smooth and nonchaotic behavior implies that different apertures maintain temporal coherence of Josephson oscillations with a well-defined frequency of $\Delta\mu / h$ in the background of externally applied phase gradients.

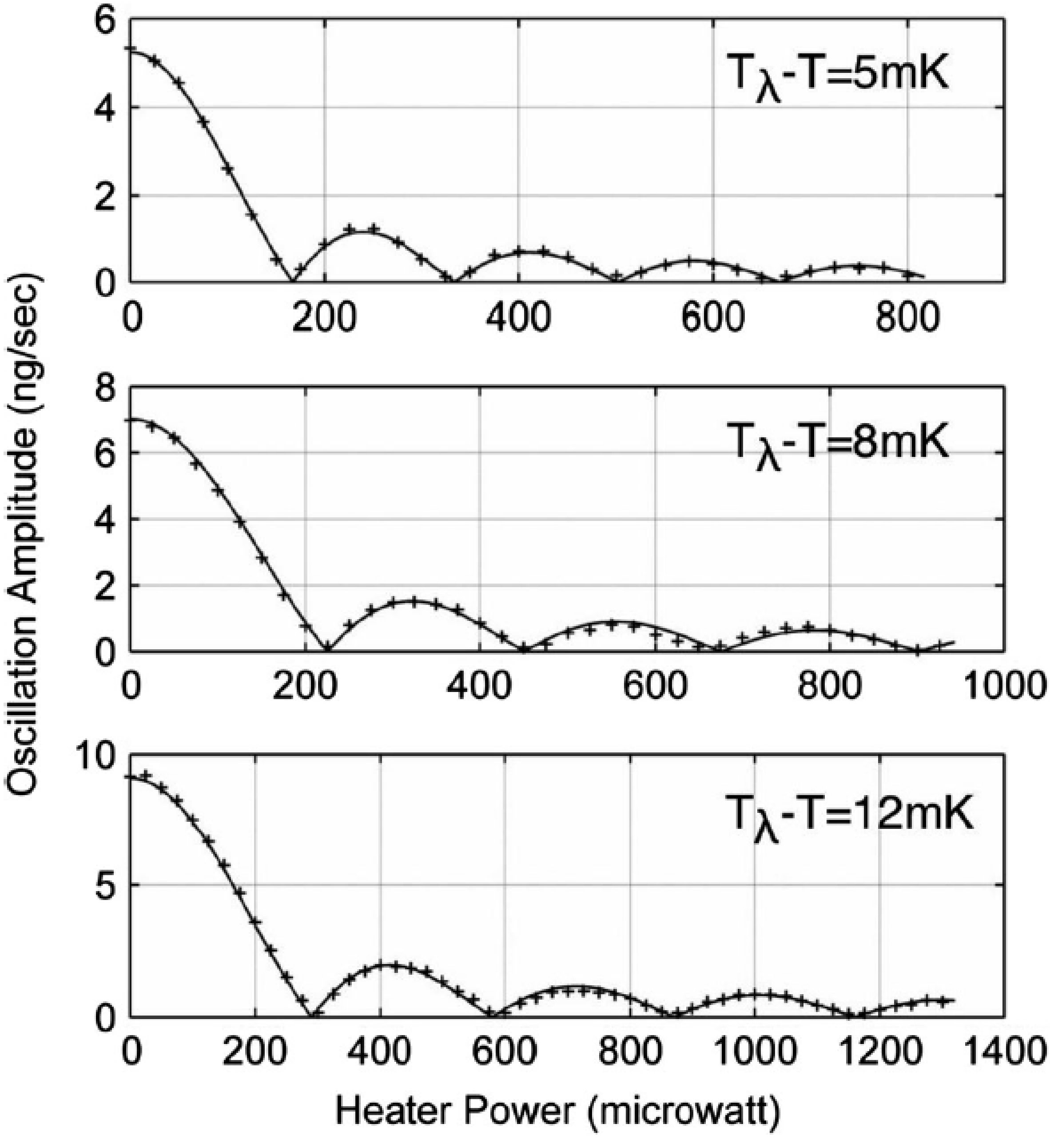


Fig.3 Oscillation amplitude as a function of heater power. The solid lines are fits. At power levels higher than what is shown here, turbulence sets in, rendering the oscillation amplitude measurement difficult

