Landau Theory of Phase Transition

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton

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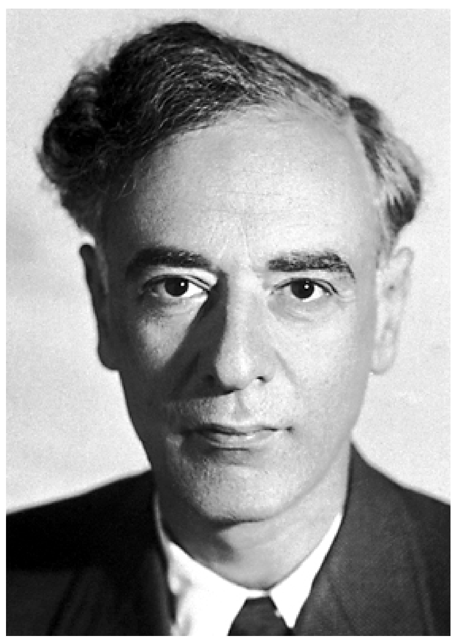
Lev Davidovich Landau (January 22, 1908 - 1 April 1968) was a Soviet physicist who made fundamental contributions to many areas of theoretical physics. His accomplishments include the independent co-discovery of the density matrix method in quantum mechanics (alongside John von Neumann), the quantum mechanical theory of diamagnetism, the theory of superfluidity, the theory of second-order phase transitions, the Ginzburg–Landau theory of superconductivity, the theory of Fermi liquid, the explanation of Landau damping in plasma physics, the Landau pole in quantum electrodynamics, the two-component theory of neutrinos, and Landau's equations for S matrix singularities. He received the 1962 Nobel Prize in Physics for his development of a mathematical theory of superfluidity that accounts for the properties of liquid helium II at a temperature below 2.17K.



https://en.wikipedia.org/wiki/Lev_Landau

The theory of phase transitions of second order is based on the thermodynamic properties of the system for given deviations from the symmetrical state. Landau introduced the concept of the order parameter. The thermodynamic properties can be expressed in terms of the order parameter σ.

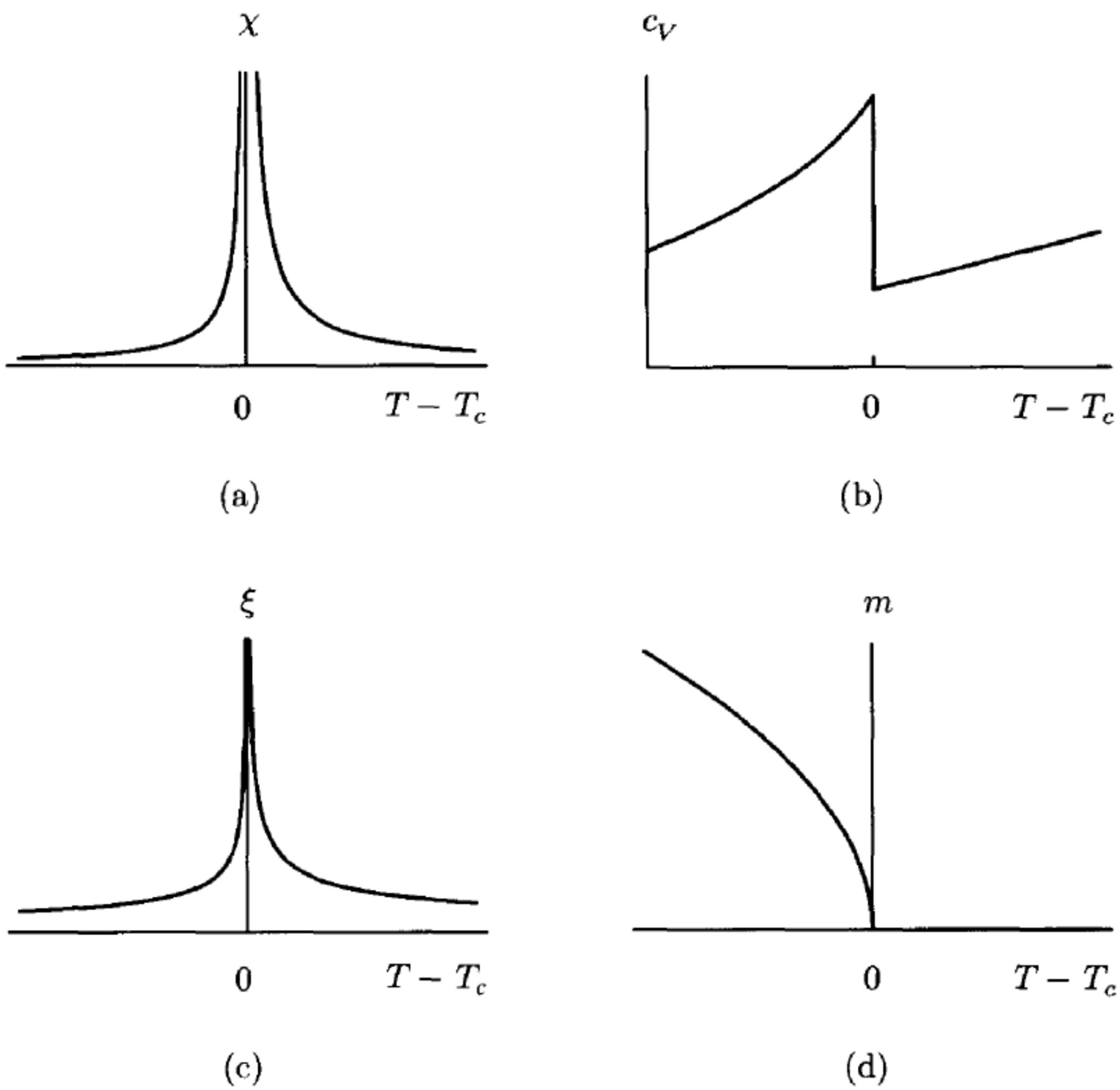


Fig. (a) Mean-field susceptibility, (b) specific heat, (c) correlation length, and (d) order parameter as a function of temperature. The mean field exponent; $\alpha = \alpha' = 0$. $\beta = \frac{1}{2}$. $\gamma = \gamma' = 1$. $\delta = 3$. $\eta = 0$. $\nu = \frac{1}{2}$.

We consider the Helmholtz free energy; $F = F(T,V)$.

$$F = U - ST$$

is a minimum in equilibrium. This free energy is a minimum with respect to an order parameter σ.

((**Example**))

σ Magnetization, dielectric polarization, fraction of superconducting electrons in a superconductor, etc.

In thermal equilibrium, σ will have a certain value $\sigma = \sigma_0(T)$. In the Landau theory, we imagine that σ can be independently specified. We consider the Landau-free energy function

$$F_L(\sigma, T) = U(\sigma, T) - TS(\sigma, T)$$

where

$$F_L(T) = F_L(\sigma_0, T) \leq F_L(\sigma, T)$$

if $\sigma \neq \sigma_0$.

$F_L(\sigma, T)$ is an even function of σ, in the absence of applied fields.

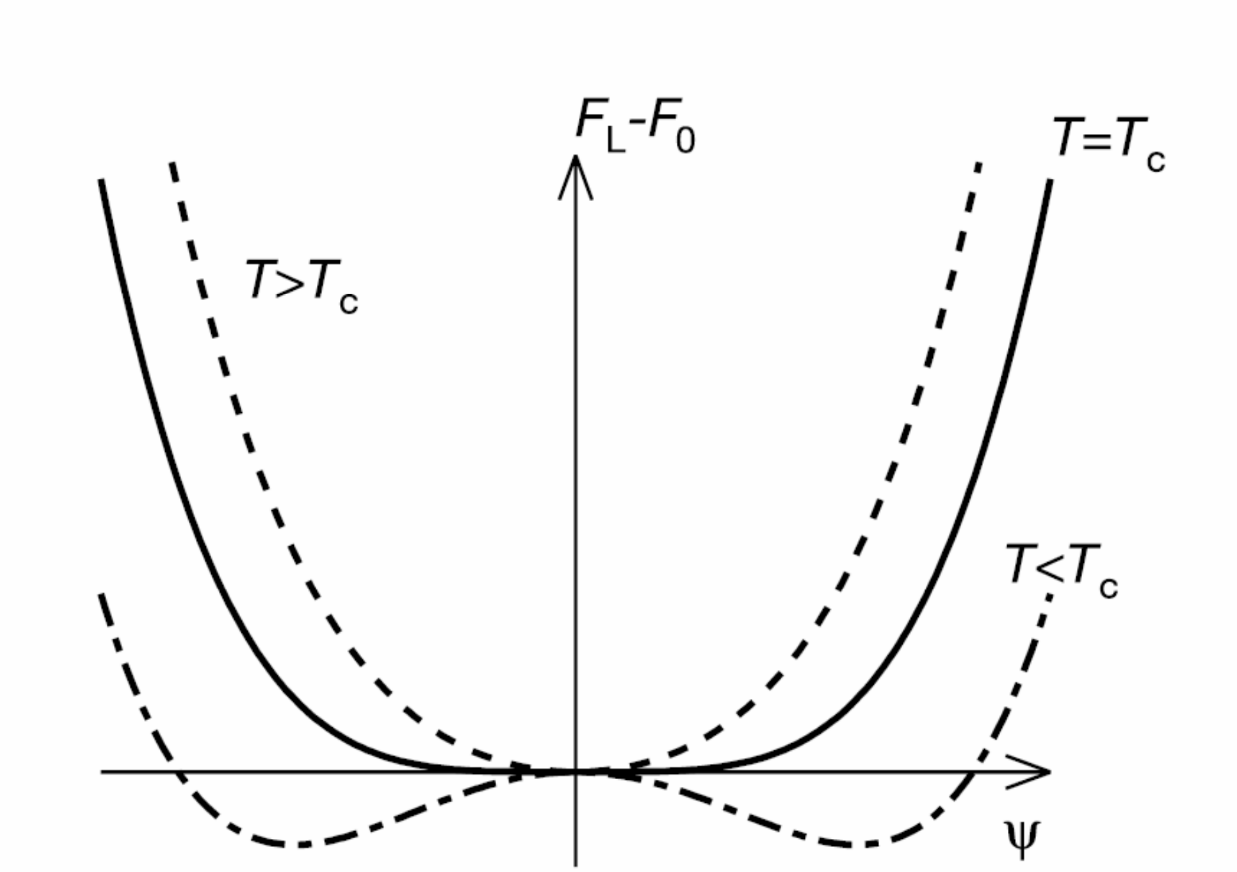


Fig. Plot of the free energy as a function of the order parameter $\psi = \sigma$

((**Assumption**))

$F_L(\sigma)$ is an even function of σ in the absence of applied fields.

$$F_L(\sigma,T) = g_0(T) + \frac{1}{2}g_2(T)\sigma^2 + \frac{1}{4}g_4(T)\sigma^4 + ...$$

For simplicity,

$$g_2(T) = a(T - T_c)$$

$$g_4(T) = \text{const>0}, \qquad g_6(T) = 0.$$

Then we have

$$F_L(\sigma, T) = g_0(T) + \frac{1}{2}a(T - T_c)\sigma^2 + \frac{1}{4}g_4\sigma^4$$

and

$$\frac{\partial F_L}{\partial \sigma} = \alpha(T - T_c)\sigma + g_4\sigma^3 = 0$$

which has the roots

$$\sigma = 0$$

and

$$\sigma^2 = \frac{a}{g_4}(T_c - T)$$

or

$$\sigma = \left(\frac{a}{g_4}\right)^{1/2}(T_c - T)^{1/2} \qquad (\beta = \frac{1}{2})$$

with

$$a > 0 \qquad \text{and} \qquad g_4 > 0$$

where β is the critical exponent of magnetization.

(i) The root $\sigma = 0$ corresponds to the minimum of F_L at T above T_c.

$$F_L(T) = g_0(T).$$

(ii) The other root $\sigma^2 = \frac{a}{g_4}(T_c - T)$ corresponds to the minimum of F_L at T above T_c.

$$F_L(T) = g_0(T) - \frac{a^2}{g_4}(T - T_c)^2.$$

2. Susceptibility

We consider the case when an external magnetic field is present.

$$F_L(\sigma,T) = g_0(T) + \frac{1}{2}a(T-T_c)\sigma^2 + \frac{1}{4}g_4\sigma^4 - B\sigma$$

in the presence of external field B.

$$\frac{\partial F_L(\sigma,T)}{\partial\sigma} = a(T-T_c)\sigma + g_4\sigma^3 - B = 0$$

$$a(T-T_c)\sigma + g_4\sigma^3 = B$$

Taking derivative of $\frac{\partial F_L(\sigma,T)}{\partial\sigma}$ with respect to B;

$$[a(T-T_c) + 3g_4\sigma^2]\frac{\partial\sigma}{\partial B} = 1$$

The susceptibility is given by

$$\chi = \frac{\partial\sigma}{\partial B} = \frac{1}{a(T-T_c) + 3g_4\sigma^2}$$

For $T > T_c$, $\sigma = 0$

$$\chi = \frac{\partial\sigma}{\partial H} = \frac{1}{a(T-T_c)} \qquad (\gamma = 1)$$

where γ is the critical exponent of the susceptibility.

For $T < T_c$

$$\sigma = \left(\frac{a}{g_4}\right)^{1/2}(T_c - T)^{1/2}. \qquad (\beta = \frac{1}{2})$$

$$\chi = \frac{1}{a(T-T_c) + 3a(T_c - T)} = \frac{1}{2a(T_c - T)} \qquad (\gamma' = 1)$$

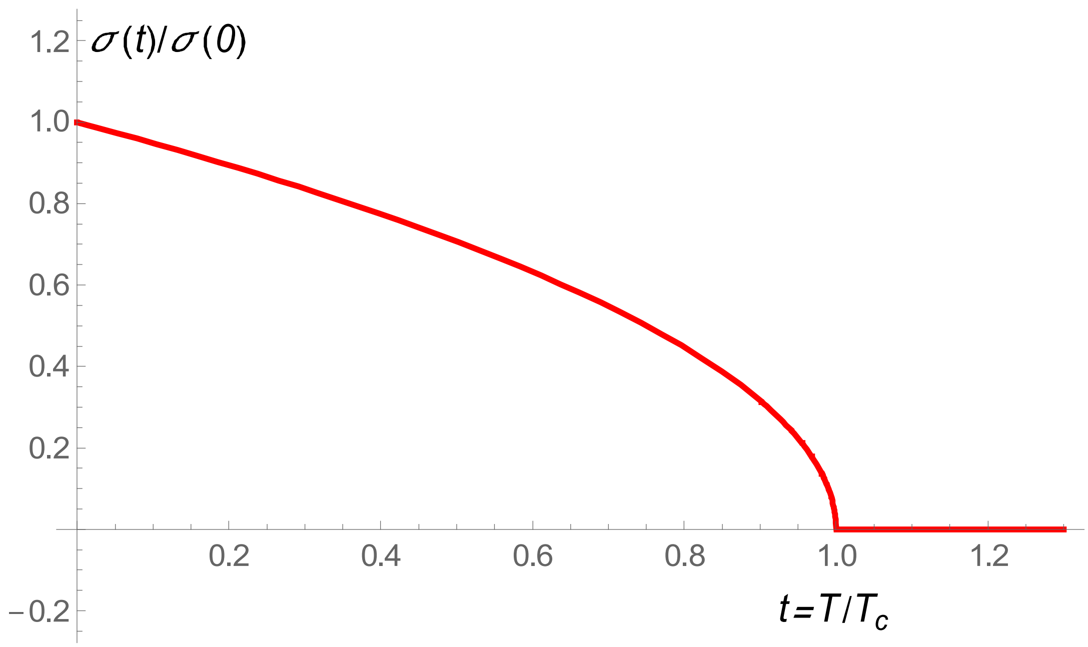


Fig. The order parameter ($\sigma(t)/\sigma_0(0)$ as a function of reduced temperature, $t = T/T_c$. $\sigma(0) = \left(\frac{aT_C}{g_4}\right)^{1/2}$.

