# Degenerate Fermi gas systems: white dwarf and neutron star (pulsar)

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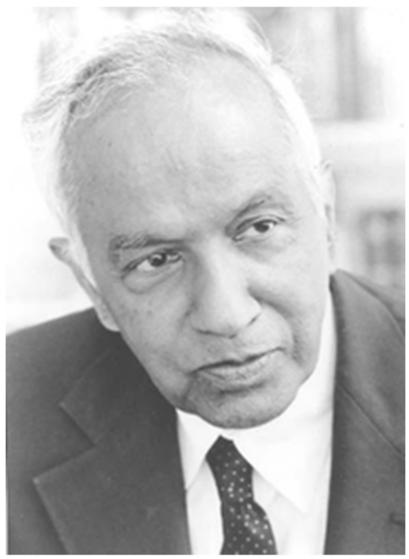
**Subrahmanyan Chandrasekhar**, FRS (October 19, 1910 – August 21, 1995) was an Indian origin American astrophysicist who, with William A. Fowler, won the 1983 Nobel Prize for Physics for key discoveries that led to the currently accepted theory on the later evolutionary stages of massive stars. Chandrasekhar was the nephew of Sir Chandrasekhara Venkata Raman, who won the Nobel Prize for Physics in 1930. Chandrasekhar served on the University of Chicago faculty from 1937 until his death in 1995 at the age of 84. He became a naturalized citizen of the United States in 1953.



http://en.wikipedia.org/wiki/Subrahmanyan_Chandrasekhar

In 1930, Subramanyan Chandrasekhar, then 19 years old, was on a sea voyage from India to Cambridge, England, where he planned to begin graduate work. Chandrasekhar was interested in exploring the consequences of quantum mechanics for astrophysics. During his trip, he analyzed how the density, pressure, and gravity in a white dwarf star vary with radius. For a star like Sirius B Chandrasekhar found that the Fermi velocity of inner electrons approaches the speed of light. Consequently he found it necessary to redo the calculation of the Fermi energy taking relativistic effects into account. Chandrasekhar deduced that a high-density, high mass star cannot support itself against gravitational collapse unless the mass of the star is less than 1.4 solar masses. This finding was quite controversial within the astronomical community and it was 54 years before Chandrasekhar was awarded the Nobel Prize for this work.

## 1. Overview

An ordinary planet, supported by material pressure will persist essentially forever. But massive stars are a different story. The pressure supporting a star comes from the heat produced by fusion of light nuclei into heavier ones. When the nuclear fuel is used up, the temperature declines and the star begins to shrink under the influence of gravity. The collapse may eventually be halted by Fermi degeneracy pressure. Electrons are pushed so close together that they resist further compression simply on the basis of the Pauli exclusion principle. A stellar remnant supported by electron degeneracy pressure is called a **white dwarf**; a typical white dwarf is comparable in size to the Earth. Lower-mass particles become degenerate at lower number densities than high-mass particles, so nucleus do not contribute appreciably to the presence in a white dwarf. White dwarfs are the end state for most stars, and are extremely common throughout the universe.

If the total mass is sufficiently high, however, the star will reach the **Chandrasekhar limit** ( $M$=1.4 $M_{sun}$), even the electron degeneracy pressure is not enough to resist the pull of gravity. When it is reached, the star is forced to collapse to an even smaller radius. At this point electrons combine with protons to make neutrons and neutrino (inverse beta decay), and the nuetrinos simply fly away. The result is a neutron star, with a typical radius of about 10 km, **Neutrons stars** have a low luminosity, but often are rapidly spinning and possess strong magnetic fields. This combination gives rise to pulsars, which accelerate particles in jets emanating from the magnetic poles, appearing to rapiddly flash as the neutron star spins.

**REFERENCES**

Sean M. Carroll, Spacetime and Geometry An Introduction to General Relativity (Addison Wesley, 2004).

R.A. Freedman and W.J. Kaufmann III, Universe, 8-th edition (W.H. Freeman, 2008). Chapter 21 neutron Star

## 2. Introduction

Electron degeneracy is a stellar application of the Pauli Exclusion Principle, as is neutron degeneracy. No two electrons can occupy identical states, even under the pressure of a collapsing star of several solar masses. For stellar masses less than about 1.4 solar masses, the energy from the gravitational collapse is not sufficient to produce the neutrons of a neutron star, so the collapse is halted by electron degeneracy to form white dwarfs. This maximum mass for a white dwarf is called the Chandrasekhar limit. As the star contracts, all the lowest electron energy levels are filled and the electrons are forced into higher and higher energy levels, filling the lowest unoccupied energy levels. This creates an effective pressure which prevents further gravitational collapse.

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(a) Earth

$M$ = 5.973610 x $10^{24}$ kg
$R$ = 6.372 x $10^6$ m

(b) Sun
$M$ = 1.988435 x $10^{30}$ kg
$R$ =6.9599 x $10^8$ m

(c) Companion of Sirius: first white dwarf (Sirius B)
$M$ = 2.0 x $10^{30}$ kg (≈ the mass of sun)
$R$ = 6.0 x $10^6$ m (a little shorter than the Earth)

(d) Crab pulsar (neutron star)
$M$ = 1.4 $M_{\text{sun}}$ = 2.78 x $10^{30}$ kg
$R$ = 1.2 x $10^3$ m.

(e) Chandrasekhar limit
The currently accepted value of the Chandrasekhar limit is about 1.4 $M_{\text{sun}}$ ($2.765\times10^{30}$ kg).

## 3. Kinetic energy of the ground state of fermion

The kinetic energy of the fermions in the ground state is given by

$$U_G = \frac{3}{5} N_f \varepsilon_F = \frac{3}{5} N_f \frac{\hbar^2}{2m_0} (3\pi^2 \frac{N_f}{V})^{2/3},$$

where $N_f$ is the number of fermions, and $m_0$ is the mass of the fermion. The pressure $P$ is calculated as

$$\begin{aligned} P &= \frac{2U_G}{3V} \\ &= \frac{2}{3}\frac{3}{5}\frac{N_f}{V}\frac{\hbar^2}{2m_0}(3\pi^2\frac{N_f}{V})^{2/3} \\ &= \frac{1}{5}(3\pi^2)^{2/3}\frac{\hbar^2}{m_0}(\frac{N_f}{V})^{5/3} \end{aligned}$$

using the formula of $P$ in the non-relativistic limit. Note that $m_0 = m_e$ for the white dwarf where electron (spin 1/2) is a fermion, and $m_0 = m_n$ for the neutron star where neutron (spin 1/2) is a fermion.

