# Liquid He three
# Pomeranchuck cooling

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**Douglas Dean Osheroff** (born August 1, 1945) is a physicist known for his work in experimental condensed matter physics, in particular for his co-discovery of superfluidity in Helium-3. For his contributions he shared the 1996 Nobel Prize in Physics along with David Lee and Robert C. Richardson.

Osheroff joined the Laboratory of Atomic and Solid State Physics at Cornell University as a graduate student, doing research in low-temperature physics. Together with David Lee, the head of the laboratory, and Robert C. Richardson, Osheroff used a Pomeranchuk cell to investigate the behaviour of $^3$He at temperatures within a few thousandths of a degree of absolute zero. They discovered unexpected effects in their measurements, which they eventually explained as phase transitions to a superfluid phase of $^3$He.[4][5] Lee, Richardson and Osheroff were jointly awarded the Nobel Prize in Physics in 1996 for this discovery.

Osheroff received a Ph.D. from Cornell University in 1973. He then worked at Bell Labs in Murray Hill, New Jersey for 15 years, continuing to research low-temperature phenomena in $^3$He. In 1987 he moved to the Departments of Physics and Applied Physics at Stanford University, where he also served as department chair from 1993-96. His research is focused on phenomena that occur at extremely low temperatures.

https://en.wikipedia.org/wiki/Douglas_Osheroff

A $^3$He atom is a fermion with nuclear spin 1/2. Although $^3$He gas liquefies at low temperatures due to the weak van der Waals force, it does not solidify even at $T = 0$ K, unless the pressure exceeds 33.5 atom (= 3.4 x $10^6$ Pa). While Bose liquid $^4$He becomes superfluid at 2.17 K, $^3$He is a Fermi-degenerate at less than the Fermi temperature $T_F$, and shows the properties of a normal Fermi gas. The specific heat is proportional to T below TF. It is only at an ultra-low temperature in the mK region that the system undergoes a transition to the superfluid state.

The BCS theory suggests that it may possible for $^3$He atoms to form Cooper pairs, just like electrons in superconductors. In this case, the attraction would come from the van der Waals forces. In 1972, using Pomeranchuk cooling, the superfluid of helium-3 was finally discovered by Osheroff_, Richardson and Lee.

## 1. Property of Liquid $^3$He

Liquid $^3$He as a Fermi gas

$$\text{spin } I = \frac{1}{2} \qquad \text{(fermion)}$$

$$\text{Density } \rho = 0.081 \text{ g/cm}^3$$

The Fermi energy

$$\varepsilon_F = \frac{\hbar^2}{2m_0}(3\pi^2 n)^{2/3}$$

where $m_0$ is the mass of $^3$He atom,

$$m_0 = \frac{3.016g}{6.022\times 10^{23}} = 0.5\times 10^{-23} = 5.0\times 10^{-24}g$$

The number density

$$n = \frac{N}{V} = \frac{N}{M}\frac{M}{V} = \frac{\rho}{m_0} = \frac{0.081}{5.0\times 10^{-24}} = 1.62\times 10^{22}/cm^3 .$$

Then the Fermi energy is

$$\varepsilon_F = 6.815\times 10^{-16}\text{erg} = 4.254\times 10^{-4} \text{ eV.}$$

The Fermi temperature

$$T_F = \frac{\varepsilon_F}{k_B} = 4.94 \text{ K.}$$

## 2. The heat capacity, entropy and Pauli spin paramagnetism

The simplest theory regards liquid $^3$He as a degenerate ideal Fermi gas. It is given by

$$C = N\frac{\pi^2 k_B}{2}\frac{T}{T_F}$$

The entropy S is

$$TdS = dQ = CdT$$

or

$$S = \int \frac{C}{T}dT$$
$$= N\frac{\pi^2 k_B}{2}\frac{T}{T_F} - \frac{17}{480}N\pi^4 k_B\left(\frac{T}{T_F}\right)^3$$

The chemical potential:

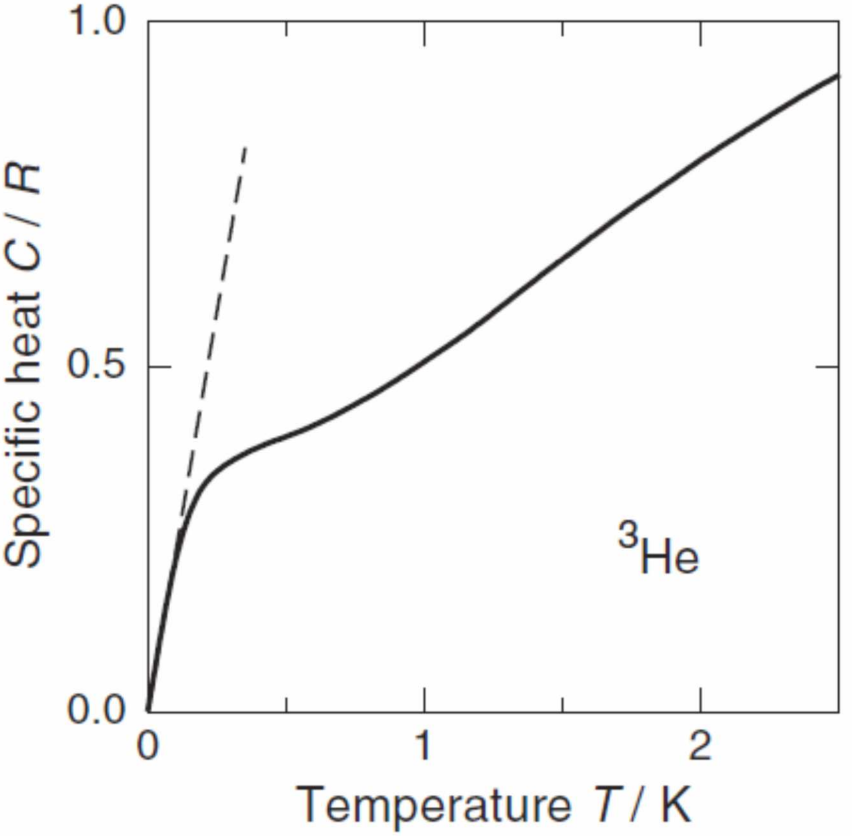
$$\mu = \varepsilon_F[1 - \frac{\pi^2}{12}(\frac{k_B T}{\varepsilon_F})^2]$$

The Pauli paramagnetic susceptibility arising from the nuclear spin is.

$$\chi_p = \mu_B{}^2 D(\varepsilon_F) = \frac{3}{2}\frac{N_A\mu_B{}^2}{k_B}\frac{1}{T_F} = \frac{9}{16}\frac{1}{T_F},$$

since $D(\varepsilon_F) = \frac{3N}{2\varepsilon_F}$, and $\frac{N_A\mu_B{}^2}{3k_B} = \frac{1}{8}$. The susceptibility $\chi_p$ is independent of T. We note that

$$\chi_p T = \frac{9}{16}\frac{T}{T_F} \qquad \text{(emu K/mol)}$$


1.0
0.5
0.0
Specific heat $C$ / $R$
0
1
2
Temperature $T$ / K
$^3$He

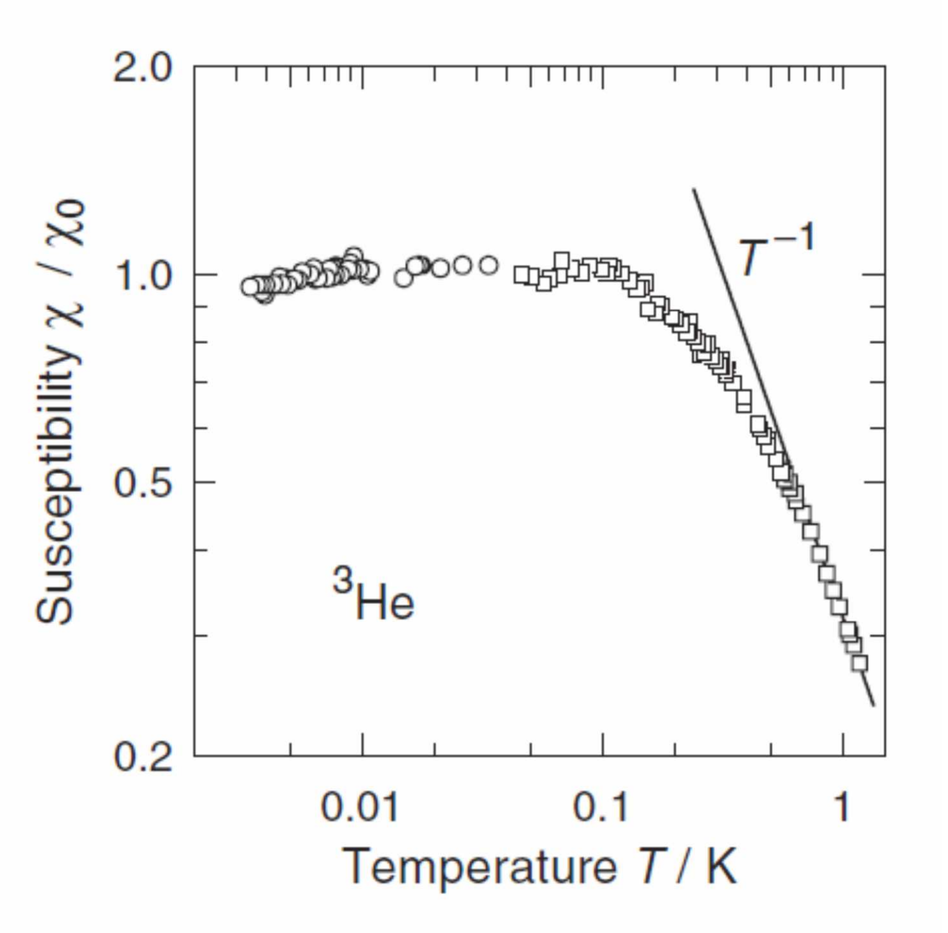
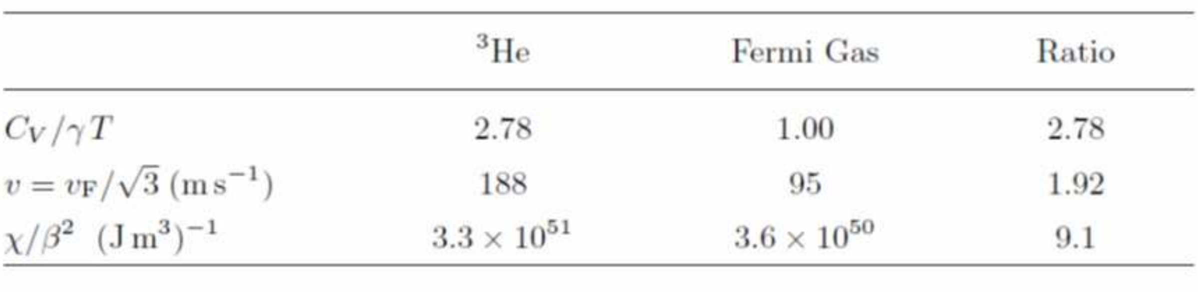


**Table:** Specific heat, sound velocity, and magnetic susceptibility of $^3$He in comparison with an ideal Fermi gas



| | $^3$He | Fermi Gas | Ratio |
|---|---|---|---|
| $C_V/\gamma T$ | 2.78 | 1.00 | 2.78 |
| $v = v_F/\sqrt{3}$ $(\mathrm{m\,s^{-1}})$ | 188 | 95 | 1.92 |
| $\chi/\beta^2$ $(\mathrm{J\,m^3})^{-1}$ | $3.3 \times 10^{51}$ | $3.6 \times 10^{50}$ | 9.1 |

## 3. Quantum concentration $n_Q(T)$

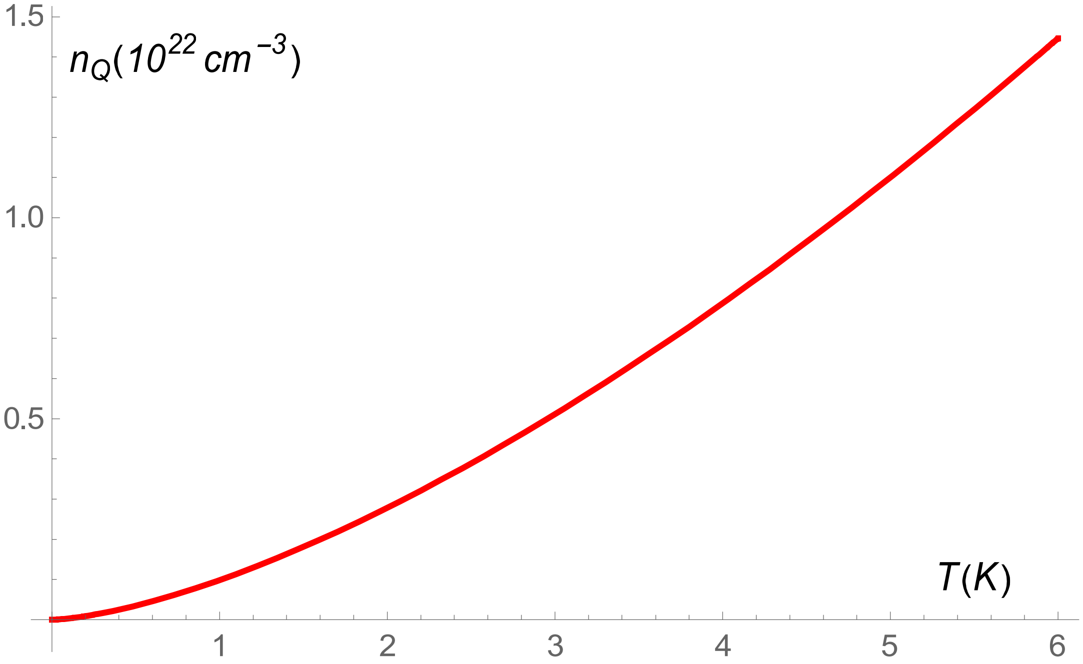
An atom $^3$He has a small mass and we are interested in temperatures $T<1$ K. Thus the quantum theory is needed. Indeed, if we treat liquid $^3$He as a ideal Fermi gas, its Fermi temperature is $T_F = 6K$. This value is based on

$$n = 2.325 \times 10^{22}\ \text{cm}^{-3} \qquad \text{at} \qquad T = 0.3\ \text{K and } P = 29\ \text{am}.$$

It is compared with the quantum concentration,

$$n_Q = \left(\frac{2\pi m_0 k_B T}{h^2}\right)^{3/2} = \left(\frac{m_0 k_B T}{2\pi\hbar^2}\right)^{3/2}$$

We make a plot of $n_Q(T)$ as a function of temperature. At temperatures below the Fermi temperature, the quantum theory is needed.



## 4. Entropy

The entropy is mainly due to the nuclear spin of the solid $^3$He around $T$ = 0.3 K. The $^3$He atoms form a lattice. Below 10 mK the nuclear spins starts to order because of the antiferromagnetic exchange interaction. The nuclear spins antiferromagnetically ordered below the Neel temperature $T_N$ = 2 mK. The antiferromagnetic exchange interaction is -0.85 mK.

For the liquid, the entropy per atom emerges as

$$\frac{S_{liq}}{k_B} = \frac{\pi^2}{2}\frac{T}{T_F}$$

On the other hand, for the solid, the entropy per atom is

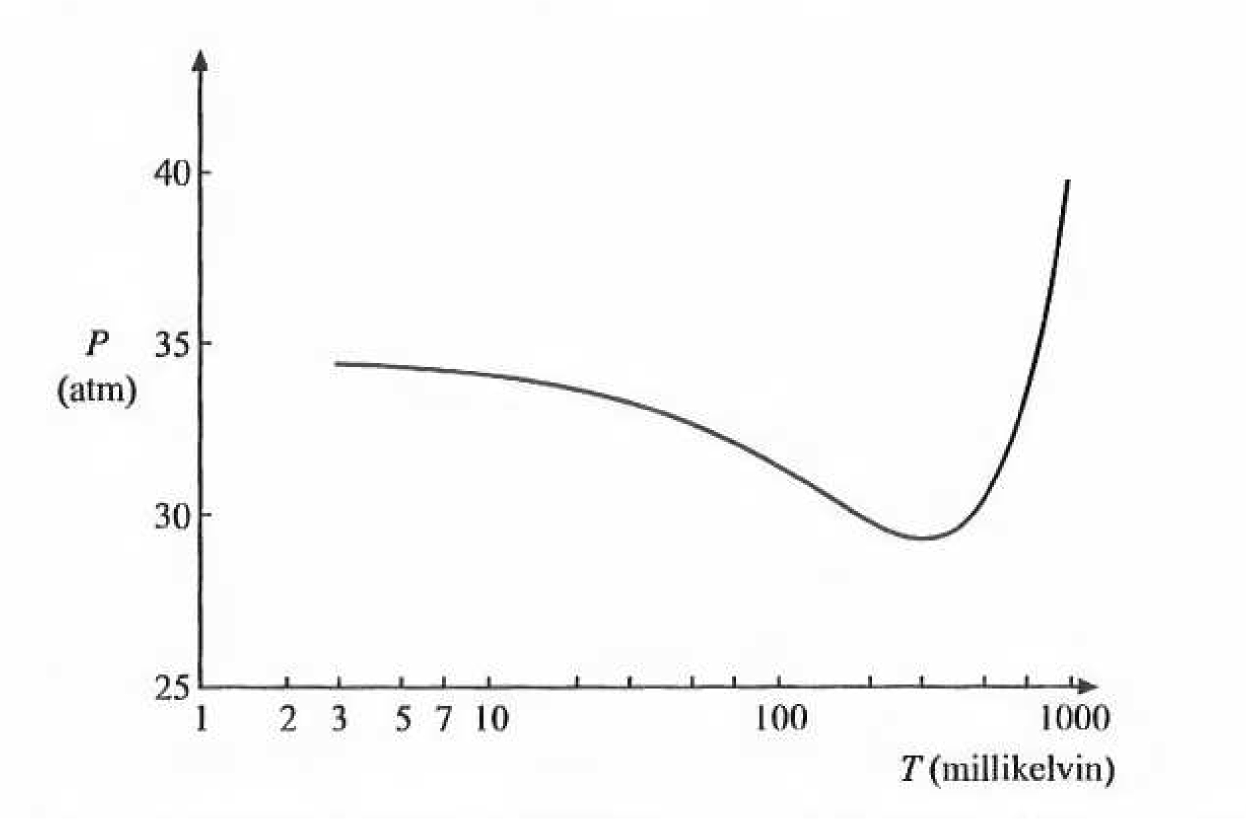
$$\frac{S_{solid}}{k_B} = \ln 2$$

since

$$F = U - ST = -k_B \ln Z_C = -k_B \ln 2$$

reflecting the spin-up state and spin-down state of the nuclear spin ( $I = 1/2$ ).

