Relativistic Fermi gas

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We discuss the relativistic Fermi gas model where the fermion has an relativistic energy dispersion. For simplicity the case at $T = 0$ K is considered.

1. Theory

Here we discuss the thermodynamics properties of the relativistic fermi gas at $T = 0$ K.

$$\ln Z_G = \sum_{\boldsymbol{p}} \ln[1 + ze^{-\beta\varepsilon_{\boldsymbol{p}}}]$$

with the relativistic energy dispersion

$$\varepsilon_{\boldsymbol{p}} = c\sqrt{p^2 + m^2c^2} - mc^2$$

and

$$\frac{\partial \varepsilon}{\partial p} = \frac{cp}{\sqrt{p^2 + m^2c^2}}.$$

Using the relation

$$\sum_{\boldsymbol{p}} \rightarrow \frac{gV}{(2\pi\hbar)^3} 4\pi p^2 dp$$

we get

$$\ln Z_G = \frac{gV}{(2\pi\hbar)^3} \int_0^\infty 4\pi p^2 dp \ln[1 + ze^{-\beta\varepsilon_{\boldsymbol{p}}}]$$

where $g = 2$ for spin 1/2. Applying a partial derivative, we have

$$\ln Z_G = -\frac{gV}{(2\pi\hbar)^3} 4\pi \int_0^\infty \frac{p^3}{3} dp \frac{ze^{-\beta\varepsilon_p}}{1+ze^{-\beta\varepsilon_p}} (-\beta) \frac{\partial \varepsilon_p}{\partial p}$$

$$= \frac{gV}{h^3} \frac{4\pi\beta}{3} \int_0^\infty p^3 dp \frac{1}{\frac{1}{z} e^{\beta\varepsilon_p} + 1} \frac{\partial \varepsilon_p}{\partial p}$$

$$N = z \frac{\partial}{\partial z} \ln Z_G$$

$$= z \frac{\partial}{\partial z} \sum_{\boldsymbol{p}} \ln[1 + ze^{-\beta\varepsilon_p}]$$

$$= \sum_{\boldsymbol{p}} \frac{ze^{-\beta\varepsilon_p}}{1+ze^{-\beta\varepsilon_p}}$$

$$= \sum_{\boldsymbol{p}} \frac{1}{\frac{1}{z} e^{\beta\varepsilon_p} + 1}$$

or

$$N = \sum_{\boldsymbol{p}} \frac{1}{\frac{1}{z} e^{\beta\varepsilon_p} + 1}$$

$$= \frac{gV}{(2\pi\hbar)^3} \int_0^\infty 4\pi p^2 dp \frac{1}{\frac{1}{z} e^{\beta\varepsilon_p} + 1}$$

$$= \frac{gV}{h^3} \int_0^\infty 4\pi p^2 dp \frac{1}{\frac{1}{z} e^{\beta\varepsilon_p} + 1}$$

At $T = 0$ K,

$$N = \frac{gV}{h^3} \int_0^{p_f} 4\pi p^2 dp = g \frac{4\pi V}{3h^3} p_f{}^3$$

The Fermi momentum is obtained from the relation

$$N = g \frac{4\pi V}{3h^3} p_f{}^3$$

as

$$p_f = \left(\frac{3}{4\pi} \frac{Nh^3}{Vg} \right)^{1/3}$$

The pressure P is evaluated from the

$$PV = k_B T \ln Z_G$$

as

$$\begin{aligned} P &= \frac{k_B T}{V} \ln Z_G \\ &= \frac{1}{\beta V} \ln Z_G \\ &= \frac{4\pi g}{3h^3} c \int_0^\infty \frac{p^4}{\sqrt{p^2 + m^2 c^2}} dp \left(\frac{1}{\frac{1}{z} e^{\beta \varepsilon_p} + 1} \right) \end{aligned}$$

At $T = 0$ K, we have

$$P = \frac{4\pi g}{3h^3} c \int_0^{p_f} dp \frac{p^4}{\sqrt{p^2 + m^2 c^2}}$$

The internal energy is given by

$$U = -\frac{\partial}{\partial \beta} \ln Z_G = \frac{4\pi g V}{h^3} \int_0^\infty p^2 dp \frac{\varepsilon_p}{\frac{1}{z} e^{\beta \varepsilon_p} + 1}$$

