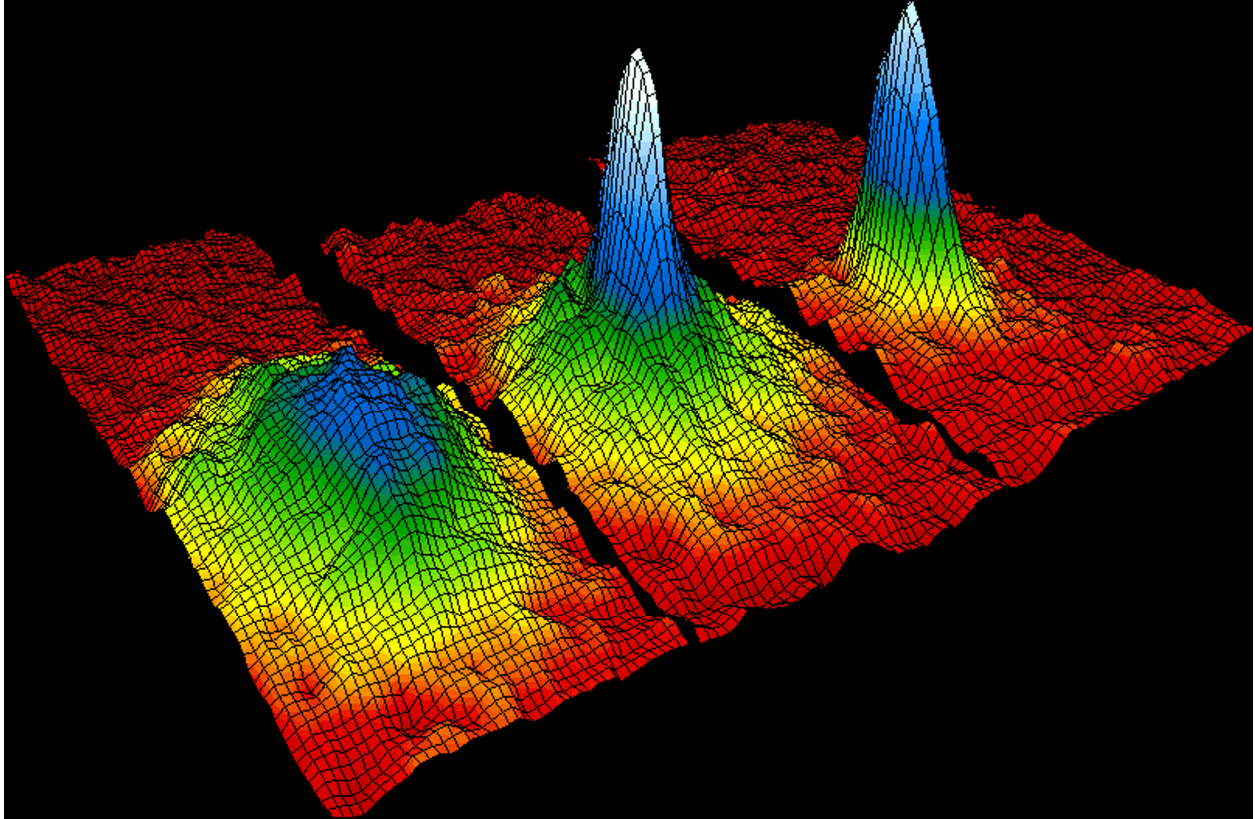


Bose-Einstein condensation of alkali atom: phase transition

Masatsugu Sei Suzuki

Department of Physics, SUNY at Binghamton

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In fact only in 1995 were actual physical examples of a BEC finally realized. These were provided not in helium, but in very dilute gases of alkali metal atoms. The techniques for trapping and cooling atoms in magnetic and laser traps had been developed and improved gradually over the preceding two decades. At first sight it may seem surprising that one can achieve the conditions of temperature and density such that BEC can occur in such systems. The densities of atoms in the traps are typically of order $10^{11} - 10^{15} \text{ cm}^{-3}$, which is many orders of magnitude less than the atomic density of ^4He , which is about $n = 2 \times 10^{22} \text{ cm}^{-3}$. Furthermore, the atomic masses of the alkalis are very much higher than for ^4He , especially for heavy alkali atoms such as ^{87}Rb . We would expect T_c values perhaps $10^{-6} - 10^{-8}$ times smaller than for the parameters of ^4He . In other words, we expect T_c values of order $10 \text{ nK} - 1 \mu\text{K}$. It is remarkable that the techniques for cooling and trapping atoms with lasers and magnetic traps can now achieve such incredibly low temperatures in the laboratory.

$$T_E = \frac{2\pi\hbar^2}{mk_B} \left(\frac{n}{\zeta_{3/2}(z=1)} \right)^{2/3} = \frac{2\pi\hbar^2}{mk_B} \left(\frac{n}{2.61238} \right)^{2/3}$$

For ^4He

$$\rho(\text{He}) = 0.145 \text{ g/cm}^3 . \quad m_{\text{He}} = 4 u ,$$

u is the atomic unit mass. $u = 1.660538782 \times 10^{-24} \text{ g}$.

$$n(\text{He}) = \rho(\text{He}) \frac{1}{m_{\text{He}}} = \frac{0.145}{4u} = 2.183 \times 10^{22} / \text{cm}^3$$

$$T_E(^4\text{He}) = 3.14 \text{ K}$$

For ^{87}Rb

$$\rho(\text{Rb}) = 1.532 \text{ g/cm}^3 . \quad m_{\text{Rb}} = 87 u .$$

$$n(\text{Rb}) = \rho(\text{Rb}) \frac{1}{m_{\text{Rb}}} = \frac{0.145}{87u} = 1.06045 \times 10^{22} / \text{cm}^3$$

$$T_E(^{87}\text{Rb}) = 89 \text{ mK}$$

We calculate the Bose-Einstein condensation temperature for ^{87}Rb

$$T_E = \frac{2\pi\hbar^2}{87uk_B} \left(\frac{n}{2.61238} \right)^{2/3}$$

when n is changed as a parameter.

$n \text{ (/cm}^3\text{)}$	T_E	
10^{10}	0.8573	nK
10^{11}	3.979	nK
10^{12}	18.47	nK
10^{13}	85.72	nK
10^{14}	397.91	nK

10^{15}	1.847	μK
10^{16}	8.573	μK
10^{17}	39.79	μK
10^{18}	184.7	μK
10^{19}	184.7	μK
10^{20}	3.979	mK
10^{21}	18.47	mK
10^{22}	85.73	mK
10^{23}	397.91	mK
10^{24}	1.847	K

((Note)) The number concentration is related to the average distance a between atoms as

$$n = \frac{1}{a^3}$$

So the distance a can be evaluated as follows.

$n \text{ (cm}^{-3}\text{)}$	a
10^{10}	4.642 μm
10^{12}	1.0 μm
10^{14}	215.4 nm
10^{16}	46.416 nm
10^{18}	100 Å = 10 nm
10^{20}	21.5 Å = 2.15 nm
10^{22}	4.64 Å = 0.464 nm
10^{24}	1 Å = 0.1 nm

Here, we shall only give a brief outline of some of the fundamental principles involved. First, how can we view a single large object, such as a rubidium atom as a boson? In quantum mechanics a particle will be a boson if it has an integer spin. Alkali metal atoms have a single valence electron in the outermost s-orbital, for example 2s for lithium (Li), 3s for sodium (Na), 4s for potassium (K), or 5s for rubidium (Rb). If the nuclear isotope is one with an odd number of protons and neutrons it will have a net half-integer spin. For example, ^7Li , ^{23}Na , and ^{87}Rb all have $S = 3/2$ nuclei. In this case the total spin of the atom is the sum of the nuclear spin and the valence electron spin, which will be an integer. The spin $I = 3/2$ nucleus and the $J = 1/2$ valence electron spin combine to give states with a total spin of either $F = 2$ or $F = 1$.

$$D_{3/2} \times D_{1/2} = D_2 + D_1$$

If we can prepare the gas so that only one of these types of states is present, then this will be a gas of particles each with an integer spin. Therefore we can view this as a gas of **Bose particles**. On the other hand, if atoms in both $S = 1$ and $S = 2$ quantum states are present in the gas, then this is effectively a mixture of two different species of bosons, since the two types of atoms are distinguishable from each other.

In order to see how these atoms can be trapped by a magnetic field we must consider the energy levels of the atom and how they are affected by a magnetic field. For definiteness, let us assume that the alkali atom has an $I = 3/2$ nucleus. It is helpful to first find the explicit spin wave functions for the different quantum states. First we find the states with maximum total spin, $F = 2$. For $F=2$, there are five different states, corresponding to z -components of total spin given by quantum numbers $m_F = 2, 1, 0, -1, -2$. The wave function corresponding to the maximum value, m_F , can be represented as follows.

The total angular momentum: addition of nuclear spin ($I = \frac{3}{2}$) and electron spin ($J = \frac{1}{2}$)

$$D_{3/2} \times D_{1/2} = D_2 + D_1$$

$F = 2$ (bosons)

$$|F = 2, m = 2\rangle = \left| \frac{3}{2}, \frac{3}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle,$$

$$|F = 2, m = 1\rangle = \frac{1}{2} \left| \frac{3}{2}, \frac{3}{2} \right\rangle \otimes \left| \frac{1}{2}, -\frac{1}{2} \right\rangle + \frac{\sqrt{3}}{2} \left| \frac{3}{2}, \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle$$

$$|F = 2, m = 0\rangle = \frac{1}{\sqrt{2}} \left| \frac{3}{2}, \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, -\frac{1}{2} \right\rangle + \frac{1}{\sqrt{2}} \left| \frac{3}{2}, -\frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle$$

$$|F = 2, m = -1\rangle = \frac{\sqrt{3}}{2} \left| \frac{3}{2}, -\frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, -\frac{1}{2} \right\rangle + \frac{1}{2} \left| \frac{3}{2}, -\frac{3}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle$$

$$|F = 2, m = -2\rangle = \left| \frac{3}{2}, -\frac{3}{2} \right\rangle \otimes \left| \frac{1}{2}, -\frac{1}{2} \right\rangle$$

$F = 1$ (bosons)

$$|F = 1, m = 1\rangle = \frac{\sqrt{3}}{2} \left| \frac{3}{2}, \frac{3}{2} \right\rangle \otimes \left| \frac{1}{2}, -\frac{1}{2} \right\rangle - \frac{1}{2} \left| \frac{3}{2}, \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle$$

$$|F = 1, m = 0\rangle = \frac{1}{\sqrt{2}} \left| \frac{3}{2}, \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, -\frac{1}{2} \right\rangle - \frac{1}{\sqrt{2}} \left| \frac{3}{2}, -\frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle$$

$$|F = 1, m = -1\rangle = \frac{1}{2} \left| \frac{3}{2}, -\frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, -\frac{1}{2} \right\rangle - \frac{\sqrt{3}}{2} \left| \frac{3}{2}, -\frac{3}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle$$

((Mathematica))

Clebsch-Gordan coefficient

```
Clear["Global`*"];

CCGG[{j1_, m1_}, {j2_, m2_}, {j_, m_}] :=
Module[{s1},
  s1 = If[Abs[m1] ≤ j1 && Abs[m2] ≤ j2 && Abs[m] ≤ j,
    ClebschGordan[{j1, m1}, {j2, m2}, {j, m}], 0]]

CG[{j_, m_}, j1_, j2_] :=
Sum[CCGG[{j1, m1}, {j2, m - m1}, {j, m}] a[j1, m1]
  b[j2, m - m1], {m1, -j1, j1}]

j1 = 3 / 2; j2 = 1 / 2
```

$$\frac{1}{2}$$

j =2;

CG[{2, 2}, j1, j2]

$$a\left[\frac{3}{2}, \frac{3}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]$$