Above 0.3 K, $S_{liq}$ is larger than $S_{solid}$, while below 0.3 K $S_{solid}$ is larger than $S_{liquid}$ .

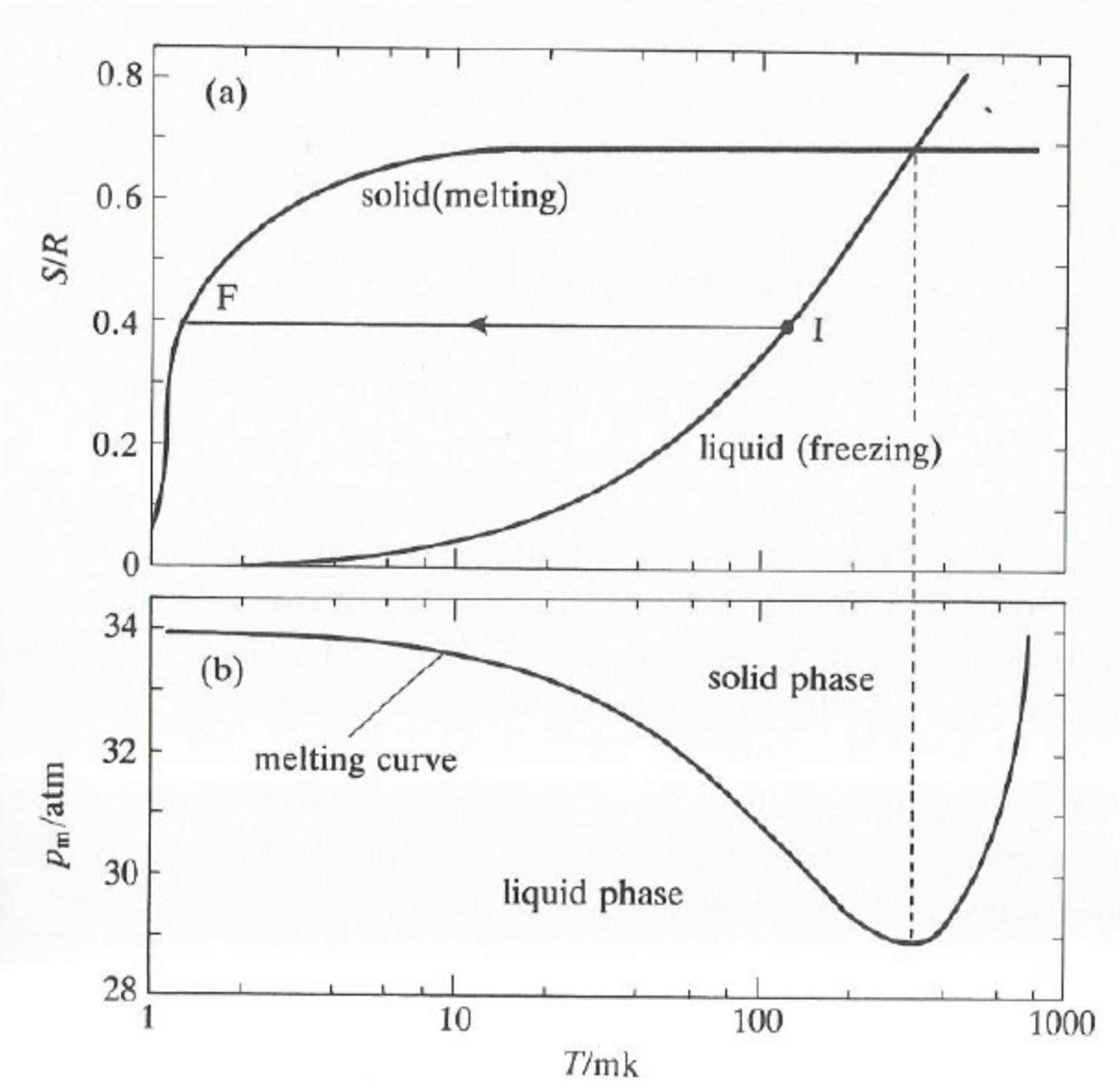


Note that

$$V_{liq} > V_{Solid}$$

For example, at $T = 0.3$ K, it is determined experimentally that

$$\frac{V_{liq} - V_{Solid}}{N} = 2.1 \times 10^{-24} cm^3 / atom$$



## 5. Clausius-Clapeyron equation

In the *P*-*V* phase diagram, the solid-liquid boundary below 0.3 K has a negative slope. This is very unusual. Note that the solid-liquid boundary for most materials (except water) has a positive slope. What are the entropies for the liquid phase and solid phase? The $^3$He atom is a fermion. The heat capacity is proportional to *T* at low temperatures

$$C_{liquid} = N_A \frac{\pi^2}{2} k_B \left( \frac{T}{T_F} \right)$$

The entropy is evaluated as

$$S_{liquid} = \int \frac{C_{liquid}}{T} dT = N_A \frac{\pi^2}{2} k_B \left( \frac{T}{T_F} \right)$$

where $T_F$ is the Fermi temperature. The entropy is proportional to $T$.

The entropy of solid $^3$He is dominated by the much larger contribution of the disordered nuclear spins (spin 1/2). Each the nuclear spin (spin 1/2) of $^3$He atom has a magnetic moment, just like a paramagnetic salts. In the paramagnetic state where the directions of spins are random, the solid entropy is given by

$$S_{soilid} = N_A k_B \ln 2 .$$

Note that these spins are antiferromagnetically ordered with a Neel temperature $T_N \approx 1$ mK and the entropy drops rapidly to zero. The liquid entropy coincides with the solid entropy at a characteristic temperature $T_1$ (=0.32 K). Below $T_1$, the solid entropy is higher (more disordered) than the liquid entropy. Above $T_1$ the liquid entropy (more disordered) is higher than the solid entropy. Such a cross-over of the liquid and solid entropies of $^3$He at about 0.32 K produces a pronounced minimum in the melting curve at about 2.93 MPa, followed by a rise in the melting pressure to 3.45 MPa at $T = 0$ K. This negative slope is used to produce adiabatic compressional cooling along the melting curve, a technique called Pomeranchuk cooling after its proposer, and used by Osheroff *et al*. (1972a) in their discovery of the superfluid phases of $^3$He.

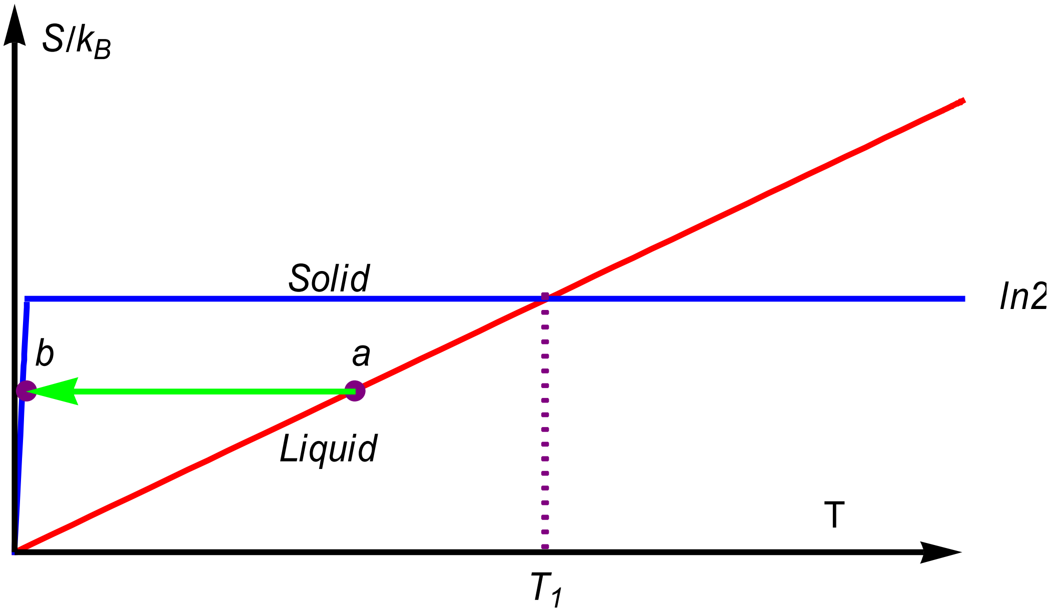
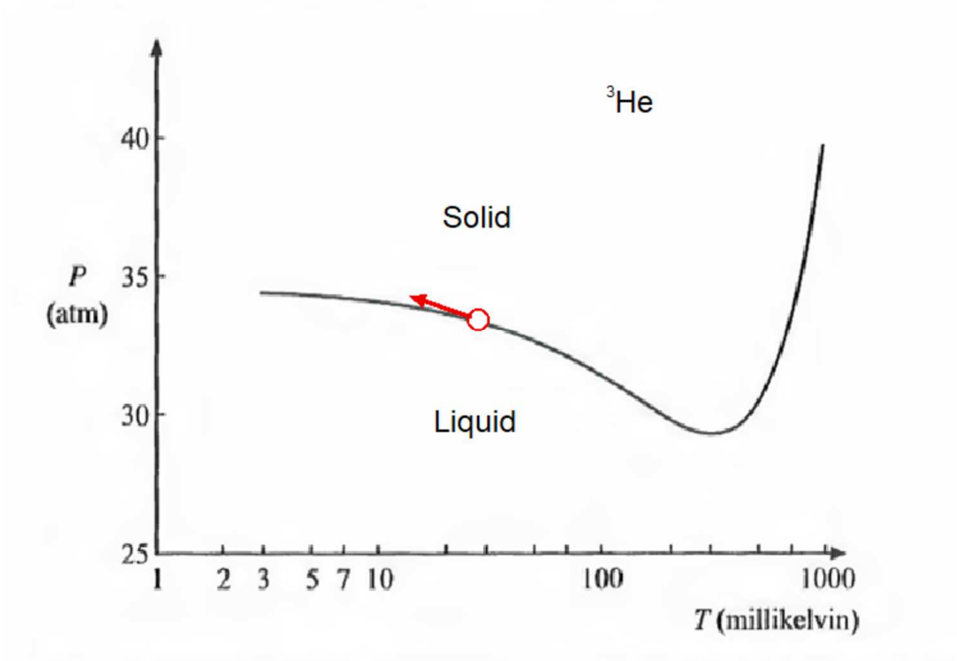


**Fig.** Entropy per atom in the coexisting solid and liquid phases of $^3$He. The entropy of the liquid phase is less than that of the solid phase below a characteristic temperature $T_1 = \frac{2\ln 2}{\pi^2} T_F$. The dotted line from *a* to *b* corresponds to slow adiabatic compression from pure liquid to pure solid.



The phase boundary is expressed by the Clausius-Clapeyron equation

$$\frac{dP}{dT} = \frac{S_{liq} - S_{solid}}{V_{liq} - V_{solid}}$$

In the phase diagram of $^3$He around 0.3 K,

$$\frac{dP}{dT} > 0 \qquad \text{for } T\text{>0.3 K} \quad \text{corresponding to } S_{liq} > S_{solid}$$

$$\frac{dP}{dT} < 0 \qquad \text{for } T\text{<0.3 K} \quad \text{corresponding to } S_{liq} < S_{solid}$$

since $V_{liq} > V_{Solid}$ in the vicinity of 0.3 K.

Consider the liquid-solid boundary (denoted by red open circle) below 0.32 K, where $S_{liq} < S_{solid}$, When temperature is increased, the entropy increases. The only way this can happen here is to solid which has higher entropy. So the liquid freezes when the temperature rises. This explains the negative slope below 0.32 K.

In order for entropy to increase, heat must be absorbed from the surrounding (like when a normal solid melts). This gives rise to a cooling effect. Higher pressure forces the liquid to become solid. This gives a cooling effect. This compressional cooling" method was proposed by Issak Pomeranchuk in 1950.

Pomenranchuk predicted that adiabatic compression of coexisting solid and liquid would cool $^3$He. If the initial temperature is below 0.32 K, then an increase in the external pressure moves the helium along the melting curve up and to the left: toward higher pressure and lower temperatures. Typically, the temperature decreases from 0.3 K to 1 mK. The solid-liquid boundary (arrow) below 0.3 K has a negative slope.

**REFERENCE**: Baierlein, Thermal Physics (Cambridge, 2001).

((**Example**))

$$S_{liq} = 22T \qquad \text{(J/mol K)}, \qquad S_{solid} = R\ln 2 \ \ \text{(J/mol K)}$$

$$T = 0.2619 \text{ K}$$

For the atoms. the main difference between solid and liquid is the ordering. In a normal material, the atoms would be more ordered in the solid where they are fixed, and more disordered in the liquid where they move randomly. Therefore, we can look for clues in the entropy of solid and liquid helium-3. Recall that the $^3$He atom is a fermion. Let us review what we know about entropy of fermions.

This means that we can use the methods for electrons in metals to calculate heat capacities and entropies in liquid helium-3 - provided we assume that the helium-3 atoms acquires a different (effective) mass. Recall that we can use the formula for heat capacity of electrons in metal:

For many years, it was believed that supefluid is not possible for $^3$He. It is a fermion, and Bose Einstein condensation cannot happen. That is, until the arrival of the BCS (Bardeen-Cooper-Schrieffer) theory.

## 6. Cornell group's experiment

We shall now look at some of the experimental evidence for $^3$He superfluid, starting with the discovery in 1972.