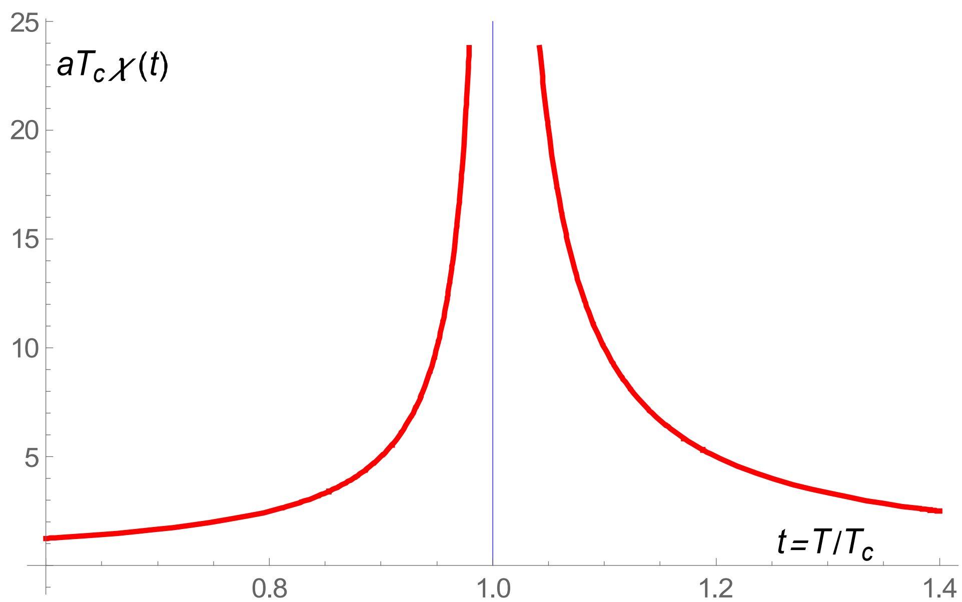


Fig. The reduced susceptibility $aT_c\chi$ as a function of reduced temperature, $t = T/T_c$.

3. Entropy and heat capacity

The entropy S is given by

$$S = -\left(\frac{\partial F}{\partial T}\right) = \begin{cases} S_0(T) & (T > T_c) \\ S_0(T) + \dfrac{a^2}{g_4}(T - T_c) & (T < T_c) \end{cases}$$

where

$$\begin{aligned} S &= -\left(\frac{\partial F}{\partial T}\right) \\ &= -\frac{\partial g_0(T)}{\partial T} - a(T - T_c)\sigma\frac{\partial \sigma}{\partial T} - \frac{1}{2}a\sigma^2 \\ &= S_0(T) - a(T - T_c)\sigma\frac{\partial \sigma}{\partial T} - \frac{1}{2}a\sigma^2 \end{aligned}$$

with

$$S_0(T) = -\frac{\partial g_0(T)}{\partial T}$$

So that there is no latent heat at the transition temperature T_0. Such a transition is called a second order phase transition.

We can evaluate the heat capacity

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V$$

of the two phases for $T > T_c$ and $T < T_c$.

$$C_V = \begin{cases} T\dfrac{dS_0(T)}{dT} & (T > T_c) \\ T\dfrac{dS_0(T)}{dT} + T\dfrac{a^2}{g_4} & (T < T_c) \end{cases}$$

At $T = T_c$, the heat capacity is discontinuous (second order phase transition).

$$\Delta C_V = T_c \frac{a^2}{g_4} \qquad \text{(Discontinuity of } C_V \text{ at } T = T_c\text{.}$$

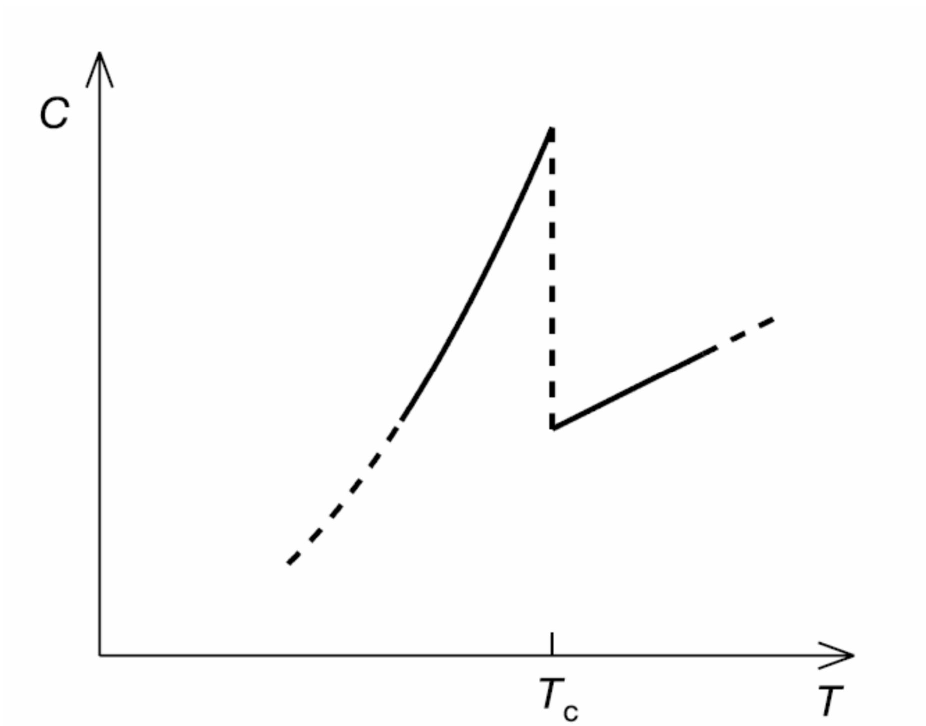


Fig. Schematic plot of the heat capacity as a function of temperature.

4. $M \approx B^{1/\delta}$

At $T = T_c$, we have

$$g_4\sigma^3 = B \qquad \text{or} \qquad \sigma = \left(\frac{B}{g_4}\right)^{1/3}$$

So we have the critical exponent $\delta = 3$.

5. Correlation length

Suppose that the order parameter is not homogenous in space. We add a term proportional to $(\nabla\sigma)^2$. The free energy is modified as

$$F(\sigma, T, \boldsymbol{r}) = F_L(\sigma, T) + \alpha(\nabla\sigma)^2$$
$$= g_0(T) + \frac{1}{2}a(T - T_c)\sigma^2 + \frac{1}{4}g_4\sigma^4 - B\sigma + \alpha(\nabla\sigma)^2$$

where H is dependent on the space. We assume the functional Hamiltonian $K(\sigma)$ which is defined by

$$K(\sigma) = \int d^3\boldsymbol{r}[\frac{1}{2}a(T - T_c)\sigma^2 + \frac{1}{4}g_4\sigma^4 - B\sigma + \frac{1}{2}\alpha(\nabla\sigma)^2]$$

To evaluate a derivative of the functional, we introduce

$$\delta K = K(\sigma + \delta\sigma) - K(\sigma)$$

where $\delta\sigma(\boldsymbol{r})$ is a small perturbation. We expand $K(\sigma + \delta\sigma)$ as

$$K(\sigma + \delta\sigma) = \int d^3\boldsymbol{r}\{\frac{1}{2}a(T - T_c)(\sigma + \delta\sigma)^2 + \frac{1}{4}g_4(\sigma + \delta\sigma)^4 - B(\sigma + \delta\sigma) + \frac{1}{2}\alpha[\nabla(\sigma + \delta\sigma)]^2\}$$

and keep only up to linear terms. Then we obtain

$$\delta K = K(\sigma + \delta\sigma) - K(\sigma) = \int d^3\boldsymbol{r}[a(T - T_c)\sigma\delta\sigma + g_4\sigma^3\delta\sigma - B\delta\sigma + \alpha(\nabla\sigma \cdot \nabla\delta\sigma)]$$

Taking the integral (the last term) by part, requiring that $\delta\sigma = 0$ at the surface and minimizing the total free energy

$$\delta K = \int d^3\boldsymbol{r}[a(T - T_c)\sigma + g_4\sigma^3 - B - \alpha\nabla^2\sigma]\delta\sigma(\boldsymbol{r}) = 0$$

leading to the equation for σ

$$-\alpha\nabla^2\sigma + a(T - T_c)\sigma + g_4\sigma^3 - B = 0.$$

((**Note**))

$$\int d^3\boldsymbol{r}[\nabla\sigma\cdot\nabla\delta\sigma)] = \iiint dxdydz(\frac{\partial\sigma}{\partial x}\frac{\partial\delta\sigma}{\partial x}+\frac{\partial\sigma}{\partial y}\frac{\partial\delta\sigma}{\partial y}+\frac{\partial\sigma}{\partial z}\frac{\partial\delta\sigma}{\partial z})$$

$$\int dx\frac{\partial\sigma}{\partial x}\frac{\partial\delta\sigma}{\partial x} = \delta\sigma\frac{\partial\sigma}{\partial x}|_{-\infty}^{\infty} - \int dx\frac{\partial^2\sigma}{\partial x^2}\delta\sigma = -\int dx\frac{\partial^2\sigma}{\partial x^2}\delta\sigma$$

$$\int dy\frac{\partial\sigma}{\partial y}\frac{\partial\delta\sigma}{\partial y} = \delta\sigma\frac{\partial\sigma}{\partial y}|_{-\infty}^{\infty} - \int dy\frac{\partial^2\sigma}{\partial y^2}\delta\sigma = -\int dx\frac{\partial^2\sigma}{\partial y^2}\delta\sigma$$

$$\int dz\frac{\partial\sigma}{\partial z}\frac{\partial\delta\sigma}{\partial z} = \delta\sigma\frac{\partial\sigma}{\partial z}|_{-\infty}^{\infty} - \int dz\frac{\partial^2\sigma}{\partial z^2}\delta\sigma = -\int dz\frac{\partial^2\sigma}{\partial z^2}\delta\sigma$$

Thus we get

$$\iiint dxdydz(\frac{\partial\sigma}{\partial x}\frac{\partial\delta\sigma}{\partial x}+\frac{\partial\sigma}{\partial y}\frac{\partial\delta\sigma}{\partial y}+\frac{\partial\sigma}{\partial z}\frac{\partial\delta\sigma}{\partial z}) = -\iiint dxdydz(\frac{\partial^2\sigma}{\partial x^2}+\frac{\partial^2\sigma}{\partial y^2}+\frac{\partial^2\sigma}{\partial z^2})\delta\sigma$$

((**Green function method**))

We solve the differential equation

$$-\alpha\nabla^2\sigma + a(T-T_c)\sigma + g_4\sigma^3 - B = 0$$

using the Green function method. Suppose that $\sigma = \sigma_0 + \phi$, where ϕ is the deviation of from each equilibrium value σ_0.

$$a(T-T_c)\sigma_0 + g_4{\sigma_0}^3 = 0$$

and

$$-\alpha\nabla^2(\sigma_0+\phi) + a(T-T_c)(\sigma_0+\phi) + g_4{\sigma_0}^3 - B = 0$$

or

$$-\alpha\nabla^2\phi + a(T-T_c)\phi + a(T-T_c)\sigma_0 + g_4\sigma_0{}^3 - B = 0$$

where σ_0 depends on T,

$$\sigma_0 = 0 \qquad\qquad \text{for } T > T_c$$

$$a(T-T_c) + g_4\sigma_0{}^2 = 0 \qquad\qquad T < T_c.$$

(a) For $T > T_c$

$$-\alpha\nabla^2\phi + a(T-T_c)\phi - B = 0$$

or

$$[\nabla^2 - \frac{a(T-T_c)}{\alpha}]\phi = -\frac{B}{\alpha}$$

(b) For $T < T_c$

$$-\alpha\nabla^2\phi - 2a(T-T_c)\phi - B = 0$$

or

$$[\nabla^2 - \frac{2a(T_c-T)}{\alpha}]\phi = -\frac{B}{\alpha}$$

Here we define the Green function such that

$$(\nabla^2 - \kappa^2)G(\boldsymbol{r}-\boldsymbol{r}') = \delta(\boldsymbol{r}-\boldsymbol{r}') \qquad\qquad \text{(Modified Helmholtz)}$$

with

$$G(\boldsymbol{r}-\boldsymbol{r}') = -\frac{\exp[-\kappa|\boldsymbol{r}-\boldsymbol{r}'|}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|},$$

where $\kappa = \frac{1}{\xi}$ is the inverse correlation length and ξ is the correlation length

$$\xi = \begin{cases} \sqrt{\dfrac{\alpha}{a(T - T_c)}} & T > T_c \\ \sqrt{\dfrac{\alpha}{2a(T_c - T)}} & T < T_c \end{cases}$$

(The critical exponent for the correlation, $\nu = \nu' = \dfrac{1}{2}$).

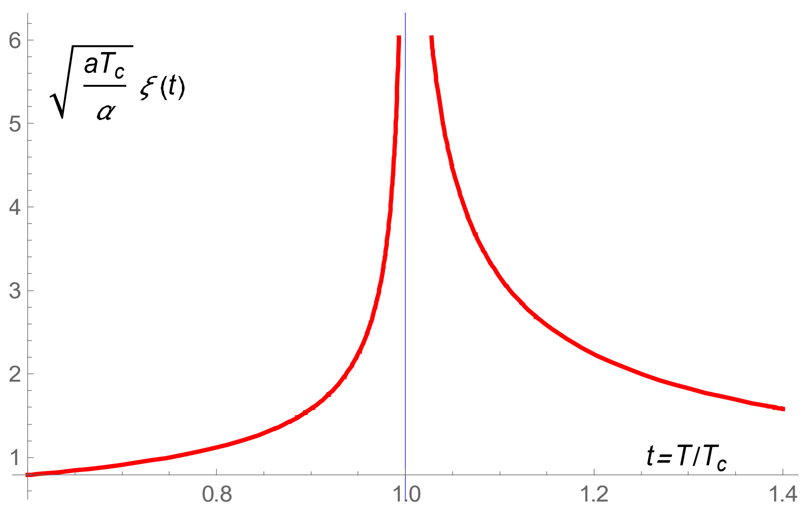


Fig. The reduced correlation length $\sqrt{aT_c/\alpha}\,\xi(t)$ as a function of reduced temperature, $t = T/T_c$.