CG[{2, 1}, j1, j2]

$$\frac{1}{2} a\left[\frac{3}{2}, \frac{3}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right] + \frac{1}{2} \sqrt{3} a\left[\frac{3}{2}, \frac{1}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]$$

CG[{2, 0}, j1, j2]

$$\frac{a\left[\frac{3}{2}, \frac{1}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right]}{\sqrt{2}} + \frac{a\left[\frac{3}{2}, -\frac{1}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]}{\sqrt{2}}$$

CG[{2, -1}, j1, j2]

$$\frac{1}{2} \sqrt{3} a\left[\frac{3}{2}, -\frac{1}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right] + \frac{1}{2} a\left[\frac{3}{2}, -\frac{3}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]$$

CG[{2, -2}, j1, j2]

$$a\left[\frac{3}{2}, -\frac{3}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right]$$

j = 1;

CG[{1, 1}, j1, j2]

$$\frac{1}{2} \sqrt{3} a\left[\frac{3}{2}, \frac{3}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right] - \frac{1}{2} a\left[\frac{3}{2}, \frac{1}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]$$

CG[{1, 0}, j1, j2]

$$\frac{a\left[\frac{3}{2}, \frac{1}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right]}{\sqrt{2}} - \frac{a\left[\frac{3}{2}, -\frac{1}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]}{\sqrt{2}}$$

CG[{1, -1}, j1, j2]

$$1 \left[\frac{3}{2}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{1}{2}\right] - \frac{1}{2} \sqrt{3} \left[\frac{3}{2}, \frac{3}{2}\right], \left[\frac{1}{2}, \frac{1}{2}\right]$$

2. Hyperfine interaction

In zero magnetic field the $F = 2$ and $F = 1$ states have slightly different energy due to the weak **hyperfine interaction** between the nucleus and the outermost unpaired valence electron. In zero magnetic field all five of the $F = 2$ states are degenerate with each other, as are the three $F = 1$ states, as shown in Fig. 1.4. But a magnetic field leads to a Zeeman splitting of these degenerate states. To a good approximation, we can write the relevant effective Hamiltonian as,

$$\hat{H} = A \left[\frac{\hat{I}_{1x}}{\hbar} \otimes \frac{\hat{J}_{2x}}{\hbar} + \frac{\hat{I}_{1y}}{\hbar} \otimes \frac{\hat{J}_{2y}}{\hbar} + \frac{\hat{I}_{1z}}{\hbar} \otimes \frac{\hat{J}_{2z}}{\hbar} \right] + 2\mu_B B (\hat{I}_1 \otimes \frac{\hat{J}_{2z}}{\hbar})$$

in the presence of an external magnetic field along the z axis, where J is the hyperfine interaction between the nuclear spin (I_1) and valence electron spins (J_2), the valence electron has a magnetic moment $-2 \frac{\mu_B}{\hbar} J_2$.

(a)

$$I_1 = \frac{3}{2}$$

$$\frac{\hat{I}_{1x}}{\hbar} = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix} \quad \frac{\hat{I}_{1y}}{\hbar} = \begin{pmatrix} 0 & \frac{-i\sqrt{3}}{2} & 0 & 0 \\ \frac{i\sqrt{3}}{2} & 0 & -i & 0 \\ 0 & i & 0 & \frac{-i\sqrt{3}}{2} \\ 0 & 0 & \frac{i\sqrt{3}}{2} & 0 \end{pmatrix}$$

$$\frac{\hat{I}_{1z}}{\hbar} = \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}$$

(b)

$J = 1/2$ (electron)

$$\frac{\hat{S}_{2x}}{\hbar} = \frac{1}{2}\hat{\sigma}_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \frac{\hat{S}_{2y}}{\hbar} = \frac{1}{2}\hat{\sigma}_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \frac{\hat{S}_{2z}}{\hbar} = \frac{1}{2}\hat{\sigma}_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Using the KroneckerProduct of the Mathematica, the matrix of the Hamiltonian is obtained as

$$\frac{\hat{H}}{A} = \begin{pmatrix} \frac{3}{4} + x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{4} - x & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{4} + x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} - x & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{4} + x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} - x & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{3}{4} + x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} - x \end{pmatrix}$$

where

$$x = \frac{\mu_B B}{A}.$$

We solve the eigenvalue problem by using the Mathematica (Eigensystem).

$F = 2$:

$$E(F = 2, m = 2) = \frac{1}{4}(3 + 4x),$$

$$E(F = 2, m = 1) = \frac{1}{4}(-1 + 4\sqrt{1 + x + x^2})$$

$$E(F = 2, m = 0) = \frac{1}{4}(-1 + 4\sqrt{1 + x^2})$$

$$E(F = 2, m = -1) = \frac{1}{4}(-1 + 4\sqrt{1 - x + x^2})$$

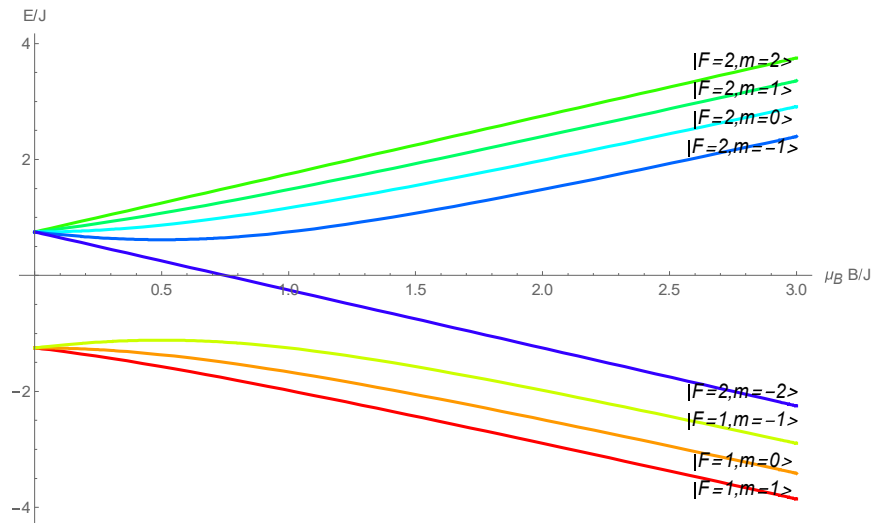
$$E(F = 2, m = -2) = \frac{1}{4}(3 - 4x)$$

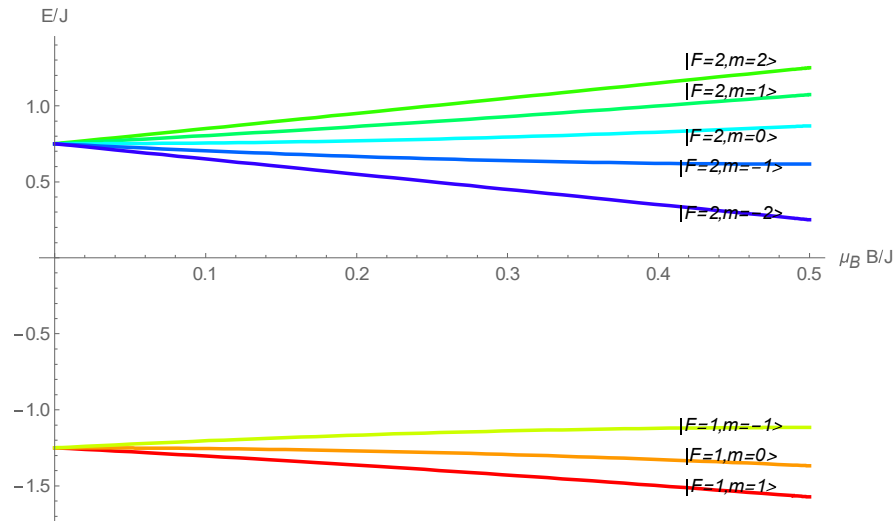
$j = 1$

$$E(F = 1, m = 1) = \frac{1}{4}[-1 - 4\sqrt{1 + x + x^2}]$$

$$E(F = 1, m = 0) = \frac{1}{4}(-1 - 4\sqrt{1 + x^2})$$

$$E(F = 1, m = -1) = \frac{1}{4}[-1 - 4\sqrt{1 - x + x^2}]$$

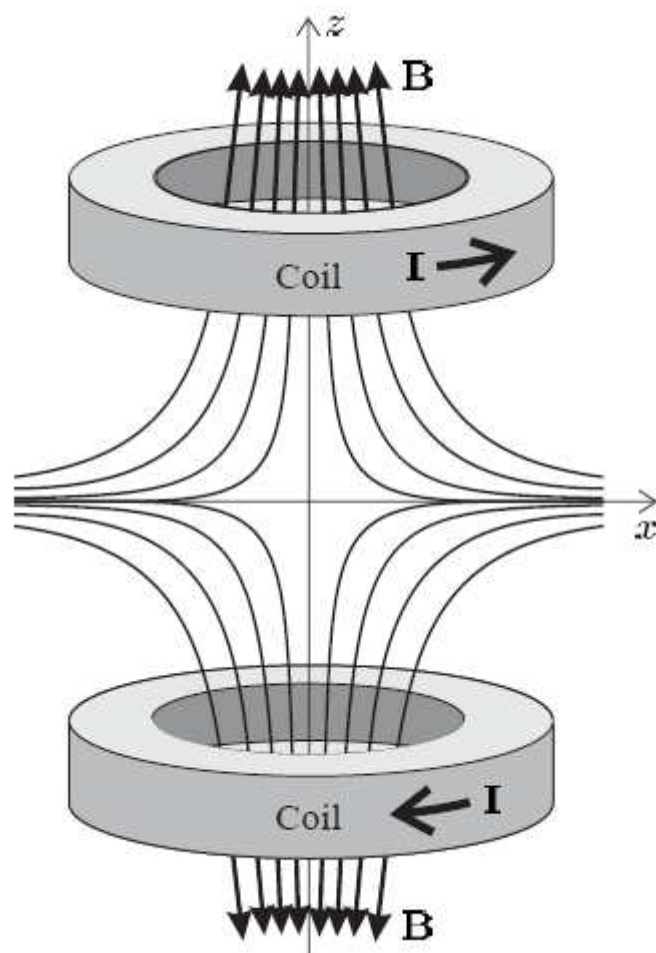




3. Magnetic trap

This magnetic field dependence of the quantum state energies is exploited in a **magnetic atom trap**. The trap is constructed by producing a region of space in which the magnetic field has a local minimum. At first sight it is surprising that it is possible to produce a local minimum in magnetic field, because the field must also obey Maxwell's equations for a region of free space

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = 0$$



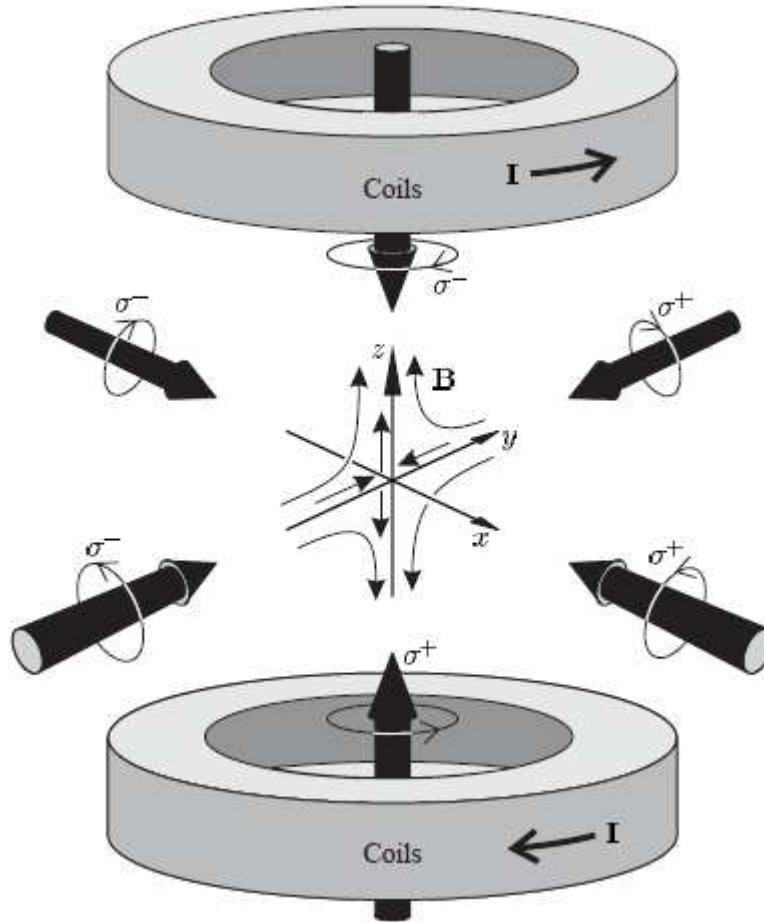


Fig. A pair of coils with currents in opposite directions produces a quadrupole magnetic field. The field is zero at the center of the coils and its magnitude increases linearly in every direction for small displacements from the zero point [C.J. Foot, Atomic Physics (Oxford, 2005)].