((**Note**))

$$\rho_n \dot{v}_n + \rho_s \dot{v}_s = 0 \qquad \text{(from } \frac{\partial j}{\partial t} = 0 \text{).}$$

$$\dot{Q} = m_4 sT\rho\dot{v}_n = m_4 sT\rho(-\frac{\rho_s}{\rho_n}\dot{v}_s)$$ (the normal component has the entropy)

$$\dot{v}_s = -\frac{\rho_n}{m_4 sT\rho\rho_s}\dot{Q}$$

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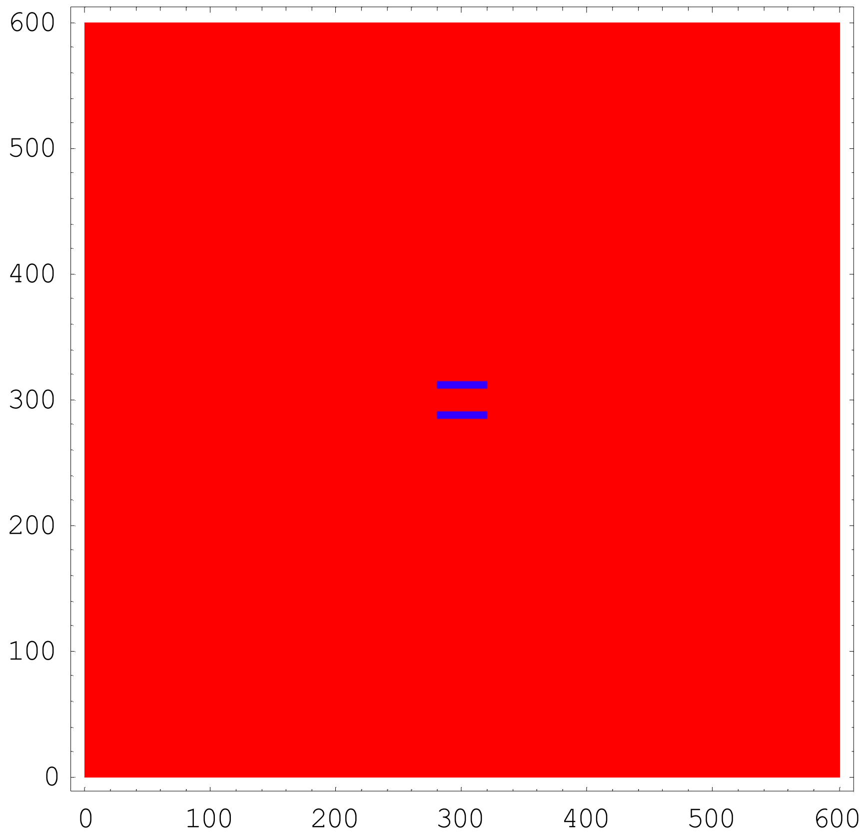
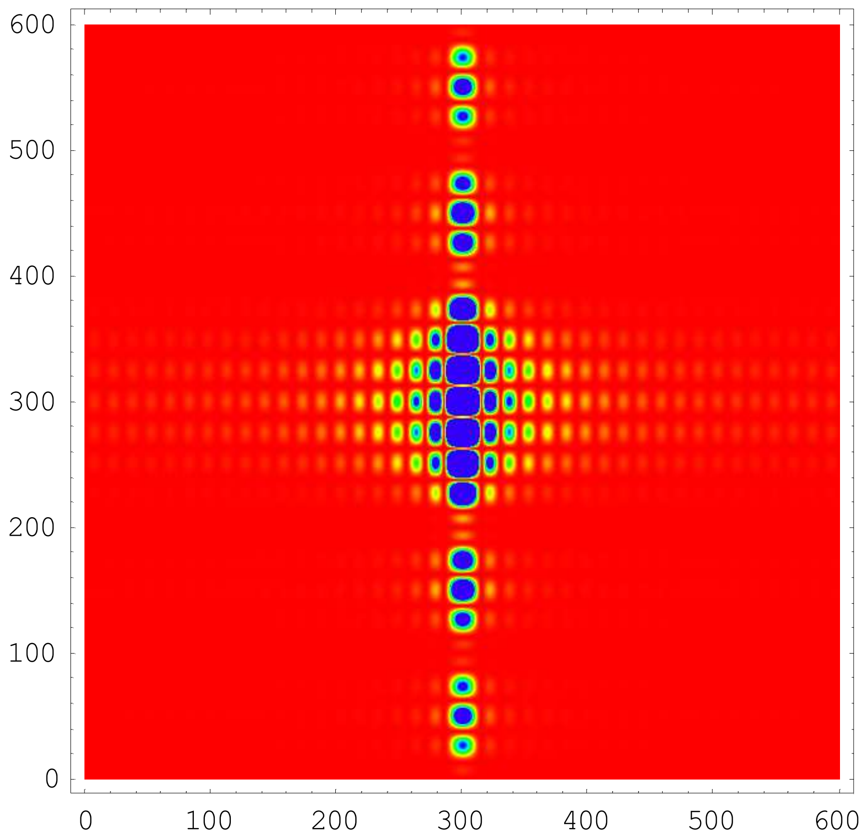
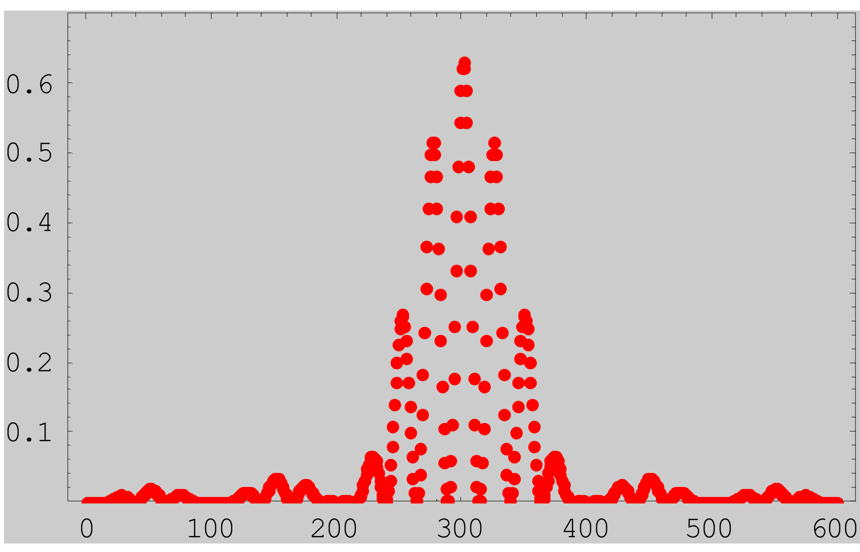
APPENDIX Fraunhofer diffraction pattern

