Fermi energy of fermion systems

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The **Fermi energy** is a concept in quantum mechanics usually referring to the energy difference between the highest and lowest occupied single-particle states in a quantum system of non-interacting fermions at absolute zero temperature. In a Fermi gas, the lowest occupied state is taken to have zero kinetic energy, whereas in a metal, the lowest occupied state is typically taken to mean the bottom of the conduction band.
https://en.wikipedia.org/wiki/Fermi_energy

Here we discuss the Fermi energy and the Fermi temperature for some typical systems are evaluated: liquid ^{3}He, white dwarf, neutron star, and conduction electron in metal, and proton and neutron in nucleus.
Using the Fermi energy ε_F, the Fermi temperature T_F is defined by

$$T_F = \frac{\varepsilon_F}{k_B}$$

Fermi momentum and Fermi wave number is defined by

$$p_F = \hbar k_F = \sqrt{2m\varepsilon_F}$$

The Fermi velocity is defined by

$$v_F = \frac{p_F}{m}.$$

1. Liquid ^{3}He

Liquid ^{3}He as a Fermi gas

spin $I = \frac{1}{2}$ (fermion)

Density $\rho = 0.081$ g/cm^3

The Fermi energy

$$\varepsilon_F = \frac{\hbar^2}{2m_0}(3\pi^2 n)^{2/3}$$

where m_0 is the mass of ^{3}He atom,

$$m_0 = \frac{3.016g}{6.022\times10^{23}} = 0.5\times10^{-23} = 5.0\times10^{-24}\,g$$

The number density

$$n = \frac{N}{V} = \frac{N}{M}\frac{M}{V} = \frac{\rho}{m_0} = \frac{0.081}{5.0\times10^{-24}} = 1.62\times10^{22}\,/\,cm^3 .$$

Then the Fermi energy is

$$\varepsilon_F = 6.815\times10^{-16}\,\text{erg} = 4.254\times10^{-4}\ \text{eV}.$$

The Fermi temperature

$$T_F = \frac{\varepsilon_F}{k_B} = 4.94\ \text{K}.$$

2. White dwarf

Suppose that the sun consists of hydrogen atoms. A hydrogen atom has 1 electron. When m_P is the mass of proton, there are N hydrogen atoms in the sun,

$$N = \frac{M_{sun}}{m_p} = 1.18888\times10^{57} .$$

The Fermi energy of electrons in the white dwarf where we use N electrons for the sun

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{\frac{4\pi R^3}{3}}\right)^{2/3} = \frac{\hbar^2}{2m}\left(\frac{9\pi N}{4R^3}\right)^{2/3}$$

where $V = \dfrac{4\pi}{3}R^3$ and R is the radius of white dwarf; $R = 2\times10^9$ cm.

$$\varepsilon_F = 6.308\times10^{-8}\ \text{erg} = 3.94\times10^{4}\ \text{eV}$$

The Fermi temperature is

$$T_F = 4.569\ \text{x}\ 10^{8}\ \text{K}$$

3. Neutron star

The Fermi energy of the neutron star is given by

$$\varepsilon_F = c\hbar\left(\frac{3\pi^2 N}{V}\right)^{1/3} \qquad \text{(relativistic limit)}$$

Suppose that R = 10 km, and ρ = 4.0 x 10^{14} g/cm^3

$$\frac{N}{V} = \frac{\rho}{m_n} = 2.39\ \text{x}\ 10^{38}/\text{cm}^3$$

The Fermi energy

$$\varepsilon_F = 6.07\ \text{x}\ 10^{-4}\ \text{erg} = 3.79\times10^{8}\ \text{eV.}$$

The Fermi temperature is

$$T_F = 4.40\times10^{12}\ \text{K.}$$

4. Conduction electron in metal

Suppose that there is one free electron per atom. There is one atom per unit cell. We consider the simple cubic crystal with the lattice constant $a = 3$ Å. So the number density of electrons is

$$n = \frac{1}{a^3} = 3.70\times10^{22}/\text{cm}^3.$$

The Fermi energy

$$\varepsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} = 6.488\ \text{x}\ 10^{-12}\ \text{erg} = 4.04\ \text{eV.}$$

The Fermi temperature is

$T_F = 4.7 \times 10^4$ K.