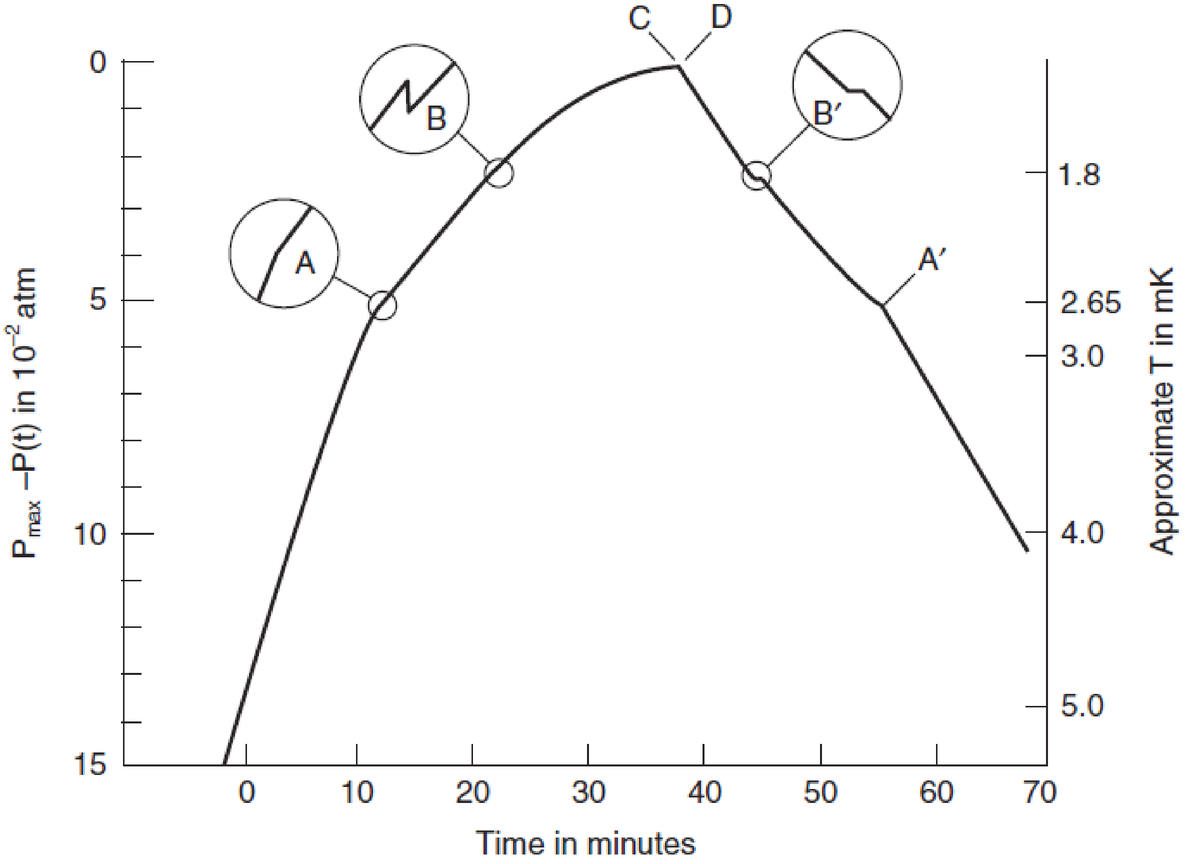
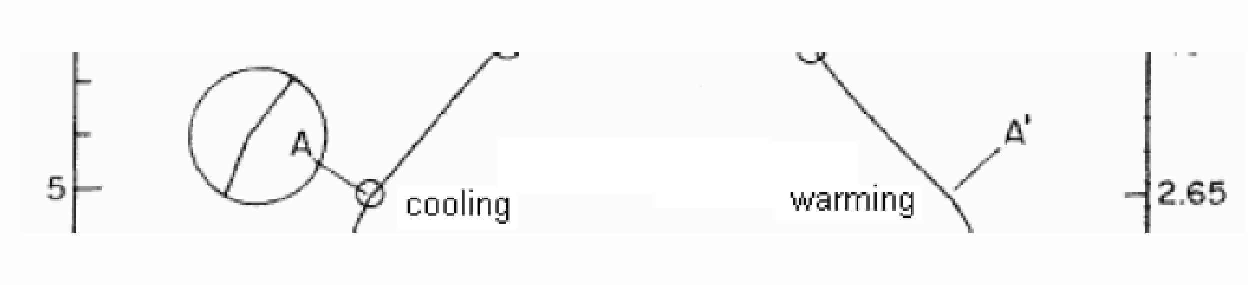


**Fig.** The original data of Osheroff, Richardson and Lee which indicated the discovery of the A- and B-phases of superfluid $^3$He.

In 1972, Osheroff, Richardson and Lee cooled liquid $^3$He using the Pomeranchuk method. They increased the pressure and recorded the falling temperature until about 1 mK. Then they release the pressure and continued recording the rising temperature. They plotted the graph and expected to see a smooth curve. They saw 2 small kinks on the way up, and 2 tiny glitches on the way down. After careful measurements, they concluded that these were due to the formation of 2 new phases, which the called A and B.

The meaning of the bends at A and A' are less obvious, but can also be explained in terms of phase transitions.



However, whereas B and B' are first order transition, A and A' are second order transition. Second order transitions are phase transitions in which latent heat is zero. It is less common, but examples include:

- transition through triple point of water
- transition from para to ferromagnet
- transition from normal to superconductor

That the transition to phase A and the transition to superconductor are both second order suggests possible connection. Soon after the discovery of the new phases, Anthony Leggett proposed that phases A and B are the result of $^3$He atoms forming Cooper pairs. The difference is that in phase A, the spins are parallel, so that the total spin is 1 ($S = 1$, $l = 1$, $^3$P). In phase B, the spins are opposite, so total spin is 0 (S = 0, $l = 0$; $^1$S). In both cases, the $^3$He atoms pair up to form bosons. This means that Bose-Einstein condensation and superfluidity are then possible.

The suggestion that phases A and B are Cooper pairs is reasonable, but needs to be checked by experiments. One experiment is to measure the magnetic energy levels. If the phase A has spin 1, it would have magnetic levels in a field. Phase B has zero spin, so there would be no levels. Photons from a high frequency magnetic field at the right energies could be absorbed and re-emitted, just as in atomic spectra. This way of measuring the magnetic energy levels is called nuclear magnetic resonance (NMR).

The NMR spectrum is taken as liquid $^3$He is cooled below 2.5 mK. A new peak from phase A appeared. On further cooling to phase B, the peak disappeared. This observation is consistent with the theory that phase A contains Cooper pairs with spin 1, and Phase B Cooper pairs with spin 0.

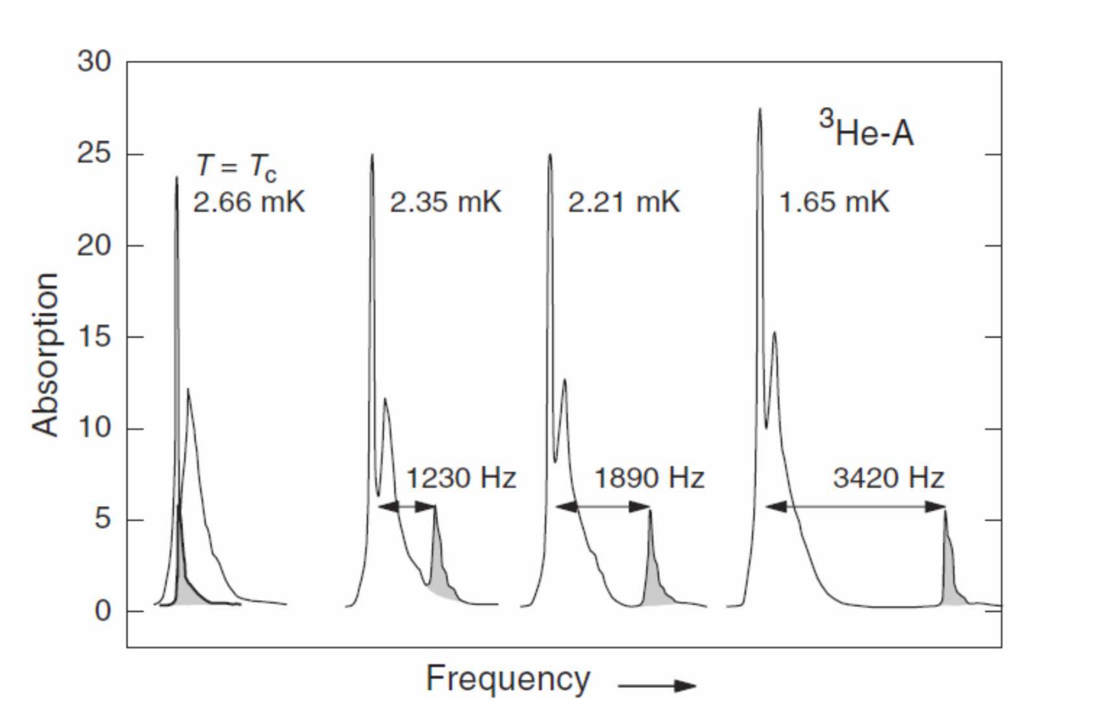


**Fig.** Transverse NMR absorption spectra observed in a mixture of solid and liquid $^3$He at the melting pressure for different temperatures. The grey tinted line corresponds to the absorption line in the liquid phase.

Another way to verify that phase B contains Cooper pair with spin 0 is to check this relation from BCS theory:

$$2\Delta = 3.52 k_B T_c .$$

Using the phase B transition temperature $T_c$, the energy gap $\Delta$ can be predicted. One experiment to measure $\Delta$ is to use ultrasound. Ultrasound is transmitted into phase B. The frequency $\omega$ is increased. If there are indeed Cooper pairs, they will break up when the phonon energy $\hbar\omega$ reaches $\Delta$. When this happens, the ultrasound will be strongly attenuated, since it has to give up its energy to break up the Cooper pairs.

The following shows the ratio of the energy gap $\Delta_{measured}$ measured from the ultrasound experiment, to the energy gap $\Delta_{BCS}$ predicted by the BCS theory

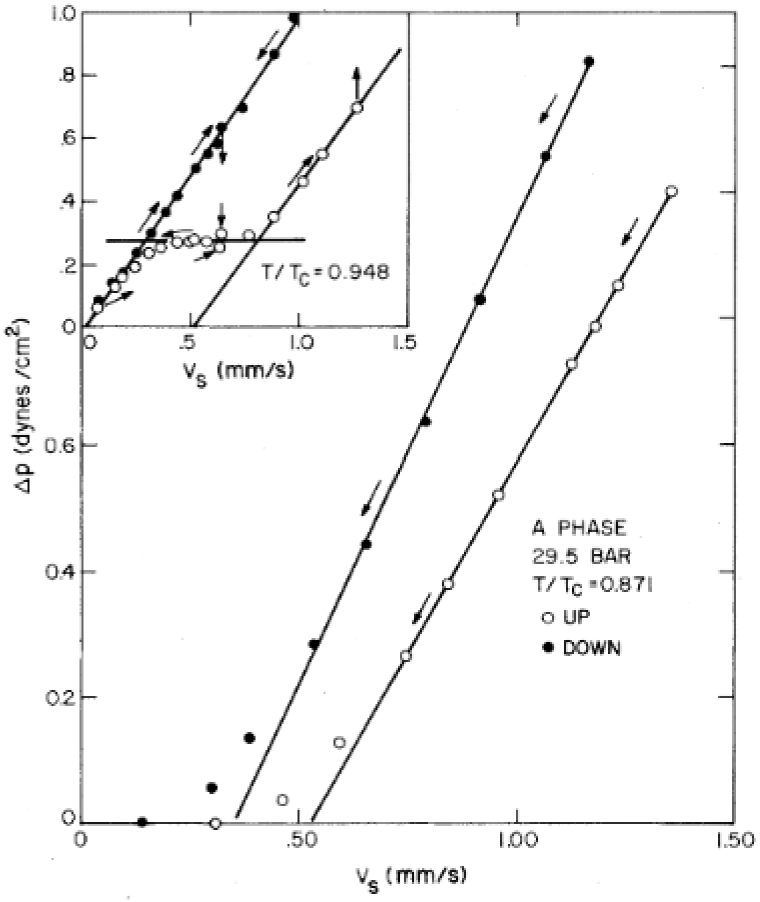
| Pressure P (bars) | Ratio $\frac{\Delta_{\exp}}{\Delta_{BCS}}$ |
|---|---|
| 4.85 | 0.994 |
| 9.80 | 1.030 |

| | |
|---|---|
| 18.10 | 1.050 |

[Movshovich, Kim, and Lee, Phys. Rev. Lett. 64, p. 431 (1990)]

The ratio is quite close to 1, showing that the BCS prediction is accurate. This provides further evidence that the Cooper pairs exist.

In another experiment, phase A flows through a narrow channel. The pressure difference between the two ends of the channel is measured. The result shows that below transition temperature, the pressure difference fall to zero. No pressure is needed to keep phase A flowing. This means it has no viscosity, confirming that it is a superfluid.



The experiments on superfluid helium-3 provide strong evidence that our understanding of helium-4 superfluid, superconductivity, and Bose-Einstein condensate is correct. Many questions remain, such as why superfluid helium-4 is only 10% BEC, why the BCS theory cannot fully explain high $T_c$ superconductors, and so on.

Order parameter:

## 7. Phase diagram

Here is the phase diagram showing the superfluid phases with variables T, P and an external magnetic field in the mK region. It should be noted that there are two thermodynamically distinct phases so-called the A phase and B phase even at $H = 0$, and that for $H \neq 0$ the A1 phase appears. The transition between the A phase and B-phase is of the first order, which suggests that these are ordered phases with different symmetries. The appearance of superfluidity has been suggested by the specific heat jump, observation of the fourth sound propagating in the :super leak: of packed fine powder, the anomaly in heat conductance, and above all, the shift in NMR frequency. The existence of the $A_1$ phase in a magnetic field is deduced from the specific heat measurement.

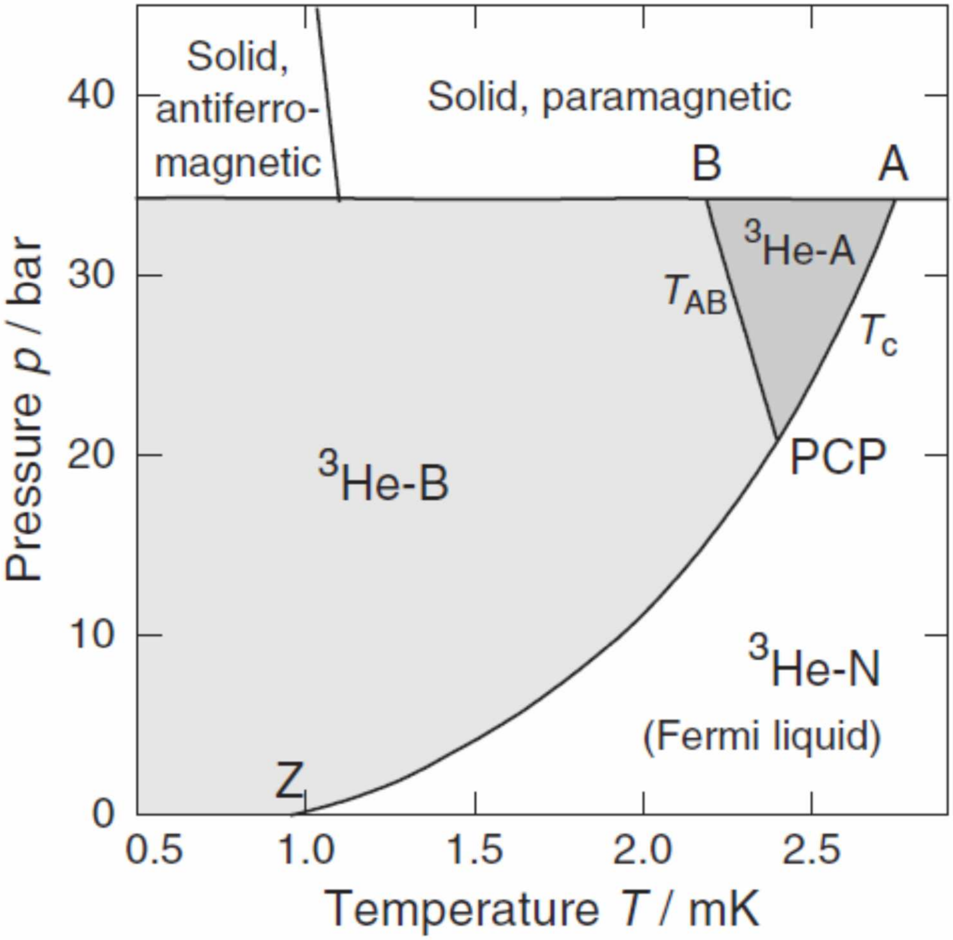


**Fig.** $P$-$T$ Phase diagram of $^3$He in the absence of a magnetic field. The point A (34.3 bar, 2.44 mK). The point B (34.3 bar, 1.90 mK). PCP (21.5 bar, 2.24 mK). Z (0 bar, 0.92 mK). The superfluid phases are grey tinted. Solid $^3$He forms an antiferromagnetic lattice with a Neel temperature $T_N$ of about 1 mK.