((**Mathematica**))



600
500
400
300
200
100
0
0
100
200
300
400
500
600





APPENDIX-II Important equations equations

$$\nabla \cdot \boldsymbol{j} + \frac{\partial \rho}{\partial t} = 0 \qquad \text{(equation of continuity)} \qquad (1)$$

$$\frac{\partial}{\partial t}\boldsymbol{j} + \nabla P = 0 \qquad \text{(Euler equation)} \qquad (2)$$

Usual hydrodynamic equation of continuity for the liquid as whole and has the effect of ensuring conservation of mass.

$$\boldsymbol{j} = \rho_n \boldsymbol{v}_n + \rho_s \boldsymbol{v}_s \qquad (3)$$

$$\rho = \rho_n + \rho_s \qquad (4)$$

$$\frac{\partial}{\partial t}\boldsymbol{v}_s = \sigma \nabla T - \frac{1}{\rho}\nabla P \qquad (5)$$

$$\nabla \cdot (\rho \sigma \boldsymbol{v}_n) + \frac{\partial}{\partial t}(\rho \sigma) = 0 \qquad (6)$$

Conservation of entropy on the assumption that viscous effects are negligible.

(i)

$$\frac{\partial^2 \rho}{\partial t^2} = \nabla^2 P \qquad (7)$$

((**Proof**))

From Eqs.(1) and (2)

$$\frac{\partial^2 \rho}{\partial t^2} = -\nabla \cdot \frac{\partial \boldsymbol{j}}{\partial t} = \nabla^2 P$$

(ii)

$$\nabla \cdot (\boldsymbol{v}_n - \boldsymbol{v}_s) = -\frac{\rho}{\rho_s \sigma}\frac{\partial \sigma}{\partial t} \qquad (8)$$

((**Proof**))

$$\begin{aligned}\rho_s\nabla\cdot(\boldsymbol{v}_n-\boldsymbol{v}_s) &= \rho_s\nabla\cdot\boldsymbol{v}_n-\rho_s\nabla\cdot\boldsymbol{v}_s \\ &= \rho_s\nabla\cdot\boldsymbol{v}_n-(-\rho_n\nabla\cdot\boldsymbol{v}_n-\frac{\partial\rho}{\partial t}) \\ &= (\rho_n+\rho_s)\nabla\cdot\boldsymbol{v}_n+\frac{\partial\rho}{\partial t} \\ &= \rho\nabla\cdot\boldsymbol{v}_n+\frac{\partial\rho}{\partial t}\end{aligned}$$

since

$$\nabla\cdot(\rho_n\boldsymbol{v}_n+\rho_s\boldsymbol{v}_s)+\frac{\partial\rho}{\partial t}=0$$

or

$$\rho_n\nabla\cdot\boldsymbol{v}_n+\rho_s\nabla\cdot\boldsymbol{v}_s+\frac{\partial\rho}{\partial t}=0$$

from Eq.(2). Then we get

$$\nabla\cdot(\boldsymbol{v}_n-\boldsymbol{v}_s)=\frac{\rho}{\rho_s}\nabla\cdot\boldsymbol{v}_n+\frac{1}{\rho_s}\frac{\partial\rho}{\partial t} \qquad \text{(a)}$$