The kinetic energy of fermions in the ground state can be rewritten as

$$U_G = \frac{3}{5} N_f \frac{\hbar^2}{2m_0} (3\pi^2 \frac{N_f}{\frac{4\pi}{3} R^3})^{2/3}$$

$$= \frac{3}{10} (\frac{9\pi}{4})^{2/3} \frac{\hbar^2}{m_0} \frac{N_f^{5/3}}{R^2} = 1.10495 \frac{\hbar^2}{m_0} \frac{N_f^{5/3}}{R^2} = \frac{B}{R^2}$$

where

$$B = 1.10495 \frac{\hbar^2}{m_0} N_f^{5/3}.$$

The volume $V$ is expressed by

$$V = \frac{4\pi}{3} R^3.$$

where $R$ is the radius of the system. We find that $P$ becomes increases as the volume $V$ decreases.

Here we note that the density of the system, $\rho$, is given by

$$\rho = \frac{M}{V} = \frac{M}{\frac{4\pi}{3} R^3}.$$

and $M$ is the total mass of the system. The number density $n_f$ for fermions is defined as

$$n_f = \frac{N_f}{V} = \frac{M}{V} \frac{N_f}{M} = \rho \frac{N_f}{m_f N_f} = \frac{\rho}{m_f}.$$

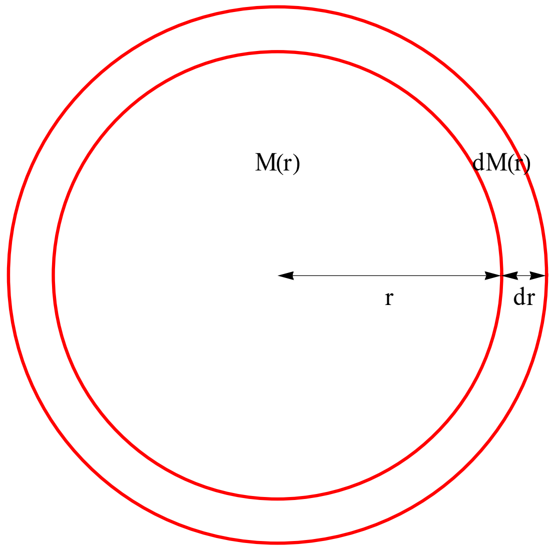
The average nearest neighbor distance between fermions can be evaluated

$$d = \left( \frac{1}{n_f} \right)^{1/3} = \left( \frac{m_f}{\rho} \right)^{1/3}$$

Note that the more detail of the mass $m_f$ will be discussed in the discussion of white dwarf. $m_f$ is the mass per fermion.

## 4. Gravitational self energy

We calculate the potential energy of the system.




Suppose that $M(r)$ is the mass of the system with radius $r$.

$$M(r) = \frac{4\pi}{3}\rho r^3, \qquad dM(r) = 4\pi r^2 \rho dr .$$

The potential energy is given by

$$U = -\int_0^R \frac{GM(r)dM(r)}{r} .$$

Noting that

$$\rho = \frac{3M}{4\pi R^3},$$

the potential energy is calculated as

$$U = -\int_0^R \frac{G(4\pi\rho)^2 r^5}{3r} dr = \frac{G}{3}(4\pi\rho)^2 \frac{1}{5} R^5 = -\frac{3GM^2}{5R} = -\frac{A}{R},$$

where

$$A = \frac{3GM^2}{5}$$

and $G$ is the universal gravitational constant.

## 5. The total energy

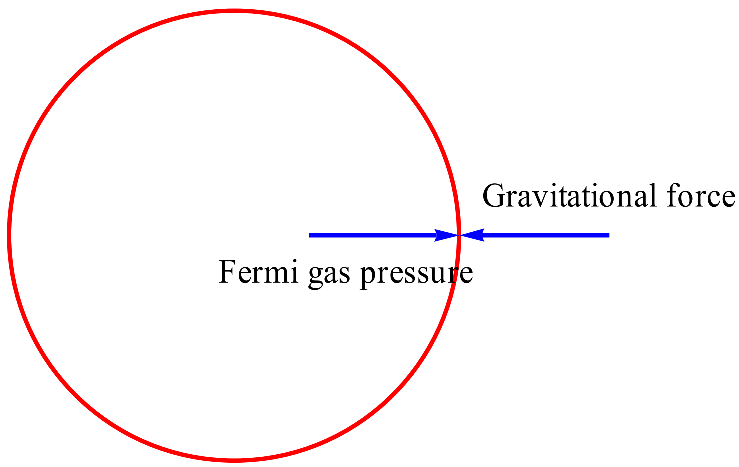



**Fig.** A balance between the gravitational force (inward) and the pressure of degenerate Fermi gas

The total energy is the sum of the gravitational and kinetic energies,

$$E_{tot} = f_{nonrel}(R) = -\frac{A}{R} + \frac{B}{R^2}$$

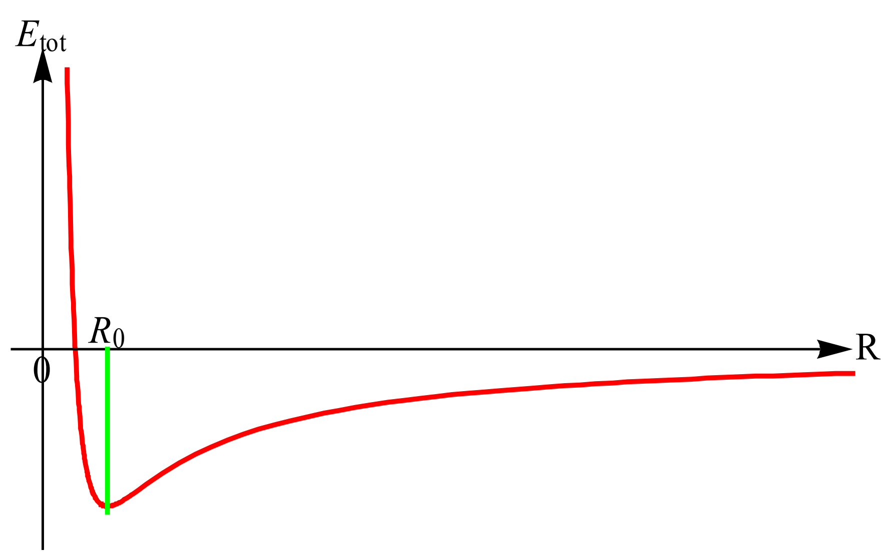


**Fig.** Non-relativistic case. The plot of $E_{tot}$ as a function of $R$. $E_{tot}$ has a local minimum at $R = R_0$, leading to the equilibrium state.

From the derivative of $f_{\text{nonrel}}(R)$ with respect to $R$, we get the distance $R$ in equilibrium.

$$\frac{d}{dR} f_{nonrel}(R) = \frac{A}{R^2} - \frac{2B}{R^3} = 0,$$

or

$$R = R_0 = \frac{2B}{A}.$$

or

$$RM^{1/3} = C = \left(\frac{81\pi^2}{16}\right)^{1/3} \frac{\hbar^2}{Gm_e} \left(\frac{Z}{Am_p}\right)^{5/3}$$ (see the detail below).