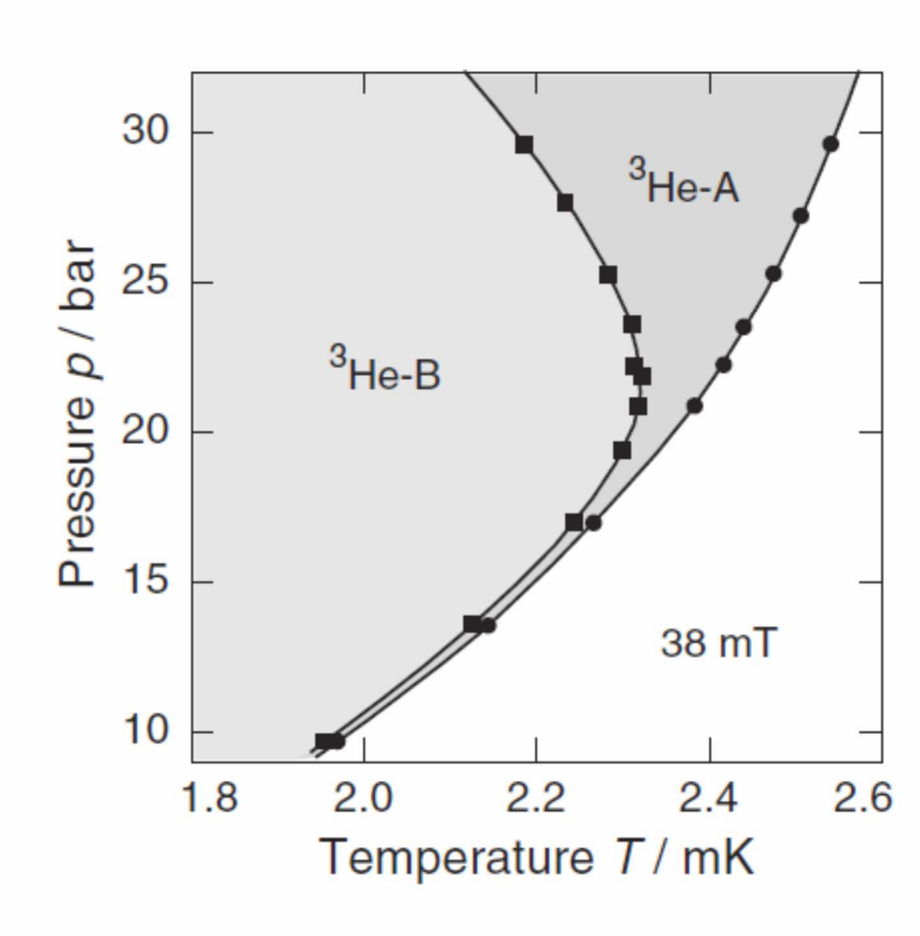
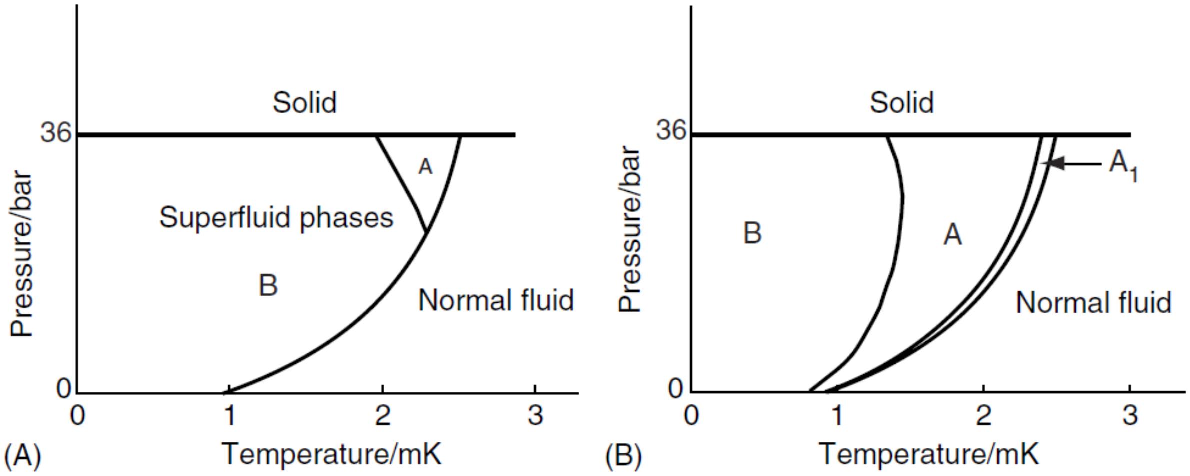


**Fig.** P-T phase diagram of liquid $^3$He below 2.6 mK in a magnetic field of 380 Oe.



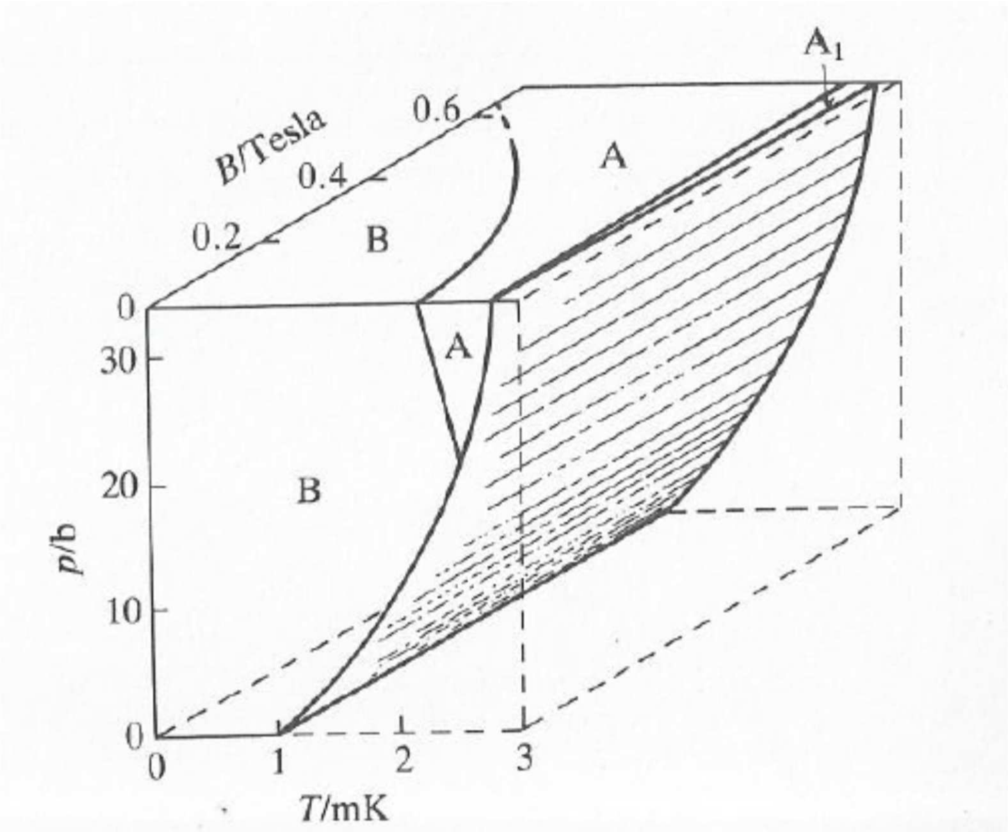


**Fig**. The phase diagram of $^3$He at millikelvin temperatures, showing the superfluid phases in zero applied magnetic field. (B) Schematic phase diagram of $^3$He in an applied magnetic field.

## 8. Critical behavior of heat capacity

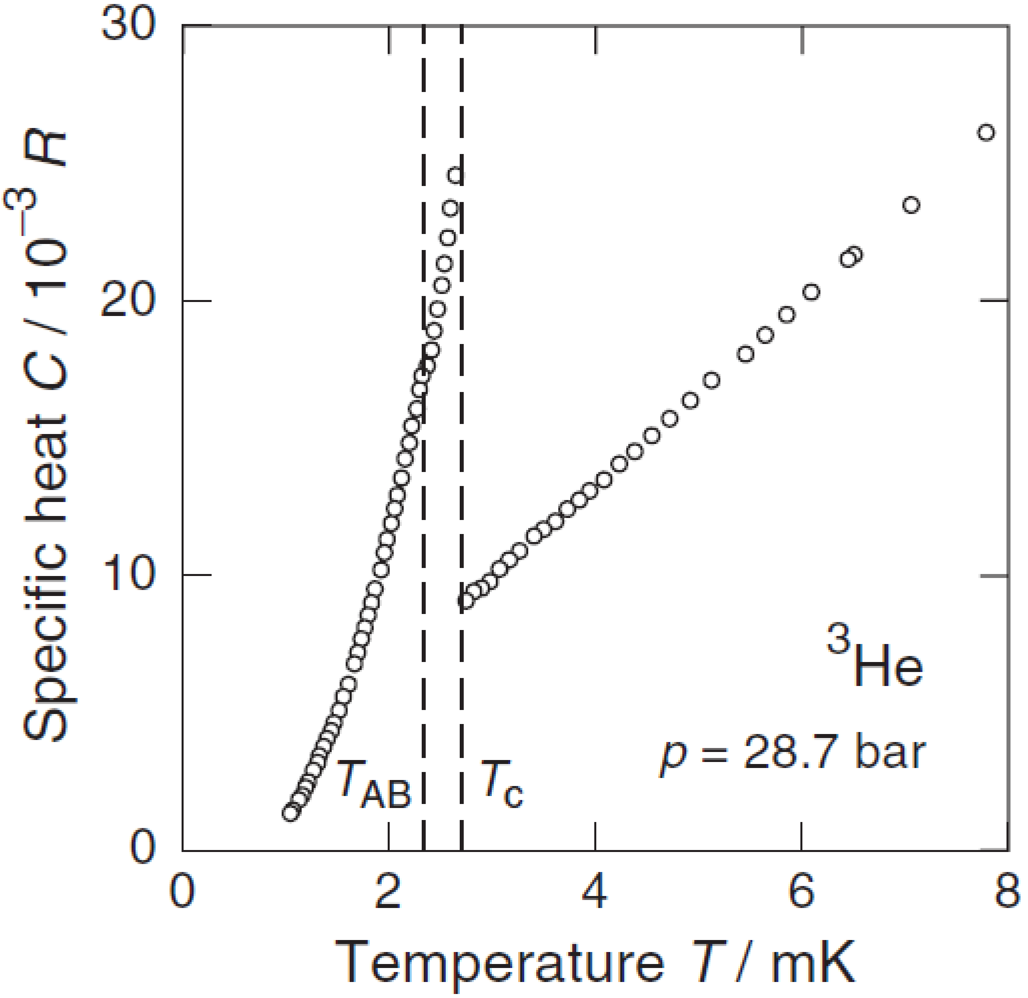


**Fig.** Reduced specific heat $C/R$ of $^3$He at a pressure of 28.7 bar as a function of temperature. At $T_c$ a clearly visible jump in $C$ exists, while a weak variation of C occurs at $T_{AB}$ .

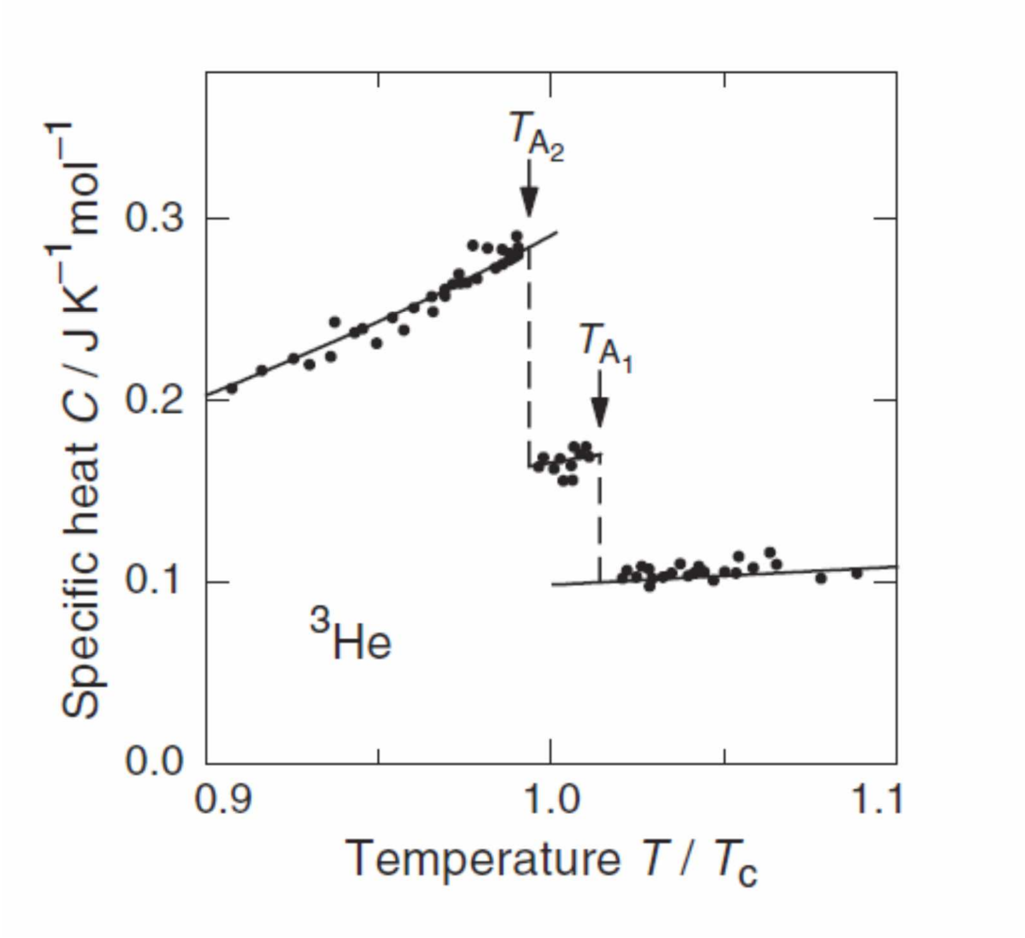


**Fig.** Specific heat of superfluid $^3$He at the melting pressure as a function of the reduced temperature $T/T_c$ in a magnetic field of 880 Oe. The solid lines above $T_{A1}$ and below $T_{A2}$ are fits to zero-field data in $^3$He-N and $^3$He-A, respectively.

## 9. Fermi liquid theory

The life time a quasi-particle excitation dressed by interactions is rather long owing to the surrounding degenerate Fermi sea. Thus the physical properties at $T << T_F$ may be well described by a theory based on quasi-particle excitations, the so-called Landau Fermi liquid theory. Since the critical temperature $T_c$ of this superfluid is of the order of $T_F/1000$, similar to that for superconductivity in a metal, one can naturally study the system on the basis of this theory.

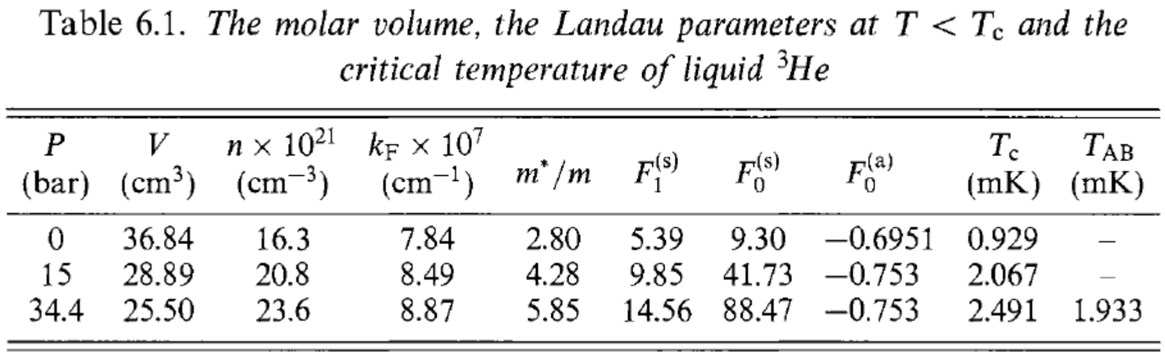


Table 6.1. *The molar volume, the Landau parameters at $T < T_c$ and the critical temperature of liquid $^3He$*

| $P$ (bar) | $V$ ($\mathrm{cm}^3$) | $n \times 10^{21}$ ($\mathrm{cm}^{-3}$) | $k_F \times 10^7$ ($\mathrm{cm}^{-1}$) | $m^*/m$ | $F_1^{(s)}$ | $F_0^{(s)}$ | $F_0^{(a)}$ | $T_c$ (mK) | $T_{AB}$ (mK) |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 36.84 | 16.3 | 7.84 | 2.80 | 5.39 | 9.30 | −0.6951 | 0.929 | – |
| 15 | 28.89 | 20.8 | 8.49 | 4.28 | 9.85 | 41.73 | −0.753 | 2.067 | – |
| 34.4 | 25.50 | 23.6 | 8.87 | 5.85 | 14.56 | 88.47 | −0.753 | 2.491 | 1.933 |

Fermi liquid relations between the Landau parameters  and experimentally measured quantities

**Effective mass:**

$$\frac{m^*}{m} = 1 + \frac{1}{3} F_1^{(s)}$$

**Specific heat:**

$$\frac{C_N}{C_g} = \frac{m^*}{m} = 1 + \frac{1}{3} F_1^{(s)}$$

where

$$C_g = \frac{1}{2}\pi^2 N k_B \frac{T}{T_F}$$

**Spin suceptibility:**

$$\frac{\chi_N}{\chi_g} = \frac{m^*}{m}\frac{1}{1+F_0^{(a)}} = \frac{1+\frac{1}{3}F_1^{(s)}}{1+F_0^{(a)}}$$

where

$$\chi_g = \mu_B{}^2 \frac{3N}{2\varepsilon_F} = \frac{3\mu_B{}^2 N}{2k_B}\frac{1}{T_F}$$

