Renormalization group for one dimensional Ising model

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Kenneth Geddes Wilson (June 8, 1936 – June 15, 2013) was an American theoretical physicist and a pioneer in leveraging computers for studying particle physics. He was awarded the 1982 Nobel Prize in Physics for his work on phase transitions—illuminating the subtle essence of phenomena like melting ice and emerging magnetism. It was embodied in his fundamental work on the renormalization group.

https://en.wikipedia.org/wiki/Kenneth_G._Wilson

Here we discuss the renormalization group (RG) for the simplest possible example- the one-dimensional Ising model. The results of the RG calculations are compared with the exact calculations of the correlation functions and the free energy of the one-dimensional Ising model; the temperature $T = 0$ can be treated as a second-order critical point.

Leo Philip Kadanoff:

Kadanoff was born in 1937 and spent his early life in New York City. He received his PhD from Harvard in 1960 and was a postdoctoral researcher at the Niels Bohr Institute in Copenhagen until he moved to the University of Illinois in 1962. His work there concerned the development of many-body theory techniques, culminating in a short but highly influential monograph with Gordon Baym, which is still widely used today. Kadanoff found himself, in the mid- 1960s, working on the problem of phase transitions. It had been known since the work of Lev Landau in 1937 that second-order phase transitions would exhibit power-law singularities or divergences as the temperature approached the critical point. But 30 years later, the exponents characterizing these singularities, which had been measured or estimated by numerical methods in a number of systems, were in clear disagreement with Landau's predictions. A crucial contribution was made in 1965 by Benjamin Widom of Cornell University, who showed how the form of the scaling laws near the critical point could be obtained from assuming mathematical homogeneity in the equation of state, but the interpretation of this assumption seemed obscure. Kadanoff made the perspicacious insight that statistical mechanics could be formulated not merely in terms of a partition function relating in one fell swoop the microscopic Hamiltonian of a system to its system-size thermodynamic free energy, but instead marching one scale at a time from small to large.

In Kadanoff's formulation, degrees of freedom on a small scale would influence degrees of freedom on a larger scale by changing or 'renormalizing' the parameters of the larger-scale Hamiltonian. This process, iterated from atomic level to system-size level, led to a mathematical structure identical to that postulated by Widom. Although Kadanoff's 'block spin construction' related the physics on different scales, it did not on its own provide a calculational method for estimating critical exponents. This last step was undertaken by Wilson through his realization that iteration of a Hamiltonian from one scale to another constituted a form of a dynamical system, whose fixed points and the linearized trajectories around them controlled the phases and phase transitions of the system. The insights derived from Widom, Kadanoff and Wilson's work revolutionized our understanding of the process of constructing theories in statistical mechanics, high-energy physics and beyond.

http://guava.physics.uiuc.edu/~nigel/REPRINTS/2015/NG%20obituary%20for%20Leo%20Kadanoff%20Nature%20Physics%202015.pdf

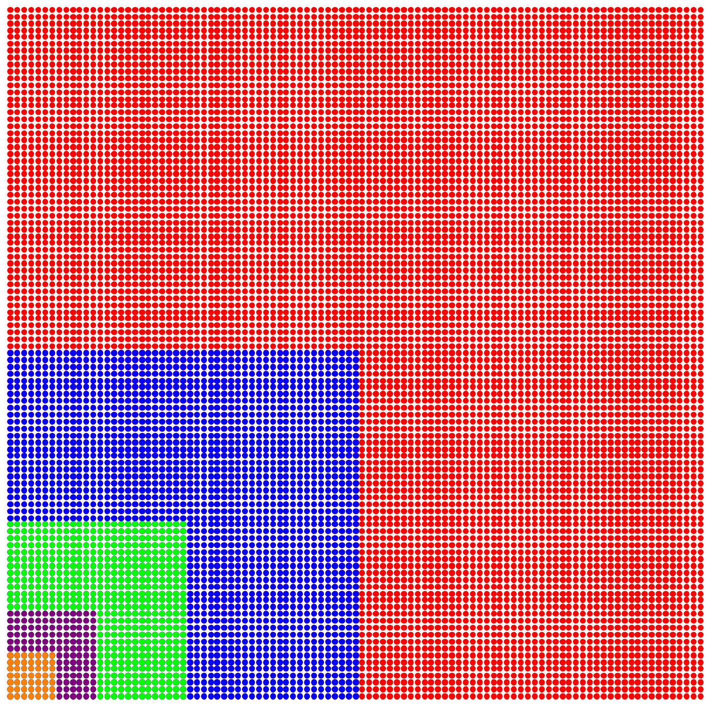


Fig. Kadanoff's construction with the combination of coarse graining procedure and rescaling procedure.

1. Renormalization group for the one-dimensional Ising model

In order to understand the essence of the renormalization group theory, we concentrate on the simple model (one-dimensional Ising model). We already know the exact solution of this model. The starting Hamiltonian is given by

$$\beta H = -K\sum_{i=1}^{N} s_i s_{i+1} - h\sum_{i=1}^{N} s_i - c$$

where c is added as an extra term.

$$K = \frac{J}{k_B T}, \qquad h = \frac{\mu_B B}{k_B T}$$

with spin $s_i = \pm 1$. The partition function Z is defined by

$$Z = \sum_{s_1=\pm 1}\sum_{s_2=\pm 1}\sum_{s_3=\pm 1}.....\sum_{s_N=\pm 1}\exp(-\beta H)$$

If we want to make several successive renormalization group (RG) transformations, the structure of H must not change after each transformation. We are thus seeking for such a transformation from the old Hamiltonian H to the new Hamiltonian H' that keeps the structure of the Hamiltonian unchanged under successive transformations. In addition, the transformation must be such that it *does not* change the partition function. That is necessary because this ensures that the physical properties of the system are not changed.

We perform a partial trace by summing over every second spin variable along the chain, leaving the alternate spins unaffected. In this way we obtain a renormalized chain with only half the number of original spins as illustrated below.