Thus, for the nonrelativistic degenerate Fermi gas, there is a balance between the gravitational force (inward) and the force due to the degenerate Fermi gas pressure, leading to a stable radius $R_0$.

((**Summary**))

Compression of a white dwarf will increase the number of electrons in a given volume. Applying the Pauli's exclusion principle, this will increase the kinetic energy of the electrons, thereby increasing the pressure. This *electron degeneracy pressure* supports a white dwarf against gravitational collapse. The pressure depends only on density and not on temperature.

Since the analysis shown above uses the non-relativistic formula $p_F^{\ 2}/(2m_0)$ for the kinetic energy, it is non-relativistic. If we wish to analyze the situation where the electron velocity in a white dwarf is close to the speed of light, $c$, we should replace $p_F^{\ 2}/(2m_0)$ by the extreme relativistic approximation $cp_F$ for the kinetic energy.

As $V$ is decreased with $N_f$ kept constant, the Fermi velocity increases,

$$v_F = \frac{\hbar}{m_0}\left(\frac{3\pi^2 N_f}{V}\right)^{1/3}.$$

in the non-relativistic case.

## 6. Relativistic degenerate Fermi gas

The Fermi energy of the non-degenerate Fermi gas is given by

$$\varepsilon_F = \frac{\hbar^2}{2m_0}(3\pi^2 \frac{N_f}{V})^{2/3}.$$

where $N_{\mathrm{f}}$ is the number of fermions. As $V \to 0$, $\varepsilon_F$ increases. Then the relativistic effect becomes important. The relativistic kinetic energy is given by

$$\varepsilon = c\sqrt{p^2 + {m_0}^2 c^2} - mc^2$$

When $p >> m_0 c$,

$$\varepsilon \approx cp = c\hbar k$$

where $m_0$ is the mass of fermion and $m_{\mathrm{f}}$ is the mass per fermion. Note that

$$dk = \frac{d\varepsilon}{c\hbar}, \qquad k = \frac{\varepsilon}{c\hbar}.$$

The density of states:

$$D(\varepsilon)d\varepsilon = \frac{2V}{(2\pi)^3}4\pi k^2 dk = \frac{2V}{8\pi^3}4\pi\left(\frac{\varepsilon}{c\hbar}\right)^2 \frac{d\varepsilon}{c\hbar} = \frac{V}{\pi^2}\left(\frac{1}{c\hbar}\right)^3 \varepsilon^2 d\varepsilon,$$

or

$$D(\varepsilon) = \frac{V}{\pi^2 c^3 \hbar^3}\varepsilon^2,$$

$$N_f = \int_0^{\varepsilon_F} D(\varepsilon)d\varepsilon = \frac{V}{\pi^2 c^3\hbar^3}\int_0^{\varepsilon_F}\varepsilon^2 d\varepsilon = \frac{V}{\pi^2}\frac{1}{3c^3\hbar^3}{\varepsilon_F}^3,$$

or

$$\varepsilon_F = \pi c\hbar\left(\frac{3N_f}{\pi V}\right)^{1/3} = \pi c\hbar\left(\frac{3n_f}{\pi}\right)^{1/3}.$$

where $n_f$ is the number density of fermions,

$$n_f = \frac{N_f}{V}.$$

The total energy in the ground state is obtained as

$$U_G = \int_0^{\varepsilon_F} \varepsilon D(\varepsilon)d\varepsilon = \frac{V}{\pi^2 c^3\hbar^3}\int_0^{\varepsilon_F} \varepsilon^3 d\varepsilon = \frac{V}{\pi^2}\frac{1}{4c^3\hbar^3}\varepsilon_F{}^4.$$

Using the expression of $N_f$, $U_G$ can be rewritten as

$$U_G = \frac{3}{4}N_f\varepsilon_F = \frac{3}{4}(3\pi^2)^{1/3}\hbar c N_f(\frac{N_f}{V})^{1/3}.$$

The pressure $P$ is calculated as

$$P = -\frac{\partial U_G}{\partial V} = \frac{U_G}{3V} = \frac{1}{4}(3\pi^2)^{1/3}\hbar c(\frac{N_f}{V})^{4/3} = \frac{1}{4}(3\pi^2)^{1/3}\hbar c(n_f)^{4/3}.$$

using the formula of $P$ in the relativistic limit. The total mass $M$ is denoted as

$$M = N_f m_f,$$

where $m_f$ is the mass per fermion (electron in white dwarf) (such as the mass of protons and neutrons per electron). Note that $m_f$ is not always equal to the mass of each fermion ($m_0$) (such as electron). Since $N_f = \frac{M}{m_f}$, we get

$$U_G = \frac{3}{4}(3\pi^2)^{1/3}\hbar c\frac{M}{m_f}(\frac{M}{\frac{4\pi}{3}R^3 m_f})^{1/3}$$

$$= \frac{3}{4}(3\pi^2)^{1/3}\left(\frac{3}{4\pi}\right)^{1/3}\frac{\hbar c}{m_f^{\ 4/3}}\frac{M^{4/3}}{R}$$

$$= \frac{3}{4}(\frac{9\pi}{4})^{1/3}\frac{\hbar c}{m_f^{\ 4/3}}\frac{M^{4/3}}{R} = A\frac{M^{4/3}}{R}$$

which is proportional to $1/R$, where

$$A = \frac{3}{4}(\frac{9\pi}{4})^{1/3}\frac{\hbar c}{m_f^{\ 4/3}} = 1.43937\frac{\hbar c}{m_f^{\ 4/3}}.$$

Since the gravitational energy is

$$-\frac{3GM^2}{5R},$$

the total energy (relativistic) is given by

$$E_{tot} = f_{rel}(R) = \frac{AM^{4/3}}{R} - \frac{3GM^2}{5R} = \frac{AM^{4/3} - \frac{3}{5}GM^2}{R}.$$