At $T = 0$ K, we get

$$\begin{aligned} U &= \frac{4\pi g V}{h^3} \int_0^\infty p^2 \varepsilon_p dp \\ &= \frac{4\pi g V}{h^3} \int_0^\infty p^2 dp [c\sqrt{p^2 + m^2 c^2} - mc^2] \end{aligned}$$

We put

$$p = mc\sinh x, \qquad y = \frac{p}{mc} = \sinh x, \qquad x = \text{arcsinh}(y)$$

Thus we have

$$\varepsilon = mc^2(\cosh x - 1)$$

and

$$\frac{d\varepsilon}{dp} = c\tanh x$$

Since

$$p_f = mc\sinh x_f \qquad y_f = \frac{p_f}{mc} = \sinh x_f \qquad x_f = \text{arcsinh}(y_f)$$

we have

$$P = \frac{4\pi g}{3h^3} m^4 c^5 \int_0^{x_f} dx \sinh^4 x$$
$$= \frac{4\pi g}{3h^3} m^4 c^5 \frac{1}{8} A(y_f)$$

with

$$\int_0^{x_f} dx \sinh^4 x = \frac{1}{8} A(y_f)$$

where

$$A(y_f) = \sqrt{1 + y_f^2}(2y_f^3 - 3y_f) + 3\text{arcsinh}(y_f).$$

The approximation:

$$A(y_f) = \frac{8}{5}y^5 - \frac{4}{7}y^7 + \frac{1}{3}y^9 - \frac{5}{22}y^{11} + \frac{35}{208}y^{13} - \frac{21}{160}y^{15} + O(y^{16}) \qquad \text{for } y << 1$$

$$A(y_f) = 2y^4 - 2y^2 + 3\ln(2y) - \frac{7}{4} + \frac{5}{4}y^{-2} + ... \qquad \text{for } y >> 1$$

The internal energy is given by

$$\begin{aligned} U &= \frac{4\pi g V}{h^3}\int_0^{p_F} p^2 dp[c\sqrt{p^2 + m^2c^2} - mc^2] \\ &= \frac{4\pi g V}{h^3} m^4 c^5 \int_0^{x_F} \sinh^2 x \cosh x(\cosh x - 1) \\ &= \frac{4\pi g V}{h^3} m^4 c^5 \frac{1}{24} B(y_f) \end{aligned}$$

where

$$\int_0^{x_F} \sinh^2 x \cosh x(\cosh x - 1) = \frac{1}{24}B(y_f)$$

$$B(y_f) = (6{y_f}^3 + 3y_f)\sqrt{1 + {y_f}^2} - 8{y_f}^3 - 3\text{arcsinh}(y_f)$$

The approximation:

$$B(y_f) = \frac{12}{5}y^5 - \frac{3}{7}y^7 + \frac{1}{6}y^9 - \frac{15}{176}y^{11} + \frac{21}{416}y^{13} - \frac{21}{640}y^{15} + O(y^{16}) \qquad \text{for } y << 1$$

$$B(y_f) = 6y^4 - 8y^3 + 6y^2 - 3\ln(2y) + \frac{3}{4} - \frac{3}{4}y^{-2} + ... \qquad \text{for } y >> 1$$