The solution of

$$(\nabla^2 - \kappa^2)\phi(\boldsymbol{r}) = -\frac{B(\boldsymbol{r})}{\alpha}$$

is given by

$$\phi(\boldsymbol{r}) = -\int d^3\boldsymbol{r}' G(\boldsymbol{r}-\boldsymbol{r}')\frac{B(\boldsymbol{r}')}{\alpha}$$
$$= \int d^3\boldsymbol{r}' \frac{\exp[-\kappa|\boldsymbol{r}-\boldsymbol{r}'|}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\frac{B(\boldsymbol{r}')}{\alpha}$$
$$= \int d^3\boldsymbol{r}' \frac{\exp[-\kappa|\boldsymbol{r}-\boldsymbol{r}'|}{4\pi\alpha|\boldsymbol{r}-\boldsymbol{r}'|}B(\boldsymbol{r}')$$

When $B(\boldsymbol{r}')$ is a highly localized magnetic field at the origin,

$$B(\boldsymbol{r}') = B_0\delta(\boldsymbol{r}'-0)$$

we have

$$\phi(\boldsymbol{r}) = \int d^3\boldsymbol{r}' \frac{\exp[-\kappa|\boldsymbol{r}-\boldsymbol{r}'|}{4\pi\alpha|\boldsymbol{r}-\boldsymbol{r}'|}B_0\delta(\boldsymbol{r}') = \frac{\exp(-\kappa r)}{4\pi\alpha r}B_0$$

In general, $\phi(\boldsymbol{r})$ has the form of

$$\phi(r) \approx \frac{1}{r^{d-2+\eta}}$$

Note that $\gamma = 1$ and $\nu = \frac{1}{2}$. Using the scaling relation $(2-\eta)\nu = \gamma$, we have

$$\eta = 0 \qquad \text{(critical exponent)}$$

where η is the critical exponent for the pair correlation function. Then we have $d-2+\eta = 1$ and

$$\phi(r) \approx \frac{1}{r}$$

for $d = 3$, which is in agreement from the prediction from the mean field theory.

6. Fluctuation

We define the partition function as

$$Z = \int \delta\sigma \exp[-\beta K(\sigma)]$$

where

$$K(\sigma) = \int d^3\boldsymbol{r}[\frac{1}{2}a(T - T_c)\sigma^2 + \frac{1}{4}g_4\sigma^4 - H\sigma + \alpha(\nabla\sigma)^2]$$

Thus we get

$$\frac{\partial Z}{\partial H} = \int \delta\sigma(\beta M)\exp[-\beta K(\sigma)] = \beta\langle M\rangle Z$$

or

$$\langle M\rangle = \frac{1}{\beta}\frac{1}{Z}\frac{\partial Z}{\partial B} = \frac{1}{\beta}\frac{\partial \ln Z}{\partial B}$$

where

$$M = \int d^3\boldsymbol{r}\sigma$$

$$\langle M\rangle = \frac{1}{Z}\int \delta\sigma\, M \exp[-\beta K(\sigma)]$$

$$\frac{\partial^2 Z}{\partial H^2} = Z" = \int \delta\sigma(\beta M)^2 \exp[-\beta K(\sigma)] = \beta^2\langle M^2\rangle Z$$

The fluctuation is given by

$$\begin{aligned}
(\Delta M)^2 &= \langle M^2\rangle - \langle M\rangle^2 \\
&= \frac{1}{\beta^2}[\frac{Z"Z - (Z')^2}{Z^2}] \\
&= \frac{1}{\beta^2}(\frac{Z'}{Z})' \\
&= \frac{1}{\beta^2}\frac{\partial^2}{\partial B^2}\ln Z
\end{aligned}$$

The susceptibility χ is defined by

$$\chi = \frac{\partial \langle M \rangle}{\partial B} = \frac{1}{\beta} \frac{\partial^2 \ln Z}{\partial B^2}$$

Then we have

$$(\Delta M)^2 = \frac{1}{\beta}\chi = k_B T \chi$$

$$\begin{aligned}
(\Delta M)^2 &= \langle M^2 \rangle - \langle M \rangle^2 \\
&= \frac{1}{\beta^2}[\frac{Z''Z - (Z')^2}{Z^2}] \\
&= \frac{1}{\beta^2}(\frac{Z'}{Z})' \\
&= \frac{1}{\beta^2}\frac{\partial^2}{\partial B^2}\ln Z
\end{aligned}$$

7. Spin correlation

We now consider the system of spin 1/2 magnetic atoms. Each atom has magnetic moment $\mu_B \sigma_i$ ($\sigma_i = \pm 1$). The magnetization is given by

$$M = \mu_B \sum_i \sigma_i$$

The fluctuation of M is obtained as

$$\begin{aligned}
(\Delta M)^2 &= \mu_B{}^2 \left\langle (\sum_i \sigma_i - \langle \sigma \rangle)^2 \right\rangle \\
&= \mu_B{}^2 \sum_i \sum_j < (\sigma_i - \langle \sigma \rangle >< (\sigma_j - \langle \sigma \rangle >
\end{aligned}$$

where $< (\sigma_i - \langle \sigma \rangle >< (\sigma_j - \langle \sigma \rangle >$ is a spin correlation function. Since $(\Delta M)^2 = \frac{1}{\beta}\chi = k_B T \chi$, the susceptibility is related to the spin correlation function as

$$\chi = \frac{\mu_B{}^2}{k_B T} \sum_i \sum_j < (\sigma_i - \langle \sigma \rangle >< (\sigma_j - \langle \sigma \rangle >$$

8. Universality and critical exponents

Critical exponents describe the behavior of physical quantities near continuous phase transitions. They are universal, i.e. they do not depend on the details of the physical system, but only on (i) • the dimension of the system, (ii) the degree of freedom such as Ising (n = 1), *XY* (n = 2) and Heisenberg (n = 3) (for spin system), and (iii) the range of the interaction.

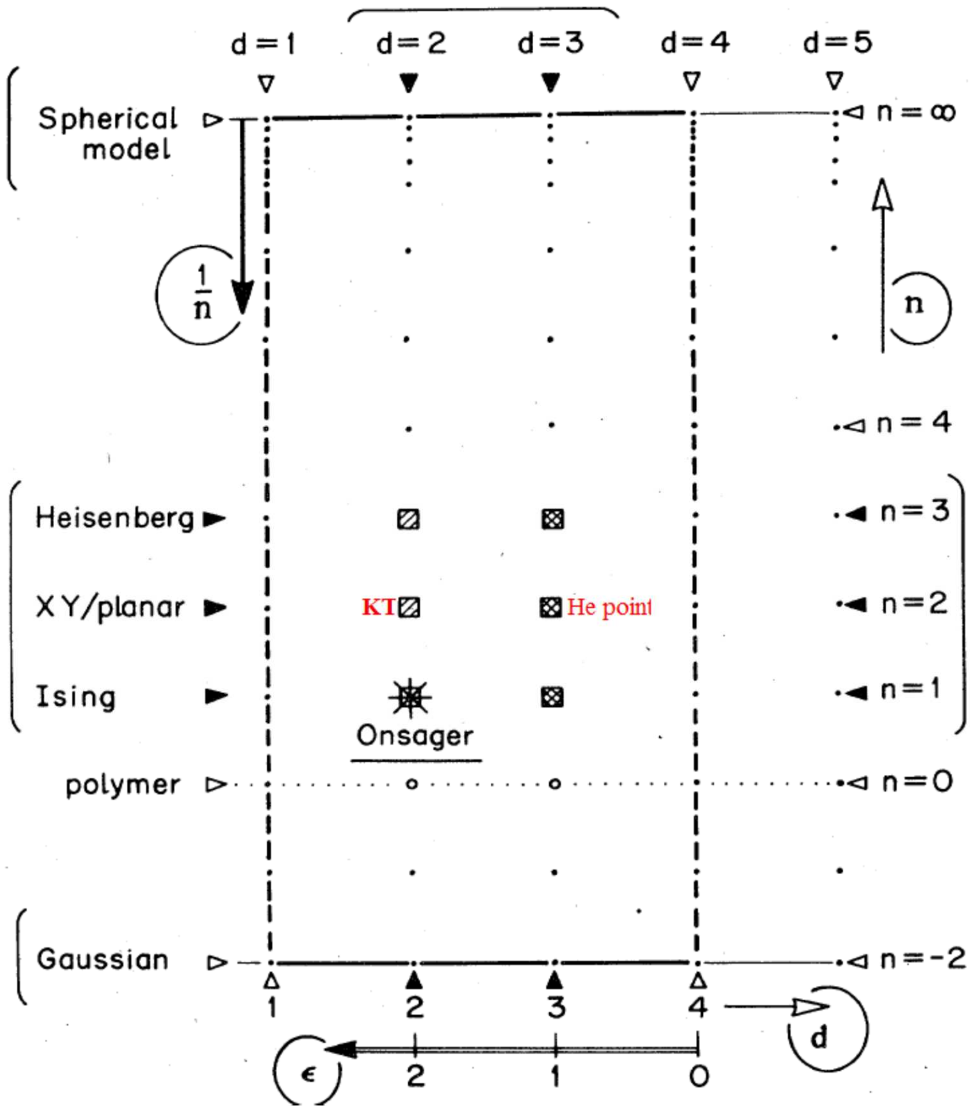


Fig. Universality class of the critical behavior, which depends on the dimension (d) and the number of freedom (n). The Onsager point(d = 2, n = 1). The Kosterlitz-Thouless (KT point) (d = 2, n = 2). The liquid ^{4}He point (d = 3, n = 2).
M.E. Fisher, Rev. Mod. Phys. 46, 597 (1974).

Magnetizion:

$$M \approx (T_c - T)^{\beta} \qquad (T < T_c)$$

Susceptibility:

$$\chi \approx (T - T_c)^{-\gamma} \qquad (T > T_c)$$

$$\chi \approx (T_c - T)^{-\gamma'} \qquad (T < T_c)$$

Heat capacity:

$$C \approx (T - T_c)^{-\alpha} \qquad (T > T_c)$$

$$C \approx (T_c - T)^{-\alpha'} \qquad (T < T_c)$$

Correlation length:

$$\xi \approx (T - T_c)^{-\nu} \qquad (T > T_c)$$

$$\xi \approx (T_c - T)^{-\nu'} \qquad (T < T_c)$$

Critical isotherm:

$$M \approx H^{1/\delta} \qquad (T = T_c)$$

((**Note**)) Mean field critical exponent

$$\alpha = \alpha' = 0, \qquad \beta = \frac{1}{2}, \qquad \gamma = \gamma' = 1, \qquad \delta = 3$$

$$\eta = 0, \qquad \nu = \frac{1}{2}$$

9. Scaling relation

There are the following scaling relations for critical exponents.

$$\alpha + 2\beta + \gamma = 2 \qquad \text{(Rushbrooke identity)}$$

$$(2-\alpha) = d\nu$$ (Josephson identity)

$$(2-\eta)\nu = \gamma$$ (Fisher identity)

$$\beta = \frac{1}{2}\nu(d-2+\eta)$$

$$\delta = \frac{d+2-\eta}{d-2+\eta}$$

$$\nu = \nu'$$

$$\alpha = \alpha'$$

$$\alpha + \beta(\delta+1) = 2$$ (Griffiths identity)

$$\gamma = \beta(\delta-1)$$ (Widom identity)

where d is the dimension of the system.

10. Universality class

http://www.sklogwiki.org/SklogWiki/index.php/Universality_classes

dimension	α	β	γ	δ	ν	η	class
							3-state Potts
							Ashkin-Teller
							Chiral
							Directed percolation
2	0	$1/8$	$7/4$		1	1/4	2D Ising
3	0.1096(5)	0.32653(10)	1.2373(2)	4.7893(8)	0.63012(16)	0.03639(15)	3D Ising
							Local linear interface
	0	$1/2$	1				Mean-field
							Molecular beam epitaxy
							Random-field
3	−0.0146(8)	0.3485(2)	1.3177(5)	4.780(2)	0.67155(27)	0.0380(4)	XY

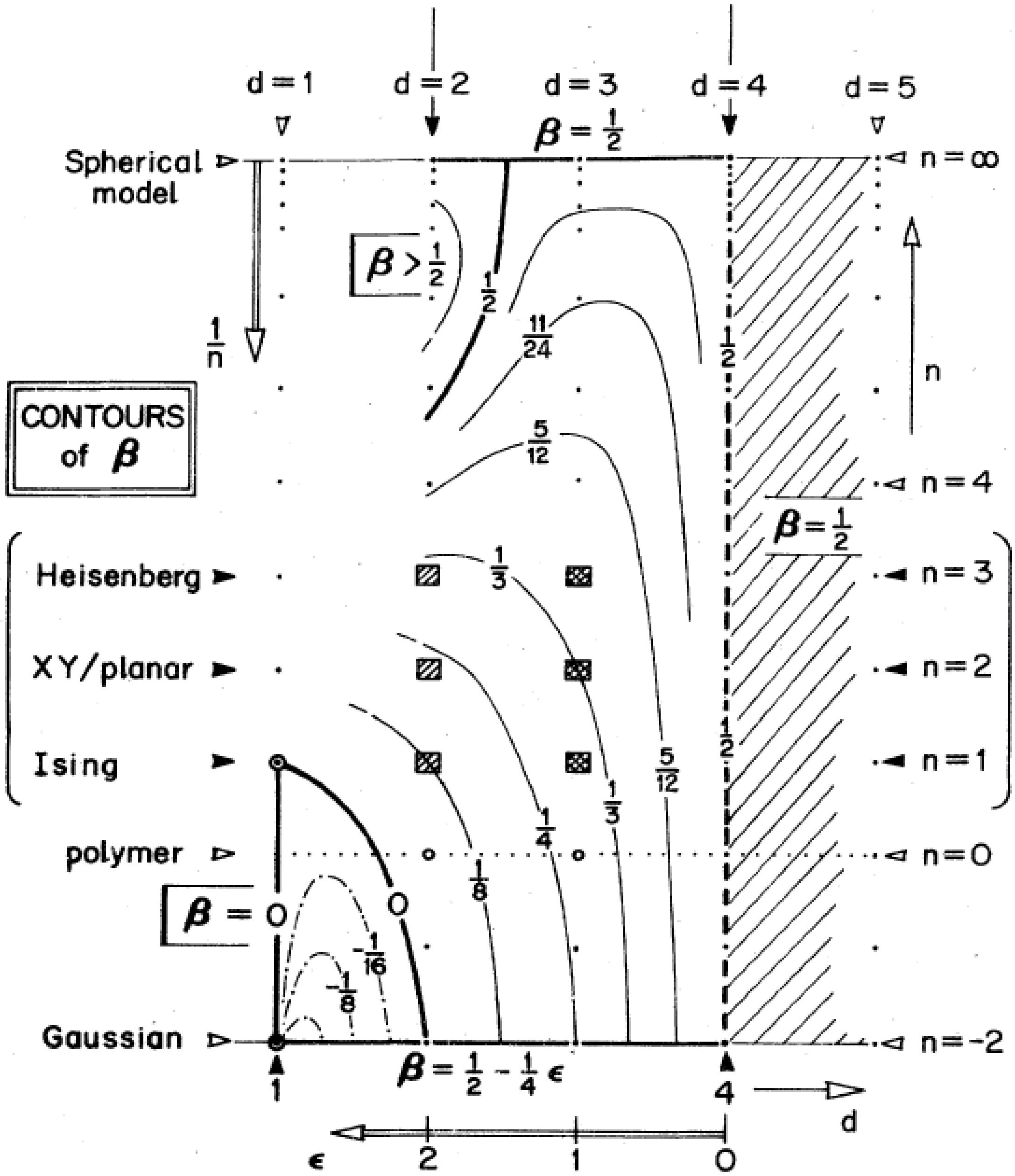


Fig. Diagram of the (d, n) plane showing contours of constant β. There is a region where $\beta > 1/2$, and a nonphysical region of negative β.
M.E. Fisher, Rev. Mod. Phys. 46, 597 (1974).