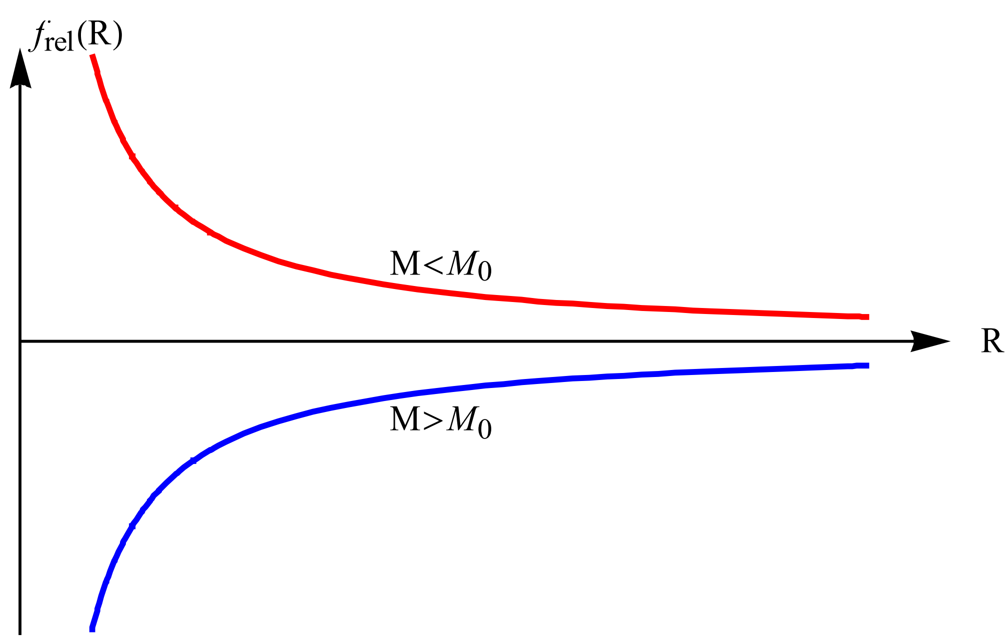


**Fig.** Relativistic case. Schematic plot of $f_{rel}(R)$ vs $R$ for $M>M_0$ and $M<M_0$. When $M = M_0$, $f_{rel}(R) = 0$. For $M>M_0$, the total energy decreases with decreasing $R$, leading to the stable state near $R = 0$. For $M<M_0$, the total energy decreases with increasing $R$, leading to the stable state near $R = \infty$.

For $M>M_0$, $R$ tends to zero, while for $M<M_0$, $R$ tends to increase. The critical mass $M_0$ is evaluated from the condition,

$$AM_0^{4/3} = \frac{3}{5}GM_0^2.$$

or

$$M_0 = \left(\frac{5\times 1.43937}{3G}\frac{\hbar c}{m_f^{4/3}}\right)^{3/2} = \left(2.39895\frac{\hbar c}{Gm_f^{4/3}}\right)^{3/2} = \frac{3.71562}{m_f^2}\left(\frac{\hbar c}{G}\right)^{3/2}.$$

((**Example**))

The interior of a white-dwarf star (electrons as fermion) is composed of atoms like $^{12}C$ (6 electrons, 6 protons, and 6 neutrons) and $^{16}O$ (8 electrons, 8 protons, and 8 neutrons), which contain equal numbers of protons, neutrons, and electrons. Thus,

$$m_f = 2m_p$$

$$\left(\frac{6m_p + 6m_n}{6} = m_p + m_n = 2m_p \quad \text{for } ^{12}\text{C}, \qquad \frac{8m_p + 8m_n}{8} = m_p + m_n = 2m_p \quad \text{for } ^{16}\text{O},\right.$$

where $m_p$ and $m_n (= m_p)$ are the proton mass and neutron mass. Then we have

$$M_0 = 1.72148\, M_{sun}.$$

The currently accepted numerical value of the limit is about $1.4\, M_{sun}$ (*Chandrasekhar limit*).

((**Mathematica**))

```
Clear["Global`*"];
rule1 = {G → 6.6742867 10^-11, me → 9.1093821545*10^-31,
   eV → 1.602176487*10^-19 , mn → 1.674927211*10^-27,
   mp → 1.672621637*10^-27, ℏ → 1.05457162853*10^-34,
   Msun → 1.988435*10^30 , c → 2.99792458*10^8};

M0 = 3.71562/(2 mp)^2 (ℏ c/G)^(3/2) /. rule1;
r = M0 / Msun /. rule1

1.72148
```

((**Note**)) **Planck mass**

The Planck mass is nature's maximum allowed mass for point-masses (quanta) – in other words, a mass capable of holding a single elementary charge. The Planck mass, denoted by $m_{\text{Planck}}$, is defined by

$$m_{Planck} = \sqrt{\frac{\hbar c}{G}} = 1.220910 \text{ x } 10^{19} \text{ GeV/c}^2.$$

where $c$ is the speed of light in a vacuum, $G$ is the gravitational constant, and $\hbar$ is the Dirac constant.

## 7. White dwarf with electron as fermion: non-relativistic case

In the white dwarf, a fermion is an electron. So we have

$$m_0 = m_e .$$

The mass $m_f$ per electron can be described in terms of atomic number $Z$, and mass number $A$ (the sum of the numbers of protons and neutrons) as follows. Since there are $Z$ electrons, mass $m_f$ per electron can be evaluated as

$$m_f = \frac{A}{Z} m_p .$$

where $m_p$ is the mass of proton and we neglect the mass of electrons.

((Note))

| | | | |
|---|---|---|---|
| Number of protons = $Z$, | | mass of protons, | $Zm_p$ |
| Number of neutron = $A-Z$ | | mass of neutron, | $(A-Z)m_n$ |
| Number of electrons = $Z$ | | mass of electron | $m_e Z$ |

where $Z$ is the atomic number and $A$ is the atomic mass. The mass $m_f$ per fermion is

$$m_f = \frac{1}{Z}[Zm_p + (A-Z)m_n + m_e Z] \approx \frac{A}{Z} m_p$$

since $m_n = m_p >> m$

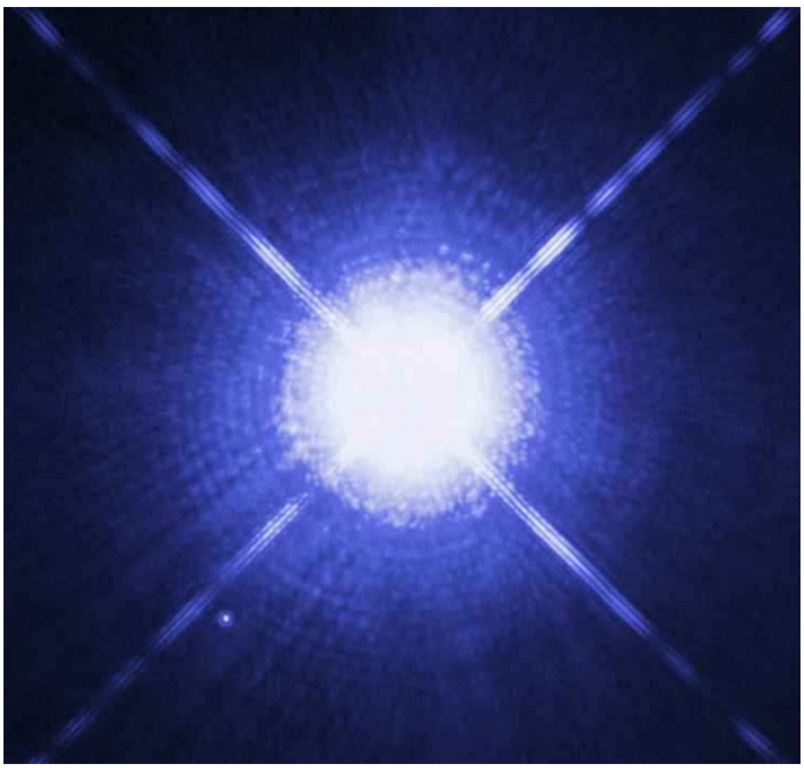


Fig. Image of Sirius A (bright star in the center) and Sirius B (white dwarf, very small spot in the figure) taken by the Hubble Space Telescope. Sirius B, which is a white dwarf, can be seen as a faint pinprick of light to the lower left of the much brighter Sirius A.
http://www.universetoday.com/wp-content/uploads/dog_star.jpg