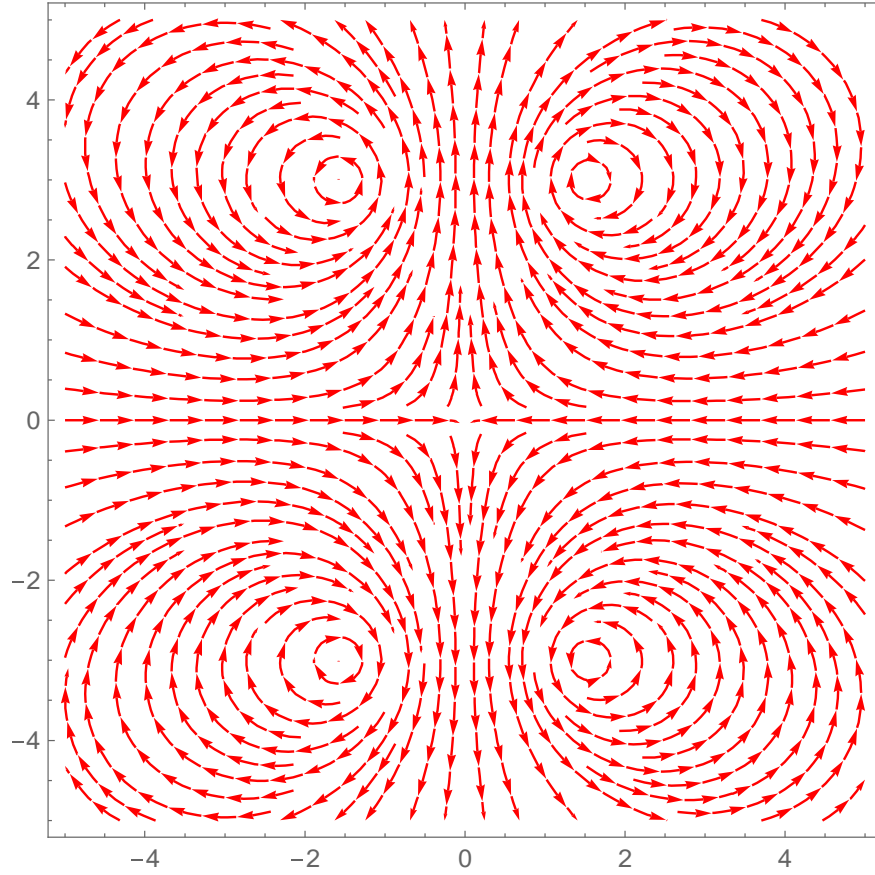


Fig. Magnetic field distribution on the x - z plane for the magnetic quadrupole configuration.

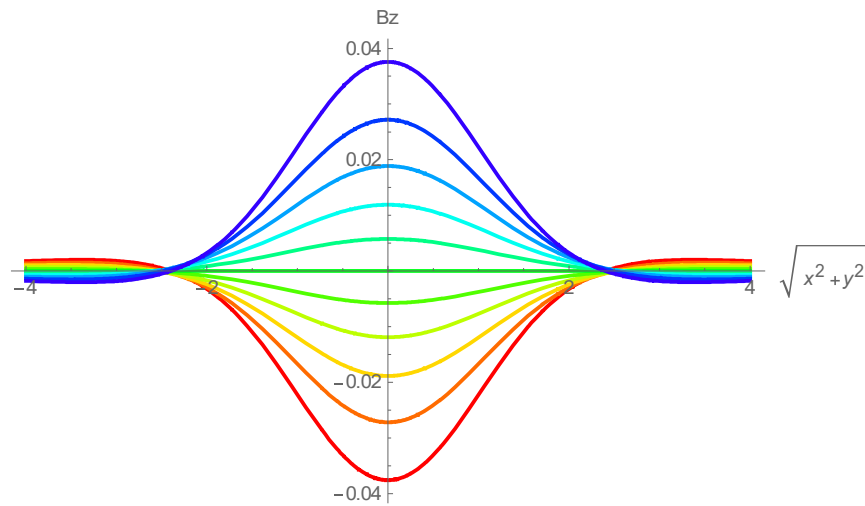
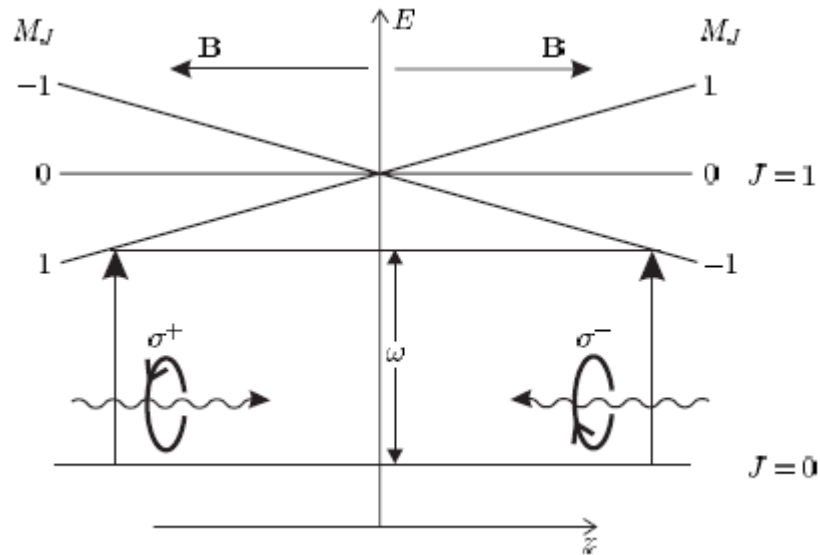
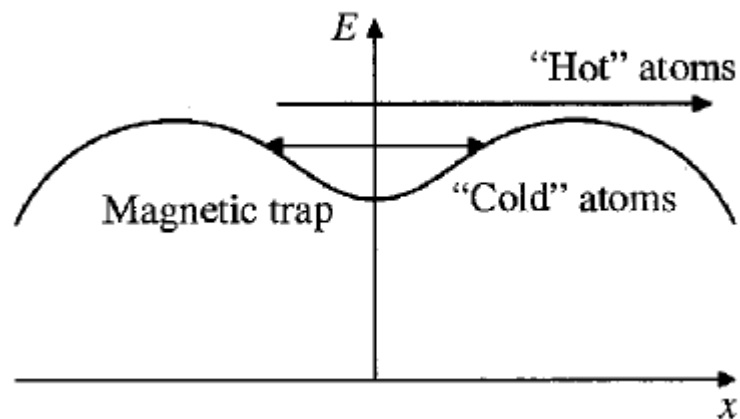
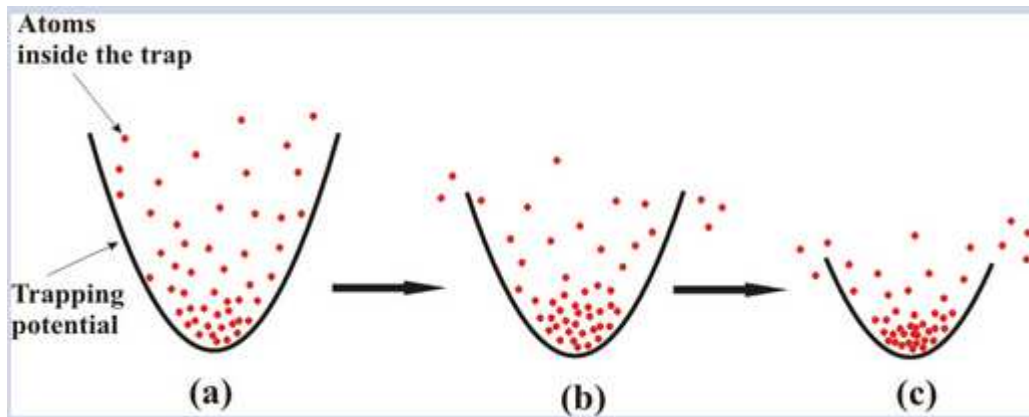


Fig. Distribution of the magnetic field along the z axis as a function of $\sqrt{x^2 + y^2}$. The magnetic field shows a local; minimum for $z < 0$ and has a local maximum for $z > 0$.



In fact it is indeed possible to both obey these and to have a local minimum in the field magnitude. Now, if we prepare an atom in a quantum state such as $F = 1$, $m_F = 1, 0, -1$, then it will lower its energy by moving toward a region of smaller magnetic field. It will therefore be attracted into the magnetic trap, which will appear to the atom as a local minimum in potential energy. This potential is only a local minimum. Atoms which have too much kinetic energy, that is, are too "hot," will not be bound by the trap and will escape. While atoms which have less kinetic energy, that is, are "cold," will be bound by the local minimum in potential energy.





5. Critical behavior

Ph.D. Thesis of J.R. Ensher

“The first experiments with Bose-Einstein condensation of ^{87}Rb ”.

University of Colorado at Boulder 1996

(a) Number density in the ground state (BEC phase)