**Comprressibility:**

$$\frac{\kappa_N}{\kappa_g} = \frac{m^*}{m}\frac{1}{1+F_0^{(s)}} = \frac{1+\frac{1}{3}F_1^{(s)}}{1+F_0^{(s)}}$$

where

$$\kappa_g = \frac{3}{2}\frac{1}{n\varepsilon_F}$$

The sound velocity:

$$\frac{c^2}{c_g} = \frac{1+F_0^{(s)}}{1+\frac{1}{3}F_1^{(s)}}$$

where

$$T_F = \frac{\hbar^2}{2mk_B}(3\pi^2 n), \qquad T_F^{\ *} = \frac{\hbar^2}{2m^* k_B}(3\pi^2 n)$$

$$p_F = \hbar(3\pi^2 n)^{2/3}$$

**Link**

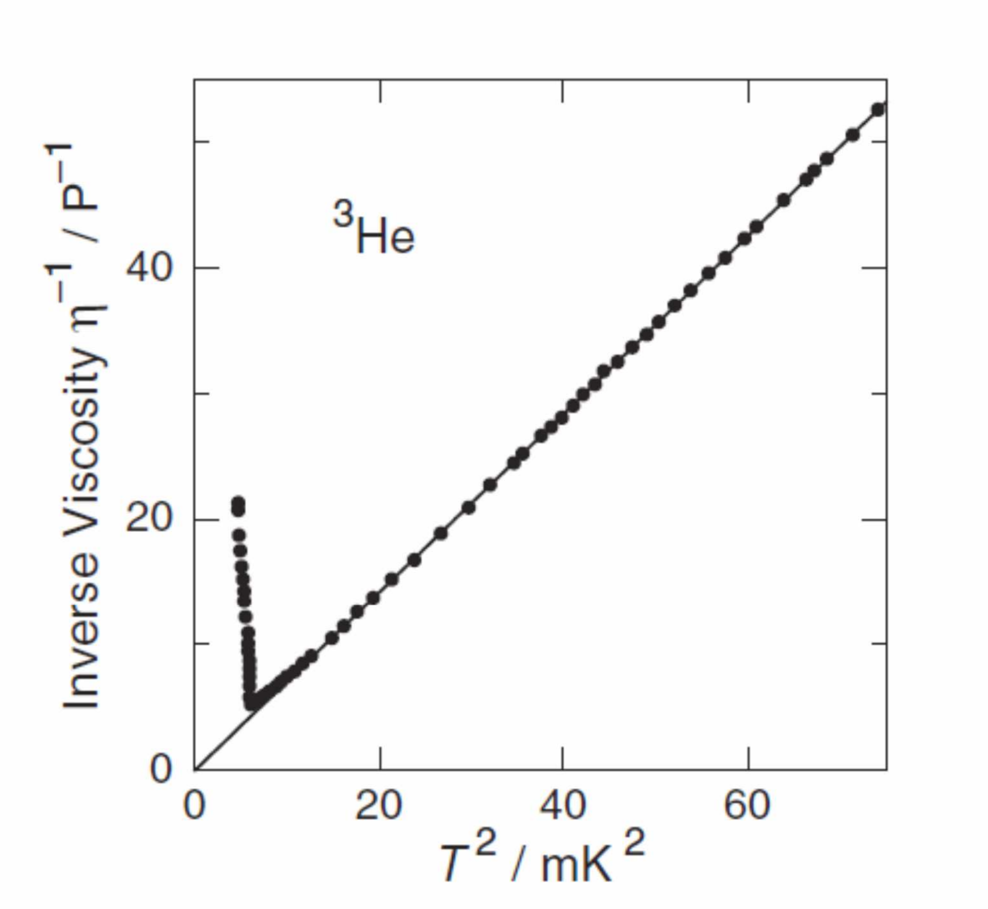
D. Osheloff's lecture on the discovery of superfluidity $^3$He
https://www.youtube.com/watch?v=XB-RMEnlG7E

In Landau's theory the frequently colliding $^3$He atoms are replaced by an equal number of nearly independent *quasiparticles*, which have an effective mass $m^* \cong 3m$ in the liquid under its saturated vapor pressure. At very low temperatures, $T << T_F$, the quasiparticles have a long mean free path and form a rarefied gas in momentum space with energies within a region of width $k_B T$ around the Fermi sphere of radius $k_B T_F$, where $T_F$ on Landau theory is lowered to 1.5 K

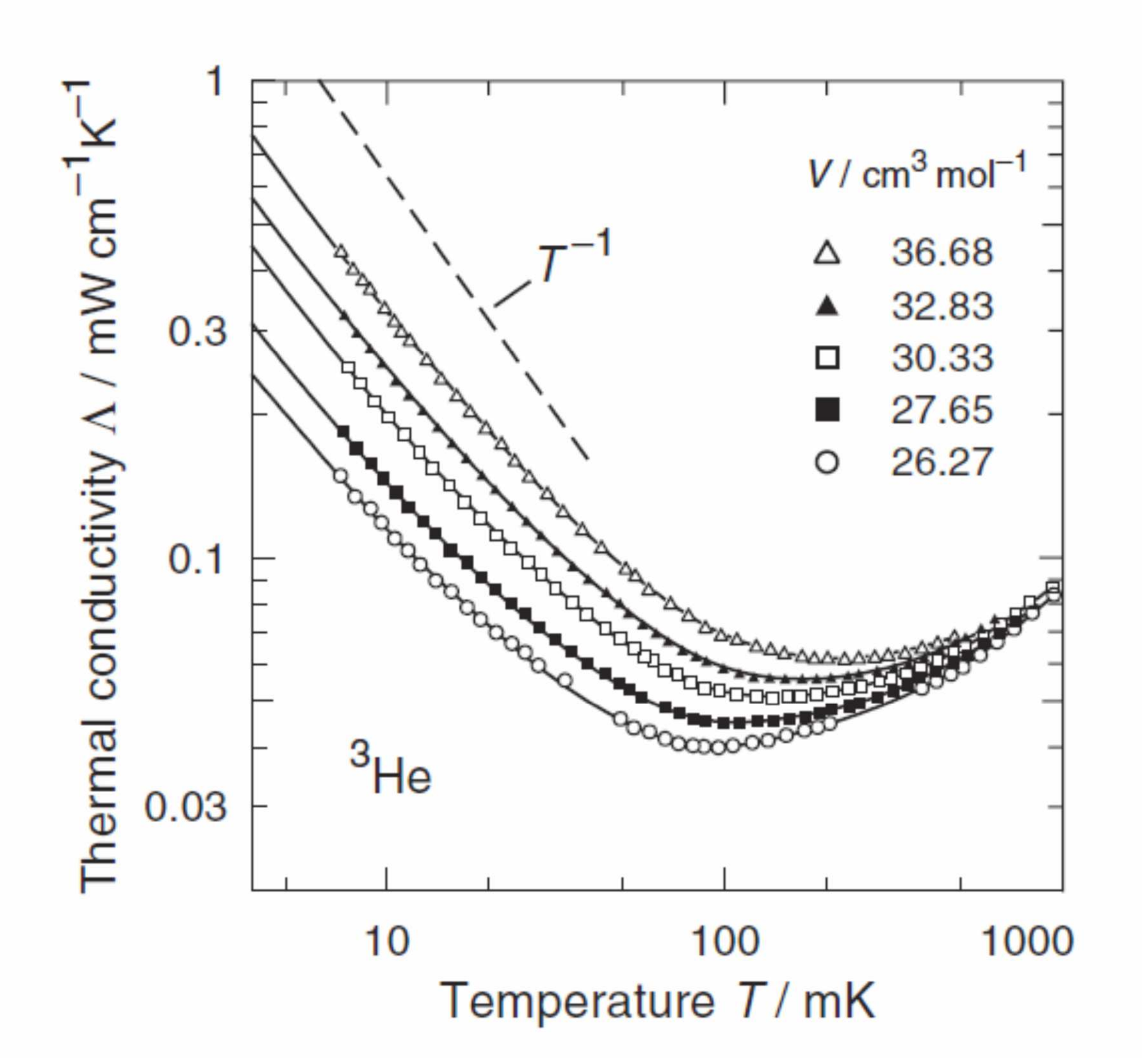
Another consequence of this model is that it can predict the temperature dependence of the viscosity of liquid $^3$He. According to elementary transport theory the viscosity of a fluid is

$$\eta = \frac{1}{3}\rho \tau v^2$$

where ρ is the fluid density, τ is the mean time between collisions, and ν is the mean speed of the atoms. For liquid $^3$He the number of quasiparticles available to scatter a given one is proportional to $k_B T$, while the number of final states is also proportional to $k_B T$. As a result the collision frequency for quasiparticle scattering is proportional to $T^2$ or the relaxation time $\tau \approx 1/T^2$. The speed of the atoms is approximately equal to the value at the Fermi surface, $v_F$, and so is independent of temperature. So the viscosity is proportional to $1/T^2$.



As a result the viscosity of liquid $^3$He, in striking contrast to that of superfluid $^4$He, rises dramatically at millikelvin temperatures, becoming 2 mP at 40 mK (Betts *et al.* 1966) and no less than 0.3 P at 3 mK—which is the same as olive oil at 40 °C! In a similar way its thermal conductivity λ rises at mK temperatures, where $\lambda \approx 1/T$.



The interaction between $^3$He atoms in liquid $^3$He which can lead to superfluidity is more subtle. The $^3$He atoms are inert and uncharged; there is no ion lattice to be polarized. The polarization therefore takes place in the Landau Fermi liquid itself, which has been shown to have a strong *spin polarization*. So we can visualize a $^3$He atom travelling through the viscous liquid leaving in its wake a partially spin-polarized track. This polarization fades slowly and so a second $^3$He atom coming near this track would be either attracted or repelled (depending on its spin) and so effectively interact with a spin dependent interaction with the first $^3$He atom. This mechanism can lead to correlated pairs of parallel-spin particles and so favors spin triplet pairing ($S = 1$) rather than the spin singlet pairing ($S = 0$) of the electrons.

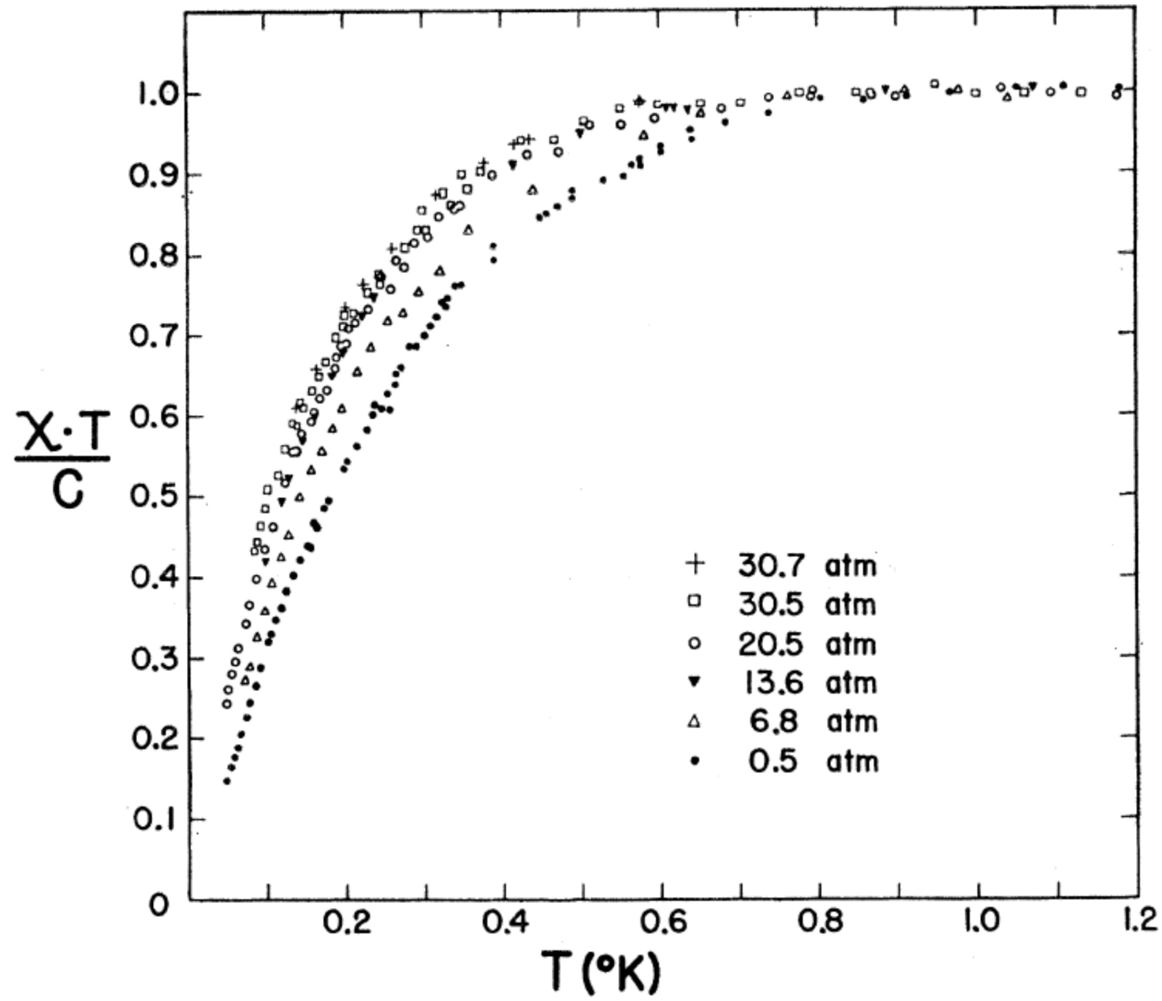


**Fig.** The susceptibility of liquid 3He, plotted as $\chi T / C$ for various pressures. A.L. Thomson et al. Phys. Rev.128, 509.

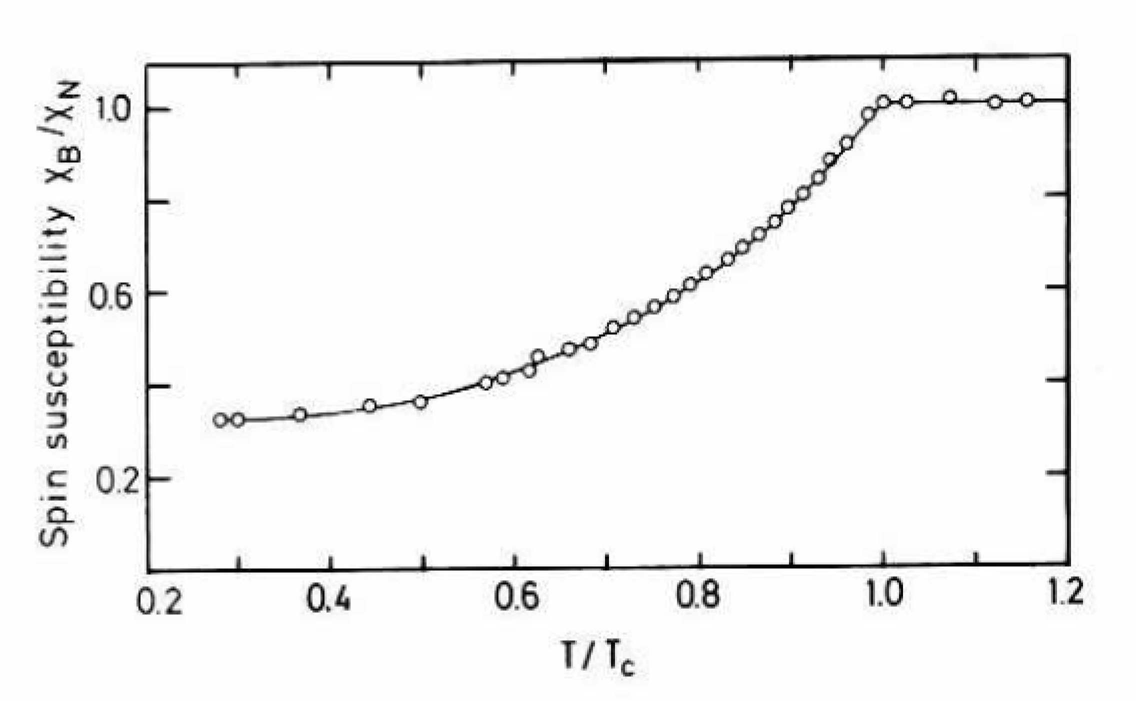


Fig. Magnetic Susceptibility of the B-phase as a function of $T/T_c$. The pressure is below the PPCP critical point.

---

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## APPENDIX-1

## 1. Approximation of C and U up to the second order

$$C = N\frac{\pi}{2}k_B\left(\frac{k_BT}{\varepsilon_F}\right)[1-\frac{3\pi^2}{20}\left(\frac{k_BT}{\varepsilon_F}\right)^2]$$

$$= N\frac{\pi}{2}k_B\left(\frac{T}{T_F}\right)[1-\frac{3\pi^2}{20}\left(\frac{T}{T_F}\right)^2]$$

The total energy

$$U = N\varepsilon_F[\frac{3}{5}+\frac{\pi^2}{4}\left(\frac{T}{T_F}\right)^2-\frac{3\pi^4}{160}\left(\frac{T}{T_F}\right)^4+...$$

## APPENDIX-II

1 atm = 1.01325 x $10^5$ Pa
1 bar = 0.98692 atm = $10^5$ bar
1 Mbar = 9.8692 atm

The nucleus of $^3$He consists of two protons, one neutrons, and two electrons. The nuclear spin is zero. The nuclear spin is 1/2. The nucleus of $^4$He consists of two protons, two neutrons, and two electrons. The nuclear spin is zero.