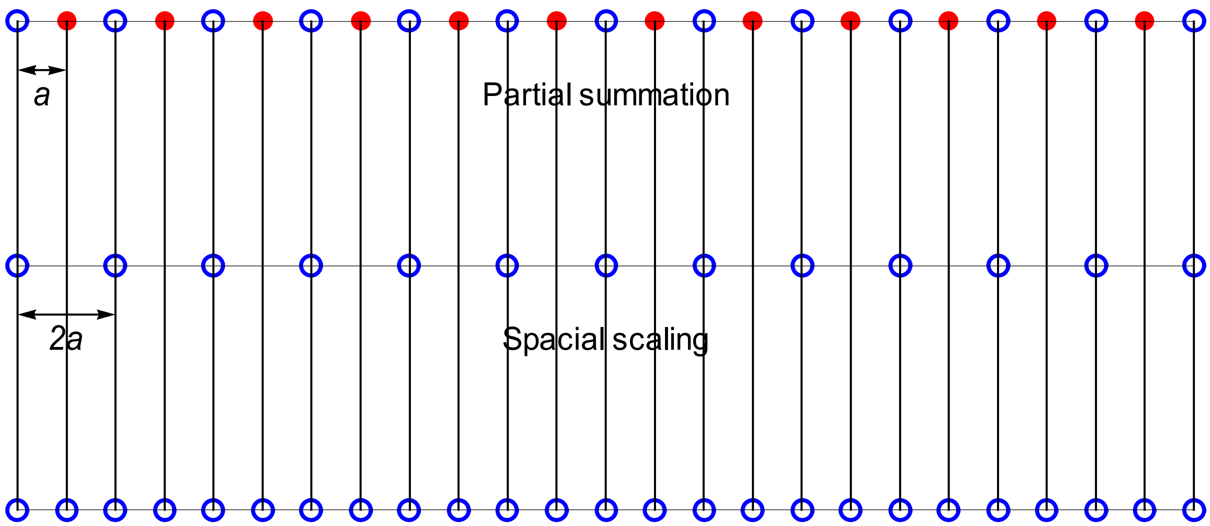


Fig. Steps involved in the RG procedure with decimation and rescaling). Renormalization of the 1D Ising model by a scale factor, $b = 2$. (a) The original lattice. (b) The renormalized lattice. (c) The renormalized lattice with the spins-re-numbered consecutively.

The partition function is given by

$$
\begin{aligned}
Z(N,K,h) &= \sum_{\{s\}} \exp(-\beta H) \\
&= \sum_{\{s\}} \exp[K s_i s_{i+1} + \frac{h}{2}(s_i + s_{i+1})] \\
&= \sum_{\{s\}} \exp[K(s_1 s_2 + s_2 s_3) + h s_2 + \frac{h}{2}(s_1 + s_3)] \exp[K(s_3 s_4 + s_4 s_5) + h s_4 + \frac{h}{2}(s_3 + s_5)] \ldots
\end{aligned}
$$

We divide the lattice into the sublattices ○ (1, 3, 5,…) and ● (2,4,6,…) denote the spins on the ● sublattice by σ_i (2, 4, 6, …)

$$Z(N,K,h)=\sum_{\{\sigma\}}\sum_{\{s\}}\exp[K(s_1\sigma_2+\sigma_2 s_3)+h\sigma_2+\frac{h}{2}(s_1+s_3)]\exp[K(s_3\sigma_4+\sigma_4 s_5)+h\sigma_4+\frac{h}{2}(s_3+s_5)]...$$

and carry out a partial trace, i.e., we sum over $2^{N/2}$ configurations of the $\{\sigma\}$ spins sitting on the ● sublattice:

$$Z(N,K,h)=\sum_{\{s\}}\{\exp[(K+\frac{h}{2})(s_1+s_3)+h]+\exp[(-K+\frac{h}{2})(s_1+s_3)-h]\}$$
$$\times\{\exp[(K+\frac{h}{2})(s_3+s_5)+h]+\exp[(-K+\frac{h}{2})(s_3+s_5)-h]\}...$$

The entire partition function must not change and *H* must keep its structure, therefore we have the condition

$$\sum_{\{s\}}\exp[K's_i s_{i+2}+\frac{h'}{2}(s_i+s_{i+2})+2g]$$

The equation given by

$$\exp[(K+\frac{h}{2})(s_i+s_{i+2})+h]+\exp[(-K+\frac{h}{2})(s_i+s_{i+2})-h]=\exp[K's_i s_{i+2}+\frac{h'}{2}(s_i+s_{i+2})+2g]$$

hold for any $s_i=\pm 1$, $s_{i+2}=\pm 1$, where g is a constant, $g=g(K,h)$.

For $s_i=1$, $s_{i+2}=1$,

$$\exp(2K+2h)+\exp(-2K)=\exp(K'+h'+2g)$$

For $s_i=-1$, $s_{i+2}=-1$,

$$\exp(2K-2h)+\exp(-2K)=\exp(K'-h'+2g)$$

For $s_i=1$ and $s_{i+2}=-1$, or $s_i=-1$ and $s_{i+2}=1$

$$\exp(h)+\exp(-h)=\exp(-K'+2g)$$

Then we have

$$e^{4K'} = \frac{\cosh(2K+h)\cosh(2K-h)}{\cosh^2(h)}$$

$$e^{2h'} = e^{2h}\frac{\cosh(2K+h)}{\cosh(2K-h)}$$

$$e^{8g} = 16\cosh^2(2K+h)\cosh^2(2K-h)\cosh^2(h)$$

or

$$K' = \frac{1}{4}\ln\frac{\cosh(2K+h)\cosh(2K-h)}{\cosh^2(h)}$$

$$h' = h + \frac{1}{2}\ln\frac{\cosh(2K+h)}{\cosh(2K-h)}$$

$$g = \frac{1}{8}\ln[16\cosh^2(2K+h)\cosh^2(2K-h)\cosh^2(h)]$$

which are the recursion relations and determine the fixed points and the flow diagram of the system.

$$Z(N.K,h) = \exp[2g(K,h)\frac{N}{2}]Z(\frac{N}{2},K',h') = \exp(Ng(K,h)]\sum_{\{s\}}\exp(-\beta H')$$

In each iteration, the number of degrees of freedom is reduced by one half, the new Hamiltonian H has only one half of the previous spins,

$$N' = N/b \qquad\qquad b = 2.$$

and the lattice spacing is increased,

$$a' = b$$

where a = 1. The correlation length has the dimension of length, but to express it as a pure number we may measure it in units of the lattice spacing. The new correlation length is measured in units of the new lattice spacing,

$$\xi' = \xi / b$$

where $\xi = \xi' b$.

The remaining spins on the decimated lattice interact with their new nearest neighbors through the renormalized coupling constants *K'* and are subject to renormalized fields *h'*.