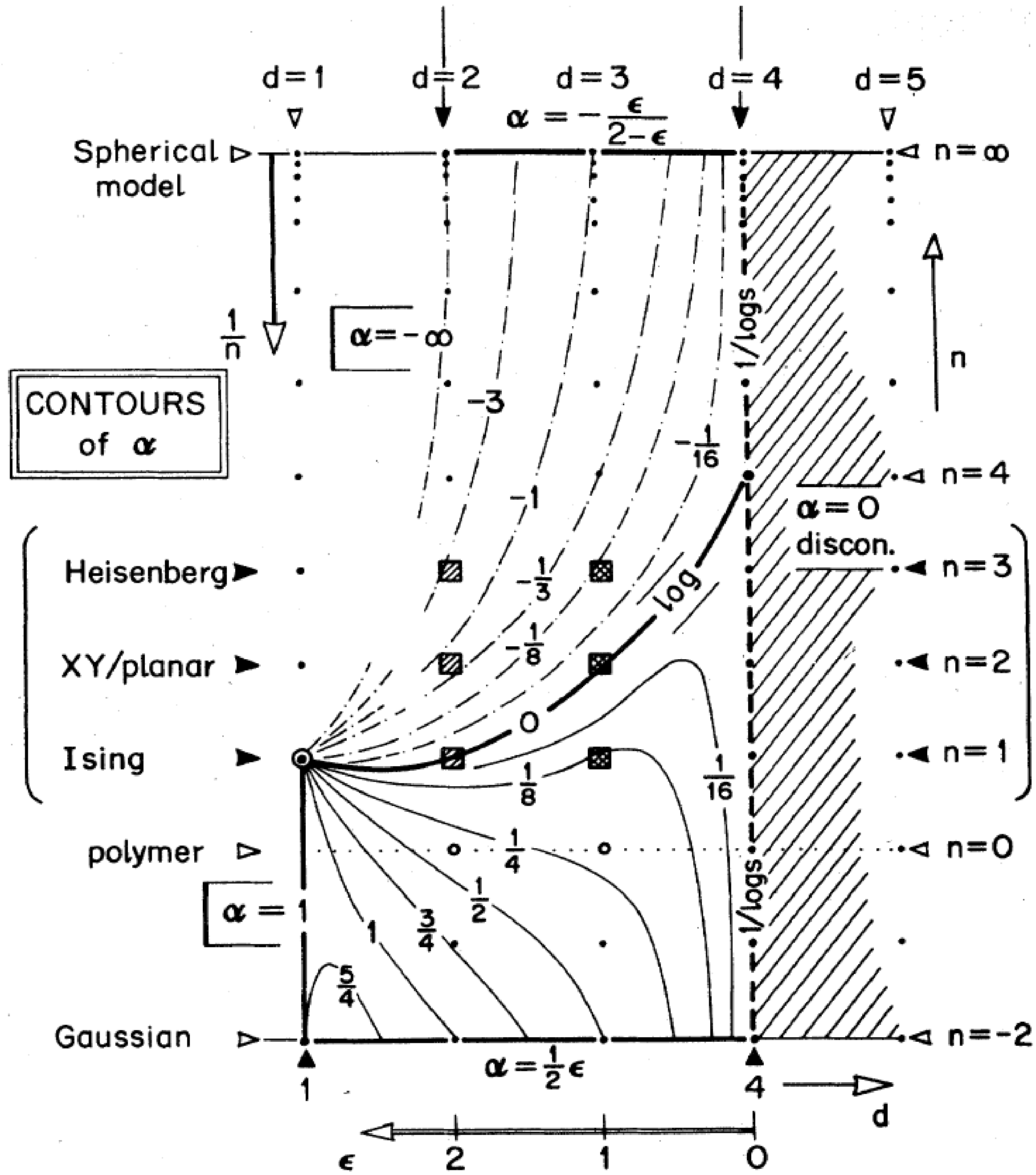


Fig. Diagram showing contours of constant exponent a in the (*d*, *n*) plane. The dash-dot contours indicate negative a; the solid contours are for $\alpha \geq 0$.
M.E. Fisher, Rev. Mod. Phys. 46, 597 (1974).

11. Example of the first order phase transition Huang 19-3

The nematic liquid crystal used in display can be described by an order parameter *S* corresponding to the degree of alignment of molecular directions. In the ordinary fluid phase $(S=0)$. The transition between the ordinary fluid phase and a nematic phase can be modeled via the mean-field Landau free energy

$$E(S)=atS^2+bS^3+cS^4,$$

where $t=T-T_0$, and a, b, c are positive constants. The third-coefficient b is usually small.

(a) Sketch $E(S)$ for range of T, from $T>>T_0$, through $T=T_0$ to $T<T_0$ Comment on the value of S at the minimum of $E(S)$ at each value of T considered.

(b) What are the conditions for $E(S)$ to be minimum?

(c) Find the transition temperarure T_c. (It is not T_0)

(d) Is there a latent heat associated with the phase transition? If so, what is it?

(e) How does the order parameter vary below T_c?

((**Solution**))

$$E(S)=atS^2+bS^3+cS^4$$

$$\frac{\partial E}{\partial S}=2atS+3bS^2+4cS^3=S(4cS^2+3bS+2at)=0$$

where $a>0$, $b>0$, $c>0$.

We make a plot of the free energy $E(S)$ with $a = 2$, $b = 0.8$, and $c = 0.2$

For $t<0$,

$E(S)$ has two local minimum at $S = \alpha$ (<0) and $S = \beta$ (>0).

$$E(\alpha) - E(\beta) = -\frac{b(9b^2 - 32act)}{256c^3}$$

with

$$\alpha = \frac{-3b - \sqrt{9b^2 - 32act}}{8c}$$

and

$$\beta = \frac{-3b - \sqrt{9b^2 - 32act}}{8c}.$$

Note that $E(\alpha) < E(\beta) < 0$

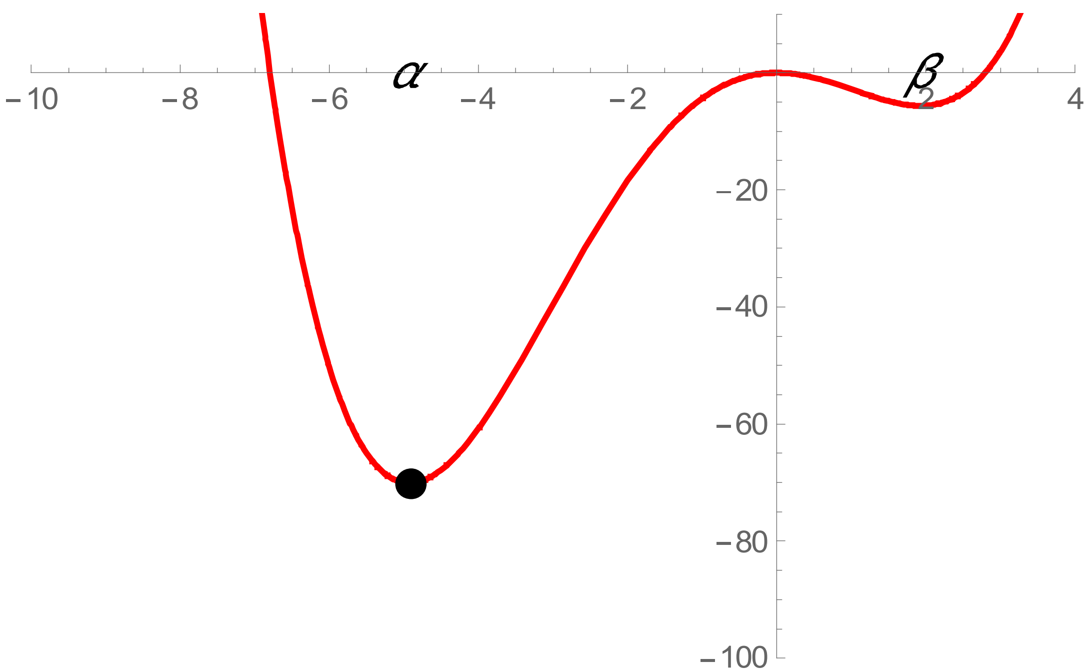
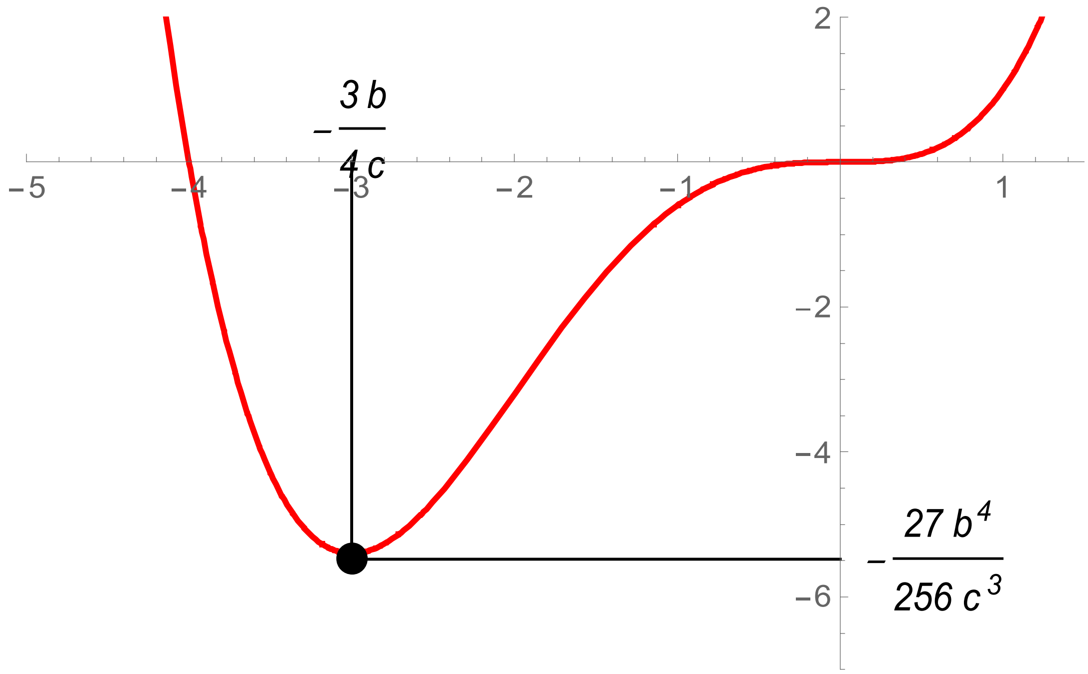


Fig. t = -1.9 (red).

When $t = 0$, there is one local minimum at $S = \alpha = -\frac{3b}{4c.}$, where $E(\alpha) = -\frac{27b^4}{256c^3}$



When $t = \frac{b^2}{4ac} = T - T_0$ or $T_c = T_0 + \frac{b^2}{4ac}$

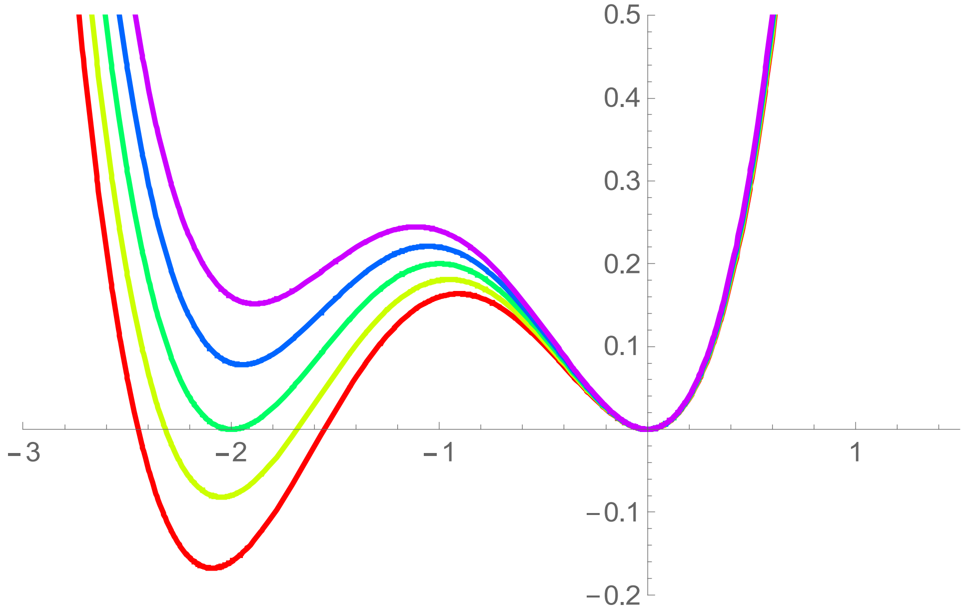
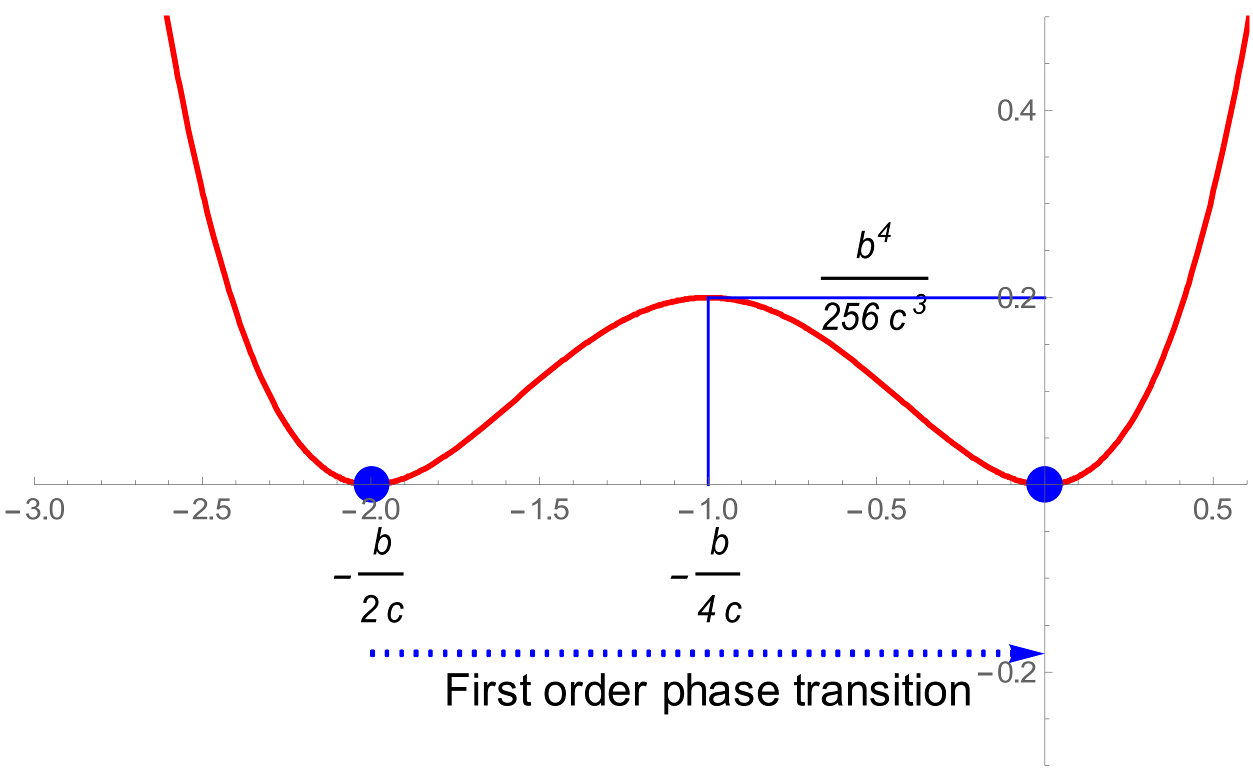


Fig. t = 0.38 (red), 0.39, 0.40 (green), 0.41, and 0.42 (purple). The green line at $t = \dfrac{b^2}{4ac}$ (in the present case t = 0.4), where $S = -\dfrac{b}{2c}$(=-2).