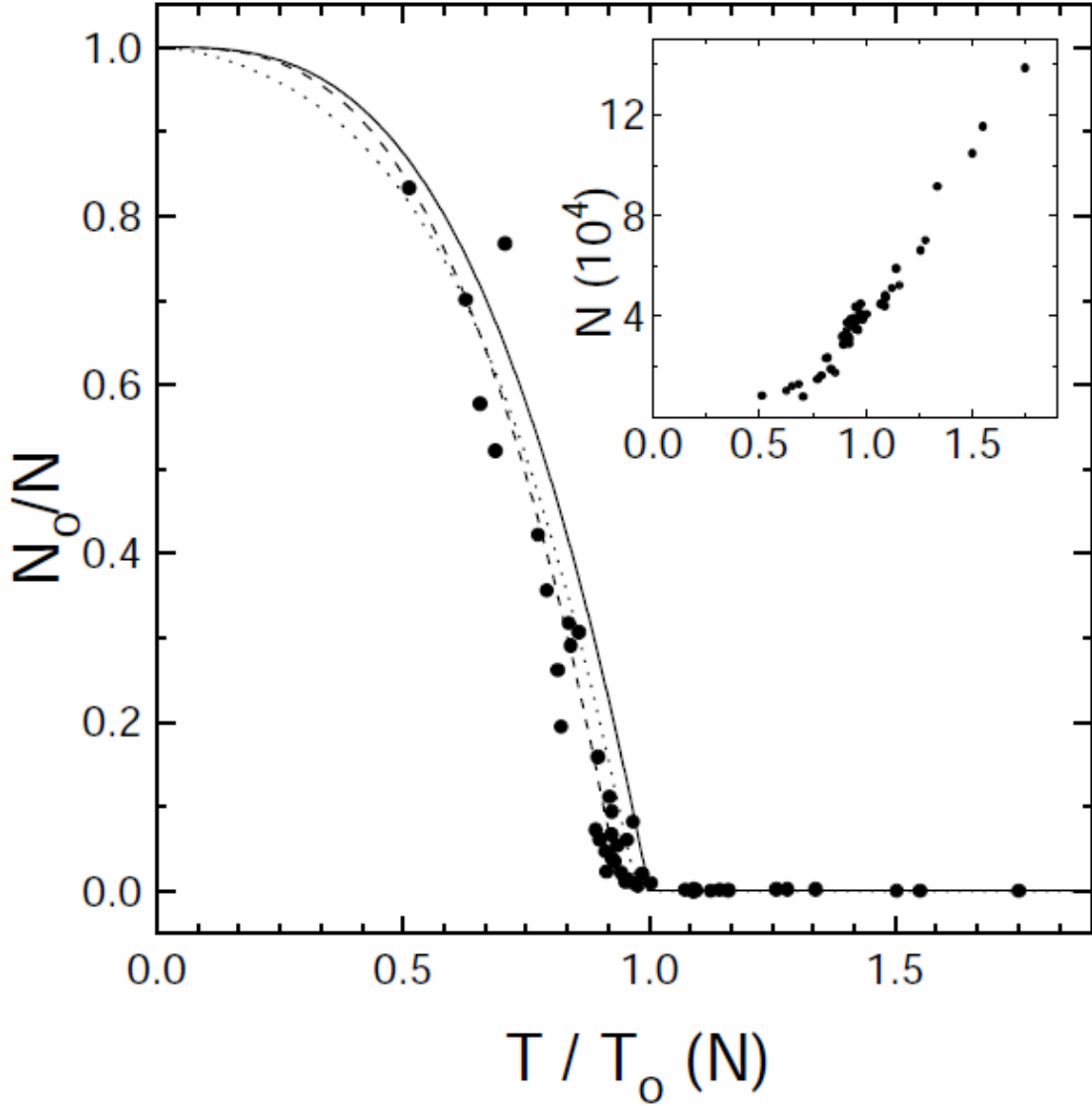


Figure 4.1: Total number N (inset) and ground-state fraction N_o/N as a function of scaled temperature T/T_o . The scale temperature $T_o(N)$ is the predicted critical temperature, in the thermodynamic (infinite N) limit, for an ideal gas in a harmonic potential. The solid (dotted) line shows the infinite (finite) N theory curves. At the transition, the cloud consisted of 40,000 atoms at 280 nK. The dashed line is a least-squares fit to the form $N_o/N = 1 - (T/T_c)^3$ which gives $T_c = 0.94(5)T_o$. Each point represents the average of three separate images. Figure taken from Ref.[13].

Huang

For a 3D harmonic oscillator potential, the number of particles in the condensate is

$$N_0 = N[1 - (\frac{T}{T_E})^3]$$

((Solution))

$$N = \sum_{n_x} \sum_{n_y} \sum_{n_z} \frac{1}{\frac{1}{z} \exp[\beta \hbar \omega (n_x + n_y + n_z + \frac{3}{2})] - 1}$$

Suppose that

$$z = \exp[\beta(\mu - \frac{3}{2} \hbar \omega)]$$

N is the number of harmonic oscillators when $z = 1$ (or $\mu = \frac{3}{2} \hbar \omega$)

$$\begin{aligned} N &= \sum_{n_x} \sum_{n_y} \sum_{n_z} \frac{1}{\exp[\beta \hbar \omega (n_x + n_y + n_z)] - 1} \\ &= \int_0^\infty dn_x \int_0^\infty dn_y \int_0^\infty dn_z \frac{1}{\exp[\beta \hbar \omega (n_x + n_y + n_z)] - 1} \end{aligned}$$

We put

$$x = \beta \hbar \omega n_x, \quad y = \beta \hbar \omega n_y, \quad z = \beta \hbar \omega n_z$$

$$\begin{aligned} N &= \left(\frac{1}{\beta \hbar \omega} \right)^3 \int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \frac{1}{\exp(x + y + z) - 1} \\ &= \left(\frac{1}{\beta \hbar \omega} \right)^3 \zeta(3) \end{aligned}$$

leading to the critical temperature T_0

$$T_0 = \frac{\hbar \omega}{k_B} \left(\frac{N}{\zeta(3)} \right)^{1/3}.$$

Note that

$$\int_0^{\infty} dx \int_0^{\infty} dy \int_0^{\infty} dz \frac{1}{\exp(x+y+z)-1} = \zeta(3) = 1.20206 \quad (\text{Mathematica})$$

$$N = N(T_E, z=1) = \left(\frac{k_B T_E}{\hbar \omega} \right)^3 \zeta(3)$$

$$N_e(T, z=1) = \left(\frac{k_B T}{\hbar \omega} \right)^3 \zeta(3)$$

where $\zeta(3) = 1.20206$.

The number of particles in the condensate phase is

$$\begin{aligned} N_0(T) &= N - N_e(T, z=1) \\ &= N \left[1 - \frac{N_e(T, z=1)}{N} \right] \\ &= N \left[1 - \left(\frac{T}{T_E} \right)^3 \right] \end{aligned}$$

or

$$N_0(T) = N \left[1 - \left(\frac{T}{T_E} \right)^3 \right]$$

((**Mathematica**))

```

Clear["Goba1`"];

f1 = 
$$\frac{1}{\text{Exp}[x1 + x2 + x3] - 1};$$


k1 =
Integrate[
  Integrate[Integrate[f1, {x3, 0, ∞}],
    {x2, 0, ∞}], {x1, 0, ∞}]

Zeta[3]

% // N

1.20206

1.202061/3 // N

1.06327

```

((Note))

$$\begin{aligned}
 N &= \sum_k \bar{n}_k \\
 &= \frac{gV}{(2\pi)^3} \int_0^\infty 4\pi k^2 dk \frac{1}{\frac{e^{\beta\varepsilon}}{z} - 1}
 \end{aligned}$$

We assume that $\varepsilon = \hbar ck$ (energy dispersion)

$$\frac{N}{V} = \frac{g}{(2\pi)^3} \left(\frac{1}{\hbar c} \right)^3 4\pi \int_0^\infty \varepsilon^2 d\varepsilon \frac{1}{\frac{e^{\beta\varepsilon}}{z} - 1}$$

$$x = \beta\varepsilon, \quad dx = \beta d\varepsilon$$

$$\begin{aligned}
n &= \frac{g}{(2\pi)^3} \left(\frac{k_B T}{\hbar c} \right)^3 4\pi \int_0^\infty \frac{x^2 dx}{e^{x/z} - 1} \\
&= \frac{g 4\pi}{(2\pi)^3} \left(\frac{k_B T}{\hbar c} \right)^3 2\zeta_3(z) \\
&= \frac{g}{\pi^2} \left(\frac{k_B T}{\hbar c} \right)^3 \zeta_3(z)
\end{aligned}$$

where

$$\int_0^\infty \frac{x^2 dx}{e^{x/z} - 1} = 2\zeta_3(z)$$

At $T = T_E$,

$$n = \frac{g}{\pi^2} \left(\frac{k_B T_E}{\hbar c} \right)^3 \zeta_3(z = 1)$$

where

$$\zeta_3(z = 1) = 1.20206$$

For $T < T_E$,

$$n_e = \frac{g}{\pi^2} \left(\frac{k_B T}{\hbar c} \right)^3 \zeta_3(z = 1).$$

The number density at the ground state is derived as

$$n_0 = n - n_e = n \left[1 - \left(\frac{T}{T_E} \right)^3 \right],$$

since

$$\frac{n_e}{n} = \left(\frac{T}{T_E} \right)^3.$$

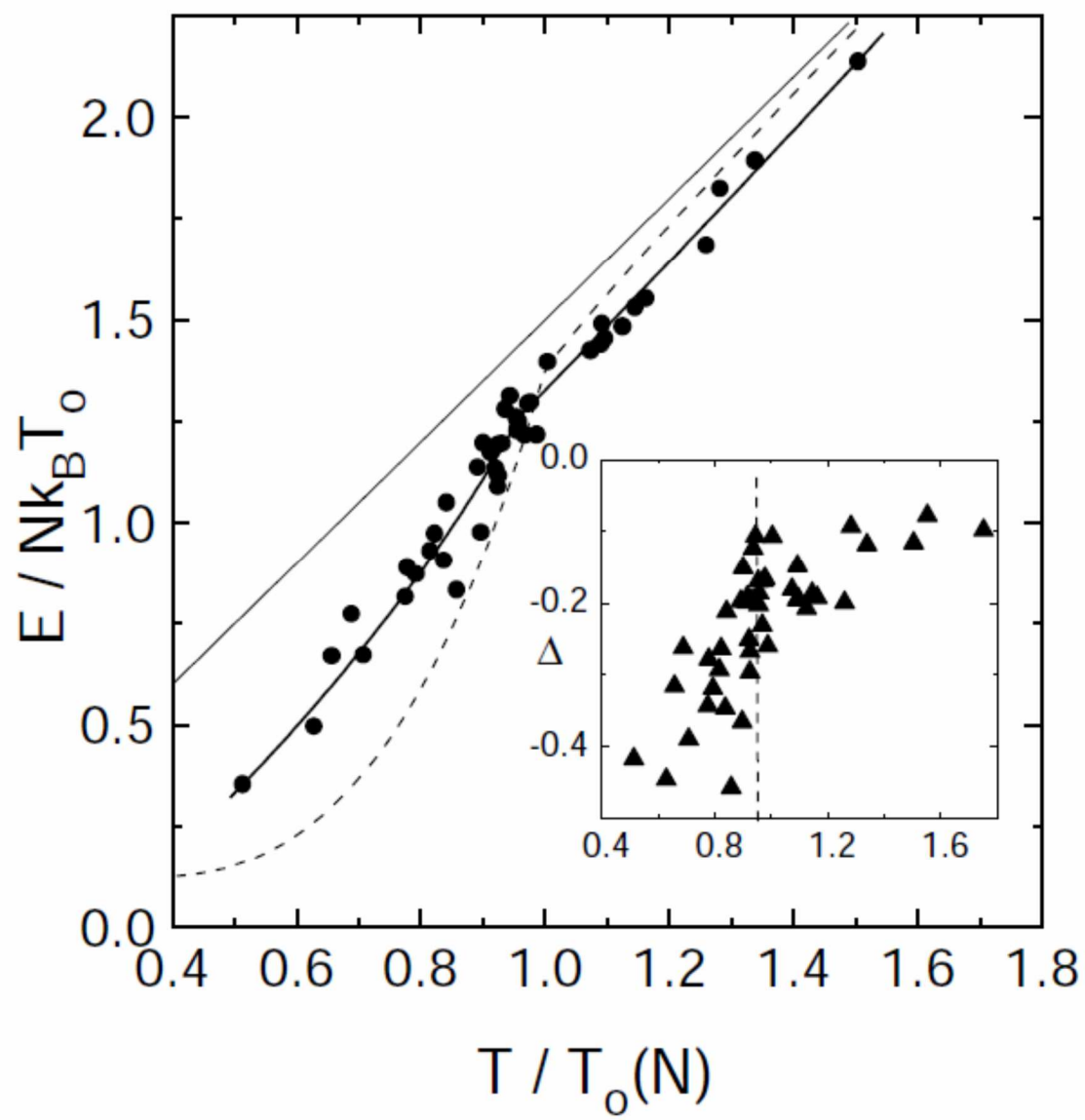
The Bose-Einstein temperature T_E :

$$T_E = \frac{\hbar c}{k_B} \left(\frac{\pi^2}{g} \right)^{1/3} \left(\frac{n}{\zeta_3(z=1)} \right)^{1/3} = \frac{\hbar c}{k_B} \left(\frac{\pi^2}{g} \right)^{1/3} \left(\frac{n}{\zeta_3(z=1)} \right)^{1/3} .$$

The expression of T_E is different from that of T_E for the system with the energy dispersion

(b) The energy density and specific heat

Specific heat, at constant external potential, vs. scaled temperature T/T_0 is plotted for various theories and experiment: theoretical curves for bosons in an anisotropic 3D harmonic oscillator and a 3D square well potential, and the data curve for liquid ^4He . The flat dashed line is the specific heat for a classical ideal gas. The derivative (bold line) of the polynomial fits to our energy data is compared to the predicted specific heat (fine line) for a finite number of ideal bosons in a harmonic potential.