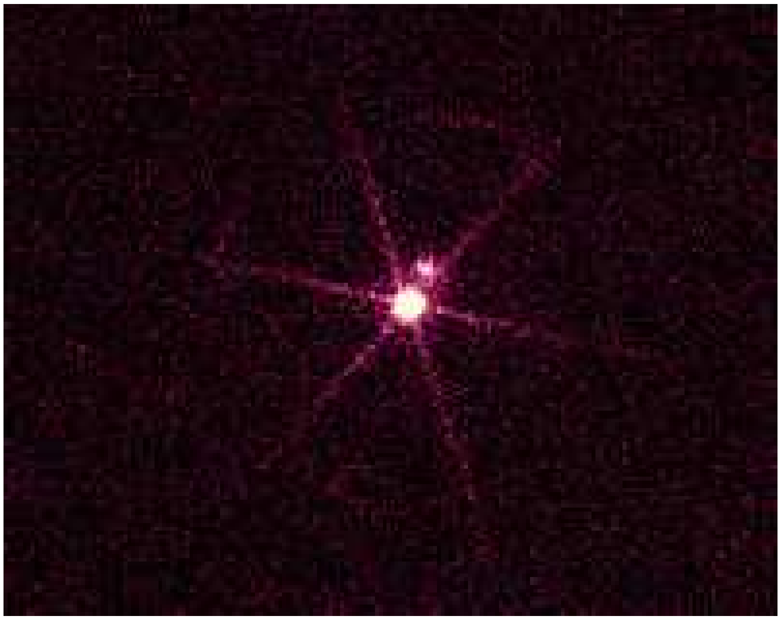


Fig. A Chandra X-ray Observatory image of the Sirius star system, where the spike-like pattern is due to the support structure for the transmission grating. The bright source is Sirius B. Credit: NASA/SAO/CXC.

http://en.wikipedia.org/wiki/File:Sirius_A_%26_B_X-ray.jpg

((**Note**))
http://chandra.harvard.edu/photo/2000/0065/

An X-ray image of the Sirius star system located 8.6 light years from Earth. This image shows two sources and a spike-like pattern due to the support structure for the transmission grating. The bright source is Sirius B, a white dwarf star that has a surface temperature of about 25,000 degrees Celsius which produces very low energy X-rays. The dim source at the position of Sirius A – a normal star more than twice as massive as the Sun – may be due to ultraviolet radiation from Sirius A leaking through the filter on the detector. In contrast, Sirius A is the brightest star in the northern sky when viewed with an optical telescope, while Sirius B is 10,000 times dimmer. Because the two stars are so close together Sirius B escaped detection until 1862 when Alvan Clark discovered it while testing one of the best optical telescopes in the world at that time. The theory of white dwarf stars was developed by S. Chandrasekhar, the namesake of the Chandra X-ray Observatory. The story of Sirius B came full cycle when it was observed by Chandra in October 1999 during the calibration or test period. The white dwarf, Sirius B, has a mass equal to the mass of the Sun, packed into a diameter that is 90% that of the Earth. The gravity on the surface of Sirius B is 400,000 times that of Earth!

Video:
https://www.youtube.com/watch?v=F0qt91rvorU

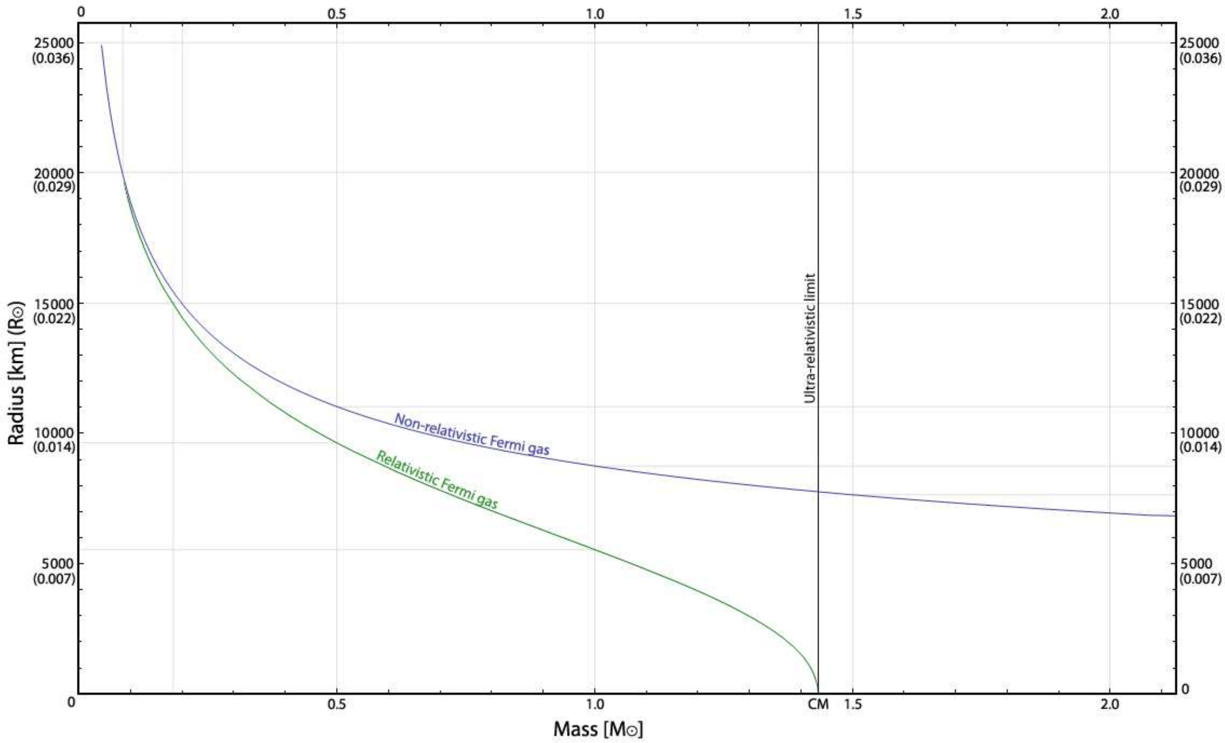



**Fig.** Mass-radius relationship. mass and radius are in the unit of the mass and radius of sun. $RM^{1/3} = const$ for the non-relativistic case.