For Eq.(6), we have

$$\nabla\cdot(\rho\sigma\boldsymbol{v}_n)=-\frac{\partial}{\partial t}(\rho\sigma)$$

or

$$\nabla\cdot\boldsymbol{v}_n=-\frac{\rho}{\rho\sigma}\frac{\partial}{\partial t}\sigma=-\frac{1}{\sigma}\frac{\partial}{\partial t}\sigma \qquad \text{(b)}$$

From (a) and (b), we get

$$\nabla\cdot(\boldsymbol{v}_n-\boldsymbol{v}_s)=-\frac{\rho}{\rho_s\sigma}\frac{\partial}{\partial t}\sigma+\frac{1}{\rho_s}\frac{\partial\rho}{\partial t}$$

When $\frac{\partial \rho}{\partial t} = 0$, we have

$$\nabla \cdot (\boldsymbol{v}_n - \boldsymbol{v}_s) = -\frac{\rho}{\rho_s \sigma} \frac{\partial}{\partial t} \sigma$$

(iii)

$$\rho_n \frac{\partial}{\partial t} \nabla \cdot (\boldsymbol{v}_n - \boldsymbol{v}_s) = -\rho \sigma \nabla^2 T \qquad (9)$$

((**Proof**))
From Eqs.(5) and (2)

$$\begin{aligned} \rho \dot{\boldsymbol{v}}_s &= \rho \sigma \nabla T - \nabla P \\ &= \rho \sigma \nabla T + \frac{\partial}{\partial t} \boldsymbol{j} \\ &= \rho \sigma \nabla T + \frac{\partial}{\partial t} (\rho_n \boldsymbol{v}_n + \rho_s \boldsymbol{v}_s) \\ &= \rho \sigma \nabla T + \rho_n \dot{\boldsymbol{v}}_n + \rho_s \dot{\boldsymbol{v}}_s \end{aligned}$$

or

$$(\rho_n + \rho_s) \dot{\boldsymbol{v}}_s = \rho \sigma \nabla T + \rho_n \dot{\boldsymbol{v}}_n + \rho_s \dot{\boldsymbol{v}}_s$$

Thus we have

$$\rho_n (\dot{\boldsymbol{v}}_n - \dot{\boldsymbol{v}}_s) = -\rho \sigma \nabla T$$

and

$$\rho_n \nabla \cdot (\dot{\boldsymbol{v}}_n - \dot{\boldsymbol{v}}_s) = -\rho \sigma \nabla^2 T$$

(iv)

$$\frac{\partial^2}{\partial t^2} \sigma = \frac{\rho_s}{\rho_n} \sigma^2 \nabla^2 T$$

((**Proof**))
From Eq.(8)

$$\nabla \cdot (\boldsymbol{v}_n - \boldsymbol{v}_s) = -\frac{\rho}{\rho_s \sigma} \frac{\partial \sigma}{\partial t} \tag{8}$$

From Eq.(9),

$$\frac{\partial}{\partial t} \nabla \cdot (\boldsymbol{v}_n - \boldsymbol{v}_s) = -\frac{\rho \sigma}{\rho_n} \nabla^2 T \tag{9}$$

Thus we have

$$\frac{\partial}{\partial t}[\frac{\rho}{\rho_s \sigma} \frac{\partial \sigma}{\partial t}] = \frac{\rho \sigma}{\rho_n} \nabla^2 T$$

If $\frac{\rho}{\rho_s \sigma}$ is independent of *t*,

$$\frac{\partial^2 \sigma}{\partial t^2} = \frac{\rho_s}{\rho_n} \sigma^2 \nabla^2 T$$

APPENDIX III

$$m \frac{\partial}{\partial t} v_s = -\nabla \mu$$

$$\nabla \mu = m(\frac{1}{\rho} \nabla p - \sigma \nabla T)$$

$$\dot{Q} = m \rho T \sigma \dot{v}_n$$

Since

$$\rho_n \dot{\boldsymbol{v}}_n + \rho_s \dot{\boldsymbol{v}}_s = 0$$

we get

$$\dot{Q} = m\rho\sigma T(-\frac{\rho_s}{\rho_n}\dot{v}_s) = -\frac{m\rho\rho_s\sigma T}{\rho_n}\dot{v}_s$$

or

$$\dot{v}_s = -\frac{\rho_n}{m\rho\rho_s\sigma T}\dot{Q}$$