2. Kadanoff construction))

The above procedure is called the Kadanoff construction. The construction can be performed by summing the configurations of every alternate spin variable in the calculation of Z. This procedure will give another Ising model (with parameters K' and h'), but with a lattice spacing which is twice the original spacing. Then, a change in the unit of length is introduced (in fact the length unit is doubled) so that the new model looks exactly like the old one. From the Hamiltonian of the new model, it will be possible to determine the parameters K' and h' in terms of K and h. Thus the two operations, partial summation of spin configurations and spatial length rescaling, can be thought of as a transformation in the parameter space.

$$\xi(K',h') = \frac{\xi(K,h)}{b}$$

Thus, spin decimation and spatial rescaling produce a new system with a smaller correlation length. Therefore, unless $\xi(k,h) = 1$, this procedure drives the system away from the critical point. Since H and H' are identical in structure, the functional forms of $\xi(K,h)$ and $\xi(K',h')$ are the same.

$$f(H) = -\frac{k_B T}{N} \ln Z(N,K,h)$$

$$\begin{aligned} f(H') &= -\frac{k_B T}{N'} \ln Z(N',K',h') \\ &= -\frac{k_B T}{N'} \ln[Z(N,K,h)\exp(-Ng(K,h)] \\ &= -\frac{k_B T}{N'} [\ln Z(N,K,h) - Ng(K,h)] \\ &= -b\frac{k_B T}{N} \ln Z(N,K,h) + k_B Tbg(K,h) \\ &= bf(H) + k_B Tbg(K,h) \end{aligned}$$

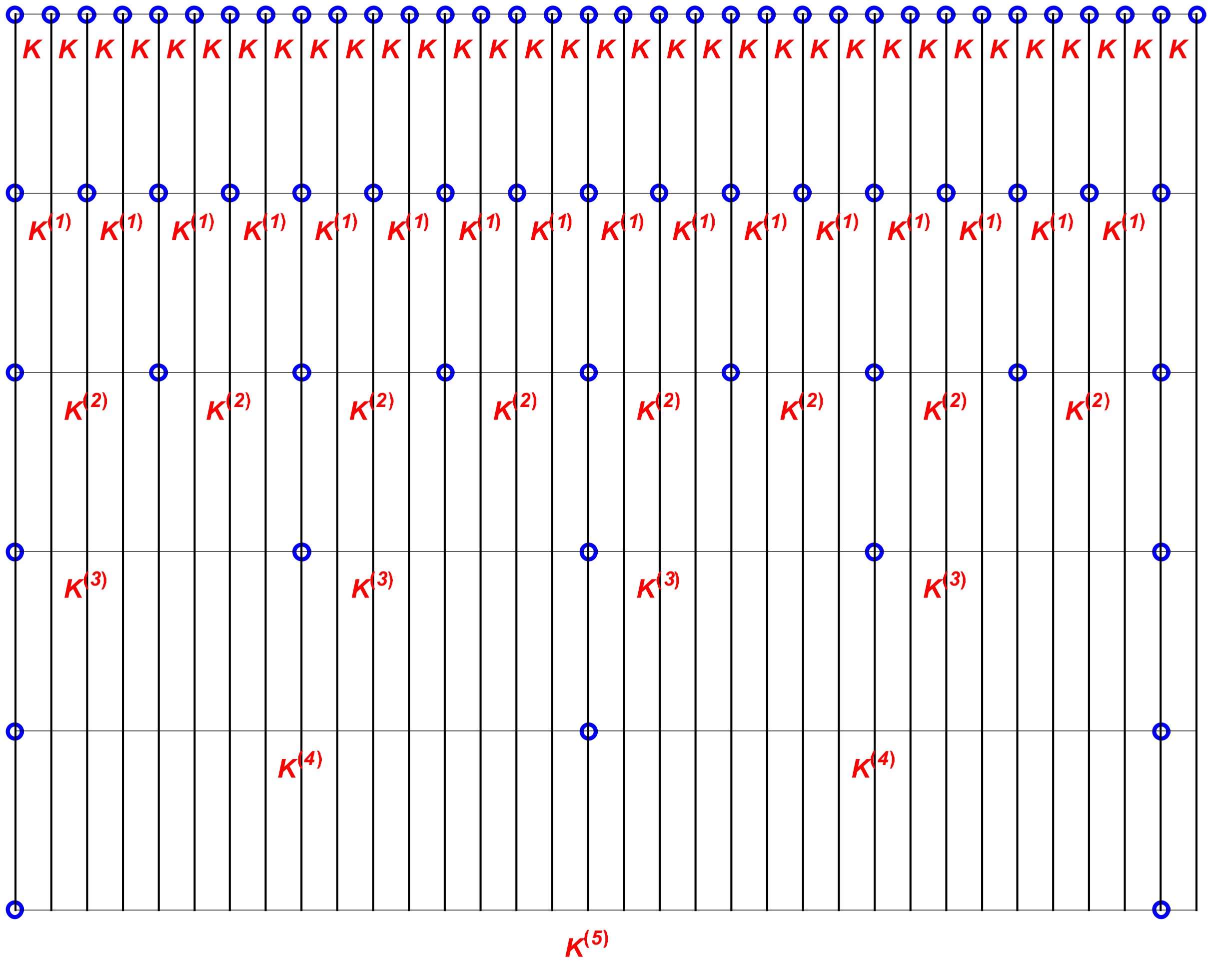
$$Z(N,K,h) = \exp[Ng(K,h)]Z(N',K',h')$$

where

$N = bN'$

The parameter g is the contribution (per site) from the decimated spins.

3. Flow equations



The equations connecting the parameter set (K', h') to (K, h) are known as flow equations. Symbolically, they are expressed as

$$K' = R_K(K, h)$$

$$h' = R_2(K, h)$$

The equation for g is not considered since it is explicitly given by K and h. Since $K = \frac{J}{k_B T}$, these equations can also be rewritten in terms of T' and h' as the formk

$$T' = R_1(T,h)$$

$$h' = R_2(T,h)$$

Thus, the application of an RG transformation produces an identical system but at a different point in the parameter space. The fixed point (T^*, h^*) of the transformation is defined as

$$T^* = R_1(T^*, h^*)$$

$$h^* = R_2(T^*, h^*)$$

By defnition, the fixed point is unaltered by the transformation. That is, if $h = h^*$, then $h' = h^*$: However, the correlation length at the fixed point must satisfy

$$\xi(T^*, h^*) = \frac{\xi(T^*, h^*)}{b}$$

where b is the scale factor. This equation has solutions

$$\xi(T^*, h^*) = 0, \qquad \xi(T^*, h^*) = \infty$$

The zero value is associated to a paramagnetic state while the infinite value corresponds to a system at the critical point. Thus (neglecting the solution $\xi(T^*, h^*) = 0$, the fixed point in the parameter space represents a critical system since the correlation length corresponding to that point is infinity. The flow equations can be linearized near the fixed point by writing

$$T' = T^* + \Delta T'$$

$$h' = h^* + \Delta h'$$

$$T = T^* + \Delta T$$

$$h = h^* + \Delta h$$

In matrix notation, the linearized transformation equations are

$$\begin{pmatrix} \Delta T' \\ \Delta h' \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} \Delta T \\ \Delta h \end{pmatrix}$$

where

$$R_{11} = \frac{\partial R_1^*}{\partial T}, \qquad R_{12} = \frac{\partial R_1^*}{\partial h}$$

$$R_{21} = \frac{\partial R_2^*}{\partial T}, \qquad R_{22} = \frac{\partial R_2^*}{\partial h}$$

4. General properties of RG transformation

The renormalization-group (RG) transformations work by changing the length scale and by reducing the number of the degrees of freedom. Only at criticality the system remains unaltered under the change in scale and the critical behavior is determined by a fixed point of the transformation. The essential condition to be satisfied by any RG transformation is that the partition function must not change,

$$Z' = R_b Z$$

The renormalization-group operator R_b reduces the number of degrees of freedom from N to N';

$$N' = \frac{N}{b^d}$$

Therefore the total free energy does not change but the free energy per unit cell (spin) increases as

$$f' = R_b f = b^d f$$

All lengths, measured in units of the new lattice spacing, are reduced by the factor b. Thus, the correlation length scales as

$$\xi' = \frac{\xi}{b}$$

We see from this that the RG group procedure has the effect of shrinking the correlation length. Hence, if we recall the central fact of critical phenomena, namely, that ξ becomes indefinitely large as $t \to 0$, we see that the RG has the effect of driving a system away from criticality (Fisher).