Entropy: $\tilde{S} = -\frac{\partial E(S)}{\partial T} = -aS^2$

Note that S is no an entropy. The entropy change:

$$\Delta\tilde{S} = a\left(\frac{b}{2c}\right)^2$$

The latent heat:

$$L = T\Delta\tilde{S} = (T_0 + \frac{b^2}{4ac})a\left(\frac{b}{2c}\right)^2$$

For $\frac{b^2}{4ac} < t < \frac{9b^2}{8ac}$

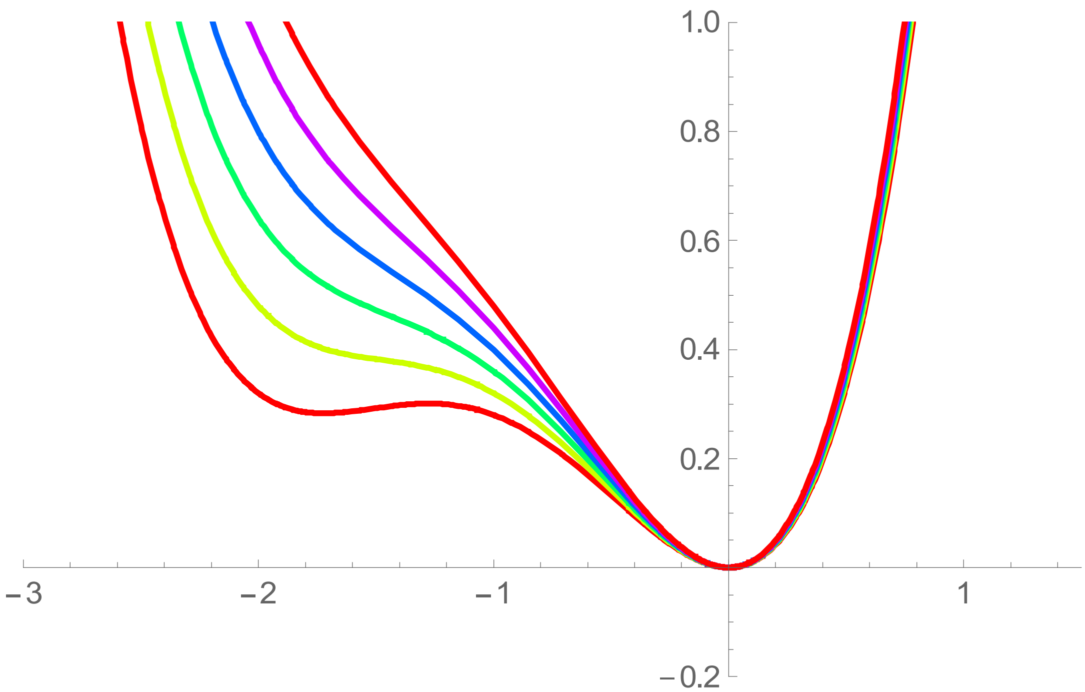


Fig. t = 0.44 (red), 0.46, 0.48 (green), 0.50, and 0.52 (purple), 0.54

When $t > \frac{9b^2}{8ac}$

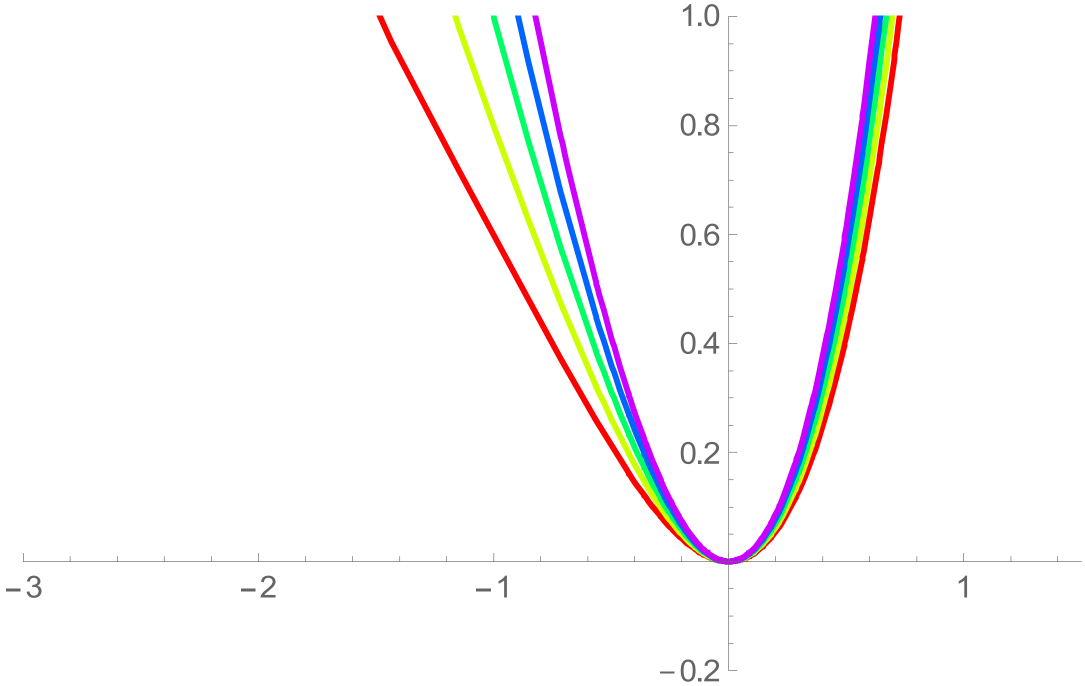


Fig. t = 0.6 (red), 0.7, 0.8 (green), 0.9, and 1 (purple).

What happens to the order parameter well below T_c?

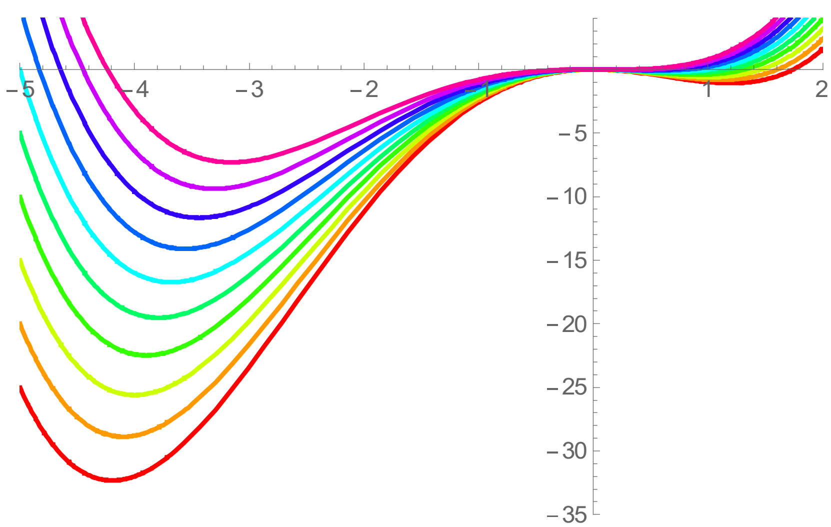
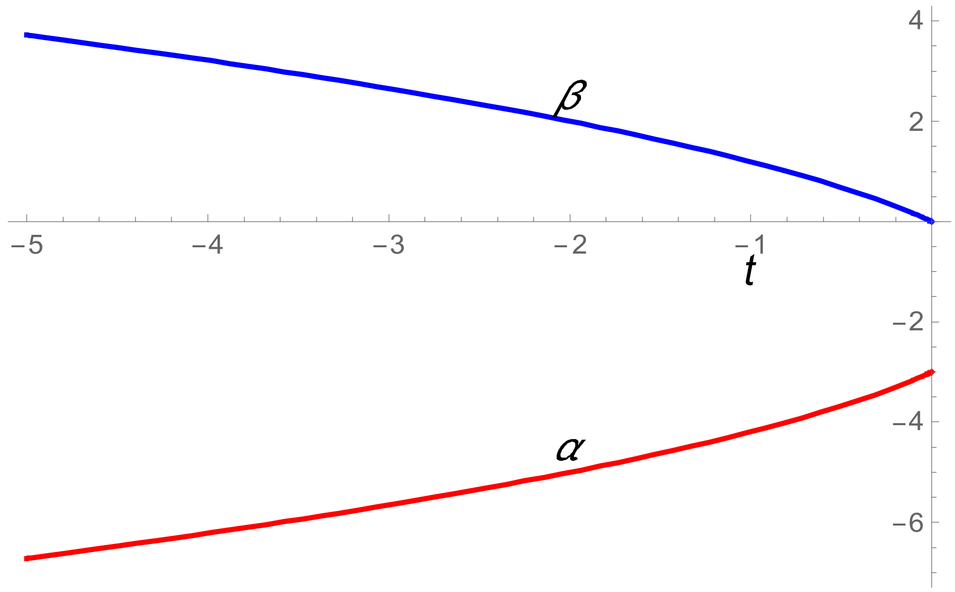


Fig. t = -1 (red), -0.9, -0.8, -0.7, -0.6, -0.5, -0.4, -0.3, -0.2, -0.1.



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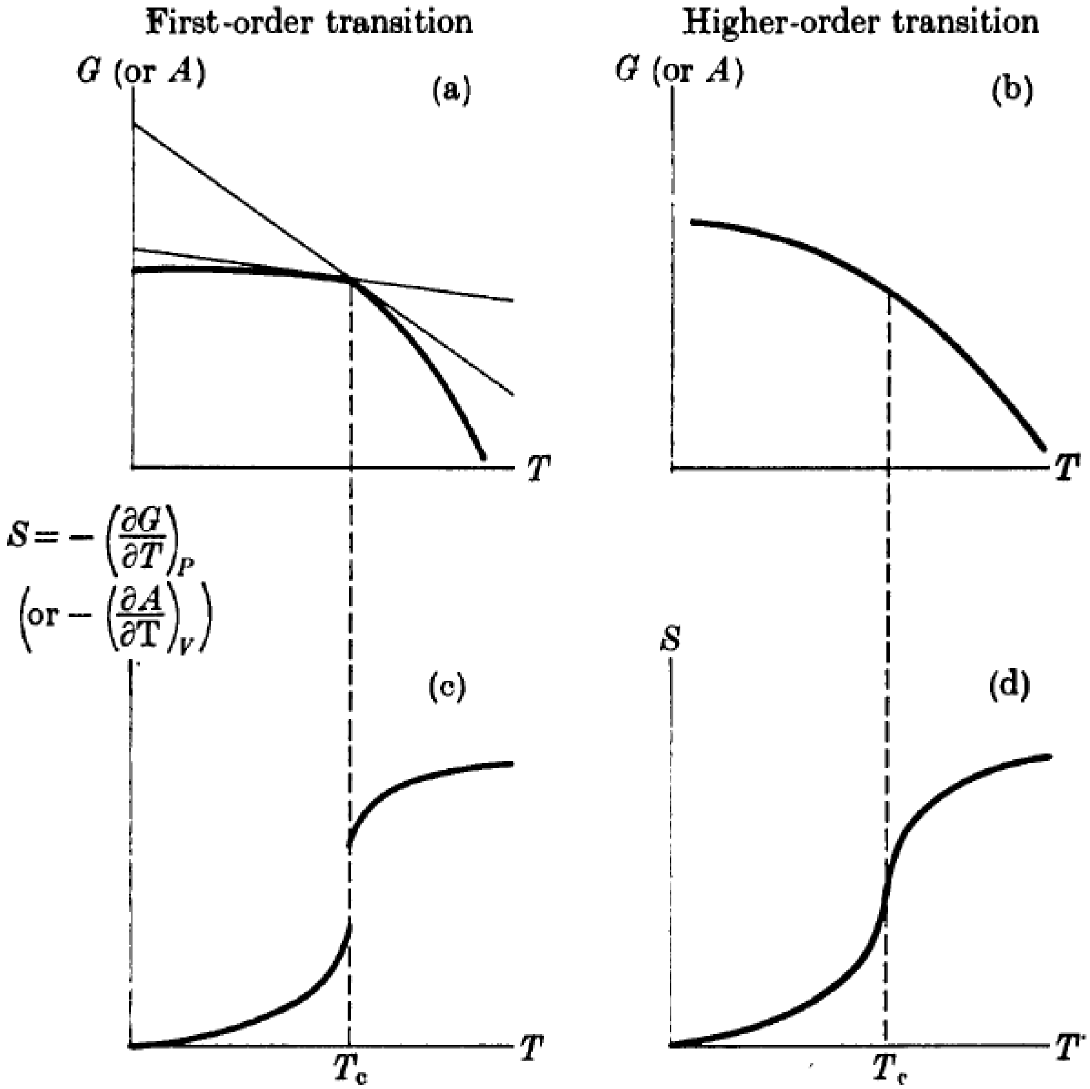
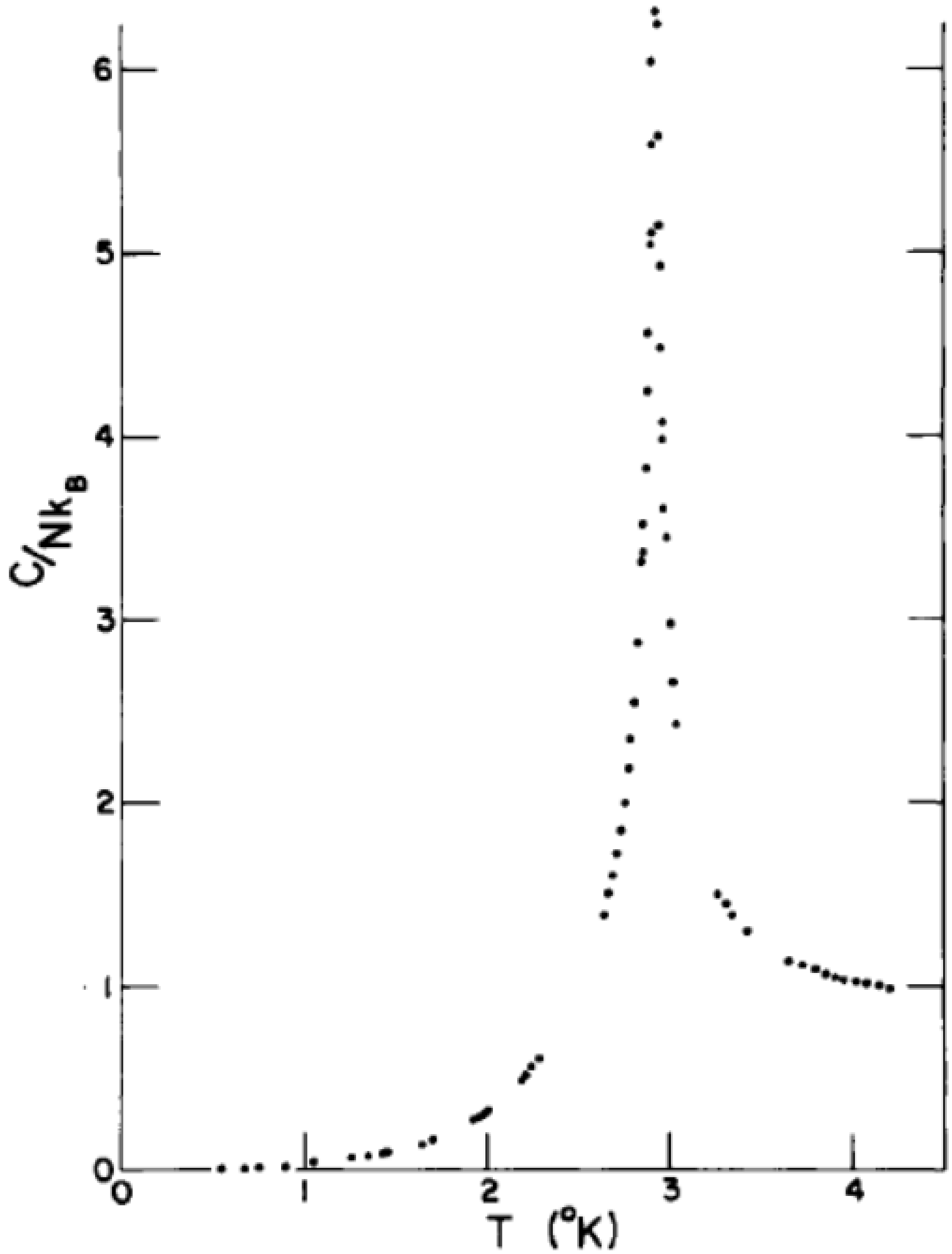
APPENDIX-I The first and second phase transitions