((R. Dobbs, Helium Three Oxford 2001))

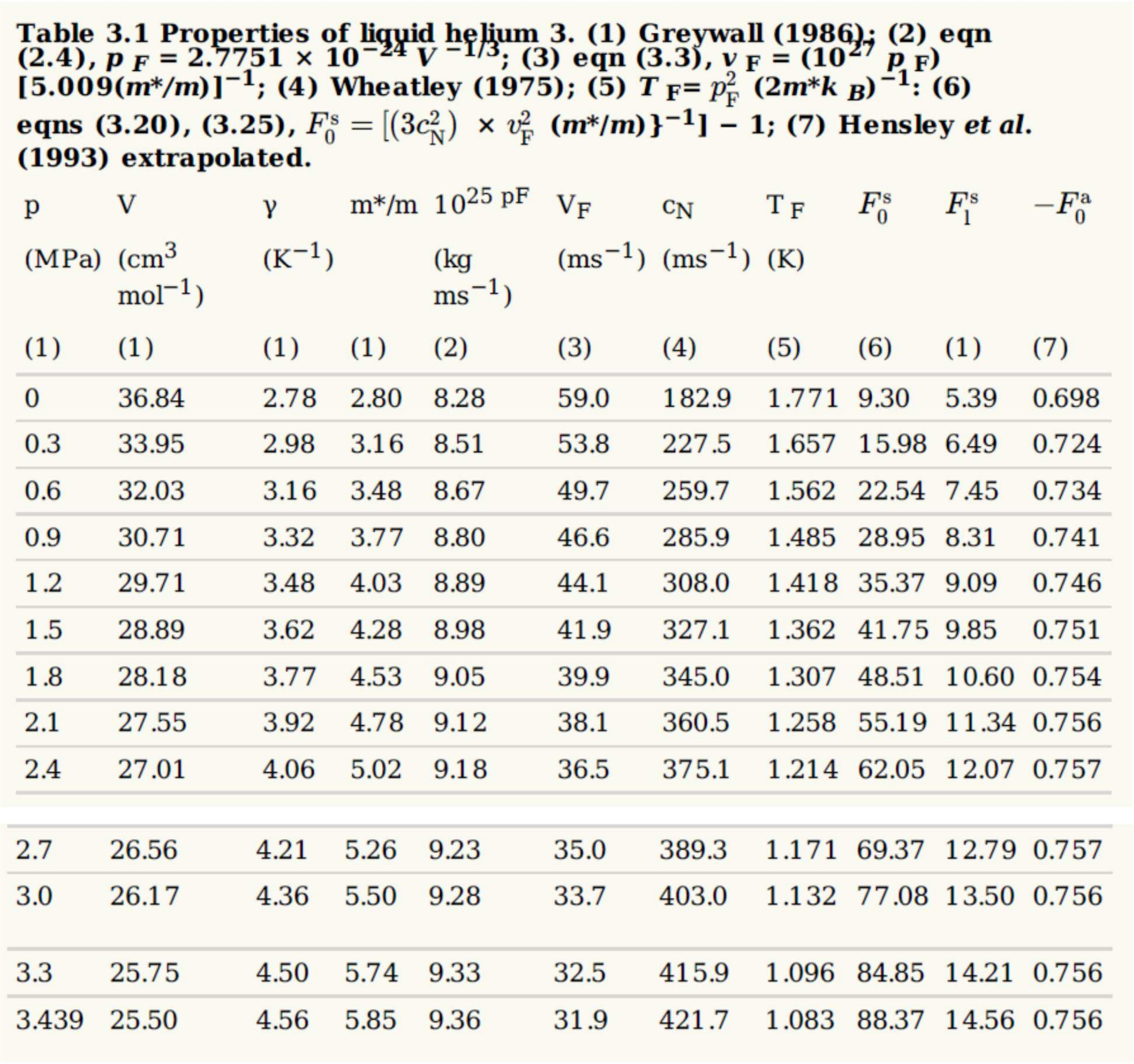


**Table 3.1 Properties of liquid helium 3. (1) Greywall (1986); (2) eqn (2.4), $p_F = 2.7751 \times 10^{-24} V^{-1/3}$; (3) eqn (3.3), $v_F = (10^{27} p_F)$ $[5.009(m^*/m)]^{-1}$; (4) Wheatley (1975); (5) $T_F = p_F^2 (2m^* k_B)^{-1}$: (6) eqns (3.20), (3.25), $F_0^s = [(3c_N^2) \times v_F^2 (m^*/m)\}^{-1}] - 1$; (7) Hensley *et al.* (1993) extrapolated.**

| p (MPa) (1) | V ($cm^3$ $mol^{-1}$) (1) | γ ($K^{-1}$) (1) | m*/m (1) | $10^{25}$ pF (kg $ms^{-1}$) (2) | $V_F$ ($ms^{-1}$) (3) | $c_N$ ($ms^{-1}$) (4) | $T_F$ (K) (5) | $F_0^s$ (6) | $F_1^s$ (1) | $-F_0^a$ (7) |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 36.84 | 2.78 | 2.80 | 8.28 | 59.0 | 182.9 | 1.771 | 9.30 | 5.39 | 0.698 |
| 0.3 | 33.95 | 2.98 | 3.16 | 8.51 | 53.8 | 227.5 | 1.657 | 15.98 | 6.49 | 0.724 |
| 0.6 | 32.03 | 3.16 | 3.48 | 8.67 | 49.7 | 259.7 | 1.562 | 22.54 | 7.45 | 0.734 |
| 0.9 | 30.71 | 3.32 | 3.77 | 8.80 | 46.6 | 285.9 | 1.485 | 28.95 | 8.31 | 0.741 |
| 1.2 | 29.71 | 3.48 | 4.03 | 8.89 | 44.1 | 308.0 | 1.418 | 35.37 | 9.09 | 0.746 |
| 1.5 | 28.89 | 3.62 | 4.28 | 8.98 | 41.9 | 327.1 | 1.362 | 41.75 | 9.85 | 0.751 |
| 1.8 | 28.18 | 3.77 | 4.53 | 9.05 | 39.9 | 345.0 | 1.307 | 48.51 | 10.60 | 0.754 |
| 2.1 | 27.55 | 3.92 | 4.78 | 9.12 | 38.1 | 360.5 | 1.258 | 55.19 | 11.34 | 0.756 |
| 2.4 | 27.01 | 4.06 | 5.02 | 9.18 | 36.5 | 375.1 | 1.214 | 62.05 | 12.07 | 0.757 |
| 2.7 | 26.56 | 4.21 | 5.26 | 9.23 | 35.0 | 389.3 | 1.171 | 69.37 | 12.79 | 0.757 |
| 3.0 | 26.17 | 4.36 | 5.50 | 9.28 | 33.7 | 403.0 | 1.132 | 77.08 | 13.50 | 0.756 |
| 3.3 | 25.75 | 4.50 | 5.74 | 9.33 | 32.5 | 415.9 | 1.096 | 84.85 | 14.21 | 0.756 |
| 3.439 | 25.50 | 4.56 | 5.85 | 9.36 | 31.9 | 421.7 | 1.083 | 88.37 | 14.56 | 0.756 |

((**Dobbs, Helium Three Oxford, 2001**)).

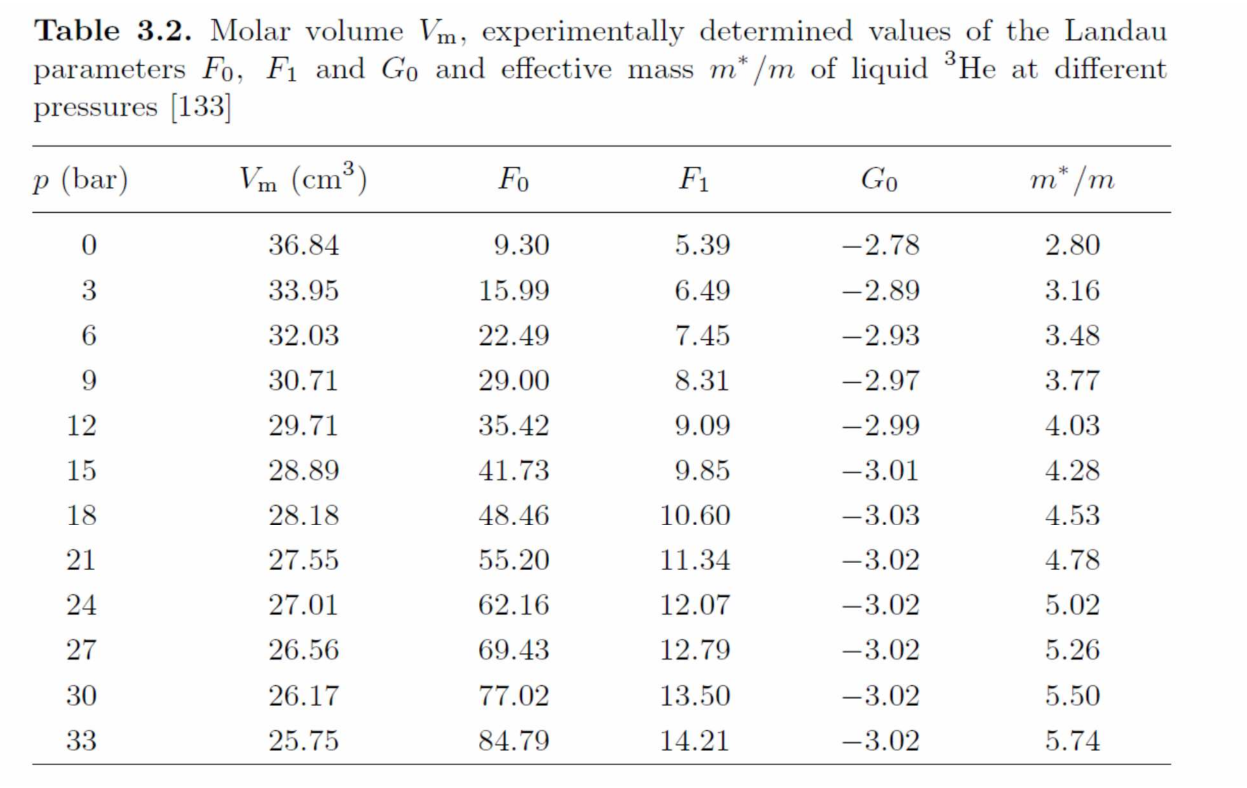


**Table 3.2.** Molar volume $V_{\mathrm{m}}$, experimentally determined values of the Landau parameters $F_0$, $F_1$ and $G_0$ and effective mass $m^*/m$ of liquid $^3$He at different pressures [133]

| $p$ (bar) | $V_{\mathrm{m}}$ (cm$^3$) | $F_0$ | $F_1$ | $G_0$ | $m^*/m$ |
|---|---|---|---|---|---|
| 0 | 36.84 | 9.30 | 5.39 | −2.78 | 2.80 |
| 3 | 33.95 | 15.99 | 6.49 | −2.89 | 3.16 |
| 6 | 32.03 | 22.49 | 7.45 | −2.93 | 3.48 |
| 9 | 30.71 | 29.00 | 8.31 | −2.97 | 3.77 |
| 12 | 29.71 | 35.42 | 9.09 | −2.99 | 4.03 |
| 15 | 28.89 | 41.73 | 9.85 | −3.01 | 4.28 |
| 18 | 28.18 | 48.46 | 10.60 | −3.03 | 4.53 |
| 21 | 27.55 | 55.20 | 11.34 | −3.02 | 4.78 |
| 24 | 27.01 | 62.16 | 12.07 | −3.02 | 5.02 |
| 27 | 26.56 | 69.43 | 12.79 | −3.02 | 5.26 |
| 30 | 26.17 | 77.02 | 13.50 | −3.02 | 5.50 |
| 33 | 25.75 | 84.79 | 14.21 | −3.02 | 5.74 |

where $F_0 = F_0^{(s)}$, $F_1 = F_1^{(s)}$, $G_0 = 4F_0^{(a)}$

(**Enns and Hunklinger**)

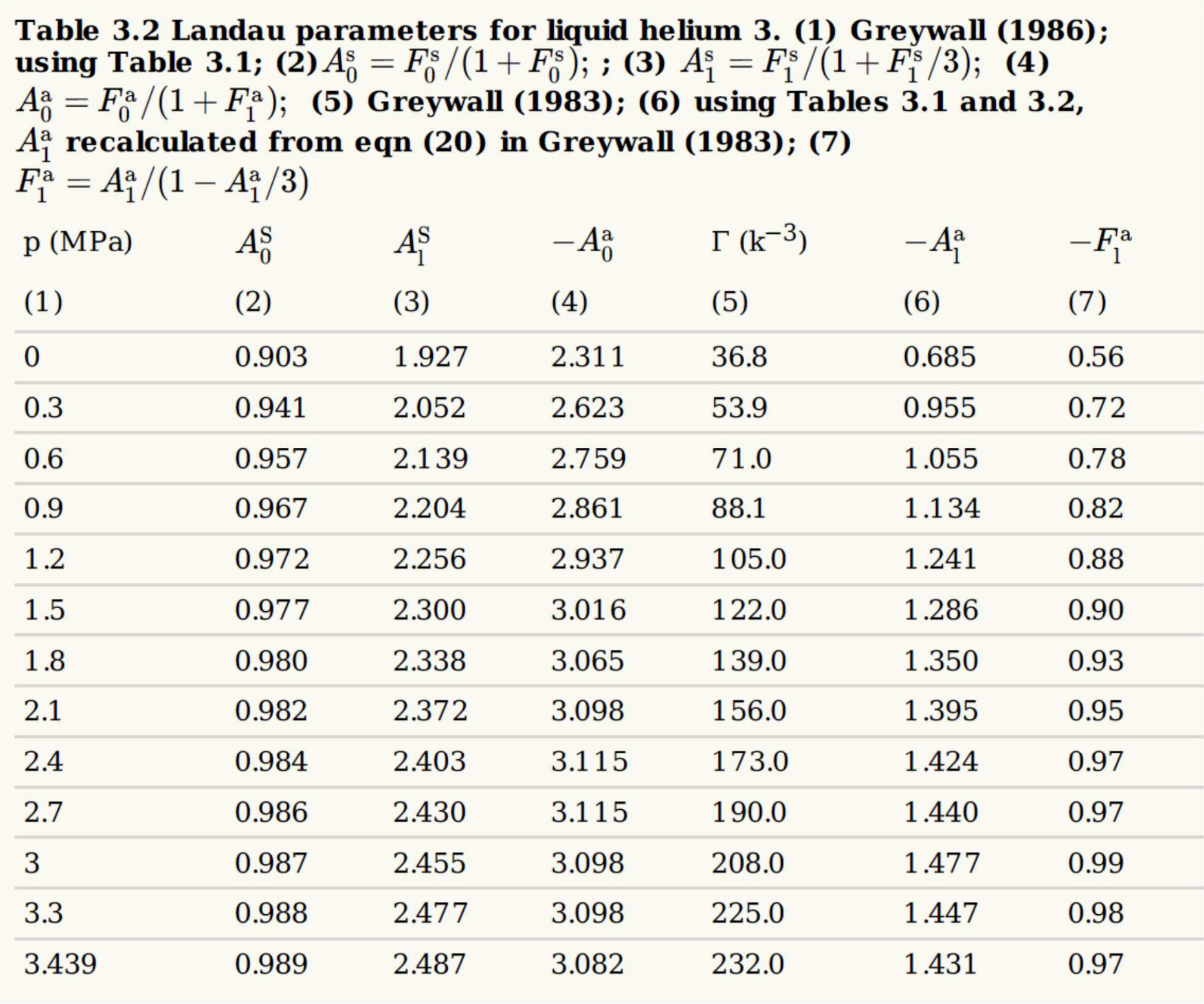


**Table 3.2 Landau parameters for liquid helium 3. (1) Greywall (1986); using Table 3.1; (2)** $A_0^s = F_0^s/(1+F_0^s)$**; ; (3)** $A_1^s = F_1^s/(1+F_1^s/3)$**; (4)** $A_0^a = F_0^a/(1+F_1^a)$**; (5) Greywall (1983); (6) using Tables 3.1 and 3.2,** $A_1^a$ **recalculated from eqn (20) in Greywall (1983); (7)** $F_1^a = A_1^a/(1-A_1^a/3)$

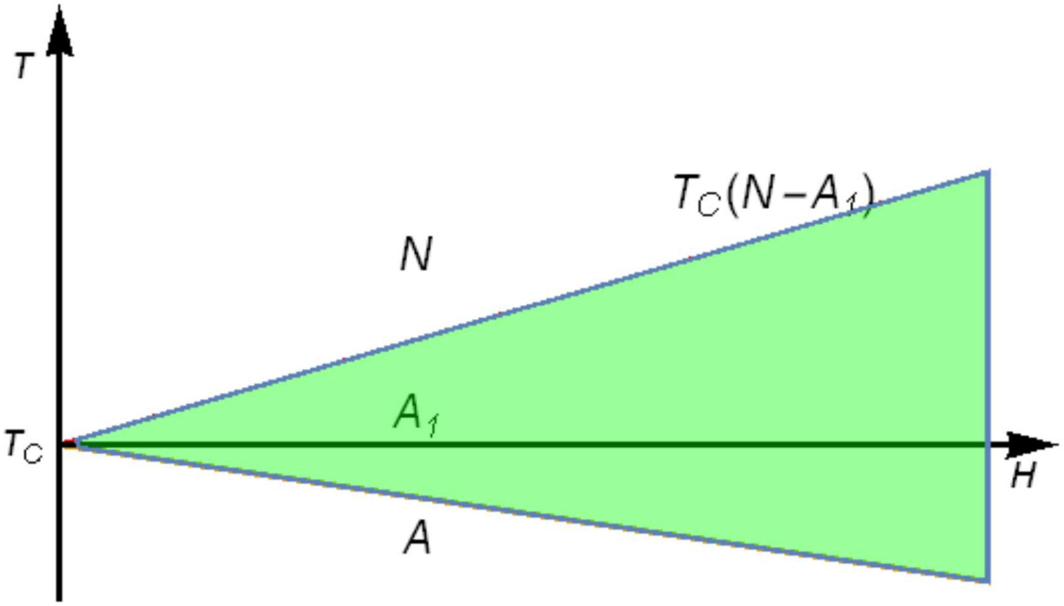
| p (MPa) | $A_0^S$ | $A_1^S$ | $-A_0^a$ | Γ ($k^{-3}$) | $-A_1^a$ | $-F_1^a$ |
|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) |
| 0 | 0.903 | 1.927 | 2.311 | 36.8 | 0.685 | 0.56 |
| 0.3 | 0.941 | 2.052 | 2.623 | 53.9 | 0.955 | 0.72 |
| 0.6 | 0.957 | 2.139 | 2.759 | 71.0 | 1.055 | 0.78 |
| 0.9 | 0.967 | 2.204 | 2.861 | 88.1 | 1.134 | 0.82 |
| 1.2 | 0.972 | 2.256 | 2.937 | 105.0 | 1.241 | 0.88 |
| 1.5 | 0.977 | 2.300 | 3.016 | 122.0 | 1.286 | 0.90 |
| 1.8 | 0.980 | 2.338 | 3.065 | 139.0 | 1.350 | 0.93 |
| 2.1 | 0.982 | 2.372 | 3.098 | 156.0 | 1.395 | 0.95 |
| 2.4 | 0.984 | 2.403 | 3.115 | 173.0 | 1.424 | 0.97 |
| 2.7 | 0.986 | 2.430 | 3.115 | 190.0 | 1.440 | 0.97 |
| 3 | 0.987 | 2.455 | 3.098 | 208.0 | 1.477 | 0.99 |
| 3.3 | 0.988 | 2.477 | 3.098 | 225.0 | 1.447 | 0.98 |
| 3.439 | 0.989 | 2.487 | 3.082 | 232.0 | 1.431 | 0.97 |

((**Dobbs, Helium Three Oxford, 2001**)).