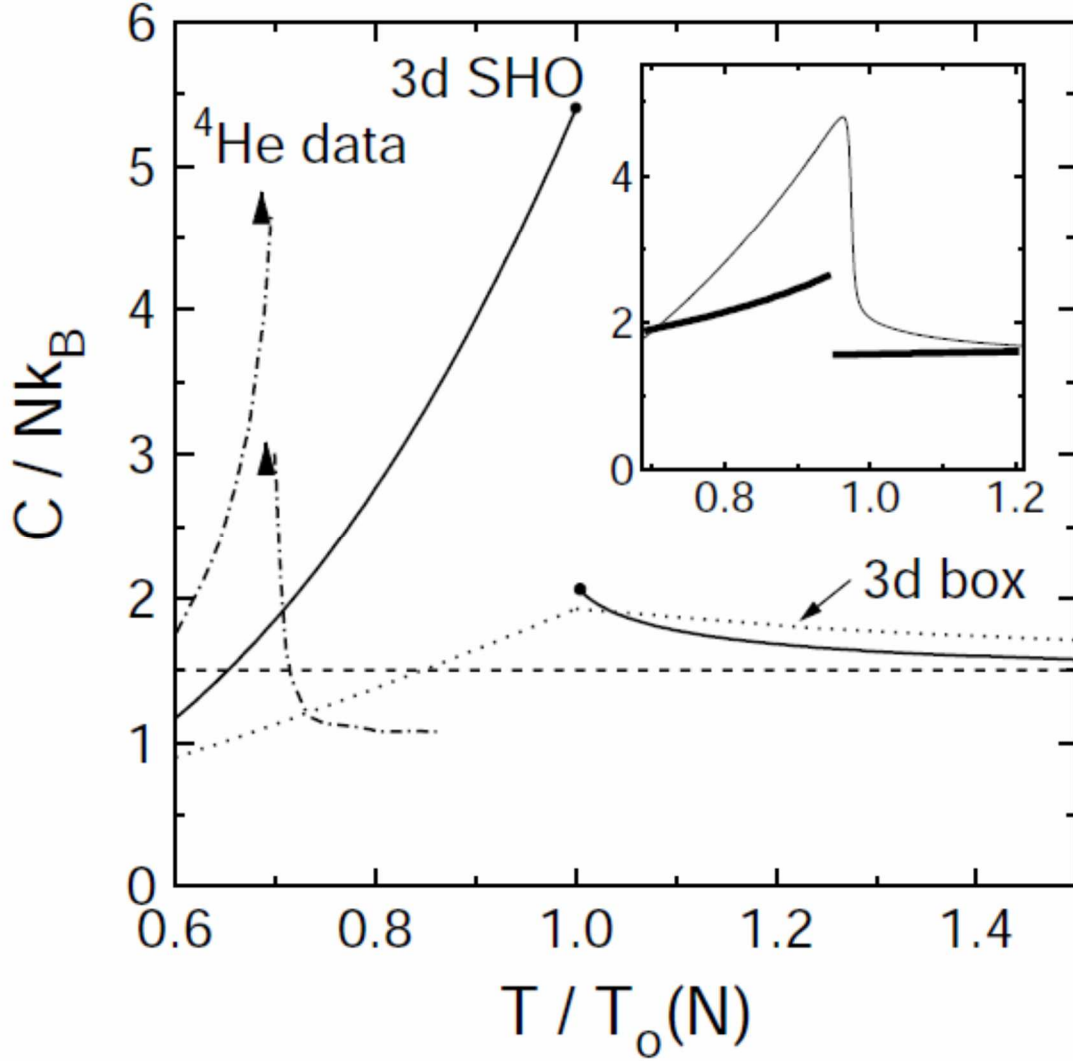


Fig. The scaled energy per particle $E/(Nk_B T_0)$ of the Bose gas is plotted vs. scaled temperature T/T_0 . The straight, solid line is the energy for a classical, ideal gas, and the dashed line is the predicted energy for a finite number of non-interacting bosons. The solid, curved lines are separate polynomial fits to the data above and below the empirical transition temperature of $0.94T_0$. The difference Δ between the data and the classical energy emphasizes the change in slope of the measured energy-temperature curve near $0.94T_0$ (vertical dashed line).

6. Interference behavior

Ph.D. Thesis

O. Garcia-Salazar

Bose-Einstein-Condensate Interferometer with Macroscopic Arm Separation

University of Virginia (1998)

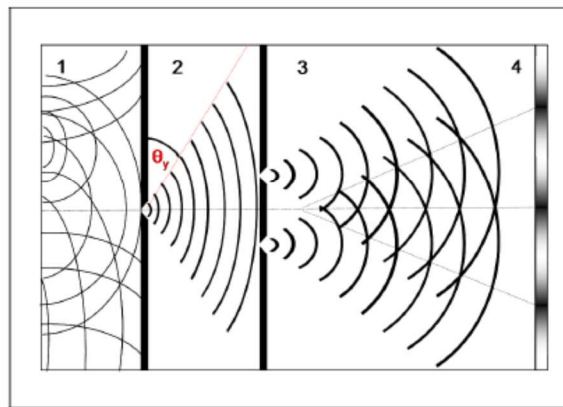


Figure 1.3: Set up for Young's double slit experiment. 1 A non coherent source of light is incident on a pinhole. 2. The pinhole outputs spatially coherent light which in turn is incident on two other pinholes. 3. Because the light is spatially coherent, the output of the two pinholes acts as two spatially coherent sources generating the light and dark fringes in 4.

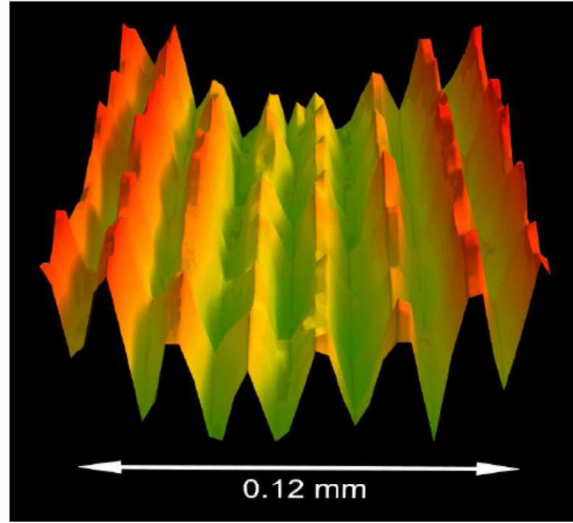


Figure 1.5: Two Bose Einstein condensates, released from a magnetic trap, free falling and overlapping while expanding. Because a condensate exhibits properties of a matter wave, as they expand they interfere. The fringes of the interference pattern are clearly resolved [22].

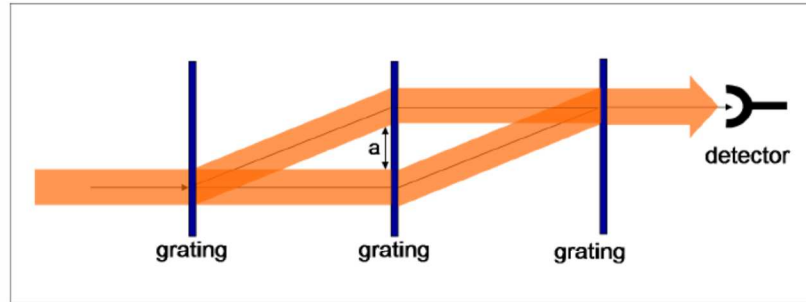


Figure 4.1: Atom beam interferometers cannot achieve large deflection angles due to the high velocity of the beam, typically atoms move at 290 m/s. This diagram depicts the small arm separation $a = 54 \mu\text{m}$ obtained. Our design can improve this limitation increasing a to $a = 250 \mu\text{m}$.

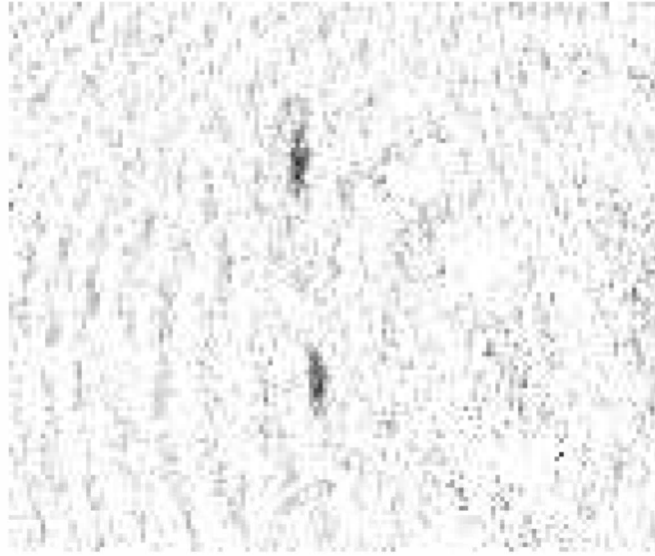


Figure 4.8: Absorption imaging showing two wave packets after applying the double pulse sequence on the condensate at rest in the waveguide. Here two wave packets are seen travelling at $\pm 2\hbar k$ 10 ms after splitting.

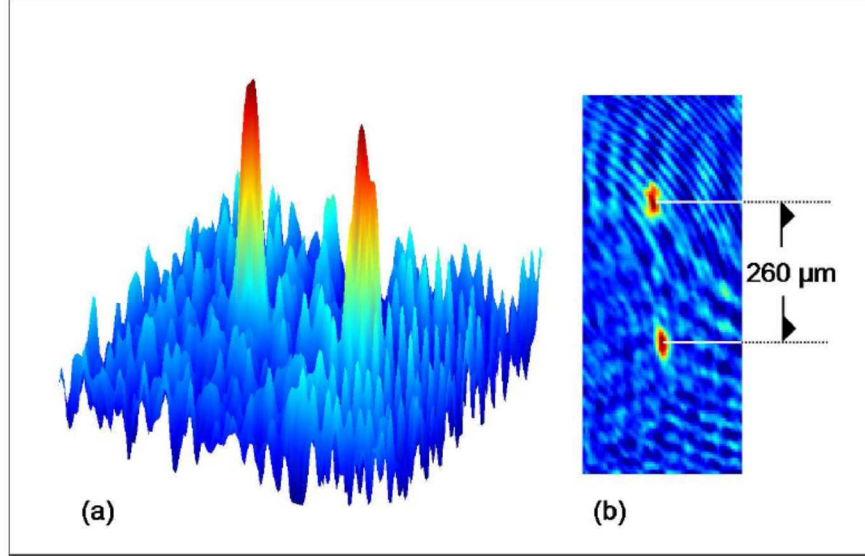


Figure 5.5: Maximum matter wave separation yielding the largest arm separation of $260 \mu\text{m}$ for our Bose-Einstein condensate interferometer. This separation was obtained from the 11-22-11 interferometer, having imaged the packets 11 ms after the initial split. Wave packets are pictured using absorption imaging, 1 shows a 3-D representation of the absorption profile and 2 shows a 2-D image. Throughout this particular sequence, the wave packets remained coherent as demonstrated by the non-zero visibility of the $T = 44$ ms interferometer. Hence the atoms in this picture were in a quantum superposition of being in both peaks. Note, red color and height indicate the highest density of atoms.