5. Recursion relation and fixed points

At a fixed point the parameters (the coupling constants – in our case *K* and *h*) do not change under successive decimations. This means that the system at a fixed point stays at this point in the parameter space. With the new variables

$$x = e^{-4K}, \qquad y = e^{-2h}, \qquad z = e^{-8g}$$

$(0 \le \{x, y, z) \le 1$

$$x' = x\frac{(1+y)^2}{(x+y)(1+xy)} \tag{1}$$

$$y' = y\frac{(x+y)}{1+xy} \tag{2}$$

$$z' = z^2xy^2\frac{1}{(x+y)(1+xy)(1+y)^2} \tag{3}$$

The equations for *x* and *y* do not depend on z and have three fixed points. We define the fixed points (x^*, y^*) which are determined by two equations

$$x^* = x^*\frac{(1+y^*)^2}{(x^*+y^*)(1+x^*y^*)} \tag{4}$$

$$y^* = y^*\frac{(x^*+y^*)}{1+x^*y^*} \tag{5}$$

We determine the values of x^* and y^* using the ContourPlot of Eqs.(4) and (5)

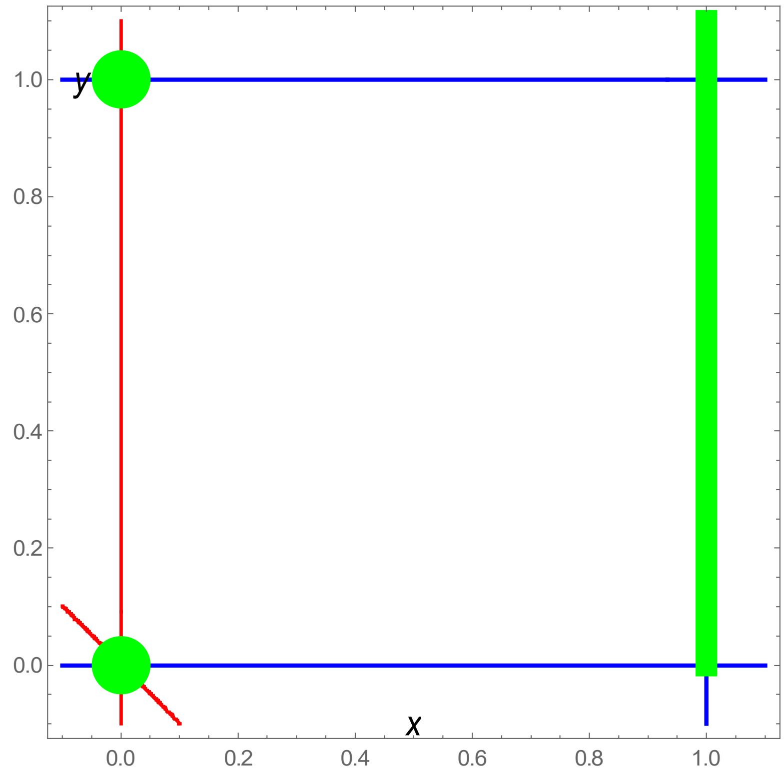


Fig.1 The fixed points (denoted by green circles and green line), which is obtained from the shared regions of Figs.2 and 3.

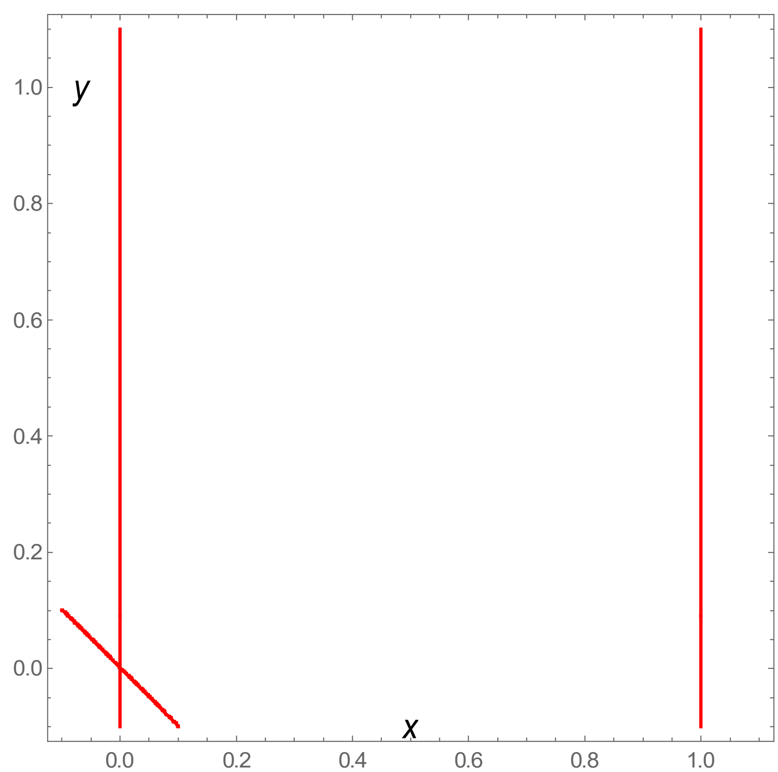


Fig.2 ContourPlot of $x^* = x^* \dfrac{(1+y^*)^2}{(x^*+y^*)(1+x^*y^*)}$ in the (x^*, y^*) plane.