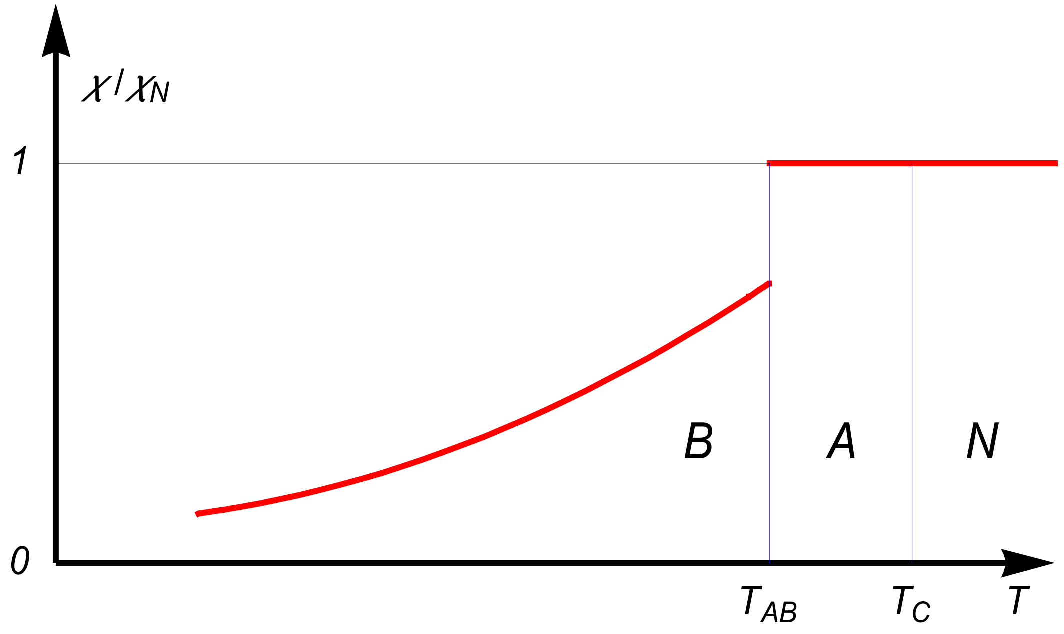
Thouless

Since the interaction between He atoms is repulsive at short distance it is not surprising that the pairing occurs in a higher relative angular momentum state rather than in the S state in which electrons are paired for conventional superconductors. In fact the pairing occurs in a P state, which, since the atoms are fermions, implies that the nuclear spins must also be couples together in a triplet state.

A triplet P state is described by 3 x 3 = 9 complex degrees of freedom. In the A phase \the orbital state



At the fixed pressure, when the magnetic field is applied, the critical temperature $T_c$ splits into two critical temperatures. In this case, the system undergoes successive phase transitions from the N phase to the phase $A$ through a new phase $A_1$ ($N \rightarrow A_1 \rightarrow A$).



What happens to the susceptibility? In the presence of the pressure, the system undergoes successive phase transitions ( $N \to A \to B$ ) as the temperature decreases, where $N$ is the normal phase. The temperature dependence of the susceptibility is schematically shown in Fig. The suspceptibility $\chi_A$ of the A phase is nearly equal to $\chi_N$. This suggests that the pair formation of nuclear spins in the A phase is different from the Cooper pair formation with antiparallel spin in the conventional superconductor (cancellation of total spin).

((Effective potential))

The effective potential between pair particles is given by the sum of the actual potential energy $V(r)$ and the centrifugal

$$V_{eff} = \frac{\hbar^2 l(l+1)}{2\mu r^2} + V(r)$$

where $\mu$ is the reduced mass

$$\frac{1}{\mu} = \frac{1}{m} + \frac{1}{m} = \frac{2}{m}, \qquad \mu = \frac{m}{2},$$

and $\hbar l$ is the orbital angular momentum. Suppose that $V(r)$ is given by a Lennard-Jones type potential

$$V(r) = 4\varepsilon[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}]$$

where $\varepsilon = 11$ kJ/mol, and $\sigma$ (the collision diameter) is $\sigma = 2.58$ Å for He. It is the hard-sphere potential. The potential energy rises abruptly to infinity as soon as the particles come within the separation $\sigma$. We consider the effective potential as a function of the reduced distance $x = \frac{r}{\sigma}$;

$$\frac{V_{eff}}{4\varepsilon} = x^{-12} - x^{-6} + \frac{\hbar^2 l(l+1)}{8\varepsilon\mu\sigma^2} x^{-2} == x^{-12} - x^{-6} + al(l+1)x^{-2}$$

For simplicity we put $a = \frac{\hbar^2}{8\varepsilon\mu\sigma^2}$, which is an adjustable parameter. It is found that equilibrium position shifts to larger distance with increasing the relative angular momentum $l$.

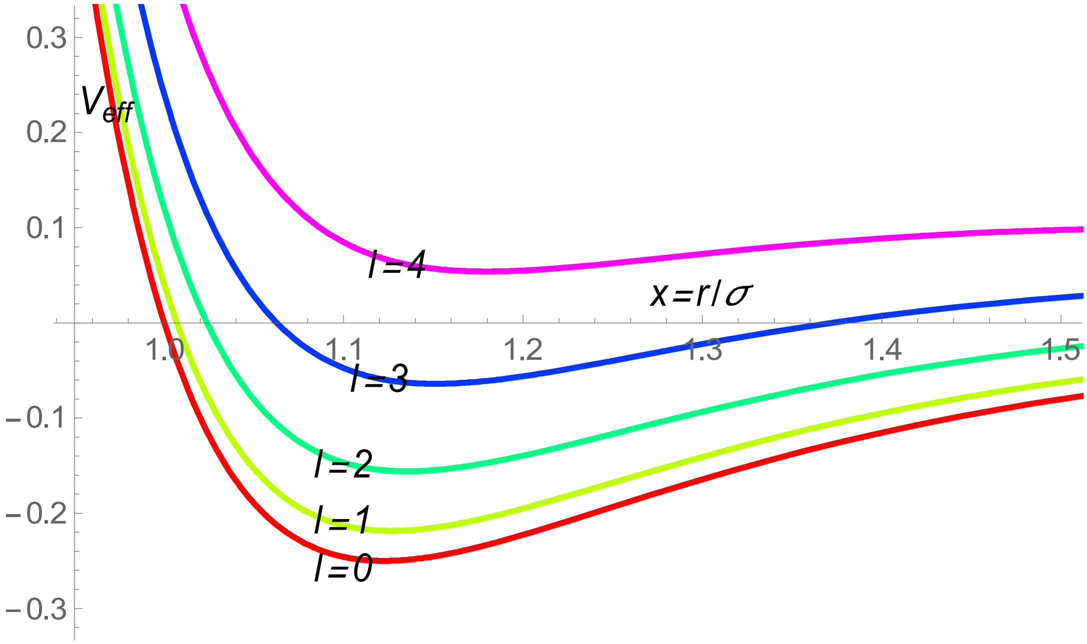


**Fig.** Effective potential $V_{eff}$ as a function of $x = r/\sigma$ with $a = \frac{\hbar^2}{8\varepsilon\mu\sigma^2} = 0.02$.

The equilibrium position is obtained as

$$\frac{d}{dx}\frac{V_{eff}}{4\varepsilon} = -\frac{12}{x^{13}} + \frac{6}{x^{7}} - \frac{2al(l+1)}{x^{3}} = 0$$

or

$$f(x) = -\frac{12}{x^{10}} + \frac{6}{x^{4}} = 2al(l+1)$$

The solution of this equation yields the distance in thermal equilibrium, which depends on the angular momentum.

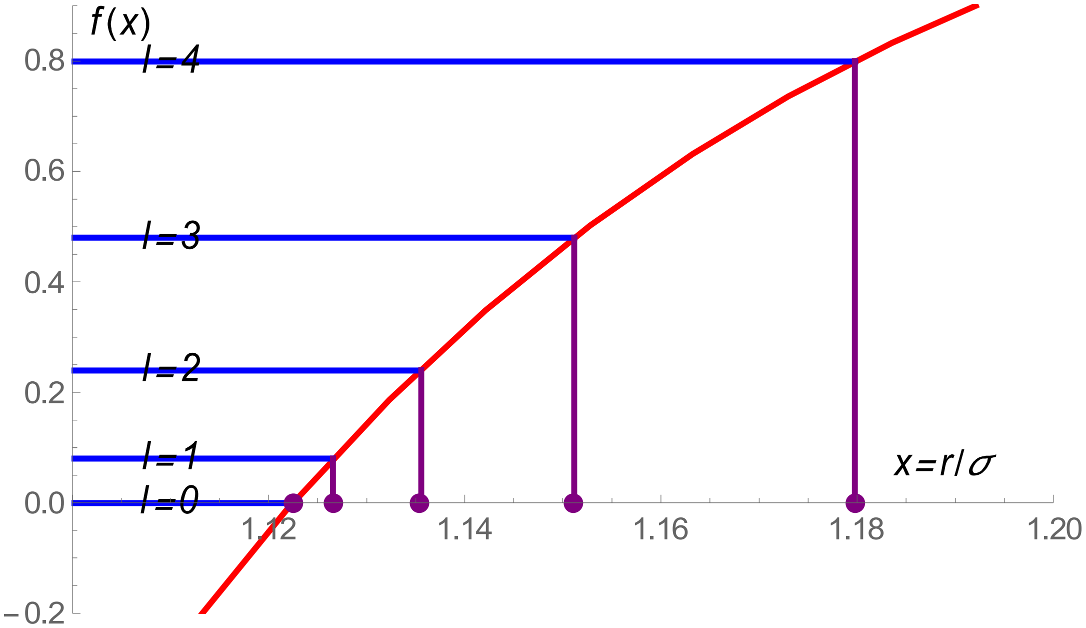


**Fig.** Plot of $f(x)$ as a function of $x = r/\sigma$. The blue lines denote $y = 2al(l+1)$ with a = 0.02. $l$ = 0, 1, 2, 3, and 4. The equilibrium distance increases with increasing $l$. $x$ = 1.12246 (l = 0. $x$ = 1.12653 ($l$ = 1). $x$ = 1.1354 ($l$ = 2). $x$ = 1.15114 ($l$ = 3), $x$ = 1.17978 ($l$ = 4).

Symmetry

Addition of two spin 1/2:

$$D_{1/2} \times D_{1/2} = D_1 + D_0$$

leading to the $S$=1 state (symmetric state) and $S$ = 0 (anti-symmetric state).

S = 1 (triplet)

$$|++\rangle, \qquad \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle), \qquad |--\rangle$$

S = 0 (singlet)

$$\frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$$

The relative orbital angular momentum;

$$\hat{\pi}|l,m\rangle = \hat{P}_{12}|l,m\rangle = (-1)^l|l,m\rangle$$

$l = 0$ (S): symmetric
$l = 1$ (P): anti-symmetric
$l = 1$ (P): symmetric
$l = 2$ (F): anti-symmetric

where $\hat{\pi}$ is the parity operator and $\hat{P}_{12}$ is the exchange operator. In this case, the exchange operator is equivalent to the parity operator for the relative angular momentum.

((**Enss**))

The most general wave function of the quasiparticle pairs in superfluid $^3$He can be represented by a linear superposition of all three spin states. Furthermore, since the angular momentum of the pair is a well-defined quantity, the pair wave function can be expanded in terms of the three states $m_z$ = -1, 0, 1 of the $l$ = 1 manifold. The general expression for the wave function can thus be written as a linear combination of 3 × 3 = 9 terms. The amplitudes of the terms are complex-valued, since each has a magnitude and phase. In other words, the wave function of quasiparticle pairs in superfluid 3He is determined by $2(2S + 1)(2L + 1) = 18$ real-valued parameters. Therefore, the order parameter of superfluid 3He is not described by a complex scalar, as in the case of helium II or in the case of conventional superconductors, but by a 3 × 3 matrix with complex-valued components.

For mathematical convenience, Balian and Werthamer introduced a vector representation $\boldsymbol{d}(\hat{\boldsymbol{k}})$ for this order parameter. Since this notation is widely used in the literature, we also adopt it. The vector $\boldsymbol{d}(\hat{\boldsymbol{k}})$ represents the pair amplitude for a particular direction $\hat{\boldsymbol{k}}$ defining a point on the Fermi surface or in other words, it defines the amplitude of the quasiparticle condensate for a point on the Fermi surface. The components $d_x$, $d_y$ and $d_z$ are each described by three complex-valued parameters and transform under rotation like a vector. The direction of ***d*** is such that its projection onto the direction of the spin of the quasiparticle pairs is zero for any point of the Fermi surface, i.e., $\boldsymbol{d} \cdot \boldsymbol{S} = 0$. We would like to point out that this not only defines the plane in which ***d*** lies, but fully determines the direction of ***d***, because ***d*** is a vector with complex components. Furthermore, we note that, in the case where ***d*** has only real-valued components, the expectation value of ***S*** is zero. The definition of ***d*** implies that it represents a unique direction in spin space, as well as the amplitude of the quasiparticle pairs for a particular direction $\hat{\boldsymbol{k}}$. Using ***d*** the general pair wave function can be written as

$$\begin{aligned} |\psi\rangle &= -d_x[|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle] + id_y[|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle] + d_z[|\uparrow\downarrow\rangle + |\uparrow\downarrow\rangle] \\ &= -(d_x - id_y)|\uparrow\uparrow\rangle + (d_x + id_y)|\downarrow\downarrow\rangle + d_z[|\uparrow\downarrow\rangle + |\uparrow\downarrow\rangle] \end{aligned} \qquad (1)$$

In principle, the internal degrees of freedom of the spin-triplet pairing allows for many different quasiparticle pair states and hence superfluid phases. Of the different states, the one with the lowest energy for a given set of external parameters will be realized. Using the general wave function (1) as a starting point, we will discuss the pair states of the different superfluid phases of $^3$He that have actually been observed.