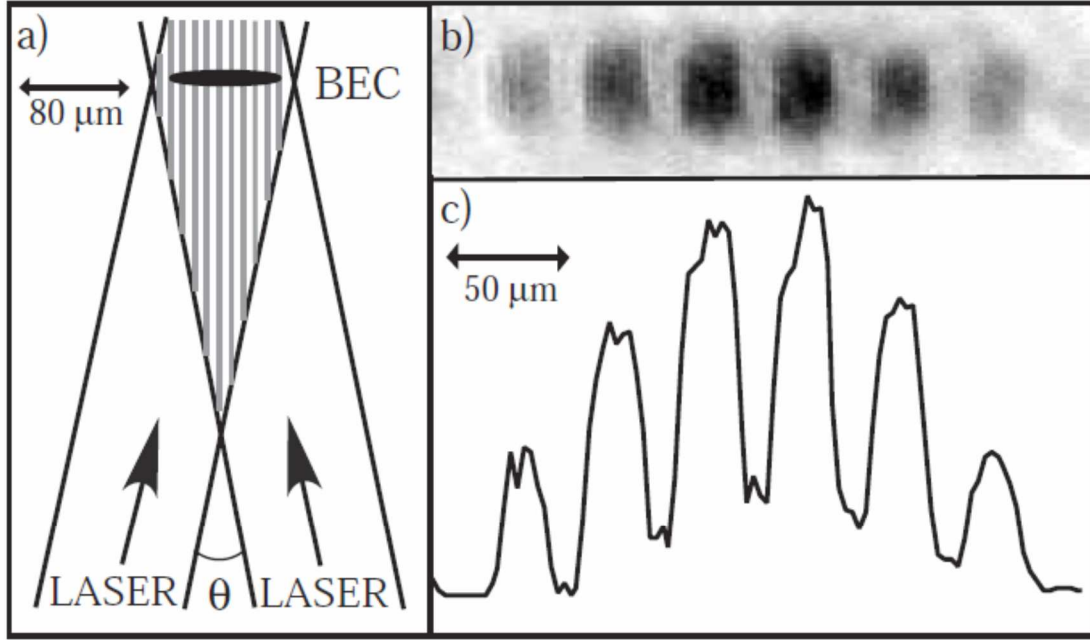


FIG. 1: Interference of Bose-Einstein condensates with uncorrelated phases. (a) A deep 1D optical lattice splits a cigar-shaped condensate into 30 independent BECs. (b) Absorption image of the cloud after 22 ms of expansion from the lattice. The density distribution shows clear interference fringes of high contrast. (c) Axial density profile of the cloud, radially averaged over the central $25\ \mu\text{m}$. The fit described in the text gives a fringe amplitude $A_1 = 0.64$ for this image.

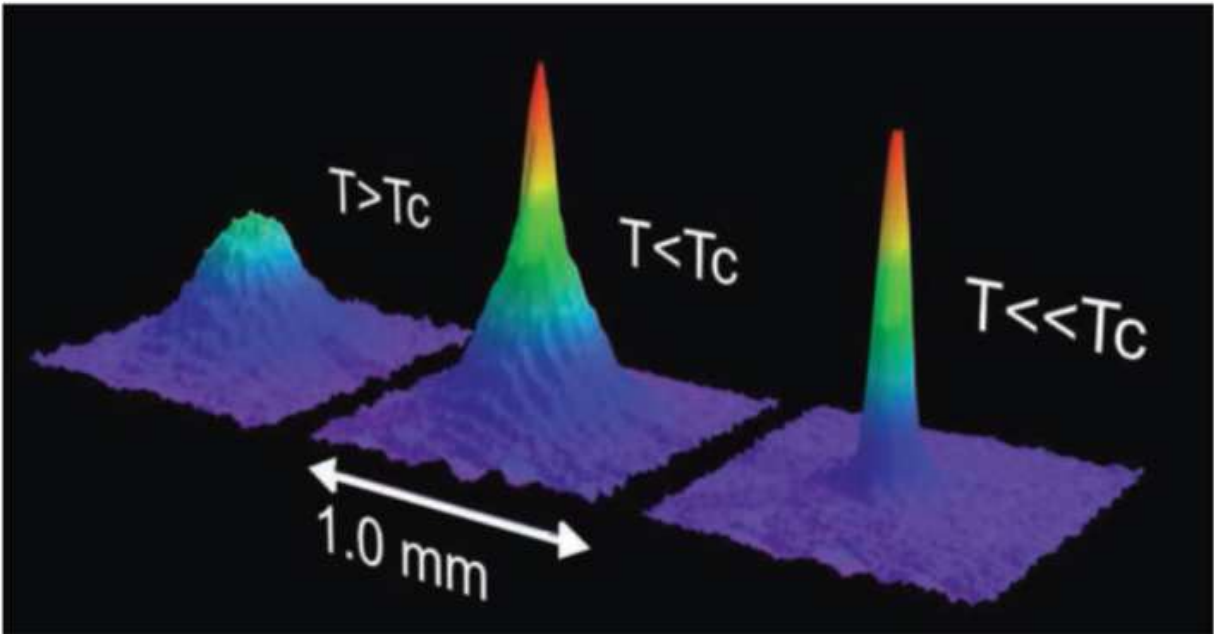


Fig. Observation of BE condensate by absorption imaging. In absorption imaging, the atom cloud is illuminated with a laser beam, and its shadow is imaged onto a digital camera. Pictures taken from expanding ultracold atom clouds reveal the onset of BE condensation below a critical temperature T_c . The BEC is characterized by its slow expansion observed after 6 ms time of flight. The left picture shows an expanding cloud cooled to just above the transition point; middle: just after the condensate appeared; right: after further evaporative cooling has left an almost pure condensate. The width of the image is 1 mm. The total number of atoms at the phase transition is about 7×10^5 , the temperature at the transition point is $2 \mu\text{K}$. The bottom picture is a 3D plot where the height of the peak represents the darkness of the shadow. The BE condensate appears as a sharp peak.

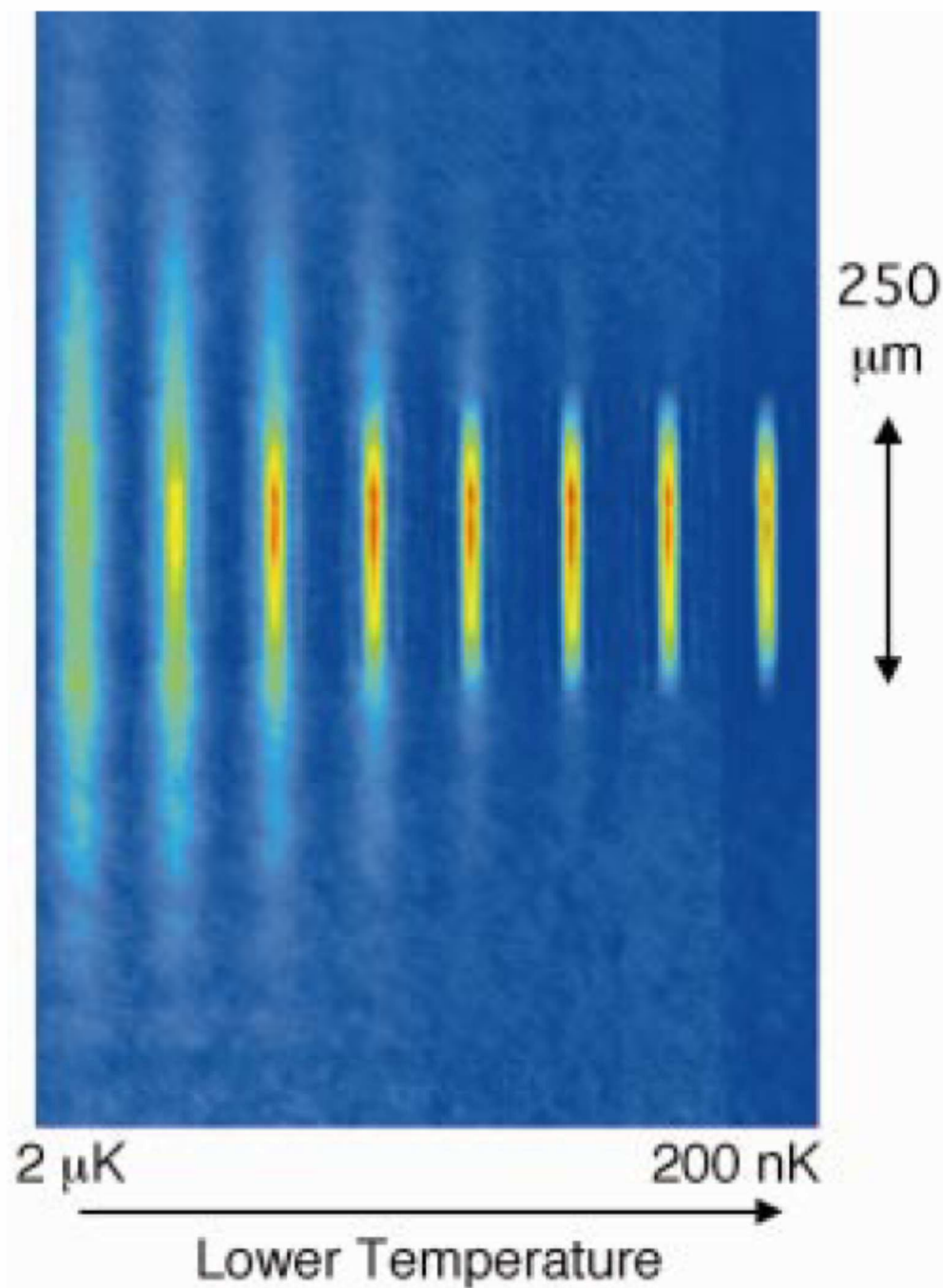


Fig. Phase contrast images of trapped Bose gases across the BEC phase transition. At high temperature, above the BEC transition temperature, the density profile of the gas is smooth. As the temperature drops below the BEC phase transition, a high-density core of atoms appears in the center of the distribution. This is the Bose-Einstein

condensate. Lowering the temperature further, the condensate number grows and the thermal wings of the distribution become shorter, Finally, the temperature drops to the point where a pure condensate with no discernible thermal fraction remains. Each image shows an equilibrated gas obtained in one complete trapping and cooling cycle. The axial and radial frequencies are about 17 and 230 Hz, respectively.