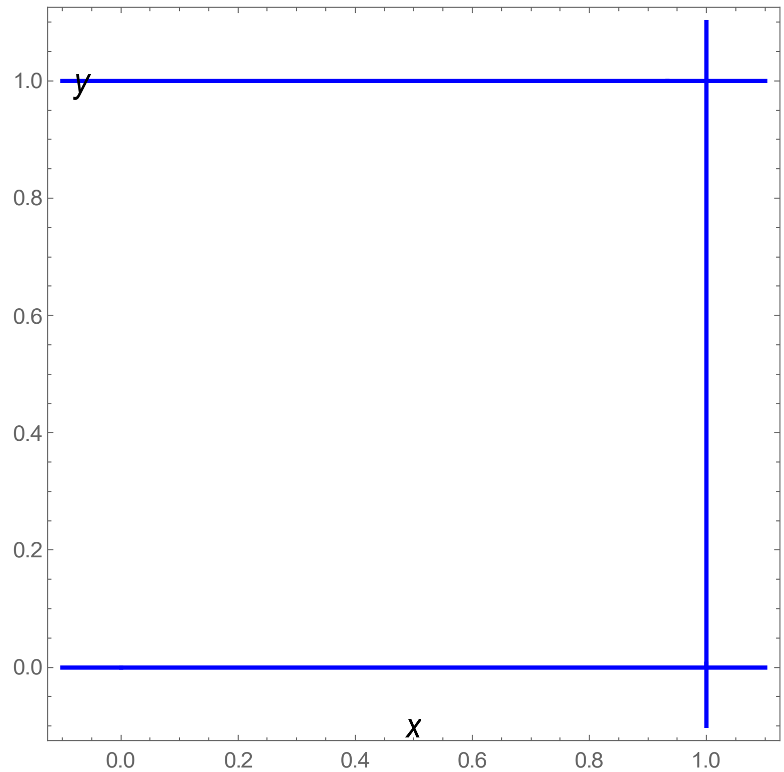


Fig.3 ContourPlot of $y^* = y^* \dfrac{(x^*+y^*)}{1+x^*y^*}$ in the in the (x^*, y^*) plane.

(i) $x^* = y^* = 0$

A frozen spin configuration (aligned spins at all temperatures as a result of an infinite external field)

(ii) $x^* = 1$, y^*; arbitrary

The high temperature paramagnetic phase

(iii) $x^* = 0$, $y^* = 1$

The $T = 0$ critical point

6. Flow diagram in the (x, y) plane

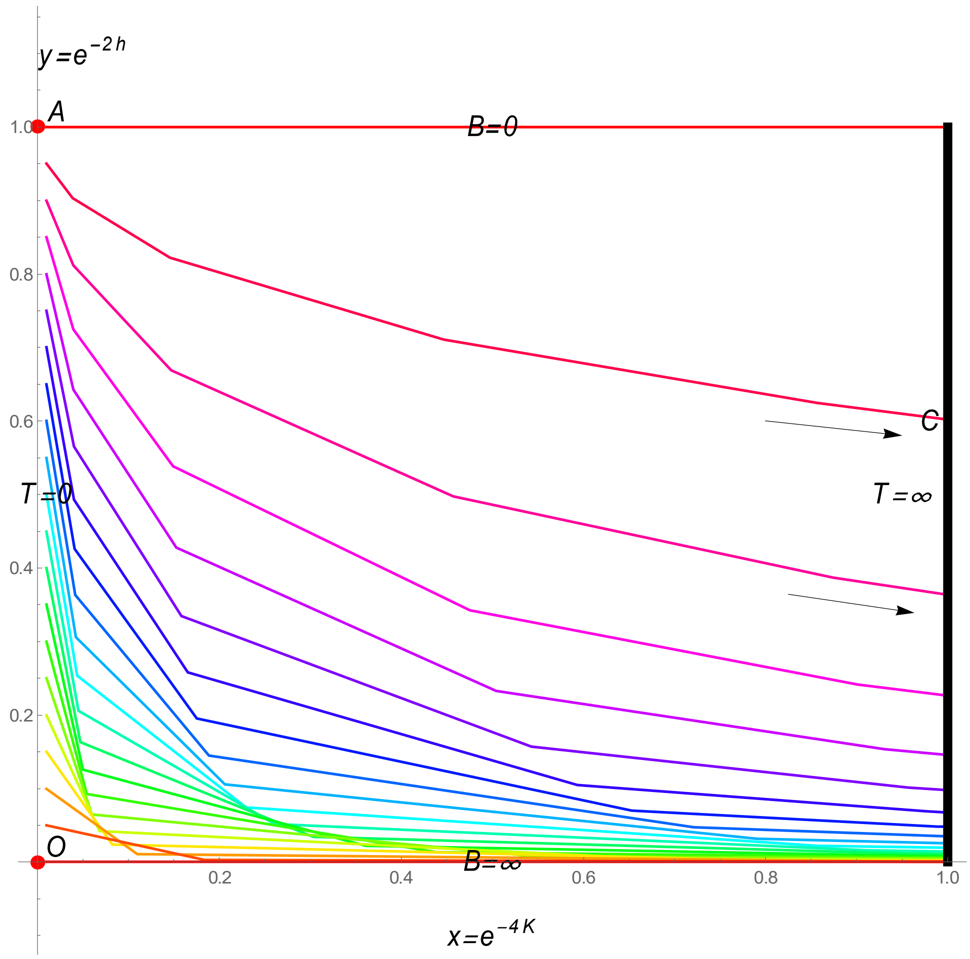


Fig. Schematic flow diagram of the d = 1 Ising model. The point O $(x^* = 0, y^* = 0)$; fully aligned configuration. The point A $(x^* = 0, y^* = 1)$; ferromagnetic fixed point. The line C; $(x^* = 1,\ 0 \le y^* \le 1)$; Infinity T sink

The recursion relations for x and y do not depend on z. Physically, this means that the singular behavior of the free energy does not depend on a shift in th energy scale. Therefore, we first investigate the flow diagram and fixed points in the (x, y) plane.

(i) There is a line of fixed points at $x^* = 1$ for $0 \le y^* \le 1$, corresponding to an infinite-temperature sink. Along this line, the spins are completely disordered and $\xi \to 0$.

(ii) The fixed point at $(x^* = 0, y^* = 1)$ (this means B = 0 and T = 0) corresponds to the critical point, where $\xi \to \infty$. This is correct because we know that the d = 1 Ising model has $T_c = 0$. This fixed point is unstable and any point in its vicinity will flow under RG transformations away from it towards the fixed points on the line at $x^* = 1$. The critical behavior of the d = 1 Ising model is governed by the least stable fixed point at $(x^* = 0, y^* = 1)$.

(iii) The point at $(x^* = 0, y^* = 0)$ is another fixed point which corresponds to $T = 0$ and $B = \infty$. At this fixed point, the spins are completely ordered.