### $^3$He-A$_1$

The simplest case is that of $^3$He-*A*$_1$, which only exists in magnetic fields. The spins are aligned parallel to the applied magnetic field ($S_z = +1$) that means only pairs in the state $|\uparrow\uparrow\rangle$ exist and that the components of ***d*** must obey the relations $d_x + id_y = 0$ and $d_z = 0$. In this case, the pair wave function reduces to

$$|\psi_{A_1}\rangle = -2d_x|\uparrow\uparrow\rangle$$

### $^3$He-A

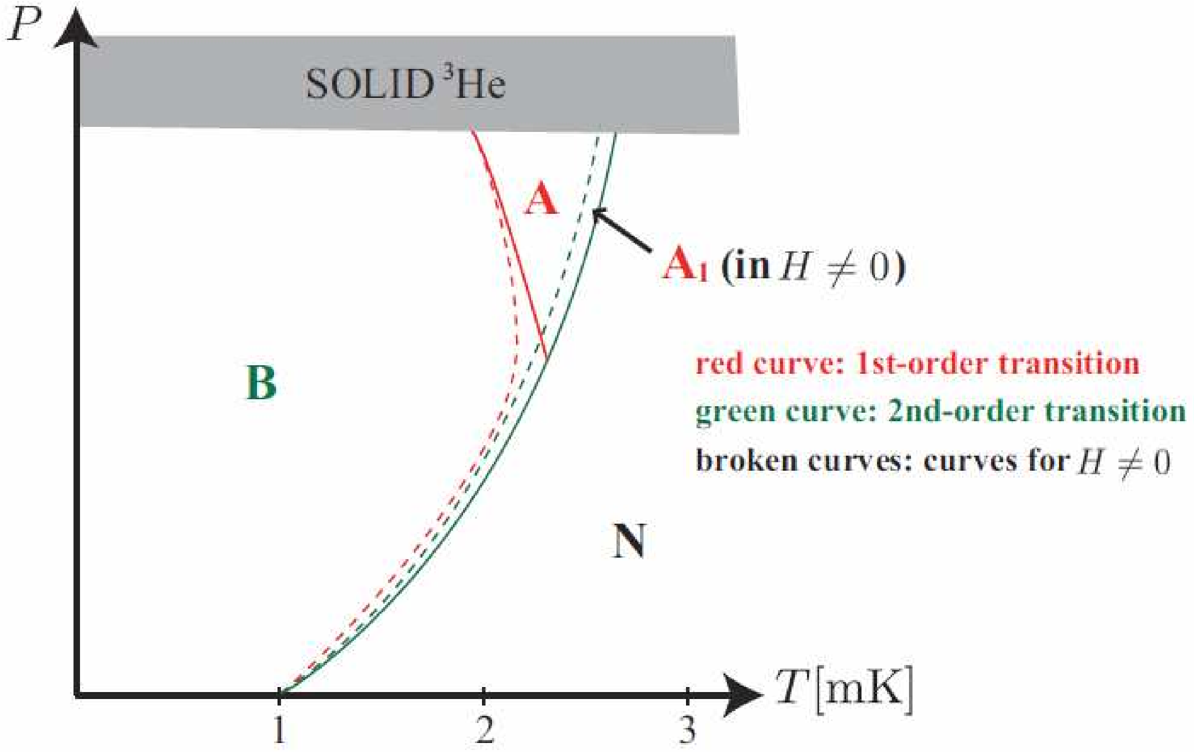
The wave function of the *A* phase is a linear combination of the states ($S_z = +1$) and thus the $d_z$ component must be zero. This corresponds to a pair state predicted by Anderson, Brinkman and Morel and is therefore often referred to as the ABM state. The wave function of this state is given by

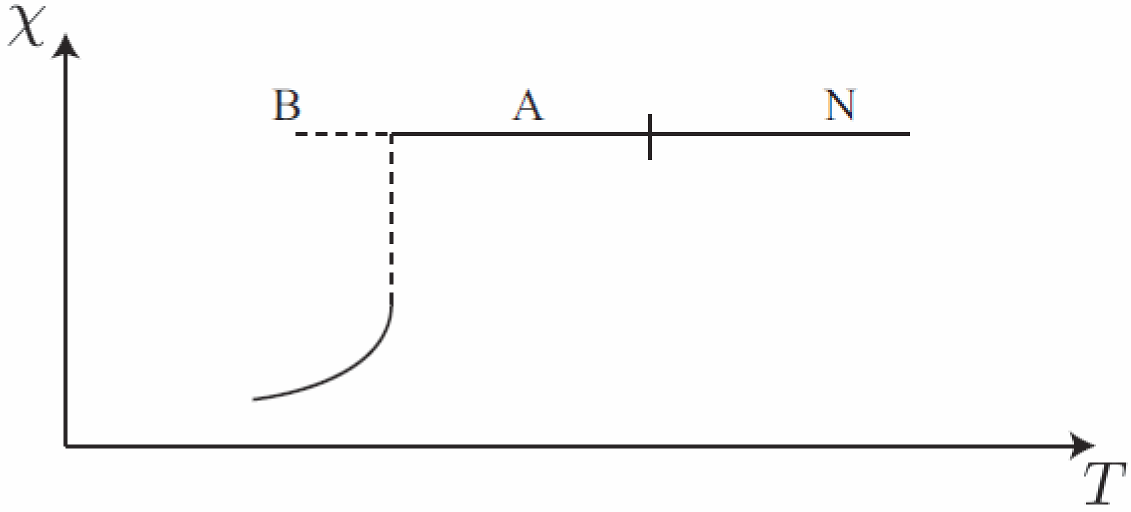
$$|\psi_A\rangle = -(d_x - id_y)|\uparrow\uparrow\rangle + (d_x + id_y)|\downarrow\downarrow\rangle$$

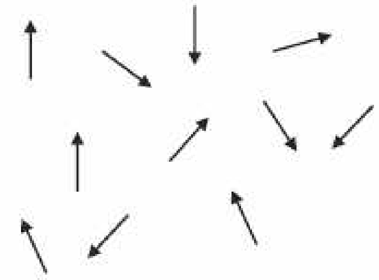
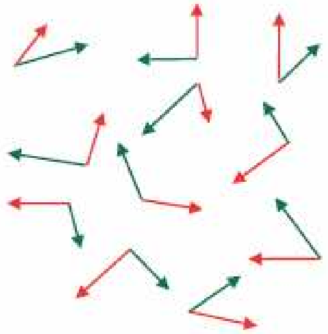
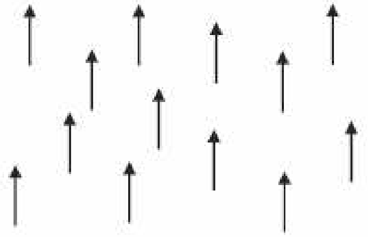
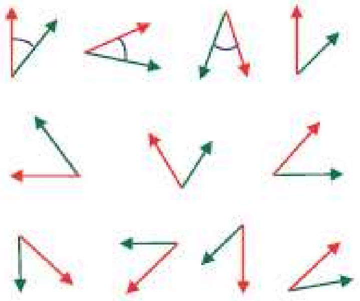
### $^3$He-B

The B phase corresponds to a pair state predicted by *Balian* and *Werthamer* in which the wave function is given by the general expression Eq.(1) that consists of all possible linear combinations. The so-called BW state is quasi-isotropic with vanishing total angular momentum

$J = L + S = 0$. It is noteworthy that both the ABM and the BW model were worked out long before the experimental discovery of superfluid $^3$He.


P
SOLID ³He
A
A₁ (in H ≠ 0)
B
N
red curve: 1st-order transition
green curve: 2nd-order transition
broken curves: curves for H ≠ 0
T[mK]
1
2
3


χ
B
A
N
T

| | Ferromagnet | Liquid $^3$He |
|---|---|---|
| | ↗ spin | ↗ total spin of pair<br>↗ relative orbital angular momentum |
| Normal phase |  |  |
| Ordered phase | <br>$\langle \boldsymbol{S} \rangle \neq 0$ | <br>$\langle \boldsymbol{S} \rangle = 0, \langle \boldsymbol{L} \rangle = 0,$<br>but $\langle \boldsymbol{L} \times \boldsymbol{S} \rangle \neq 0$ |

# Clebsch-Gordan coefficient

```
Clear["Global`*"];
CCGG[{j1_, m1_}, {j2_, m2_}, {j_, m_}] :=
 Module[{s1},
  s1 = If[Abs[m1] ⩽ j1 && Abs[m2] ⩽ j2 && Abs[m] ⩽ j,
    ClebschGordan[{j1, m1}, {j2, m2}, {j, m}], 0]];
CG[{j_, m_}, j1_, j2_] :=
 Sum[CCGG[{j1, m1}, {j2, m - m1}, {j, m}] a[j1 , m1] b[j2, m - m1], {m1, -j1, j1}];
```

j1 = 1 and j2 = 1;

D1 x D1 = D2+D1+D0

## D2 (j = 2, m = 2, 1, 0, -1, -2)

```
j1 = 1; j2 = 1; CG[{2, 2}, j1, j2]
```

$$a[1,1]\, b[1,1]$$

```
CG[{2, 1}, j1, j2]
```

$$\frac{a[1,1]\, b[1,0]}{\sqrt{2}}+\frac{a[1,0]\, b[1,1]}{\sqrt{2}}$$

```
CG[{2, 0}, j1, j2]
```

$$\frac{a[1,1]\, b[1,-1]}{\sqrt{6}}+\sqrt{\frac{2}{3}}\, a[1,0]\, b[1,0]+\frac{a[1,-1]\, b[1,1]}{\sqrt{6}}$$

```
CG[{2, -1}, j1, j2]
```

$$\frac{a[1,0]\, b[1,-1]}{\sqrt{2}}+\frac{a[1,-1]\, b[1,0]}{\sqrt{2}}$$

```
CG[{2, -2}, j1, j2]
```

$$a[1,-1]\, b[1,-1]$$

## D1 (j = 1, m = 1, 0, -1)

```
CG[{1, 1}, j1, j2]
```

$$\frac{a[1,1]\,b[1,0]}{\sqrt{2}} - \frac{a[1,0]\,b[1,1]}{\sqrt{2}}$$

```
CG[{1, 0}, j1, j2]
```

$$\frac{a[1,1]\,b[1,-1]}{\sqrt{2}} - \frac{a[1,-1]\,b[1,1]}{\sqrt{2}}$$

```
CG[{1, -1}, j1, j2]
```

$$\frac{a[1,0]\,b[1,-1]}{\sqrt{2}} - \frac{a[1,-1]\,b[1,0]}{\sqrt{2}}$$

## D0 (j = 0, m = 0)

```
CG[{0, 0}, j1, j2]
```

$$\frac{a[1,1]\,b[1,-1]}{\sqrt{3}} - \frac{a[1,0]\,b[1,0]}{\sqrt{3}} + \frac{a[1,-1]\,b[1,1]}{\sqrt{3}}$$

### Orbital angular momentum

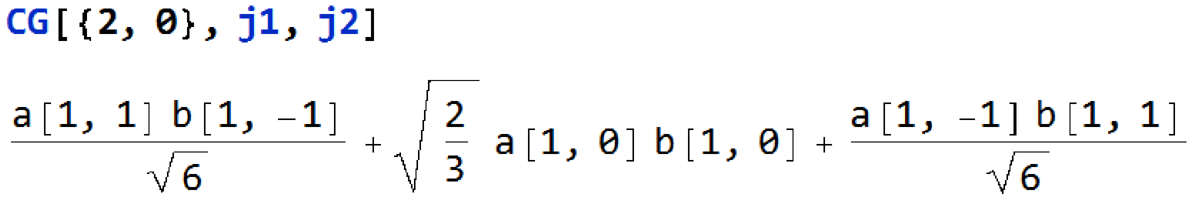
$$Y_1^1 = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{i\phi}\sin\theta = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}(d_x + id_y)$$

$$Y_1^0 = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta = \frac{1}{2}\sqrt{\frac{3}{\pi}}d_z = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sqrt{2}d_z$$

$$
\begin{aligned}
Y_1^{-1} &= (-1)^1[-\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{i\phi}\sin\theta]^* \\
&= \frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{-i\phi}\sin\theta \\
&= \frac{1}{2}\sqrt{\frac{3}{2\pi}}(d_x - id_y)
\end{aligned}
$$

$$d_x = \sin\theta\cos\phi, \qquad d_y = \sin\theta\sin\phi, \qquad d_z = \cos\theta$$

$$d_x{}^2 + d_y{}^2 + d_z{}^2 = 1$$



```
CG[{2, 0}, j1, j2]
```

$$\frac{\mathtt{a[1,\ 1]\ b[1,\ -1]}}{\sqrt{6}} + \sqrt{\frac{2}{3}}\ \mathtt{a[1,\ 0]\ b[1,\ 0]} + \frac{\mathtt{a[1,\ -1]\ b[1,\ 1]}}{\sqrt{6}}$$

$$
\begin{aligned}
&\frac{1}{\sqrt{6}}|\uparrow\uparrow\rangle\frac{1}{2}\sqrt{\frac{3}{2\pi}}(d_x - id_y) + \sqrt{\frac{2}{3}}\frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sqrt{2}d_z - \frac{1}{\sqrt{6}}|\downarrow\downarrow\rangle\frac{1}{2}\sqrt{\frac{3}{2\pi}}(d_x + id_y) \\
&= -\frac{1}{4\sqrt{\pi}}[-|\uparrow\uparrow\rangle(d_x - id_y) + |\downarrow\downarrow\rangle(d_x + id_y) - (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)2d_z]
\end{aligned}
$$

which is almost equivalent to the expression given by Enns

$$
\begin{aligned}
|\psi\rangle &= -d_x[|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle] + id_y[|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle] + d_z[|\uparrow\downarrow\rangle + |\uparrow\downarrow\rangle] \\
&= -(d_x - id_y)|\uparrow\uparrow\rangle + (d_x + id_y)|\downarrow\downarrow\rangle + d_z[|\uparrow\downarrow\rangle + |\uparrow\downarrow\rangle]
\end{aligned}
$$

Tony Guenault

### 5.2.2 p-wave pairing

Having seen that simple intuition based on ideal gases can work for the normal fluid phase of $^3$He, we are now equipped to talk about the nature of the superfluid. As we have noted, it is now widely accepted that p-wave pairing occurs. This is strongly suggested since:

1. An ideal looking second-order transition visible in the heat capacity of the Fermi system suggests formation of “weak” Cooper pairs by analogy with conventional BCS superconductivity.
2. Experimentally there are magnetic properties, ruling out $S = 0$, since the simple BCS s-wave pairing ties up opposite spins in a nonmagnetic pair.
3. For $^3$He, it might somewhat stretch one’s credulity to imagine pairs having $L = 0$, even for a true believer in the indistinguishability of identical particles according to the ideas of quantum mechanics. This is because $L = 0$ means no angular momentum of the pair, so that they must pass right through each other (electrons, point particles, yes; $^3$He, enormous atoms, no !?).
4. In any case, if we accept that $S = 1$ from the observed magnetic properties, then we must have $L$ odd; and $L = 1$ is by far the simplest option. This result comes from the wave function symmetry for identical fermions; the total wave function must be antisymmetric for co-ordinate exchange, $S = 1$ means a spin symmetric function, so the spatial wave function must be antisymmetric, and that means $L$ odd. When we decide on p-wave pairing, the theoretical situation is fascinating, yet very complex and quite a challenge.

In s-wave theory, there is just one sort of Cooper pair, constructed from the $\boldsymbol{k},\uparrow$ and $-\boldsymbol{k},\downarrow$ one-particle states. The pair is a spin singlet and an orbital (spatial) singlet. The superfluid ordering can be described by an order parameter, the macroscopic wave function of the form $\Psi = C \exp(i\varphi)$ with $C$ and $\varphi$ simple scalars. This is the recipe we have adopted in discussing both superfluid 4He and (conventional BCS) superconductivity. We have seen that it leads to a simple isotropic energy gap $\Delta$ between the superfluid ground state and the excitations. But now, in p-wave pairing, we have triplets in both space and spin coordinates. There are three possible spin configurations, giving $S_z = +1, 0, -1$. In an obvious terminology, these spin states may be referred to a $\uparrow\uparrow$, $\uparrow\downarrow$ and $\downarrow\downarrow$. The $Sz = 0$ state, here written simply as $\uparrow\downarrow$ is of course the symmetric spin pairing more fully written as $(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle / \sqrt{2}$. We also have three orbital configurations corresponding to $L_z = +1, 0, -1$. Therefor there are in all no fewer than 9 types of Cooper pair, and the order parameter must in general be written as a 3 × 3 matrix, $A_{\mu j}$ where $\mu$ cover the three spin states and $j$ the three orbital states. Since each of the $A$ is a complex number, that means that a Cooper pair has 18 number associated with it.

In principle, therefore, there is a fantastic variety of possibilities for the most stable state. Fortunately, however, there is a simplification in that the various spin components in the Cooper pair wave function behave almost independently. Thus the observed magnetic properties of the $A$, and $A_1$ phases can be used to give imp ortant clues about the make-up of these phases.

1 We saw that the A-phase susceptibility is the same as in the unpaired normal state. This is a signal that only ↑↑ and ↓↓ states are involved, since the ↑↓ state has no magnetic properties. Thus the A-phase is referred to as an “equal spin pairing” state.

2 The $A_1$-phase only occurs near $T_C$ and when a magnetic field is applied. It is thought to involve ↑↑ spins only, i.e. those spins aligned to the applied field.

3 The B-phase susceptibility is lower than that in the normal state, a signal that the ↑↓ state is now involved. In fact it is believed that all three spin components are equally involved. At first sight this might seem to imply that the susceptibility should reduce by a factor of 2/3 rather than the observed factor of about 1/3. However the interactions, as described by the Landau parameters, need to be considered. In the normal fluid, we saw above that $\chi_N$ is enhanced by the large ferromagnetic interaction measured by $F_0^{(a)}$. But this spin-dependent interaction is decreased when non-magnetic ↑↓ pairs are involved, and hence the theory based on Landau’s Fermi liquid theory turns out to give good agreement with the observed $\chi_B = \chi_N / 3$

.