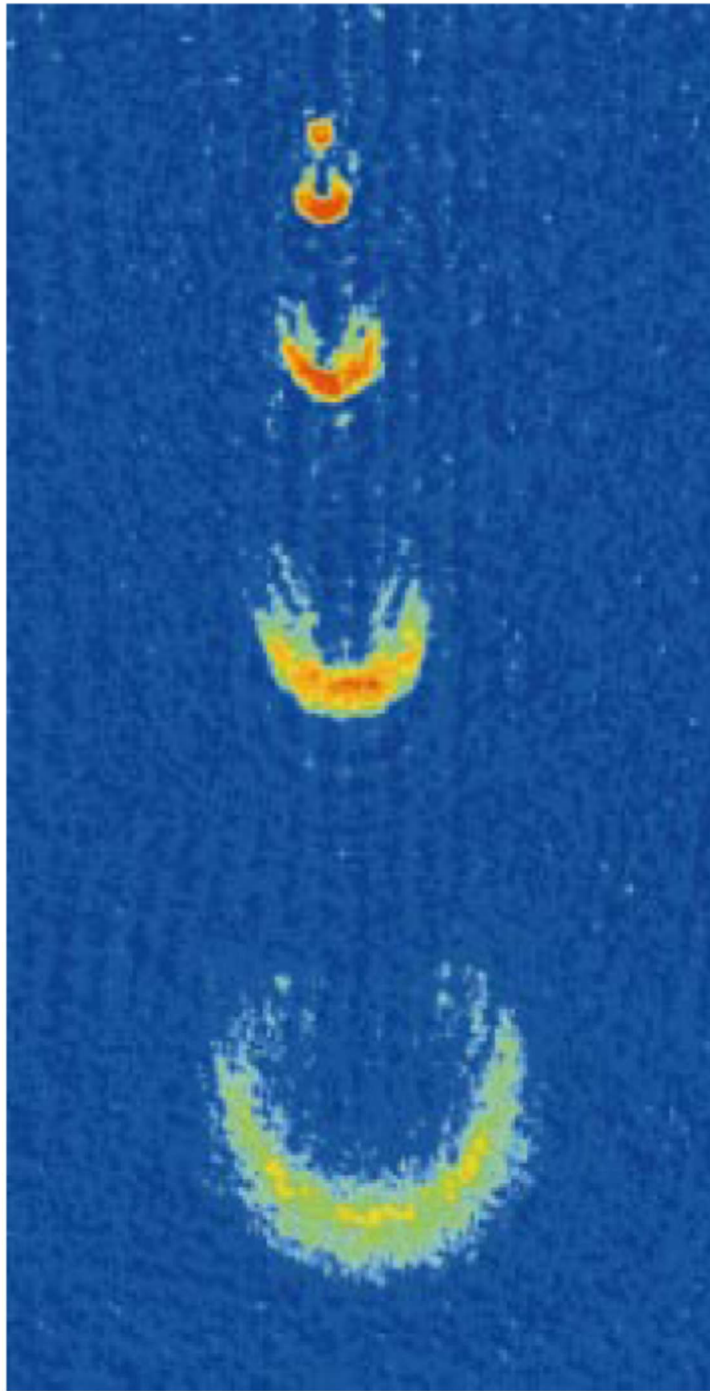


Fig. MIT atom laser operating at 200 Hz. Pulse of coherent Na atoms are coupled out from a BEC (Bose-Einstein condensate) confined in a magnetic trap (field of view 2.5×5.0 mm²). Every 5 ms, a short rf pulse transferred a fraction of these atoms into an

unconfined quantum state. These atoms were accelerated downward by gravity and spread out due to repulsive interactions. The atom pulses were observed by absorption imaging. Each pulse contained between 10^5 and 10^6 atoms.

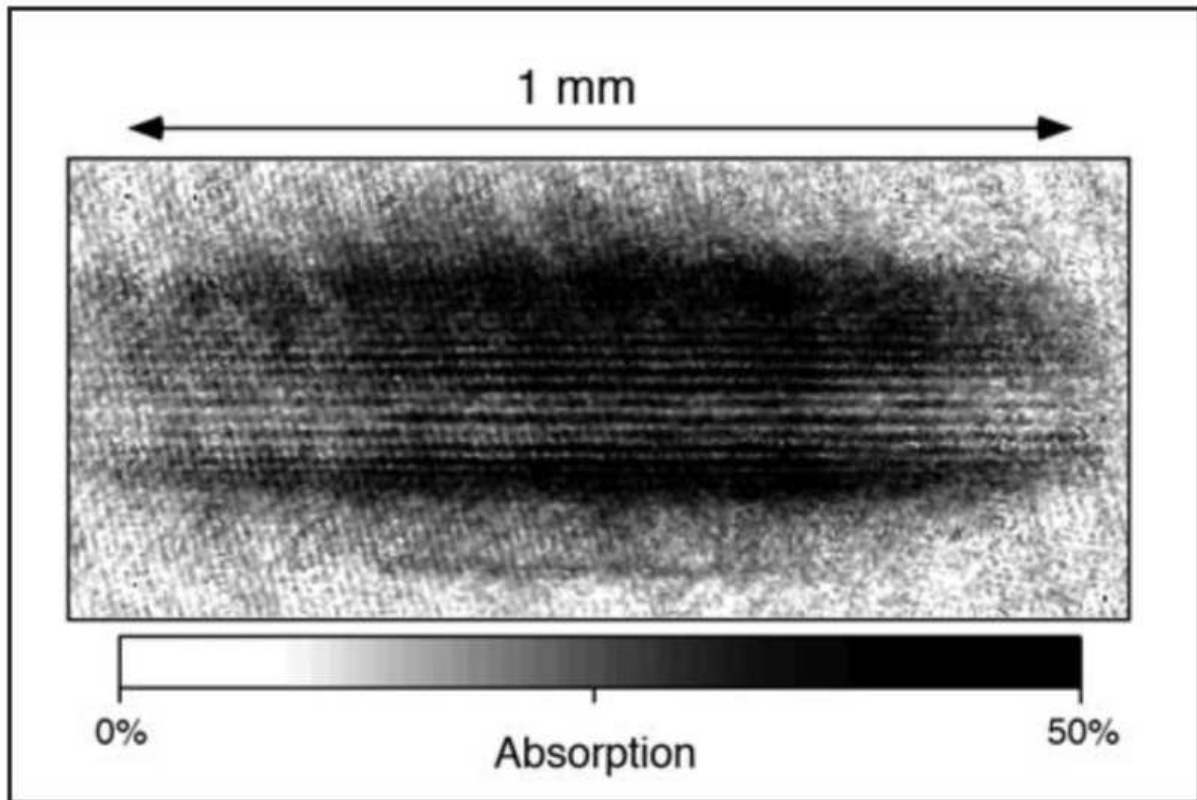


Fig. Interference pattern of two expanding condensates observed after 40 ms time of flight using absorption imaging. This observation proved that the atoms in a Bose-Einstein condensate form one big matter wave that can interfere with another Bose-Einstein condensate in the same way as two beams interfere and create a pattern of dark and bright fringes.

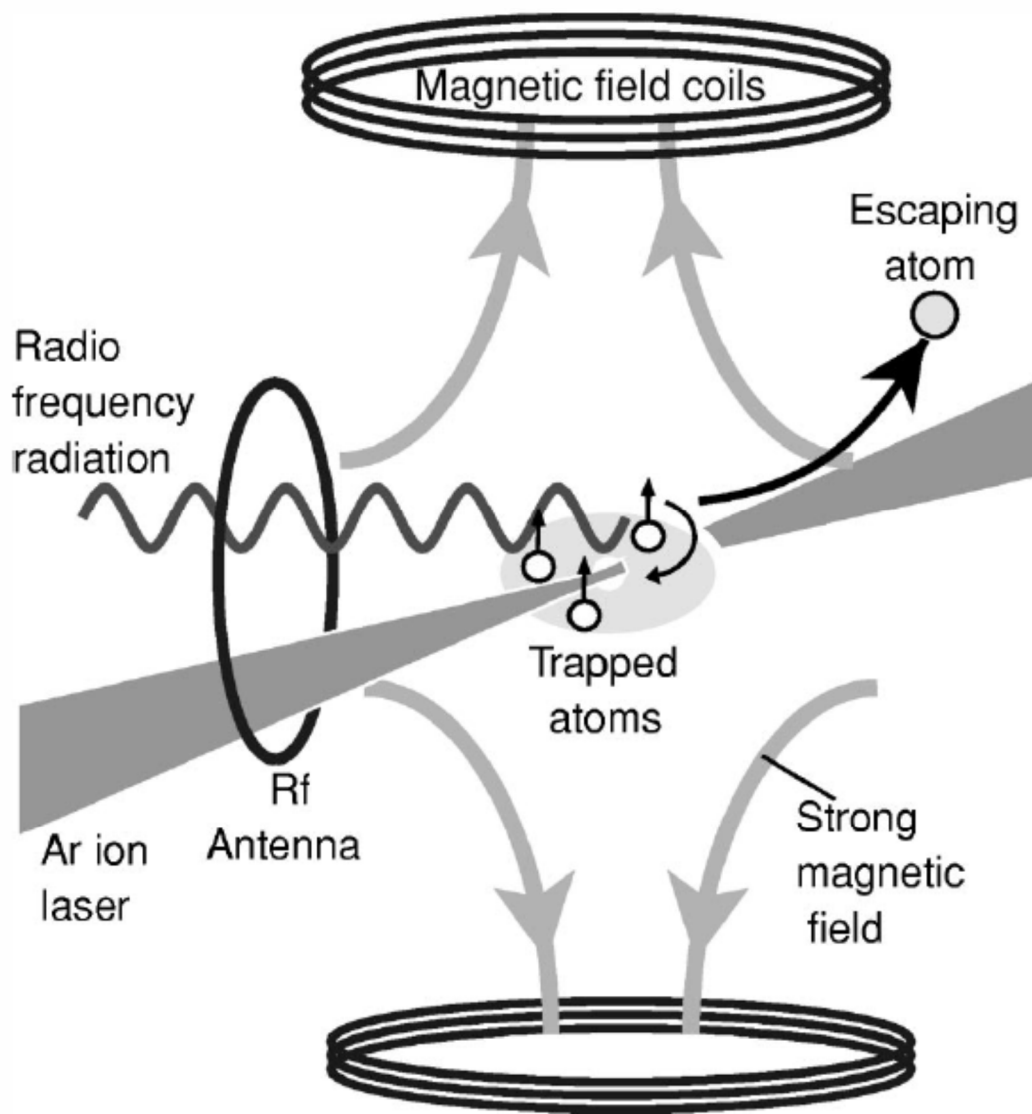
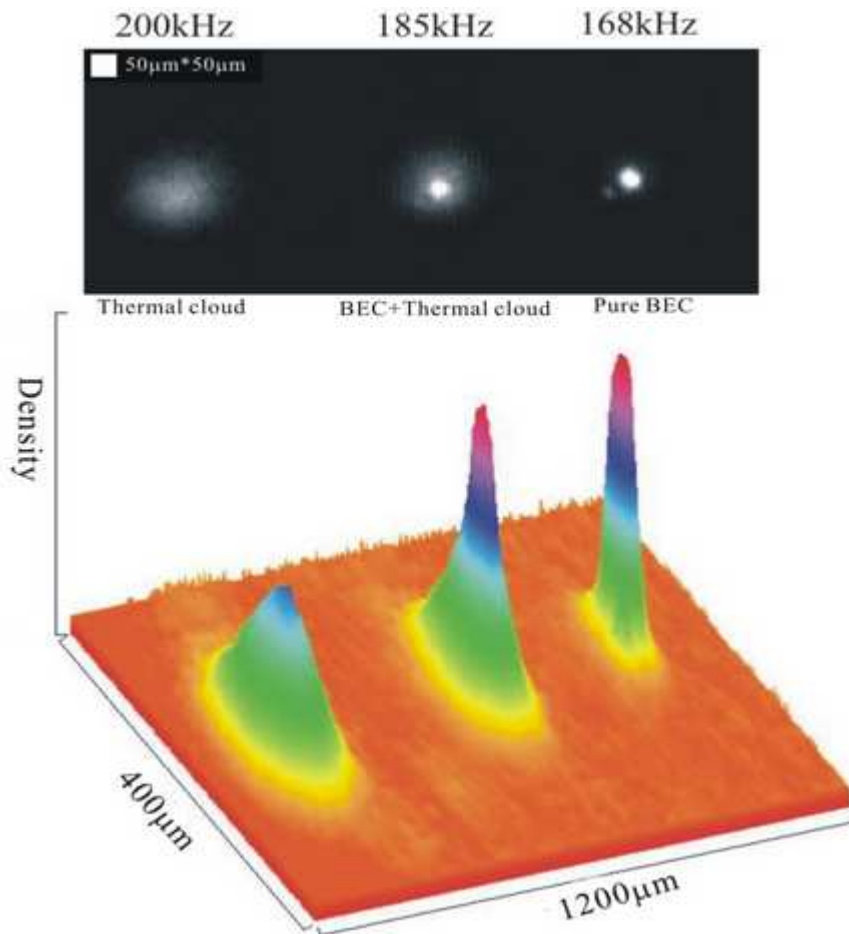
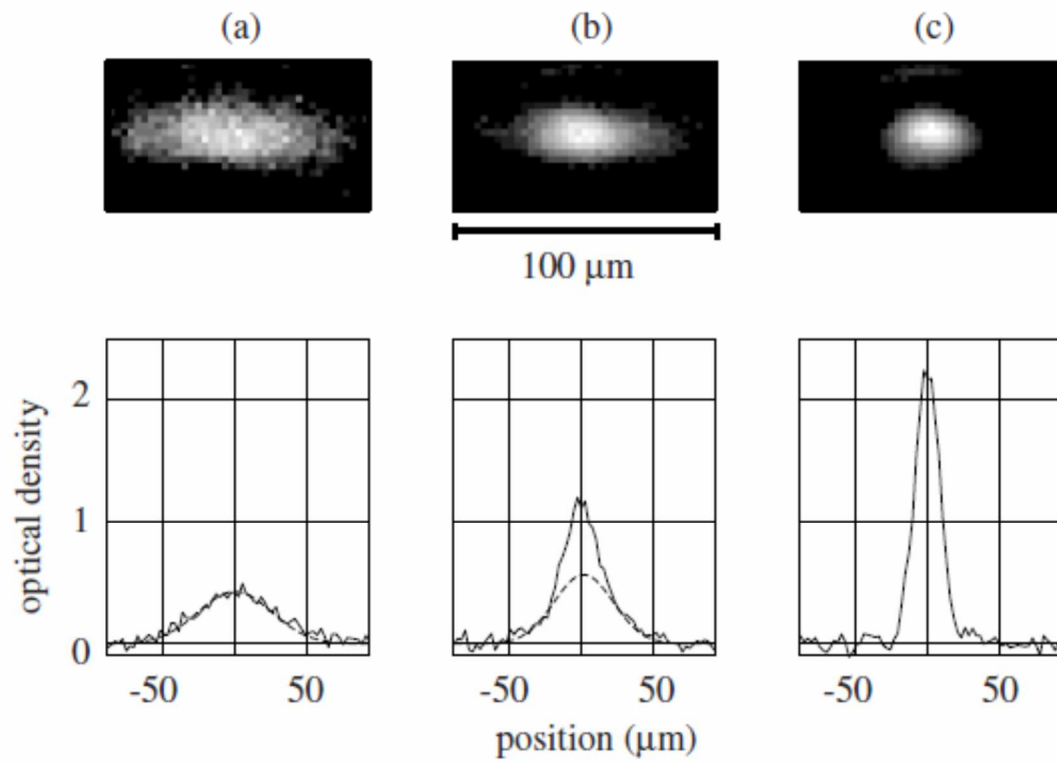