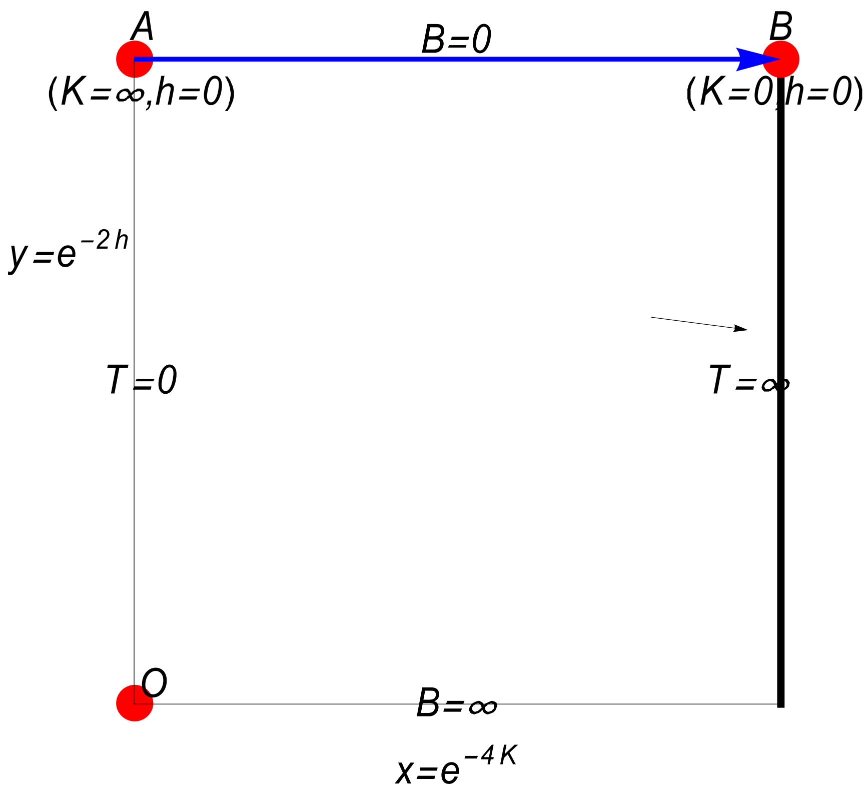


Fig. Fixed points A (K=∞, h=0; T = 0, B = 0), and B (K = 0, h = 0; T = ∞, B = 0)

When h = 0, there are two fixed points A (K=∞, h=0), and B (K = 0, h = 0, as shown above. The point A does not move after the renormalization procedures, showing the unstable fixed point. The point B is a stable fixed point. When the initial state is at a point between the line AB, the point moves toward the point B after the renormalization group processes.

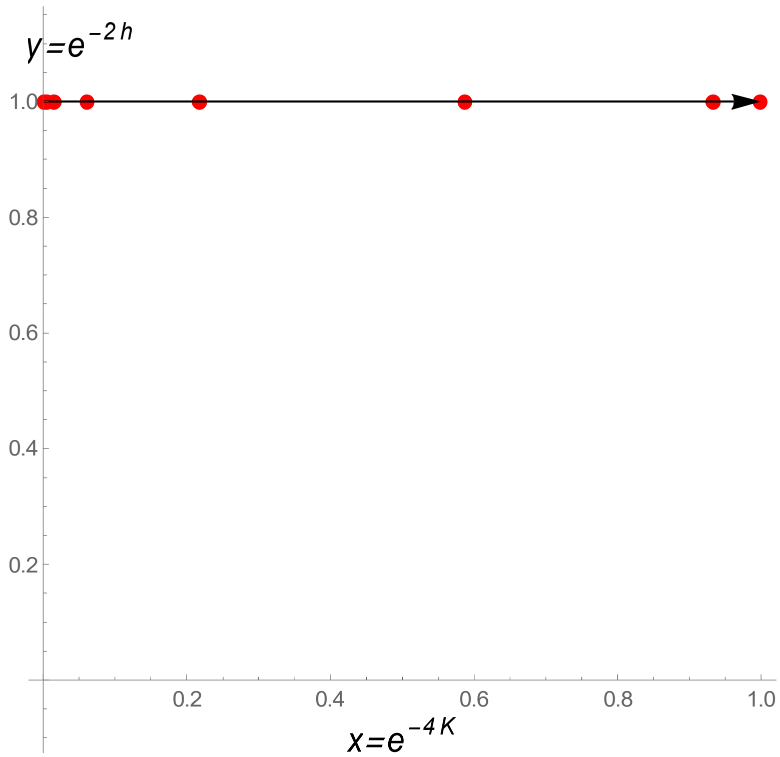


Fig. Flow diagram starting from the point (0.001, 1).

7. Fixed points and flow diagram in the (x, h) plane.

We have the recursion relations in the (x, h) plane,

$$x' = x\frac{2[1+\cosh(2h)]}{x^2+2x\cosh(h)+1}, \tag{1}$$

$$h' = 2h + \frac{1}{2}[\ln(1+xe^{-2h}) - \ln(1+xe^{2h})] \tag{2}$$

(a) Determination of the fixed points.

Using Eqs.(1) and (2), we determine the fixed point in the (x, h) plane. This can be done by using the ContourPlot (Mathematica).

There are two fix points;

(i) $x^* = 0$, $h^* = 0$

(ii) $x^* = 1$, $h^* = 0$

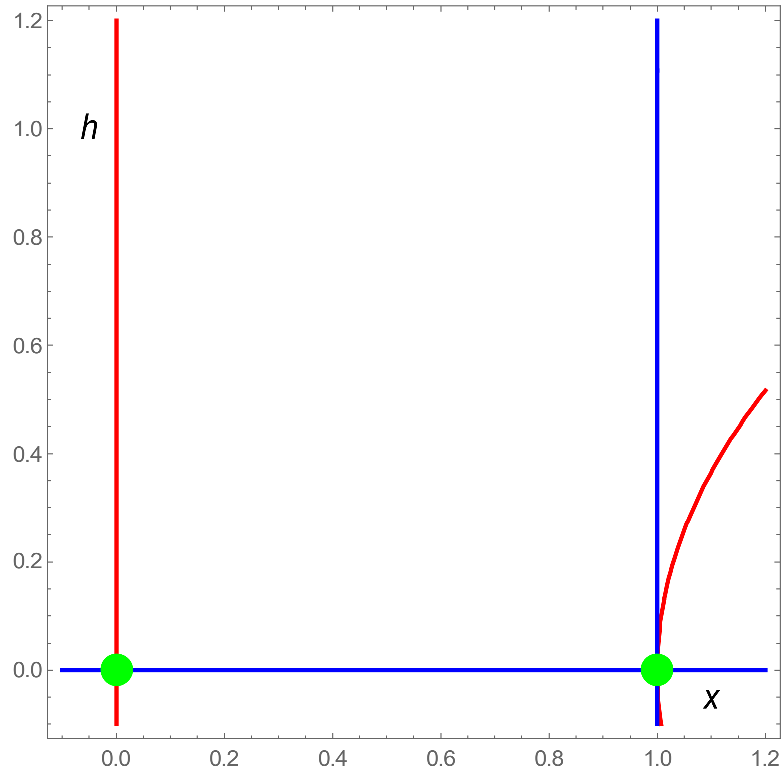


Fig. ContouPlot of Eqs.(1) and (2);Eq.(1) (denoted by red lines) and Eq.(2) (denoted by blue line). The intersection of the two ContourPlots leads to the fixed points; (0,0), and (1,0) (denoted by green points).

(b) The flow diagram in the (x, h) plane

The flow diagram in the (x, h) plane is shown below, where the starting point is at (0.001, 0). The point moves toward the point (1,0).