http://upload.wikimedia.org/wikipedia/commons/8/81/WhiteDwarf_mass-radius.jpg

The number of fermions is

$$N_f = \frac{M}{m_f}.$$

Then the kinetic energy $U_G$ (in the non-relativistic case) can be given by

$$\begin{aligned} U_G &= \frac{3}{10}(\frac{9\pi}{4})^{2/3}\frac{\hbar^2}{m_e}\frac{N_f^{5/3}}{R^2} \\ &= \frac{3}{10}(\frac{9\pi}{4})^{2/3}\frac{\hbar^2}{m_e R^2}\left(\frac{M}{m_f}\right)^{5/3}, \\ &= \frac{B}{R^2} \end{aligned}$$

and

$$B = \frac{3}{10}(\frac{9\pi}{4})^{2/3}\frac{\hbar^2 M^{5/3}}{m_e m_f^{5/3}}.$$

The equilibrium distance $R$ is given by

$$\begin{aligned} R &= R_0 \\ &= \frac{2B}{A} \\ &= (\frac{3}{2})^{4/3}\frac{\pi^{2/3}\hbar^2}{GM^2 m_e}\left(\frac{M}{m_f}\right)^{5/3} \\ &= \left(\frac{81\pi^2}{16M}\right)^{1/3}\frac{\hbar^2}{Gm_e m_f^{5/3}}{}^{5/3} \\ &= \left(\frac{81\pi^2}{16M}\right)^{1/3}\frac{\hbar^2}{Gm_e}\left(\frac{Z}{Am_p}\right)^{5/3} \end{aligned}.$$

or

$$RM^{1/3} = \left(\frac{81\pi^2}{16}\right)^{1/3}\frac{\hbar^2}{Gm_e}\left(\frac{Z}{Am_p}\right)^{5/3}$$

where

$$m_f = \frac{A}{Z}m_p$$

Thus we have the relation

$$RM^{1/3} = \text{constant}$$

for the non-relativistic case. The more massive a white dwarf is, the smaller it is. The electrons must be squeezed closer together to provide the greater preasure needed to a more massive white dwarf.

((**Example**))

The interior of a white-dwarf star is composed of atoms like $^{12}$C (6 electrons, 6 protons, and 6 neutrons) and $^{16}$O (8 electrons, 8 protons, and 8 neutrons), which contain equal numbers of protons, neutrons, and electrons.

$m_f = 2m_p \qquad m_0 = m_e.$

In this case we have

$$C = M^{1/3}R_0 = \left(\frac{81\pi^2}{16}\right)^{1/3} \frac{\hbar^2}{Gm_e(2m_p)^{5/3}} = 9.00397 \text{ x } 10^{16} \text{ kg}^{1/3} \text{ m}$$

The radius $R$ is proportional to $M^{-1/3}$. When $M$ is equal to the mass of sun, $M_{sun}$, then we have

$$R_0 = \frac{C}{M_{sun}^{1/3}} = 7.16028 \text{ x } 10^3 \text{ km.}$$

which is almost equal to the radius of Earth (6371 km). The number density is

$$n_f = \frac{N_f}{V} = \frac{\rho}{m_f} = \frac{1}{m_f}\frac{M}{\frac{4\pi}{3}R_0^3} = 3.86549 \text{ x } 10^{35} \text{ /m}^3.$$

The Fermi energy of the electrons is

$$\varepsilon_F = \frac{\hbar^2}{2m_e}(3\pi^2 n_f)^{2/3} = 1.93497 \text{ x } 10^5 \text{ eV.}$$

The Fermi velocity is

$$v_F = \frac{\hbar}{m_e}k_F = \frac{\hbar}{m_e}(3\pi^2 n_f)^{2/3} = 2.6089 \text{ x } 10^8 \text{ m/s}$$

The Fermi temperature is

$$T_F = \frac{\varepsilon_F}{k_B} = 2.245 \times 10^9 K$$

The density is

$$\rho = \frac{M}{V} = \frac{m_f N_f}{V} = 1.2931 \text{ x } 10^9 \text{ kg/m}^3.$$

The average distance between fermions is

$$d = \left(\frac{m_f}{\rho}\right)^{1/3} = 1.37277 \text{ x } 10^{-12} \text{ m}.$$

((**Mathematica**)) Numerical calculation for white dwarf

```
Clear["Global`*"];
rule1 = {G → 6.6742867 10^-11, me → 9.1093821545*10^-31,
  eV → 1.602176487*10^-19 , kB → 1.3806504*10^-23, mn → 1.674927211*10^-27,
  mp → 1.672621637*10^-27, ℏ → 1.05457162853*10^-34,
  Msun → 1.988435*10^30 , c → 2.99792458*10^8};

C1 = ((81 π^2)/16)^(1/3) ℏ^2 / (2^(5/3) G me mp^(5/3)) /. rule1

9.00397×10^16

R0 = C1 / Msun^(1/3) /. rule1

7.16028×10^6

n1 = Msun / (2*4 π (mp/3) R0^3) /. rule1

3.86549×10^35
```

```
EF1 = ℏ^2/(2 me) (3 π^2 n1)^(2/3);  EF1/eV /. rule1 // ScientificForm
```

$1.93497 \times 10^{5}$

```
TF = EF1 / kB //. rule1
```

$2.24543 \times 10^{9}$

```
vF = ℏ/me (3 π^2 n1)^(1/3) /. rule1 // ScientificForm
```

$2.60893 \times 10^{8}$

```
ρ1 =  mp n1 /. rule1
```

$6.46551 \times 10^{8}$

```
d1 = (2 mp/ρ1)^(1/3) /. rule1
```

$1.72958 \times 10^{-12}$

## 8. Neutron star with neutron as fermion: relativistic case

**J.S. Townsend, Quantum Physics A Fundamental Approach to Modern Physics (University Science Books, 2019).**

"The natural question to raise is what happens if the mass of the star exceeds this 1.4 solar mass limit. As the star collapses, the size of the box confining the electrons decreases and, consequently, the energy of the electrons confined in the box increases. When the energy of the electrons reaches the point that is sufficient to initiate the reaction

$$e^{-} + p \rightarrow \nu_e + n,$$

that is, an electron combines with a proton to produce the more massive neutron (a spin-1/2 particle, fermion) and an associated neutrino, the inner core of the star collapses to a neutron star.

Calculations similar in spirit to the ones Chandrasekhar did for white dwarf stars show that for neutron stars, a typical radius is on the order of 10 km and the density is on the order of $10^{14}$ g/cm$^3$ (the density of nuclear matter). Moreover, there is an upper limit on the mass of a neutron star of roughly 1.5 to 3 solar masses. Beyond that limit, the star either collapses to a black hole or ejects mass in a catastrophic explosion known as a supernova, often leaving a neutron star surrounded by eject gas as a remnant, as is the case for the Crab nebula."

---

We consider the case of neutron star. The system consists of only neutron (spin 1/2 fermion). We use

$m_0 = m_n$, $\quad m_f = m_n$, $\quad M = M_{\text{sun}}$ (for convenience).