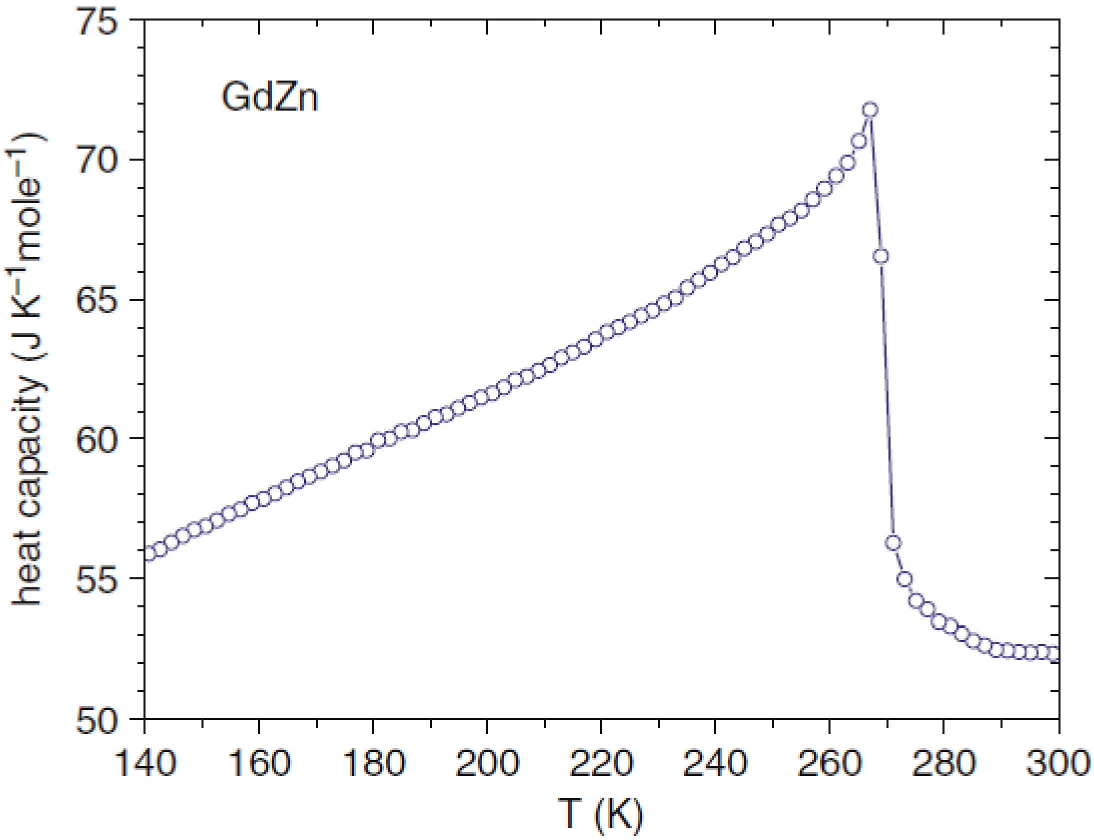
APPENDIX II Magnetic phase transitions. Typical experimental results

[U. Köbler and A. Hoser, Renormalization Group Theory, Impact on Experimental Magnetism (Springer, 2010).]

(1) Specific heat of ^{4}He adsorbed on grafoil at critical coverage for $\sqrt{3}\times\sqrt{3}$ epitaxial structure.



(2) Specific heat of GdZn



(3) Sublattice magnetization of K_2NiF_4

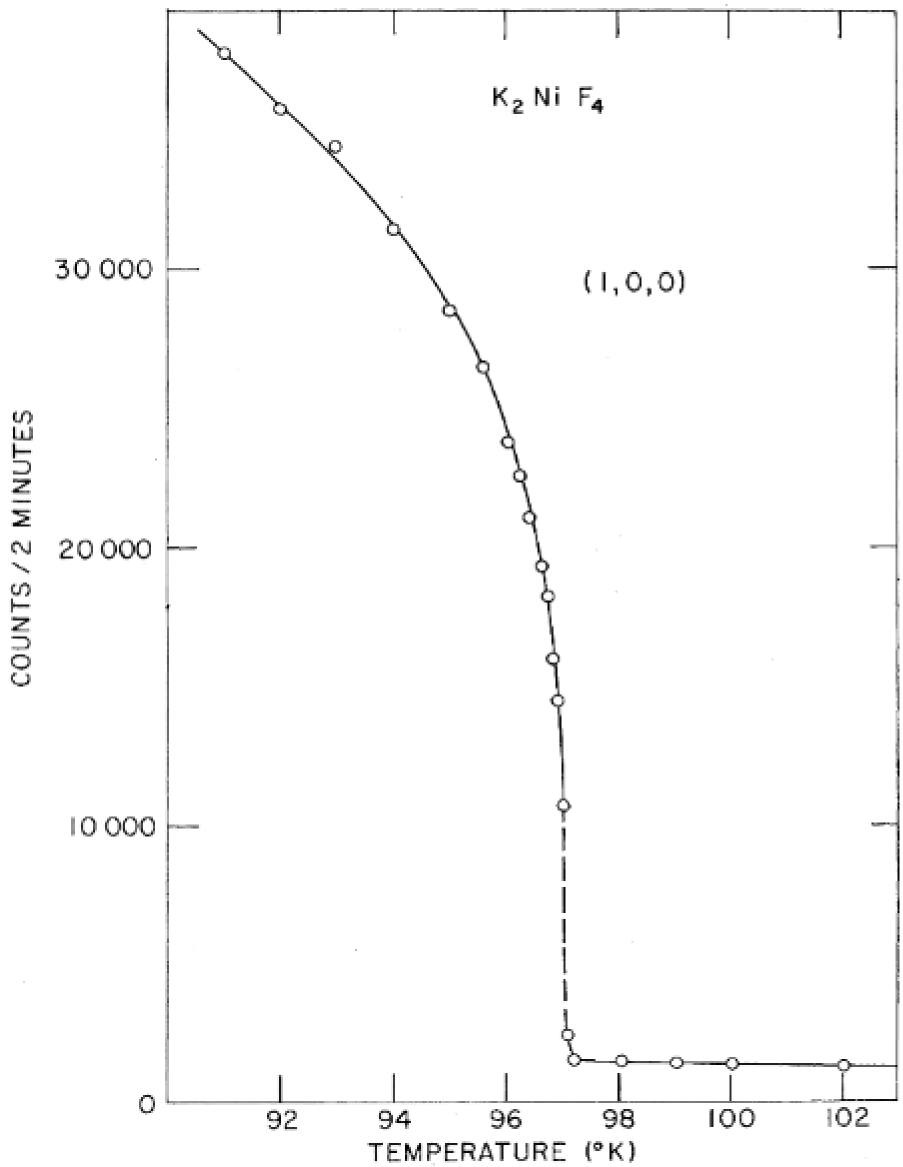
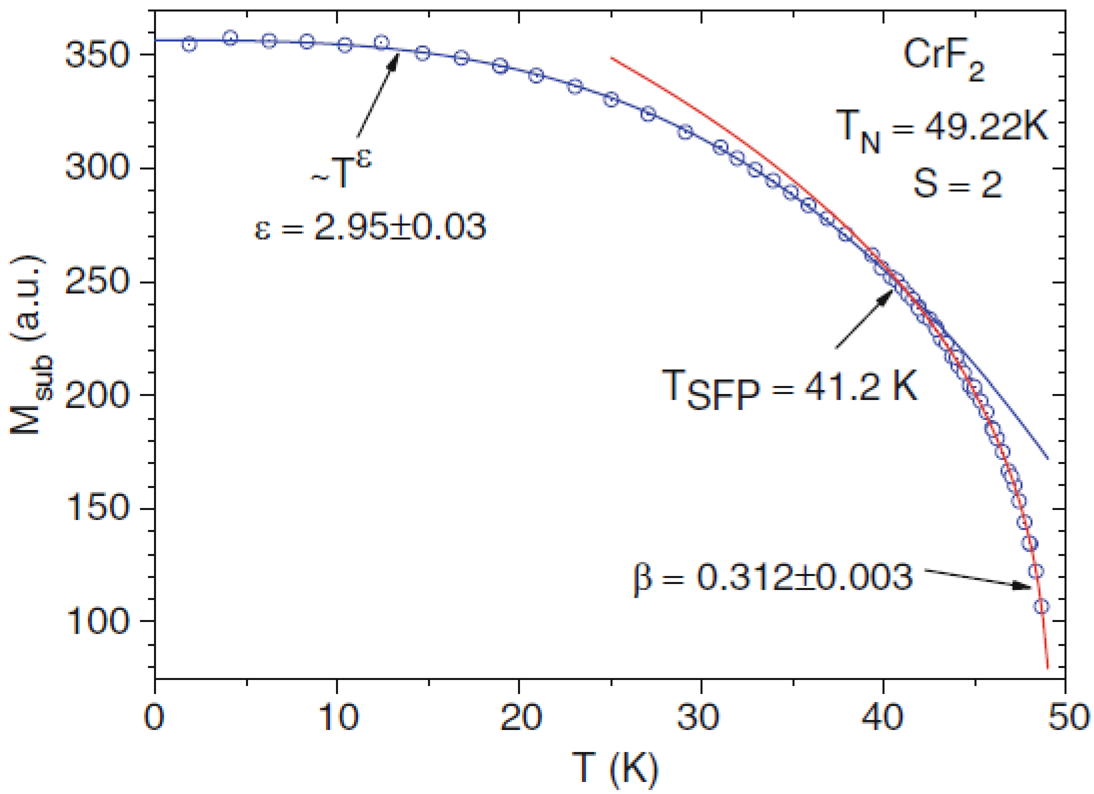
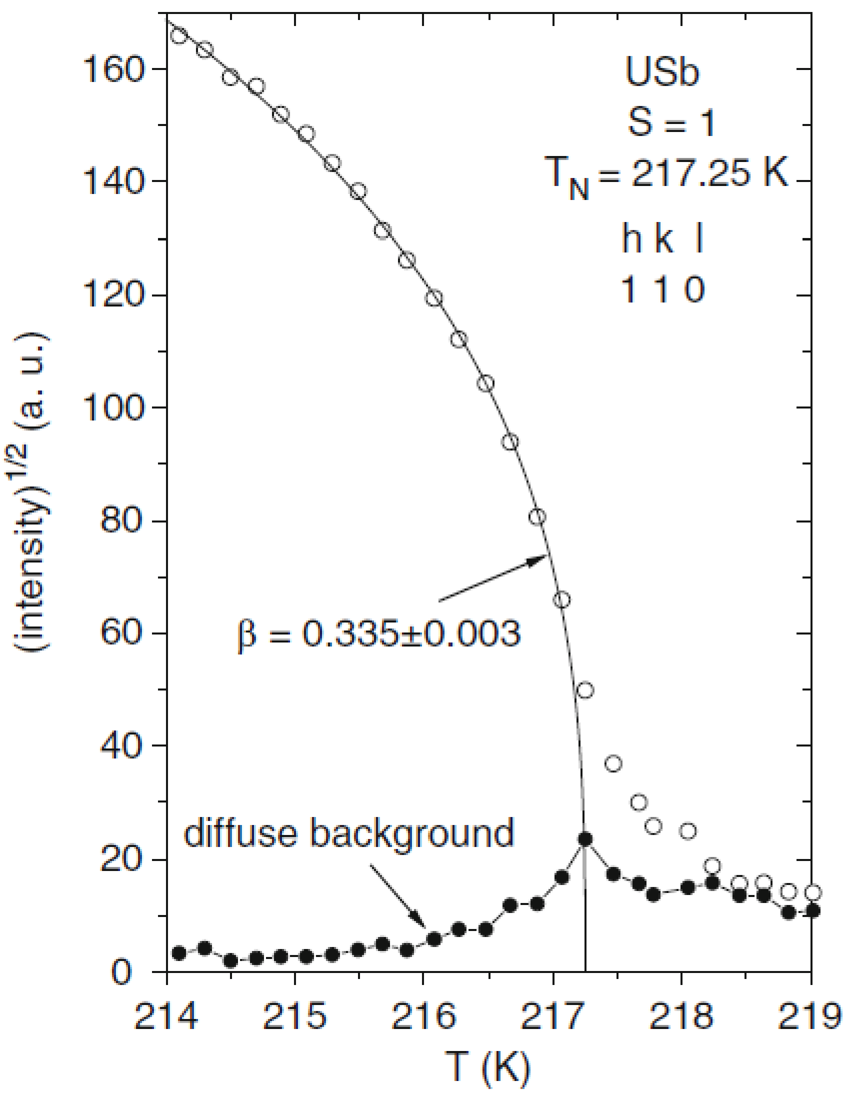


Fig. Neutron scattering intensity at the (1, 0, 0) peak of K_2NiF_4 as a function of temperature (Birgeneau et al. 1970).