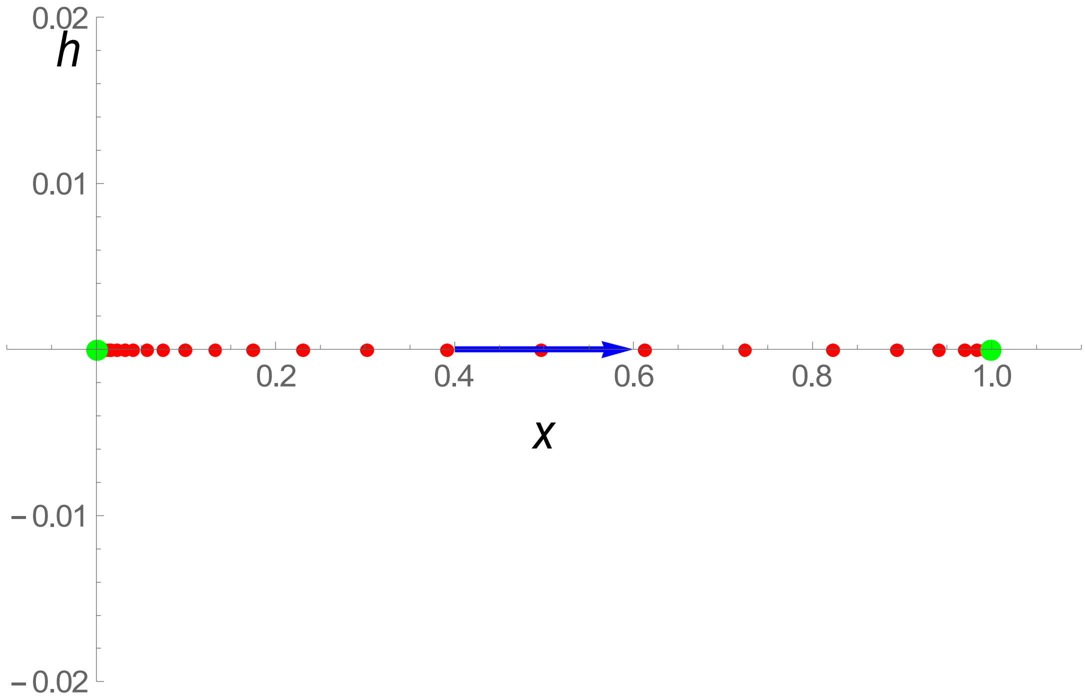


Fig. Flow diagram in the (x, h) plane. The starting point at at (0.001, 0). There are two fixed points at the points (0,0) and (1,0).

8. General properties of eigenvalue problem:

$$\hat{A}|a_1\rangle = a_1|a_1\rangle, \qquad \hat{A}|a_2\rangle = a_2|a_2\rangle$$

$$|a_1\rangle = \hat{U}|1\rangle = \begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix}, \qquad |a_2\rangle = \hat{U}|2\rangle = \begin{pmatrix} U_{12} \\ U_{22} \end{pmatrix}$$

The RG transformation

$$|\psi^{(n)}\rangle = \hat{A}^n|\psi\rangle$$

with

$$\begin{aligned} |\psi\rangle &= \sum_a |a_i\rangle\langle a_i|\psi\rangle \\ &= |a_1\rangle\langle a_1|\psi\rangle + |a_2\rangle\langle a_2|\psi\rangle, \\ &= z_1|a_1\rangle + z_2|a_2\rangle \end{aligned}$$

$$
\begin{aligned}
\left|\psi^{(1)}\right\rangle &= \hat{A}\left|\psi\right\rangle \\
&= \sum_{a}\left|a_i\right\rangle\left\langle a_i\right|\hat{A}\left|\psi\right\rangle \\
&= a_1\left|a_1\right\rangle\left\langle a_1\right|\psi\rangle + a_2\left|a_2\right\rangle\left\langle a_2\right|\psi\rangle \\
&= a_1 z_1\left|a_1\right\rangle + a_2 z_2\left|a_2\right\rangle
\end{aligned}
$$

Similarly,

$$\left|\psi^{(2)}\right\rangle = \hat{A}^2\left|\psi\right\rangle = a_1^{\,2} z_1\left|a_1\right\rangle + a_2^{\,2} z_2\left|a_2\right\rangle$$

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$$\left|\psi^{(n)}\right\rangle = \hat{A}^n\left|\psi\right\rangle = a_1^{\,n} z_1\left|a_1\right\rangle + a_2^{\,n} z_2\left|a_2\right\rangle$$

where $\qquad z_1 = \left\langle a_1\right|\psi\rangle, \quad z_2 = \left\langle a_2\right|\psi\rangle.$

((**Renormalization group**))

$$\left|\psi\right\rangle = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \left|\psi^{(1)}\right\rangle = \begin{pmatrix} x' \\ y' \end{pmatrix}, \qquad \left|\psi^{(2)}\right\rangle = \begin{pmatrix} x'' \\ y'' \end{pmatrix}$$

$$a_1 = \rho_1(b), \qquad a_2 = \rho_2(b)$$

The linearlized RG equations can be written as

$$\left|\psi^{(1)}\right\rangle = \begin{pmatrix} x' \\ y' \end{pmatrix} = z_1\rho_1(b)\begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix} + z_2\rho_2(b)\begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix}$$

If the RG transformation is applied again, we get

$$\left|\psi^{(2)}\right\rangle = \begin{pmatrix} x'' \\ y'' \end{pmatrix} = z_1\rho_1^{\,2}(b)\begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix} + z_2\rho_2^{\,2}(b)\begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix}$$

However, $\left|\psi^{(2)}\right\rangle$ can be obtained from $\left|\psi\right\rangle$ by decimating b^2 and then employing a spatial rescaling by b^2. Then we can write

$$\left|\psi^{(2)}\right\rangle = \begin{pmatrix} x'' \\ y'' \end{pmatrix} = z_1\rho_1(b^2)\begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix} + z_2{\rho_2}^2(b^2)\begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix}$$

The two expressions obtained for $\left|\psi^{(2)}\right\rangle$ show that

$$\rho_1(b^2) = {\rho_1}^2(b), \qquad \rho_2(b^2) = {\rho_2}^2(b)$$

For this relation to be obeyed for arbitrary b, we should have

$$\rho_1(b) = b^{\lambda_i} \qquad (i = 1, 2)$$

Thus the dependence of the eigenvalues on the scaling b is obtained.