Then $U_G$ (in the relativistic case) can be given by

$$E_G = \frac{3}{10}\left(\frac{9\pi}{4}\right)^{2/3}\frac{\hbar^2 M^{5/3}}{R^2 m_n^{8/3}} = \frac{B}{R^2},$$

where

$$B = \frac{3}{10}\left(\frac{9\pi}{4}\right)^{2/3}\frac{\hbar^2 M^{5/3}}{m_n^{8/3}}.$$

The radius $R_0$ is obtained as

$$R_0 = \left(\frac{81\pi^2}{16M}\right)^{1/3}\frac{\hbar^2}{Gm_n^{8/3}}$$

Then we have

$$M^{1/3}R_0 = C = \left(\frac{81\pi^2}{16}\right)^{1/3}\frac{\hbar^2}{Gm_n^{8/3}} = 1.55112 \text{ x } 10^{14} \text{ kg}^{1/3} \text{ m}.$$

If $M = M_{sun}$, then we get

$$R_0 = \frac{C}{M_{sun}^{1/3}} = 12.335 \text{ km}.$$

The Fermi energy is given by

$$\varepsilon_F = \pi c\hbar(\frac{3n_f}{\pi})^{1/3} = 3.25082 \text{ x } 10^8 \text{ eV}$$

and the Fermi temperature is

$$T_F = \frac{\varepsilon_F}{k_B}\ T_F = \frac{\varepsilon_F}{k_B} = 3.7724 \text{ x } 10^{12} \text{ K.}$$

where

$$n_f = \frac{N_f}{V} = \frac{\rho}{m_f} = \frac{1}{m_f}\frac{M}{\frac{4\pi}{3}R^3} = 1.51008 \text{ x } 10^{44} \text{ /m}^3.$$

with $m_f = m_n$. The average distance between fermions is

$$d = \left(\frac{1}{n_f}\right)^{1/3} = \left(\frac{m_n}{\rho}\right)^{1/3} = 1.87788 \text{ x } 10^{-15} \text{ m.}$$

The density $\rho$ is

$$\rho = m_p n_f == 2.5293 \text{ x } 10^{17} \text{ kg/m}^3.$$

((**Mathematica**))        Numerical calculation for neutron star

```
Clear["Global`*"];
rule1 = {G → 6.6742867 10^-11, me → 9.1093821545■10^-31,
   kB → 1.3806504■10^-23, eV → 1.602176487■10^-19 ,
   c → 2.99792458■10^8, mn → 1.674927211■10^-27,
   mp → 1.672621637■10^-27, ℏ → 1.05457162853■10^-34,
   Msun → 1.988435■10^30, Rsun → 6.9599■10^8, AU → 1.49597870 ■10^11};
```

$$C1 = \frac{\left(\frac{81\,\pi^2}{16}\right)^{1/3} \hbar^2}{G\, mn^{8/3}} \;/.\; rule1$$

$1.55112 \times 10^{14}$

$$R0 = \frac{C1}{Msun^{1/3}} \;/.\; rule1$$

12 335.1

$$nf1 = \frac{Msun}{\frac{mn}{3}\, 4\,\pi\, R0^3} \;/.\; rule1$$

$1.51008 \times 10^{44}$

$$EF1 = \pi\, c\, \hbar \left(\frac{3\ nf1}{\pi}\right)^{1/3} \Big/ eV \;//.\; rule1$$

$3.25082 \times 10^{8}$

$$TF1 = \pi\, c\, \hbar \left(\frac{3\ nf1}{\pi}\right)^{1/3} \Big/ kB \;//.\; rule1$$

$3.77242 \times 10^{12}$

$$d1 = nf1^{-1/3} \;/.\; rule1$$

$1.87788 \times 10^{-15}$

$$\rho 1 = mn\ nf1 \;/.\; rule1$$

$2.52927 \times 10^{17}$

((**Note**))

The electrons are captured by nucleus. $N$ decreases. However, $V$ is also decreased. Then the number density $n$ remains unchanged. Thus $P$ does not change. When the system is further compressed, then all electrons are captured by nucleus.

$$A_z + e^- = A_{z-1} + \nu$$
$$A_{z-1} + e^- = A_{z-2} + \nu$$
.............................
$$A_1 + e^- = A_0 + \nu$$

where $A_0$ is a neutron and $\nu$ is a neutrino. $z$ is the number of protons. Finally, nucleus is composed of only neutrons.

## 9. Crab pulsar (neutron star)

The Crab Pulsar (PSR B0531+21) is a relatively young neutron star. The star is the central star in the Crab Nebula, a remnant of the supernova SN 1054, which was widely observed on Earth in the year 1054. Discovered in 1968, the pulsar was the first to be connected with a supernova remnant. The optical pulsar is roughly 25 km in diameter and the pulsar "beams" rotate once every 33 ms, The outflowing relativistic wind from the neutron star generates synchrotron emission, which produces the bulk of the emission from the nebula, seen from radio waves through to gamma rays. The most dynamic feature in the inner part of the nebula is the point where the pulsar's equatorial wind slams into the surrounding nebula, forming a termination shock. The shape and position of this feature shifts rapidly, with the equatorial wind appearing as a series of wisp-like features that steepen, brighten, then fade as they move away from the pulsar into the main body of the nebula. The period of the pulsar's rotation is slowing by 36.4 ns per day due to the large amounts of energy carried away in the pulsar wind.