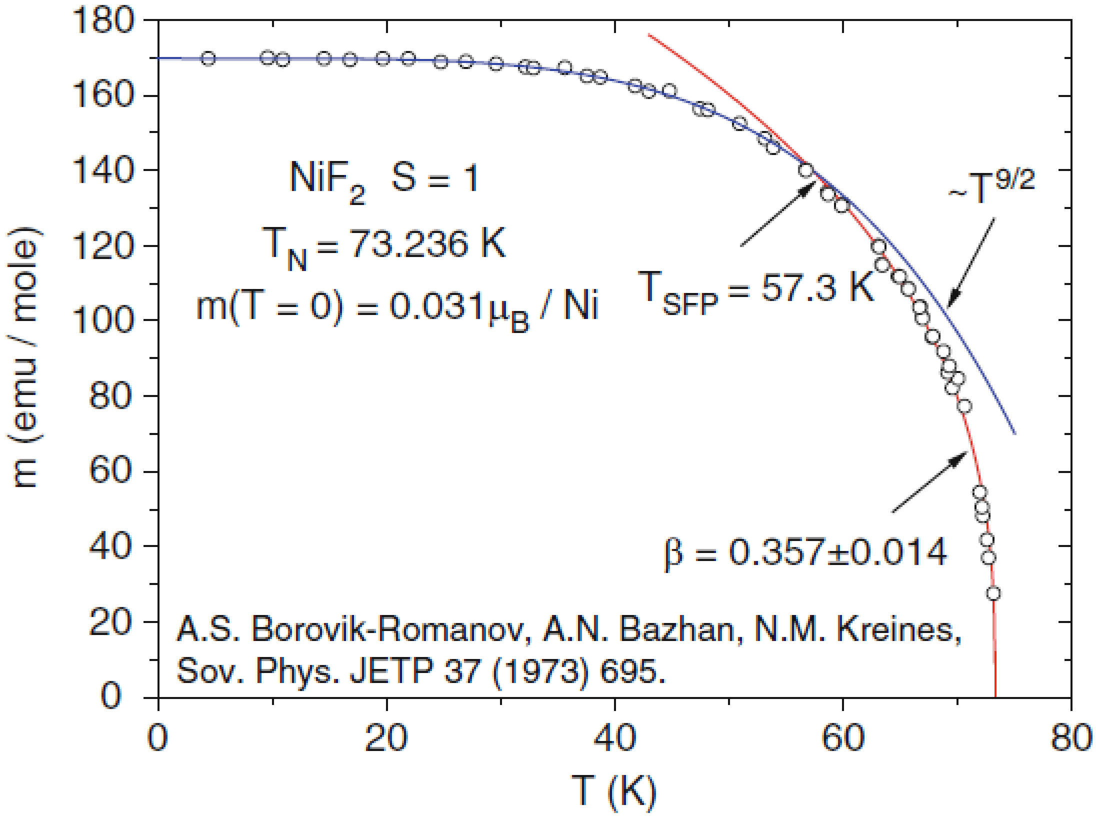
(4) Sublattice magnetization of CrF_2



(5) Sublattice magnetization and staggered suceptibility of USb

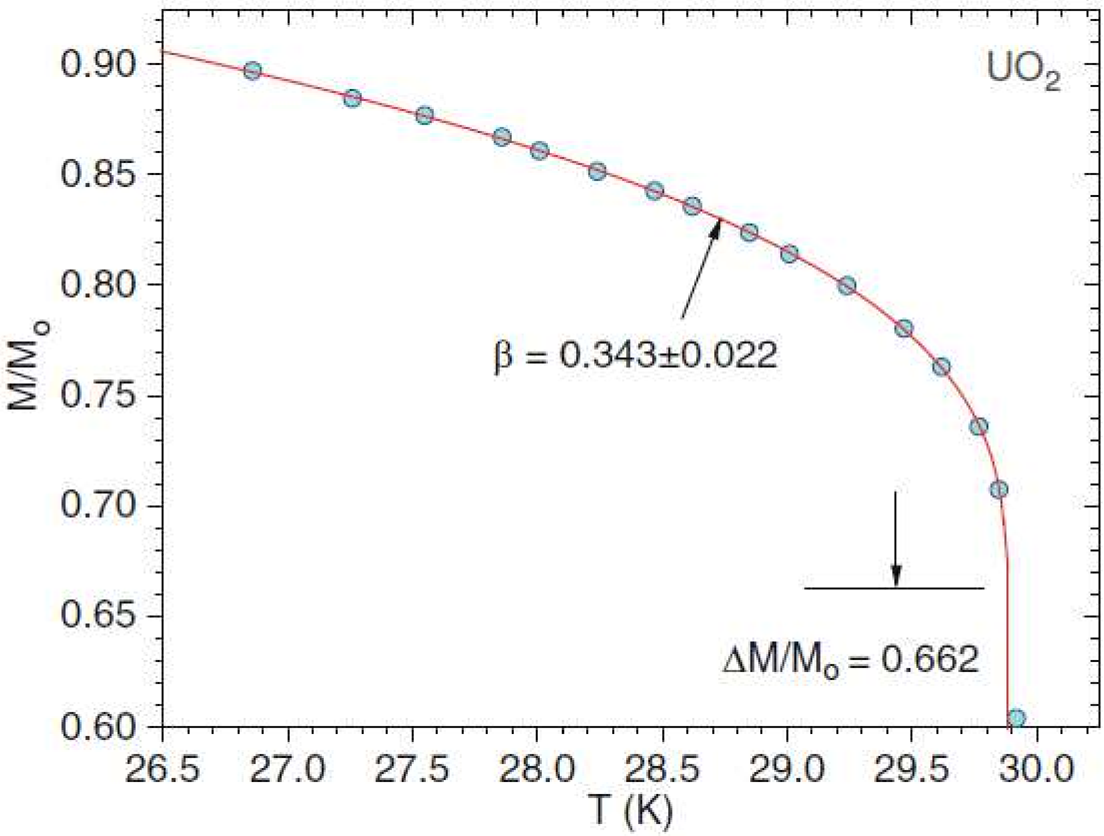


(6) Sublattice magnetization of NiF_2

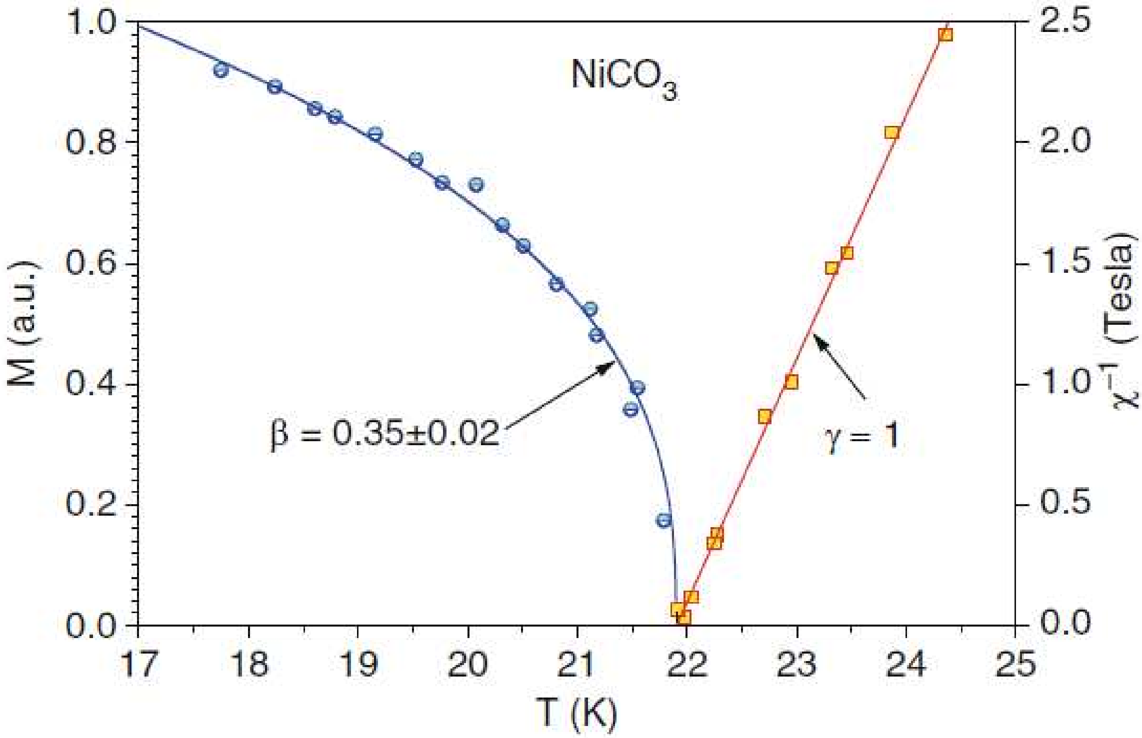


NiF2 S = 1
TN = 73.236 K
m(T = 0) = 0.031μB / Ni
TSFP = 57.3 K
~T9/2
β = 0.357±0.014
A.S. Borovik-Romanov, A.N. Bazhan, N.M. Kreines,
Sov. Phys. JETP 37 (1973) 695.
m (emu / mole)
T (K)

(7) UO_2

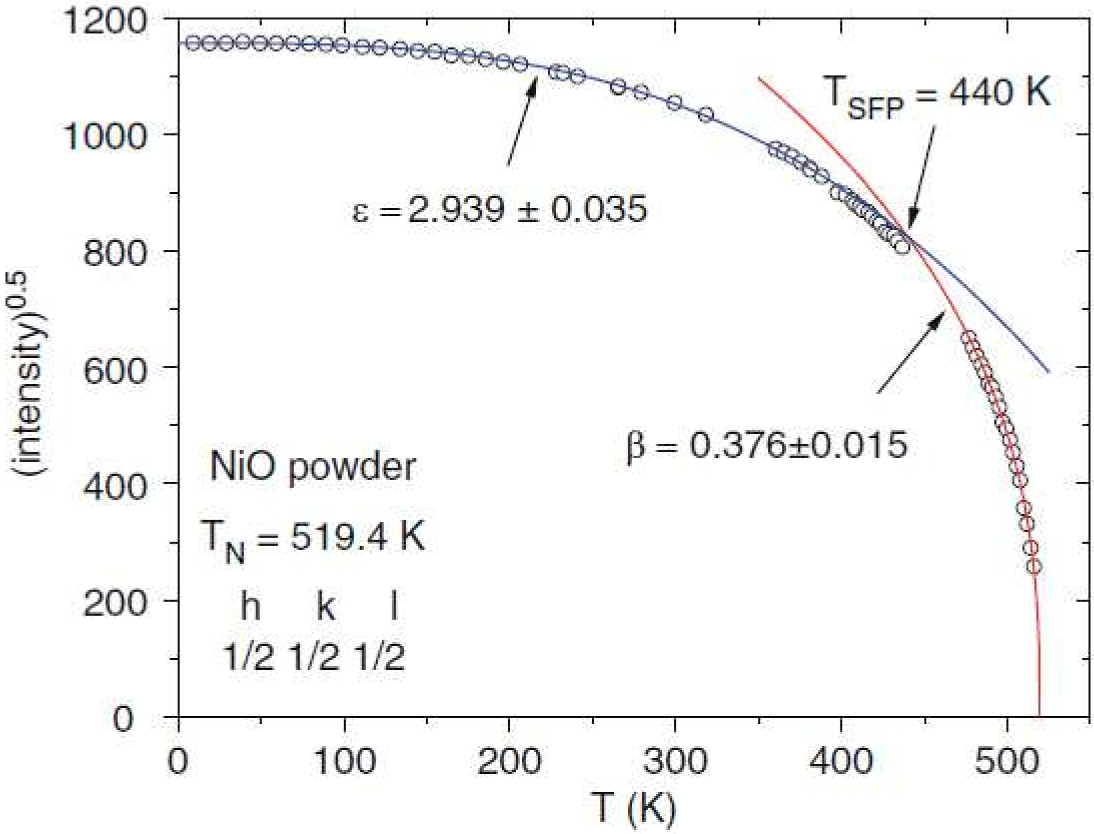

UO2
0.90
0.85
0.80
0.75
0.70
0.65
0.60
M/Mo
β = 0.343±0.022
ΔM/Mo = 0.662
26.5
27.0
27.5
28.0
28.5
29.0
29.5
30.0
T (K)

(8) NiCo₃



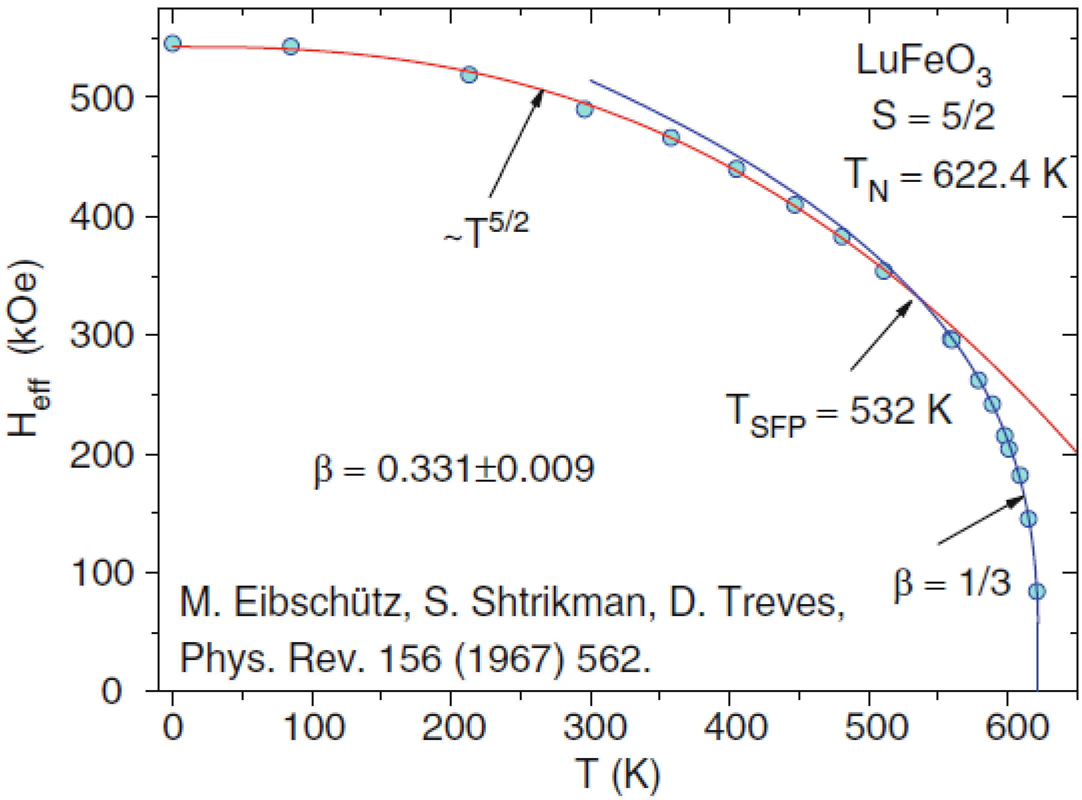
NiCO$_3$
M (a.u.)
χ^{-1} (Tesla)
T (K)
$\beta = 0.35 \pm 0.02$
$\gamma = 1$
0.0
0.2
0.4
0.6
0.8
1.0
0.5
1.5
2.0
2.5
17
18
19
20
21
22
23
24
25