9. Linearized recursion relation

We consider the third fixed point describes the $T = 0$ critical point ($x^* = 0$ and $y^* = 1$). Suppose that x is close to 0 and y is close to 1. The recursion relation is given by

$$\begin{aligned} x' &= x\frac{(1+y)^2}{(x+y)(1+xy)} \\ &\approx x\frac{(1+1)^2}{(x+1)(1+x)} \\ &= 4x \end{aligned}$$

and

$$\begin{aligned} h' &= h + \frac{1}{2}\ln\frac{\cosh(2K+h)}{\cosh(2K-h)} \\ &= 2h + \frac{1}{2}[\ln(1+xe^{-2h}) - \ln(1+xe^{2h})] \\ &\approx 2h \end{aligned}$$

Then we have the recursion relation

$$x' = 4x, \qquad h' = 2h$$

The fixed point satisfies the equations given by

$$x^{*} = 0,\ y^{*} = 1\ (h^{*} = 0)$$

From these equations, we get

$$x' = 2^{2} x,$$

$$h' = 2^{1} h$$

10. Solving the eigenvalue problem

We start with the linearized recursion relation,

$$x' = 4x, \qquad h' = 2h \qquad (b = 2).$$

Here we use the temperature and field variables as

$$x = e^{-4K} = t \qquad \text{(temperature variable)}$$

$$h \qquad \text{(reduced magnetic field)}$$

$$\left|\psi^{(1)}\right\rangle = \begin{pmatrix} t' \\ h' \end{pmatrix} = \hat{A}\left|\psi\right\rangle = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} t \\ h \end{pmatrix}$$

where

$$\hat{A} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalue of the matrix A is

$$a_1 = \rho_1(b) = 2^{2}. \qquad a_2 = \rho_2(b) = 2^{1}$$

with

$$\lambda_1 = 2, \qquad \lambda_2 = 1$$

Note that

$$|a_1\rangle = \begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |a_2\rangle = \begin{pmatrix} U_{12} \\ U_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

11. Finite reduced temperature

We assume that t is the reduced temperature deviation from the fixed point A (0, 1). After one renormalization procedure (scaling factor b), the resulting temperature t' is related to the original t by

$$t' = \rho_1(b)t$$

After the l-times repetition of the RG procedure, we have

$$t^{(l)} = \rho_1^{\,l}(b)t$$

which is equivalent to one RG procedure with scale factor b^l such that

$$t^{(l)} = \rho_1(b^l)t$$

Then we have the form of $\rho_1(b)$ as

$$\rho_1(b) = b^{\lambda_1}$$

leading to

$$t^{(l)} = b^{\lambda_1 l}t$$

The correlation length ξ is

$$\xi' = \xi / b$$

$$\xi^{(l)} = \xi(t^{(l)}) = \frac{\xi}{b^l} = b^{-l}\xi(t)$$

Suppose that for the effective temperature $t^{(l)} = 1$, $\xi(t^{(l)}) = \xi(1)$

$$\xi(t) = b^l \xi(1) = (t^{-\frac{1}{\lambda_1 l}})^l = t^{-\frac{1}{\lambda_1}},$$

since

$$t^{(l)} = b^{\lambda_1 l} t = 1, \qquad b = t^{-\frac{1}{\lambda_1 l}}.$$

The critical exponent for the correlation length ξ is

$$\nu = \frac{1}{\lambda_1}.$$

The free energy:

$$f^{(l)} = f(t^{(l)}) = b^{dl} f(t),$$

or

$$f(t) = t^{\frac{d}{\lambda_1}} f(1) = t^{d\nu} f(1).$$

$f(1)$ is the free energy density at the effective temperature. Since $f(t) \approx t^{2-\alpha}$

$$d\nu = 2 - \alpha \qquad \text{(hyper-scaling law)}$$

12. Finite reduced field

After one renormalization procedure (scaling factor b), the resulting field h' is related to the original h by

$$h' = \rho_2(b) h.$$

After the l-times repetition of the RG procedure, we have

$$h^{(l)} = [\rho_2(b)]^l h,$$

which is equivalent to one RG procedure with scale factor b^l such that

$$h^{(l)} = \rho_2(b^l) h.$$

Then we have the form of $\rho_2(b)$ as

$$\rho_2(b) = b^{\lambda_2},$$

leading to

$$h^{(l)} = b^{\lambda_2 l} h.$$

The free energy:

$$f^{(l)} = f(t^{(l)}, h^{(l)}) = b^{dl} f(t, h)$$

or

$$f(t,h) = b^{-dl} f\ (1, b^{\lambda_2 l} h) = b^{-dl} f(1, \frac{h}{t^{\nu\lambda_2}}) = t^{\frac{d}{\lambda_1}} \bar{f}(\frac{h}{t^{\nu\lambda_2}}) = t^{d\nu} \bar{f}(\frac{h}{t^{\nu\lambda_2}})$$

where $f(1,x) = \bar{f}(x)$

Then we have

$$\Delta = \lambda_2 \nu$$

leading to the relation

$$\lambda_2 = \frac{\Delta}{\nu} = \frac{\alpha + \beta}{\nu}$$

13. Scaling for the correlation length

$$\xi(t,h) = b\xi(t', h') = b\xi(b^{\lambda_1} t, b^{\lambda_2} h)$$

We consider the system with $y = 0$. Then ξ is given by

$$\xi(t) = b\xi(t') = b\xi(b^{\lambda_1} t)$$

for arbitrary b. Putting $b^{\lambda_1} t = 1$ (or $b = t^{-1/\lambda_1}$), it is found that

$$\xi(t) = t^{-1/\lambda_1} \xi(1) = x^{-\nu} \xi(1)$$

where ν is the correlation length exponent. Since $x = e^{-4K}$, we have

$$\xi(t) \approx \exp(4K\nu) = \exp(\frac{4J\nu}{k_B T})$$

Thus, for the 1D Ising model, an exponential divergence is obtained, as compared to the usual power law divergence.

14. Scaling law for the free energy per spin

$$f(t,h) = b^{-d} f(t',h') = b^{-d} f(b^{\lambda_1} t, b^{\lambda_2} h)$$

where the term *c*' is omitted. We again choose $b = t^{-1/\lambda_1}$. Thus we have

$$f(t,h) = t^{d/\lambda_1} f(1, \frac{h}{x^{\lambda_2/\lambda_1}})$$

which is of the scaling form if

$$\frac{d}{\lambda_1} = 2 - \alpha = d\nu, \qquad \frac{\lambda_2}{\lambda_1} = \Delta.$$

REFERENCES

I. Vilfan, *Lecture Notes in Statistical Mechanics* (2002).

J.M. Yeomans, *Statistical Mechanics of Phase Transitions* (Oxford, 1992).

F.J.W. Hahne edited, *Critical Phenomena*, M.E. Fisher, Scaling, Universality and Renormalization Group Theory (Springer-Verlag, 1983).

H.S. Robertson, *Statistical Thermodynamics* (P T R Prentice Hall, 1993).

S.V.G. Menon. *Renormalization Group Theory of Critical Phenomena* (New Age International Limited, 1995).

P.M. Chaikin and T.C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge, 1995).

M.E. Fisher, *Rev. Mod. Phys*. 46, 597 (1974).

H.J. Maris and L.P. Kadanoff, *Am. J. Phys*. 46, 652 (1978).