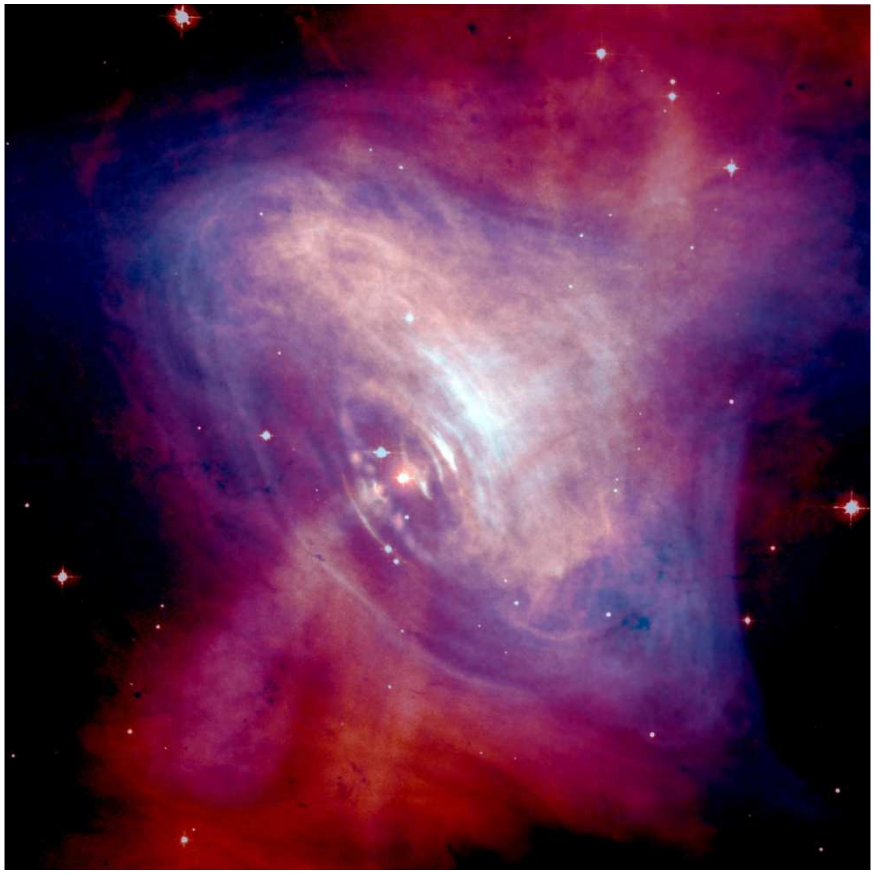


**Fig.** The Crab Nebula, which contains the Crab Pulsar. Image combines optical data from Hubble (in red) and X-ray images from Chandra (in blue). NASA/CXC/ASU/J. Hester

Video:
https://www.youtube.com/watch?v=pLivjAoDrTg

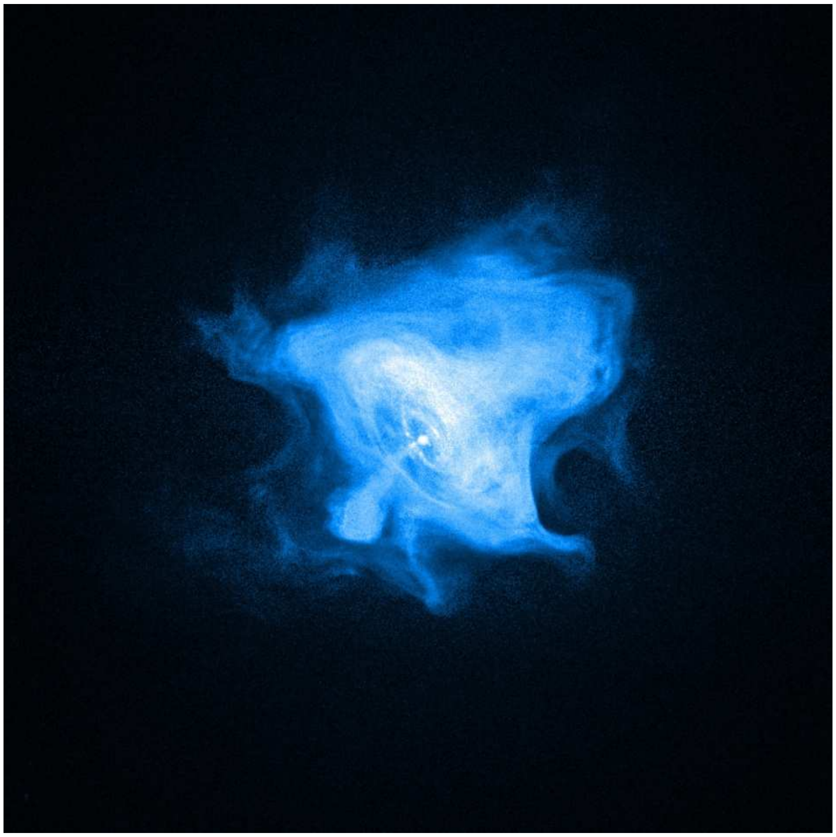


**Fig.** X-ray picture of Crab pulsar, taken by Chandra

http://en.wikipedia.org/wiki/Crab_Pulsar

Data of Crab pulsar:

$\nu = 30/\text{s}$ ($T = 33$ ms), $M = 1.4\ M_{\text{sun}}$, $R = 12$ km. $\Delta T = 36.4$ ns.

The density $\rho$ is

$$\rho = \frac{M}{\frac{4\pi}{3}R^3} = 3.84598 \text{ x } 10^{17} \text{ kg/m}^3.$$

The moment of inertia $I$ is calculated as

$$I = \frac{2}{5}MR^2 = 1.60347 \times 10^{38} \text{ kg m}^2.$$

The rotational kinetic energy is

$$K_{rot}=\frac{1}{2}I\omega^2=\frac{1}{2}\frac{2}{5}MR^2\omega^2 = 2.91 \text{ x } 10^{42} \text{ J},$$

where the angular frequency $\omega$ is

$$\omega=\frac{2\pi}{T} = 190.4 \text{ rad/s}.$$

The loss of energy per day is

$$P=\frac{dK_{rot}}{dt}=\frac{d}{dt}\left(\frac{1}{2}I\omega^2\right)=-\frac{4\pi^2}{T^3}I\frac{\Delta T}{\Delta t} = 7.42 \text{ x } 10^{31} \text{ W},$$

where $\Delta T$ = 36.4 ns per $\Delta t$ = 1 day = 24 x 3600 s. The age of the Crab pulsar can be estimated as

$$t=\frac{1}{2}\frac{T}{\Delta T}= 3.9 \text{ x } 10^{10} \text{ s} = 1240 \text{ years}.$$

The Crab pulsar is thought to be about 930 years, so the age we have just estimated is roughly correct..

**REFERENCES**

R.C. Bless, Discovering the Cosmos, second edition (University Science Book, 2013)

((**Note-1**)) **Density**

For a rotating object to remain bound, the gravitational force at the surface must exceed the centripetal acceleration:

$$m\frac{GM}{r^2}>mr\omega^2 \Rightarrow \frac{GM}{r^3}>\omega^2=\frac{4\pi^2}{T^2}\Rightarrow \frac{G\rho}{r^3}\frac{4\pi}{3}r^3>\frac{4\pi^2}{T^2}\Rightarrow \rho>\frac{3\pi}{T^2G}.$$

For $T$ = 33 ms, the density must be greater than $1.3\times10^{11}$ g/cm$^3$ = 1.3x$10^{14}$ kg/m$^3$. This exceeds the maximum possible density for a white dwarf.

((**Note-2**)) **Angular momentum conservation**

Suppose that the Sun ($T$ = 25 days, radius $7\times10^8$ m, mass $1.988\text{x}10^{30}$ kg) were to collapse to a neutron star with a radius of 16 km. Using the angular momentum conservation law, we have

$$R_i^{\,2}\omega_i = R_f^{\,2}\omega_f ,$$

or

$$\frac{\omega_f}{\omega_i} = \frac{R_i^{\,2}}{R_f^{\,2}} = \left(\frac{7\times10^8}{16\times10^3}\right)^2 = \frac{49\times10^{10}}{256} = 2\times10^9$$

In other words, the star is rotating $2\times10^9$ faster after the collapse than it was before.

$$\frac{T_f}{T_i} = \frac{1}{2\times10^9} ,$$

or

$$T_f = \frac{25\times(24\times3600)}{2\times10^9} = 1ms .$$