(9) NiO powder



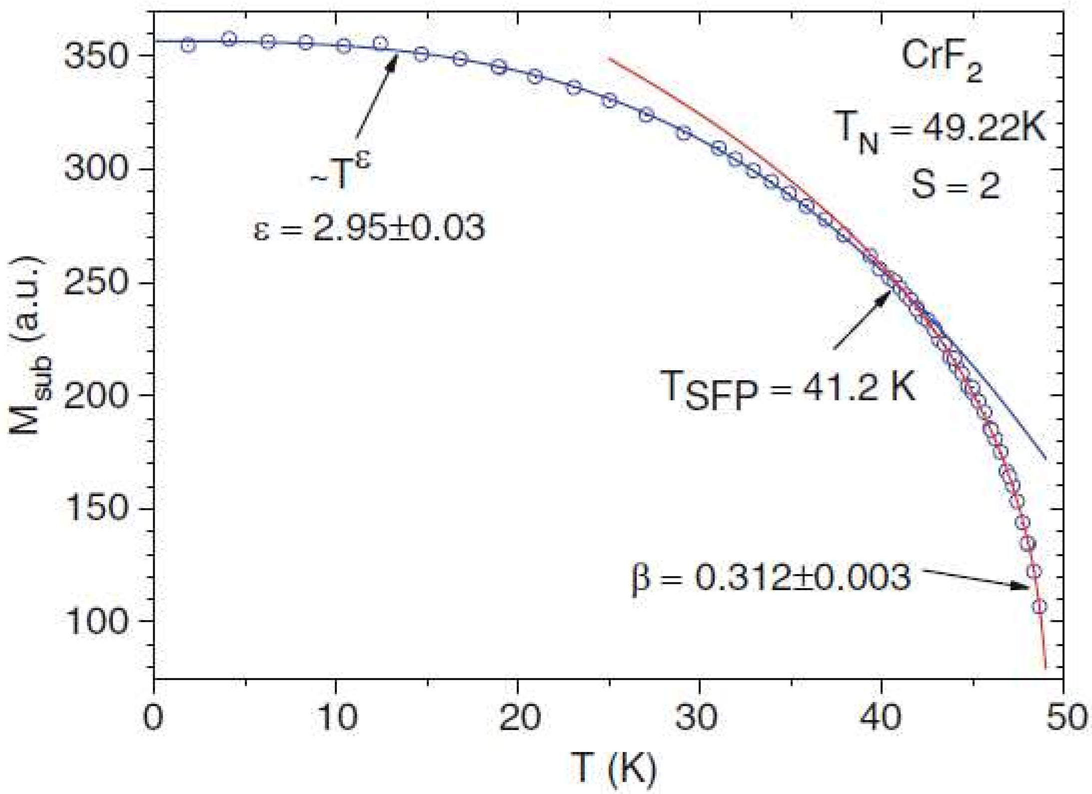
T_{SFP} = 440 K
ε = 2.939 ± 0.035
β = 0.376±0.015
NiO powder
T_N = 519.4 K
h k l
1/2 1/2 1/2
(intensity)^0.5
T (K)

(10) $LuFeO_3$



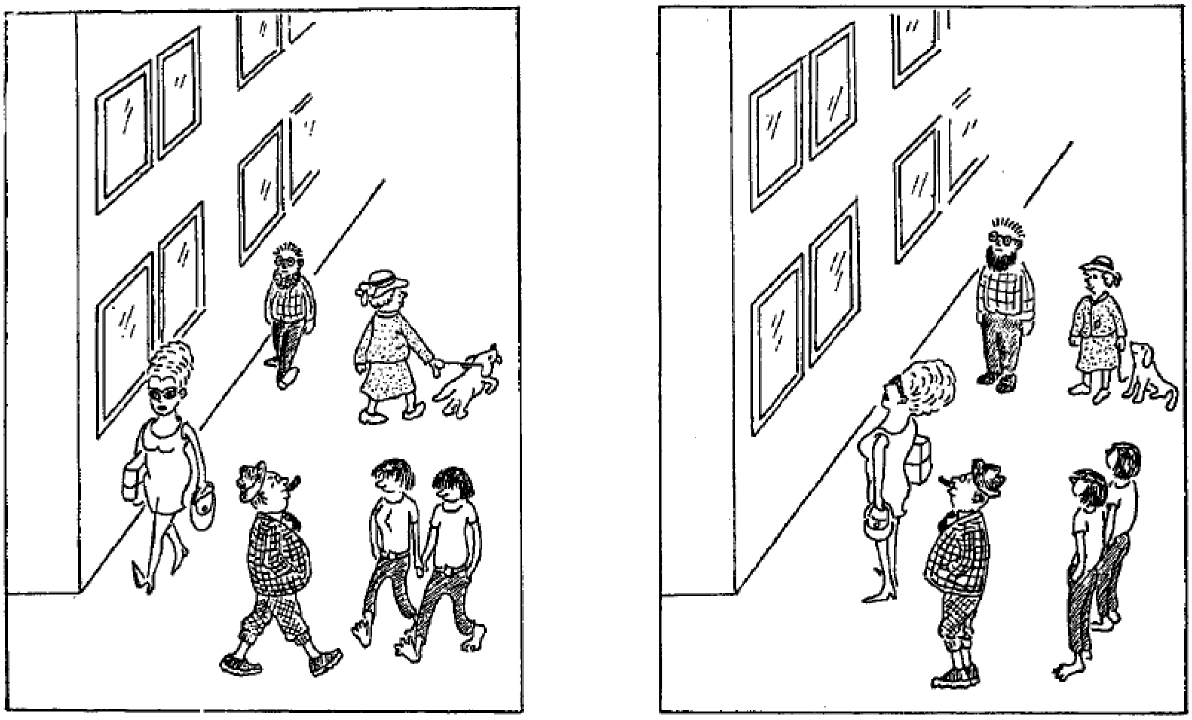
LuFeO$_3$
S = 5/2
T_N = 622.4 K
~$T^{5/2}$
T_{SFP} = 532 K
β = 0.331±0.009
β = 1/3
M. Eibschütz, S. Shtrikman, D. Treves,
Phys. Rev. 156 (1967) 562.
H_{eff} (kOe)
T (K)
0
100
200
300
400
500
600

(11) CrF_2



APPENDIX-III **Phase transition**

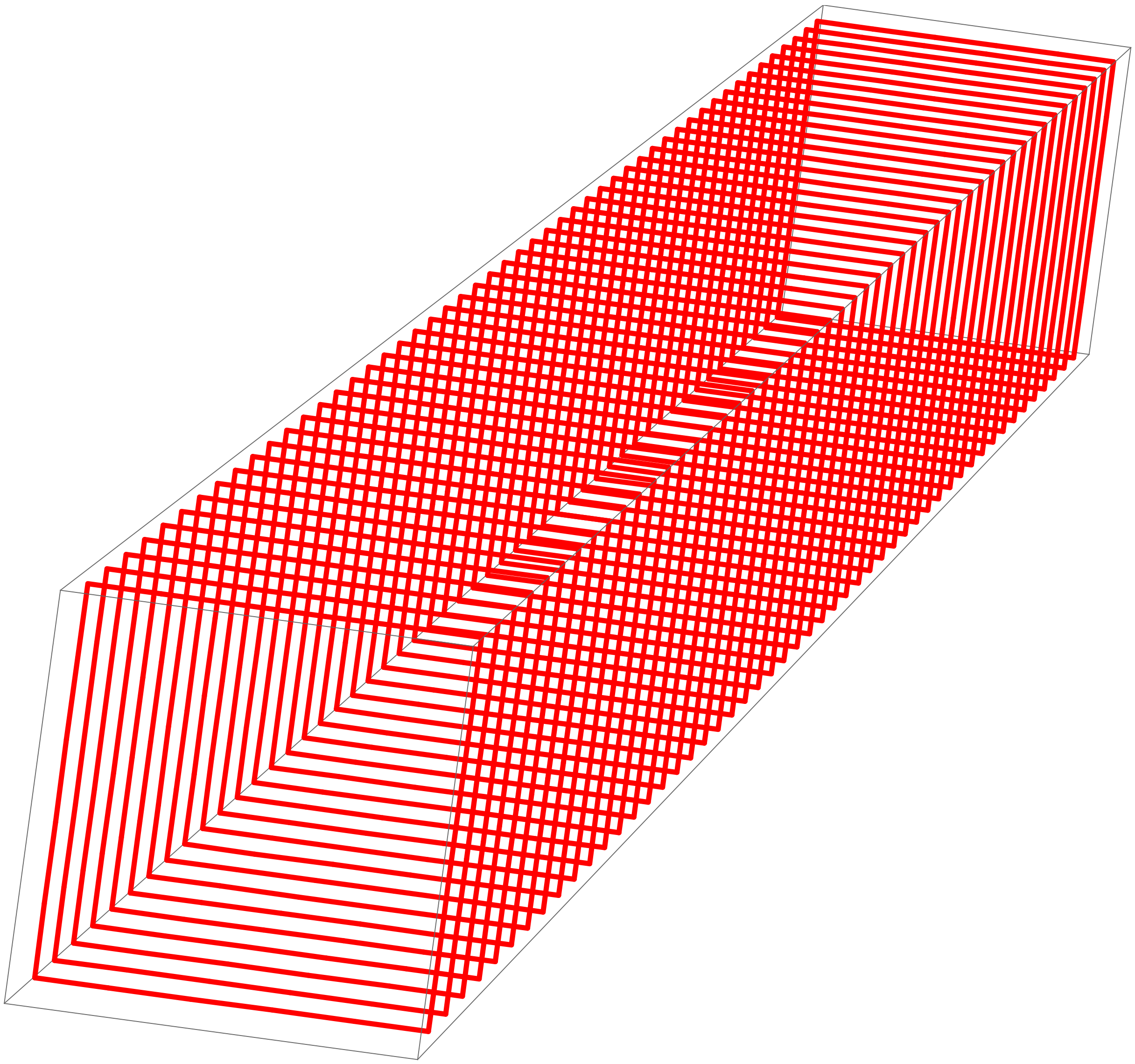
L.J. de Jongh and A.R. Miedema, Advances in Physics, 50, 947 (2001). Experiments on simple magnetic model systems



(a) (b)

Figure 95. The 'staring crowd' phenomenon (Mattuck and Johansson 1968) is an illustration of the fact that phase transitions are certainly not confined to physics but are a more general phenomenon that can be found everywhere in Nature, whenever one deals with a system consisting of elements between which there exists some sort of feed-back mechanism (exchange). If one of the elements (spins, human beings) gets conditioned in a certain fashion—in the above example the attention of one of the persons is attracted by something at the window—the neighbouring elements become conditioned in a similar way, even though there is no external force present that compels them to do so (they may see nothing at all at the window in question). Another example from daily life is the 'spontaneous buying' of luxury goods such as colour-television sets, new cars, etc., which occurs when a given person has enough neighbours around him that possess such an item.

APPENDIX-II Railroad track analogy of renormalization group



Mean field theory

For spin 1/2 system,

$$\begin{aligned} M &= \frac{N}{V}\mu_B \tanh(\beta\mu_B B) \\ &= n\mu_B \tanh(\beta\mu_B B) \end{aligned}$$

or

$$M = ng\mu_B \langle S \rangle \qquad (g = 2,\ S = \frac{1}{2}).$$

We assume that

$$B = B_{eff} = \lambda M$$

When $y = \frac{M}{N\mu_B}$, we have

$$\begin{aligned} y &= \frac{M}{n\mu_B} \\ &= \tanh(\lambda\beta\mu_B M) \\ &= \tanh(\lambda n\beta\mu_B{}^2 y) \end{aligned}$$

We note that

$$\lambda n\beta\mu_B{}^2 y = \frac{\lambda n\mu_B{}^2}{k_B T} y = \frac{y}{x}$$

or

$$x = \frac{k_B T}{\lambda n\mu_B{}^2}$$

The critical temperature is defined by

$$x = 1 = \frac{k_B T_c}{\lambda n\mu_B{}^2}$$

or

$$k_B T_c = \lambda n\mu_B^2$$

The reduced temperature is

$$x = \frac{T}{T_c}.$$

So we make a ContourPlot of

$$y = \tanh(\frac{y}{x})$$

for $x > 0$.

We note that the critical temperature T_c is expressed by

$$T_c = \frac{2}{3k_B} zJS(S+1)$$

where J is the exchange interaction and z is the number of the nearest neighbor spins. When S = 1/2,

$$T_c = \frac{zJ}{2k_B}$$

Using the relation

$$k_B T_c = \frac{zJ}{2} = \lambda n \mu_B{}^2$$

or

$$\lambda = \frac{zJ}{2n\mu_B{}^2}$$

((**Note**))
Spin Hamiltonian:

$$H = -2zJ\langle S\rangle S_i = -g\mu_B B_{eff} S_i$$

where $g = 2.$ The effective magnetic field is

$$B_{eff} = \frac{2zJ}{g\mu_B}\langle S \rangle$$

When g =2 and $\langle S \rangle = \frac{1}{2}$, we have

$$B_{eff} = \frac{zJ}{2\mu_B}$$

which is equal to

$$\lambda M = \frac{zJ}{2n{\mu_B}^2} ng\mu_B \langle S \rangle = \frac{zJ}{2\mu_B}.$$

```
Clear["Global`*"];
f1 = ContourPlot[y == Tanh[y/x], {x, 0, 1.2},
   {y, 0, 1.2}, ContourStyle → {Red, Thick},
   PlotPoints → 150];
f2 =
  Graphics[
   {Text[Style["t=T/Tc", Italic, 12, Black],
     {1.1, 0.02}],
    Text[Style["y=M/(NμB)", Italic, 12, Black],
     {0.1, 1.1}]}];
f3 = Plot[√(1 - t), {t, 0, 1}, PlotStyle → {Blue, Thick}];
Show[f1, f2]
```