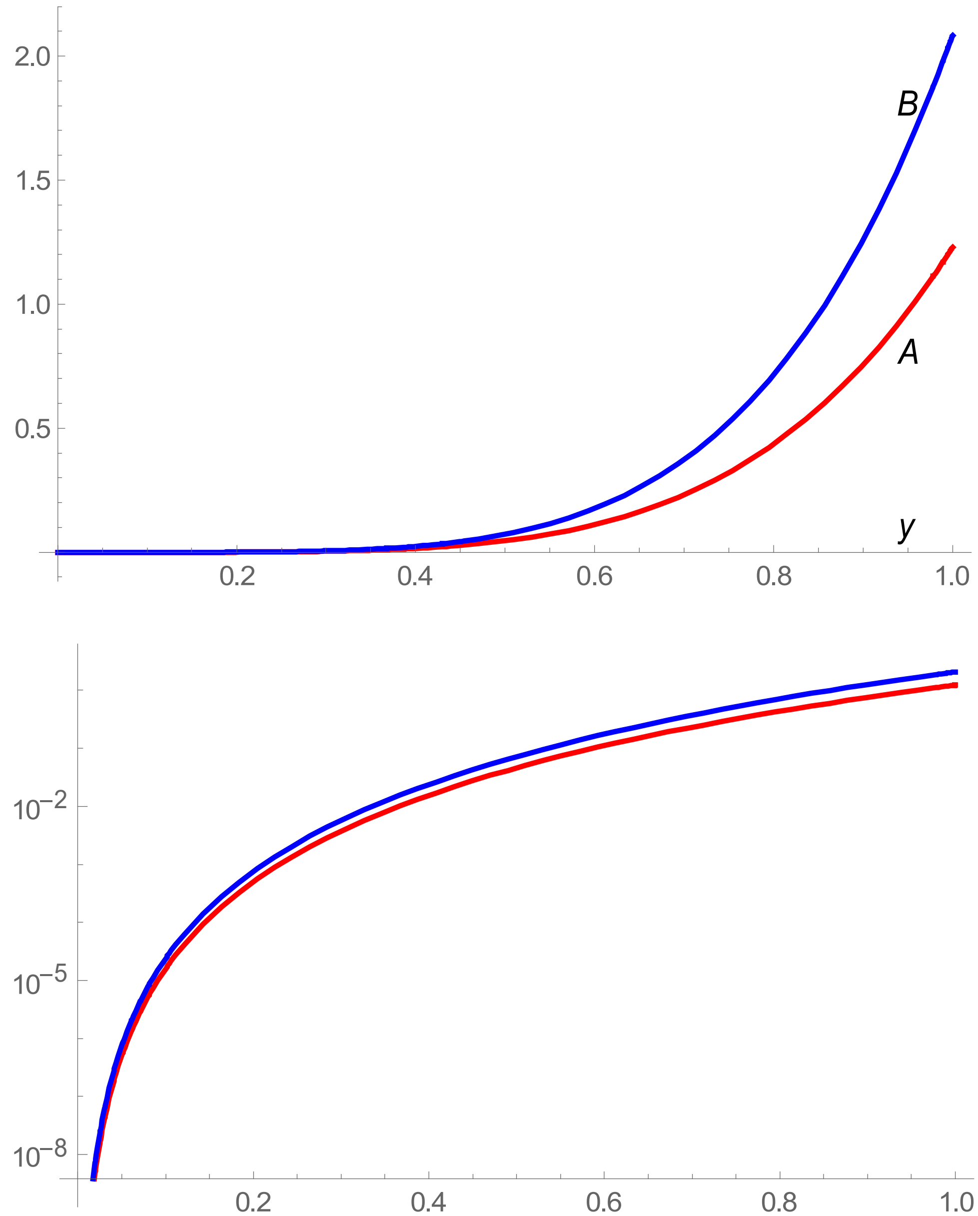

Fig. Functions $A(y)$ (denoted by red) and $B(y)$ (denoted by blue) as a function of y.

2. Ultra-relativistic limit

The value of y_f can be determined by

$$N = g\frac{4\pi V}{3h^3}p_f{}^3 = g\frac{4\pi V}{3h^3}m^3c^3y_f{}^3$$

or

$$\frac{N}{V} = g\frac{4\pi}{3h^3}m^3c^3{y_f}^3$$

The value y_f can be determined as

$$y_f = \frac{h}{mc}\left(\frac{3n}{4\pi g}\right)^{1/3}$$

The pressure:

$$\begin{aligned} P &= \frac{4\pi g}{3h^3}m^4c^5\frac{1}{4}{y_f}^4 \\ &= \frac{1}{4}\frac{4\pi g}{3h^3}m^4c^5\frac{h^4}{m^4c^4}\left(\frac{3n}{4\pi g}\right)^{4/3} \\ &= \frac{1}{4}ch(\frac{4\pi g}{3})^{-1/3}n^{4/3} \end{aligned}$$

leading to the relation

$$PV^{4/3}\text{=const.}$$

The internal energy:

$$\begin{aligned} U &= \frac{4\pi gV}{h^3}m^4c^5\frac{1}{4}{y_f}^4 \\ &= \frac{3}{4}\frac{4\pi gV}{3h^3}m^4c^5\frac{h^4}{m^4c^4}\left(\frac{3n}{4\pi g}\right)^{4/3} \\ &= \frac{3}{4}Vch(\frac{4\pi g}{3})^{-1/3}n^{4/3} \end{aligned}$$

