The Radiation Pressure

Masatsugu Sei Suzuki,
Department of Physics, SUNY at Binghamton
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1. Introduction

There is a very simple connection between the radiation pressure P and the energy density u. This can be derived from the classical electromagnetic theory, but it is simpler to use some elementary idea from both quantum theory and the kinetic theory of gases to obtain the result. However, the physical origin of this pressure can be understood from the electromagnetic theory.

It is known from the kinetic theory that the pressure in a gas is

$$PV = \frac{2}{3}E = \frac{2}{3}N\varepsilon$$

where ε $(= \frac{1}{2}mv^2)$ is an energy of each particle. Then we get

$$P = \frac{2}{3}\frac{N}{V}\frac{1}{2}m(v_{rms})^2 = \frac{1}{3}nm(v_{rms})^2$$

where n is he number density, m is the mass of particle, and v_{rms} is the root-mean square velocity. Since $\rho = nm$ is the density, the pressure P can be rewritten as

$$P = \frac{1}{3}\rho(v_{rms})^2 .$$

If we consider the radiation as a photon gas where the photons are all moving with the velocity c, we have

$$P = \frac{1}{3}\rho c^2 .$$

According to the Einstein energy-mass relation, the energy density is expressed by

$$u = \frac{E}{V} = \frac{Nmc^2}{V} = \rho c^2 .$$

Thus we have

$$P = \frac{1}{3}\rho c^2 = \frac{1}{3}u = \frac{1}{3}\frac{E}{V}.$$

or

$$PV = \frac{1}{3}E.$$

This is our required relation.

2. Canonical ensemble for the photon gas

Using the concept of microcanonical ensemble, we can directly show that

$$PV = \frac{1}{3}E$$

where the energy of photon is given by $\varepsilon = cp$.

We consider an ideal gas consisting of N particles obeying classical statistics. Suppose that the energy of one particle is given by $\varepsilon = cp$, where p is the linear momentum. We find the thermodynamic functions of this ideal gas without considering the internal structure of the particles.

First we calculate the one-particle partition function based on the canonical ensemble,

$$\begin{aligned}
Z_{C1} &= \frac{V}{h^3}\int d^3\boldsymbol{p}\exp(-\beta cp) \\
&= \frac{V}{h^3}\int_0^\infty 4\pi p^2 \exp(-\beta cp)dp \\
&= \frac{4\pi V}{h^3}\int_0^\infty p^2 \exp(-\beta cp)dp \\
&= \frac{4\pi V}{h^3}\frac{2!}{(\beta c)^3} \\
&= \frac{8\pi V}{h^3c^3}\frac{1}{\beta^3}
\end{aligned}$$

where $\beta = \frac{1}{k_B T}$, c is the velocity of light, h is the Planck's constant.

$$\int_0^\infty p^2 \exp(-\alpha p)dp = \frac{2!}{\alpha^3} \qquad \text{(Laplace transformation)}$$

The N-particle partition function:

$$Z_{CN} = \frac{1}{N!}(Z_{C1})^N = \frac{1}{N!}(\frac{8\pi V}{h^3 c^3}\frac{1}{\beta^3})^N = \frac{1}{N!}(\frac{8\pi V}{h^3 c^3})^N \beta^{-3N}$$

$$\ln Z_{CN} = -\ln N! + N\ln(\frac{8\pi}{h^3 c^3}) + N\ln V - 3N\ln\beta$$

Using the Stirling's law in the limit of large N,

$$\ln Z_{CN} = N[\ln\frac{V}{N} - 3\ln\beta + \ln(\frac{8\pi}{h^3 c^3}) + 1]$$

The internal energy

$$E = -\frac{\partial}{\partial\beta}\ln Z_{CN} = \frac{3N}{\beta} = 3Nk_B T \tag{1}$$

The Helmholtz free energy:

$$F = -k_B T\ln Z_{CN} = -Nk_B T[\ln\frac{V}{N} - 3\ln\beta + \ln(\frac{8\pi}{h^3 c^3}) + 1]$$

The entropy S is obtained as

$$S = \frac{1}{T}(E - F) = Nk_B[\ln\frac{V}{N} - 3\ln\beta + \ln(\frac{8\pi}{h^3 c^3}) + 4]$$

S can be also derived from the relation as

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = Nk_B[\ln\frac{V}{N} - 3\ln\beta + \ln(\frac{8\pi}{h^3 c^3}) + 4]$$

The pressure P:

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{Nk_B T}{V} \qquad \text{or} \qquad PV = Nk_B T \qquad (2)$$

From Eqs.(1) and (2) we get the relation

$$PV = \frac{1}{3}E$$

or

$$P = \frac{1}{3}\frac{E}{V} = \frac{1}{3}u$$

where u is the energy density of photon gas.

3. Stefan-Boltzmann law

Suppose that E is given by

$$E = \sigma T^4 V , \qquad PV = \frac{1}{3}\sigma T^4 V$$

leading to the pressure P as

$$P = \frac{1}{3}\sigma T^4$$

where

$$\sigma = \frac{4\sigma_B}{c}$$

and σ_B is the Stefan-Boltzmann constant.

We note that

$$dF = -PdV - SdT$$

Since

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{1}{3}\sigma T^4$$

we get the expression of F as

$$F = -\frac{1}{3}\sigma T^4 V$$

and the entropy as

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = \frac{4}{3}\sigma T^3 V$$

(b)

Suppose that R is the radius of the universe,

$$V = \frac{4\pi}{3}R^3$$

Then we get the expression for S as

$$S = \frac{4}{3}\sigma T^3 \frac{4\pi}{3}R^3 = \frac{16}{9}\pi\sigma T^3 R^3$$

When S is kept constant, the temperature is inversely proportional to T,

$$T \propto \frac{1}{R}.$$

REFERENCES

C.B.P. Finn, Thermal Physics, second edition (Nelson-Thornes, 2001).

R.E. Kelly, Am. J. Phys. 49, 714 (1981)