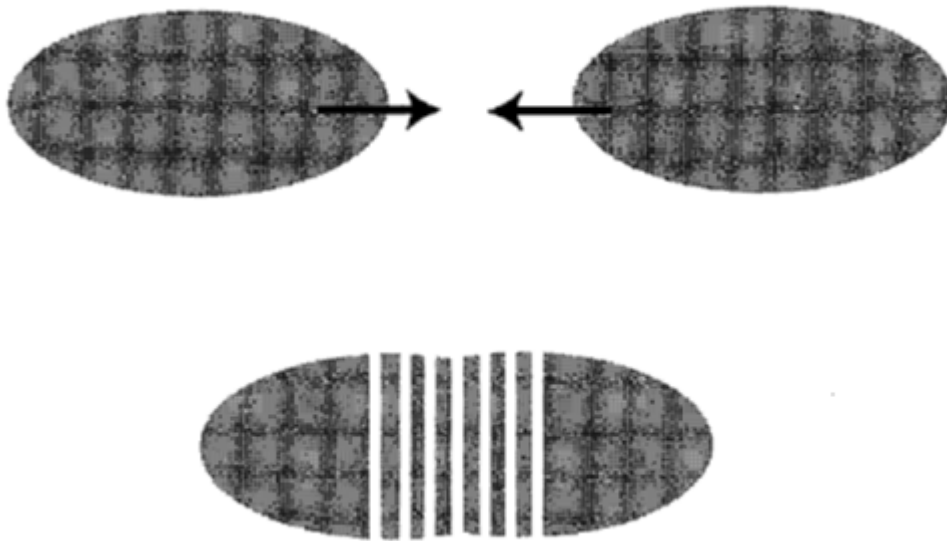
FIG. 3. Experimental setup for cooling atoms to Bose-Einstein condensation. Sodium atoms are trapped by a strong magnetic field, generated by two coils. In the center, the magnetic field vanishes, which allows the atoms to spin flip and escape. Therefore the atoms are kept away from the center of the trap by a strong (3.5-W) argon ion laser beam (“optical plug”), which exerts a repulsive force on the atoms. Evaporative cooling is controlled by radio-frequency radiation from an antenna. The rf selectively flips the spins of the most energetic atoms. The remaining atoms rethermalize (at a lower temperature) by collisions among themselves. Evaporative cooling is forced by lowering the rf frequency.





Time-of-flight:

The cloud falls with gravity and expands as no force applies on it. After some time (4-20 millisecond) we image the atoms by shining the laser beam through the atom cloud and into the CCD camera.



Huang:

K. Huang, Introduction to Statistical Physics, second edition (CRC Press, 2010)

Chapter 20 Superfluidity

20.1 In the photograph in Figs.1 and 2, the interference fringes between two Bose-Einstein condensates have a spacing of 1.5×10^{-3} cm. Find the relative velocity between the two condensates.

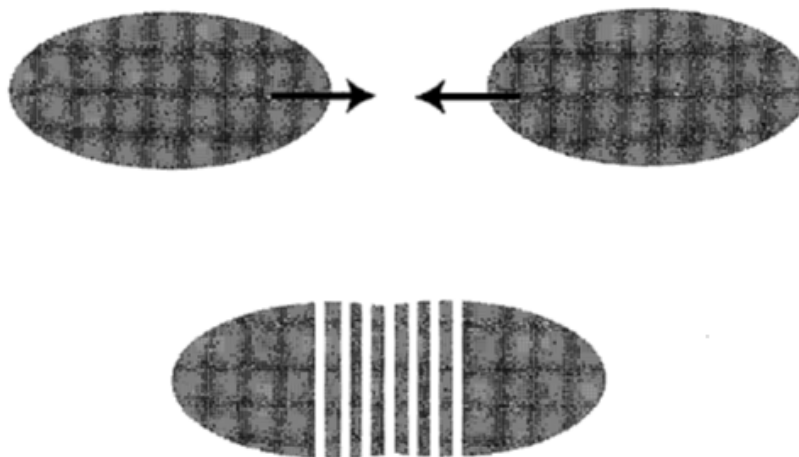


Fig.1 Two condensates approach each other, overlap, and exhibit interference fringes.

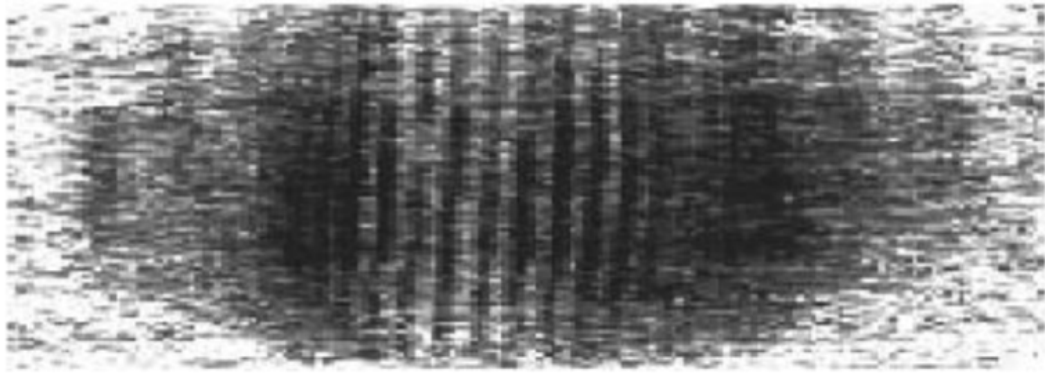


Fig.2 Photograph of interference fringes produced by two overlapping condensates of Na atoms. The separation between fringes is 1.5×10^{-3} cm (Ketterle)

In the paper of Ketterle, there is no detail on the experiment. I think that Bragg reflection occurs in the BEC. I am not sure that my solution is appropriate.

((Solution))

Suppose that two BE condensates (BEC) undergoes an elastic collision. The wave vector of one BEC is k before the collision and $k' = -k$ after the collision. On the other hand, the wave vector of the other BEC is $-k$ before the collision and $k' = k$ after the collision. We just consider the Bragg reflection for the one BEC.

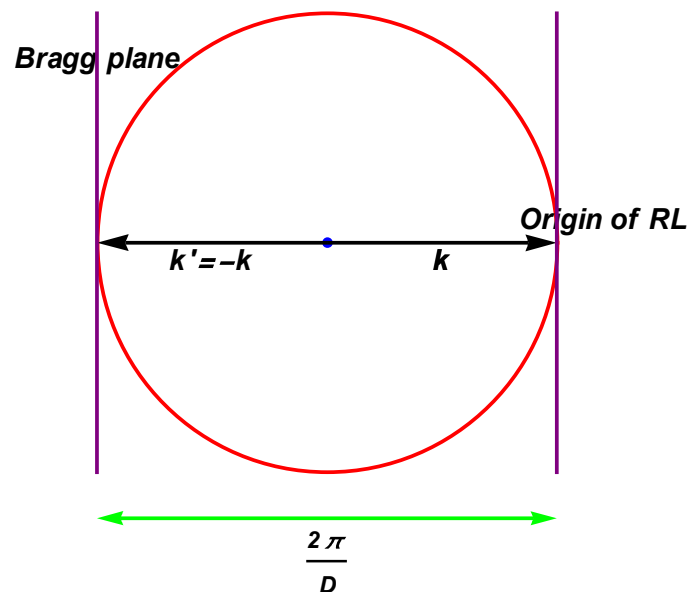


Fig. Ewald sphere for the Bragg reflection for the 1D system. $k = \frac{2\pi}{\lambda} = \frac{\pi}{D}$, or $\lambda = 2D$

We consider the Bragg reflection for the 1D system. In the reciprocal lattice, there appears the Bragg plane at $G = \frac{2\pi}{D}$, where D is the lattice constant. The red circle denotes the Ewald sphere.

The Bragg reflection occurs when the 2D Bragg plane intersects the Ewald sphere (denoted by purple color). The Bragg reflection occurs when

$$\mathbf{k} - \mathbf{k}' = \mathbf{G},$$

where $|\mathbf{k}'| = |\mathbf{k}| = \frac{2\pi}{\lambda}$, and $\mathbf{k}' = -\mathbf{k}$. $\mathbf{G}$ is the reciprocal lattice vector and. The magnitude of G is given by

$$G = |\mathbf{G}| = \frac{2\pi}{D}$$

where $D = 1.5 \times 10^{-3}$ cm. Then we have the relation

$$2k = G = \frac{2\pi}{D}.$$

The relative velocity is

$$v = \frac{\hbar}{m} k = \frac{\hbar}{m} \frac{\pi}{D} = 0.057 \text{ cm/s}$$

where the mass of Na atom is given by $m(\text{Na}) = 22.98976 \text{ u}$.

((Note))

We consider the two waves (incident wave and reflected wave).

$$\psi = e^{ikx} - e^{-ikx} = 2i \sin(kx)$$

From $\sin(kx) = 0$, we have

$$kx = n\pi \quad \text{or} \quad \Delta x = \frac{\pi}{k}$$

When $\Delta x = D$, we have

$$k = \frac{\pi}{D} = \frac{mv}{\hbar}$$

Thus we get the velocity v as

$$v = \frac{\pi \hbar}{mD} = 0.057 \text{ cm/s}$$

REFERENCES

C.J. Foot, Atomic Physics (Oxford, 2005).