We note that

$$\frac{PV}{U} = \frac{1}{3}$$

or

$$PV = \frac{1}{3}U$$

3. Non-relativistic limit

The pressure:

$$\begin{aligned}
P &= \frac{4\pi g}{3h^3} m^4 c^5 \frac{1}{5} y_f^{\ 5} \\
&= \frac{4\pi g}{3h^3} m^4 c^5 \frac{1}{5} \frac{h^5}{m^5 c^5} \left(\frac{n}{4\pi g} \right)^{5/3} \\
&= \frac{4\pi g}{15} \frac{h^2}{m} \left(\frac{n}{4\pi g} \right)^{5/3} \\
&= \frac{(4\pi g)^{-2/3}}{15} \frac{h^2}{m} n^{5/3}
\end{aligned}$$

leading to the relation

$$PV^{5/3} = \text{const.}$$

The internal energy:

$$\begin{aligned}
U &= \frac{4\pi g V}{h^3} m^4 c^5 \frac{1}{10} y_f^{\ 5} \\
&= \frac{4\pi g V}{h^3} m^4 c^5 \frac{1}{10} \frac{h^5}{m^5 c^5} \left(\frac{n}{4\pi g} \right)^{5/3} \\
&= \frac{4\pi g V}{10} \frac{h^2}{m} \left(\frac{n}{4\pi g} \right)^{5/3} \\
&= V \frac{(4\pi g)^{-3/2}}{10} \frac{h^2}{m} n^{5/3}
\end{aligned}$$

We note that

$$\frac{PV}{U} = \frac{2}{3}$$

or

$$PV = \frac{2}{3}U \qquad \text{(non-relativistic limit)}$$

4. Approach from the energy dispersion $\varepsilon_p = c\sqrt{p^2 + m^2c^2}$

At $T = 0$ K, we have

$$P = \frac{4\pi g}{3h^3} c \int_0^{p_f} dp \frac{p^4}{\sqrt{p^2 + m^2c^2}}$$

which is the same as that obtained from the dispersion relation

$$\varepsilon_p = c\sqrt{p^2 + m^2c^2} - mc^2$$

The internal energy is given by

$$U = -\frac{\partial}{\partial \beta} \ln Z_G = \frac{4\pi g V}{h^3} \int_0^\infty p^2 dp \frac{\varepsilon_p}{\frac{1}{z} e^{\beta \varepsilon_p} + 1}$$

At $T = 0$ K, we get

$$\begin{aligned} U &= \frac{4\pi g V}{h^3} \int_0^\infty p^2 \varepsilon_p dp \\ &= \frac{4\pi g V}{h^3} \int_0^\infty p^2 dp [c\sqrt{p^2 + m^2c^2}] \end{aligned}$$

We put

$$p = mc \sinh x, \qquad y = \frac{p}{mc} = \sinh x, \qquad x = \text{arcsinh}(y)$$

Thus we have

$$\varepsilon = mc^2 \cosh x$$

and

$$\frac{d\varepsilon}{dp} = \frac{cp}{\sqrt{m^2c^2 + p^2}} = c\tanh x$$

Since

$$p_F = mc\sinh x_F \qquad y_F = \frac{p_F}{mc} = \sinh x_F \qquad x_F = \text{arcsinh}(y_F)$$

we have

$$P = \frac{4\pi g}{3h^3} m^4 c^5 \int_0^{x_F} dx \sinh^4 x$$
$$= \frac{\pi g}{6} \frac{m^4 c^5}{h^3} [\frac{1}{4}\sinh(4x_F) - 2\sinh(2x_F) + 3x_F]$$

with

$$\int_0^{x_F} dx \sinh^4 x = \frac{1}{32}[\sinh(4x_F) - 8\sinh(2x_F) + 12x_F]$$

The internal energy is given by

$$U = \frac{4\pi gV}{h^3}\int_0^{p_F} p^2 dp[c\sqrt{p^2 + m^2c^2}]$$
$$= \frac{4\pi gV}{h^3} m^4 c^5 \int_0^{x_F} \sinh^2 x \cosh^2 x$$
$$= \frac{\pi gV}{8h^3} m^4 c^5 [\sinh(4x_F) - 4x_F]$$

where

$$\int_0^{x_F} \sinh^2 x \cosh^2 x = \frac{1}{32}[\sinh(4x_F) - 4x_F]$$

and

$$\varepsilon_F = mc^2 \cosh x_F$$

The number density:

$$\frac{N}{V} = g\frac{4\pi}{3h^3}m^3c^3{y_F}^3 = g\frac{4\pi}{3h^3}m^3c^3\sinh^3 x_F$$

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