The Fermi velocity is

$$v_F = 1.19 \text{ x } 10^8 \text{ cm/s.}$$

5. Fermi energy of nucleon

Model a heavy nucleus of mass number A (= $Z + N$) as a free Fermi gas of an equal number of protons and neutrons, containing in a sphere of radius

$$R = r_0 A^{1/3} \qquad (1)$$

where $r_0 = 1.4 \times 10^{-13}$ cm. We calculate the Fermi energy and the average energy per nucleon in MeV.

The stable nucleus has approximately a constant density and therefore the nuclear radius R can be approximated by the following formula (1), where A = Atomic mass number (= $Z + N$, Z: the number of protons, N: the number of neutrons N) and

$$r_0 = 1.4 \times 10^{-13} \text{ cm.}$$

The volume V_0 is

$$V_0 = \frac{4\pi}{3} R^3 = \frac{4\pi}{3} r_0^{\ 3} A$$

After assuming that the proton and neutron potential wells have the same radius, we find for a nucleus with

$$N_0 = Z = N = \frac{A}{2}$$

Since the Fermi energy only applies to fermions of the same type, one must divide this density in two. This is because the presence of neutrons does not affect the Fermi energy of the protons in the nucleus, and vice versa. The Fermi momentum is given by

$$N_0 = \frac{2V_0}{(2\pi\hbar)^3} \frac{4\pi}{3} p_F^{\ 3}$$

Then the Fermi momentum p_F for proton and neutron can be rewritten as

$$p_F = p_F{}^n = p_F{}^p = \hbar\left(\frac{3\pi^2 N_0}{V_0}\right)^{1/3} = \hbar\left(\frac{3\pi^2 \frac{A}{2}}{\frac{4\pi}{3} r_0{}^3 A}\right)^{1/3} = \frac{\hbar}{r_0}\left(\frac{9\pi}{8}\right)^{1/3} = 214.7\ \text{MeV/c}$$

The Fermi energy is

$$\varepsilon_F = \frac{1}{2m_p} p_F^p = \frac{1}{2m_n} p_F^n = 24.564\ \text{MeV}$$

using the mass of proton and neutron (m_p and m_n). The average energy is

$$\frac{\langle E \rangle}{N_0} = \frac{3\varepsilon_F}{5} = 14.738\ \text{MeV}$$

((**Mathematica**))

```
Clear["Global`*"];
rule1 = {kB → 1.3806504 × 10^-16, c → 2.99792 × 10^10, ℏ → 1.054571628 10^-27,
   me → 9.10938215 10^-28, mp → 1.672621637 × 10^-24, mn → 1.674927211 × 10^-24,
   qe → 4.8032068 × 10^-10, eV → 1.602176487 × 10^-12,
   meV → 1.602176487 × 10^-15, keV → 1.602176487 × 10^-9,
   MeV → 1.602176487 × 10^-6, Å → 10^-8};
r0 = 1.4 × 10^-13;
```

$$\texttt{pF} = \frac{\hbar}{\texttt{r0}}\left(\frac{9\,\pi}{8}\right)^{1/3} \;\texttt{/. rule1}$$

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1.1474 × 10^-14
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$$\frac{\texttt{pF c}}{\texttt{MeV}} \;\texttt{/. rule1}$$

```
214.697
```

$$\texttt{EF1} = \frac{1}{2\,\texttt{mp}}\,\frac{\texttt{pF}^2}{\texttt{MeV}} \;\texttt{//. rule1}$$

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24.5637
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$$\frac{3\,\texttt{EF1}}{5}$$

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14.7382
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APPENDIX

N_f : number of fermion

m_f : mass per fermion

m: mass of fermion

The number density:

$$n = \frac{N_f}{V} = \frac{N_f}{M}\frac{M}{V} = \frac{1}{m_f}\rho\,.$$

The Fermi energy:

$$\varepsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$$