Energy state of Binary Magnetic System

Masatsugu Sei Suzuki
Department of Physics
SUNY at Binghamton
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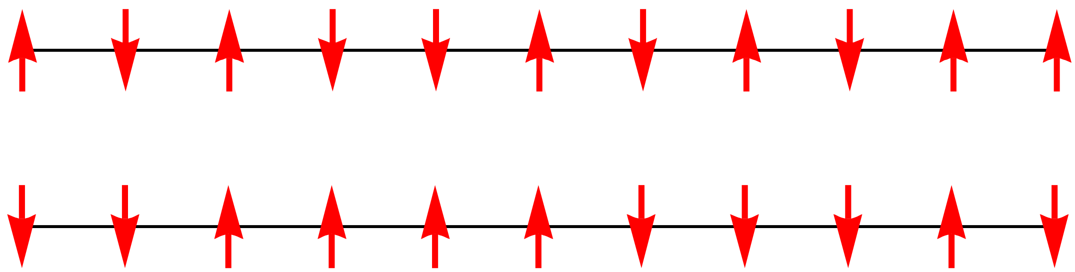
1. Energy of the binary magnetic system



When the magnetic field is applied to this system, the energies of the different states are no longer all equal. The energy of a single magnet m with an external magnetic field B is,

$$E = -\boldsymbol{m}\cdot\boldsymbol{B} \qquad \text{(Zeeman energy)}$$

For the model system of N elementary magnets, the total energy is

$$\begin{aligned}E &= -\boldsymbol{B}\cdot\sum_{i=1}^{N}\boldsymbol{m}_i \\ &= -B[mN_\uparrow + (-m)N_\downarrow] \\ &= -Bm(N_\uparrow - N_\downarrow)\end{aligned}$$

Since $N_\uparrow - N_\downarrow = 2s$, we have

$$E(s) = -2smB = -MB$$

Here

$$M = 2sm$$

for the total magnetic moment $2sm$. We note that

$$N_\uparrow + N_\downarrow = N, \qquad N_\uparrow - N_\downarrow = 2s$$

or

$$N_\uparrow = \frac{1}{2}(N+2s) = \frac{1}{2}N + s$$

$$N_\downarrow = \frac{1}{2}(N-2s) = \frac{1}{2}N - s$$

Since $N_\uparrow \geq 0$ and $N_\downarrow \geq 0$, we have

$$-\frac{1}{2}N \leq s \leq \frac{1}{2}N$$

The total magnetic moment of the system of *N* magnets each of magntic moment m well be defined by *M*. The set of possible values is given by

$$M = Nm, \qquad (N-2)m, \qquad (N-4)m, \ldots, -Nm.$$

There are $(N+1)$ possible values of the total moment, whereas there are 2^N states. If $N = 10$, there are $2^{10} = 1024$ states distributed among 11 different values of the total magnetic moment.

$$(1+x)^{10} = 1 + 10x + 45x^2 + 120x^3 + 210x^4 + 252x^5 + 210x^6 + 120x^7 + 45x^8 + 10x^9 + x^{10}$$

((Summary))

$$E = -Bm(N_\uparrow - N_\downarrow) = -MB = -2smB$$
$$M = (N_\uparrow - N_\downarrow)m = 2sm$$
$$2s = N_\uparrow - N_\downarrow$$

2. Example-1: *N* = 2 case

$$(1+x)^2 = 1 + 2x + x^2$$

(1) $N_\uparrow = 2$, $N_\downarrow = 0$, $2s = 2 - 0 = 2$ $(s = 1)$

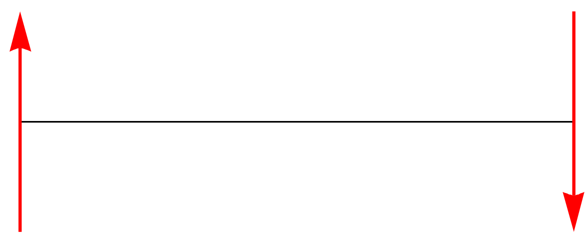
There is one state.

$E = -2mB$

(2) $N_{\uparrow} = 1$, $N_{\downarrow} = 1$, $2s = 1 - 1 = 0$ $(s = 0)$

There are two states.

$E = 0$



(3) $N_{\uparrow} = 0$, $N_{\downarrow} = 2$, $2s = 0 - 2 = -2$ $(s = \text{-}1)$

There is one state.

$E = 2mB$

((**Energy and multiplicity**))

total states = $2^2 = 4$ states
1+2+1 = 4 (the same as 2^3)

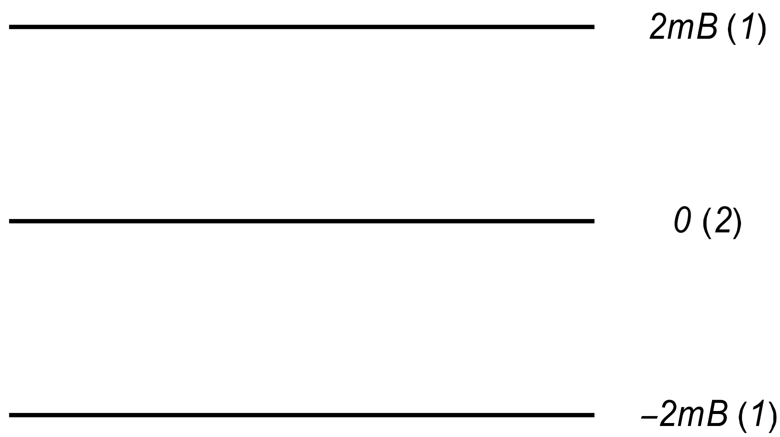


Fig. Energy level and multiplicity for $N = 2$

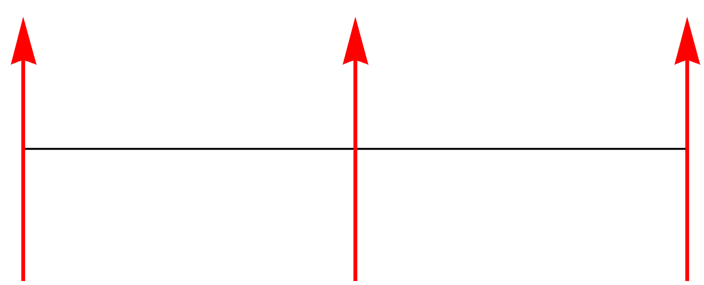
3. Example-2 $N = 3$ case

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3$$

(i) $N_{\uparrow} = 3$, $N_{\downarrow} = 0$, $2s = 3 - 0 = 3$ $(s = 3/2)$

$E = -3mB$

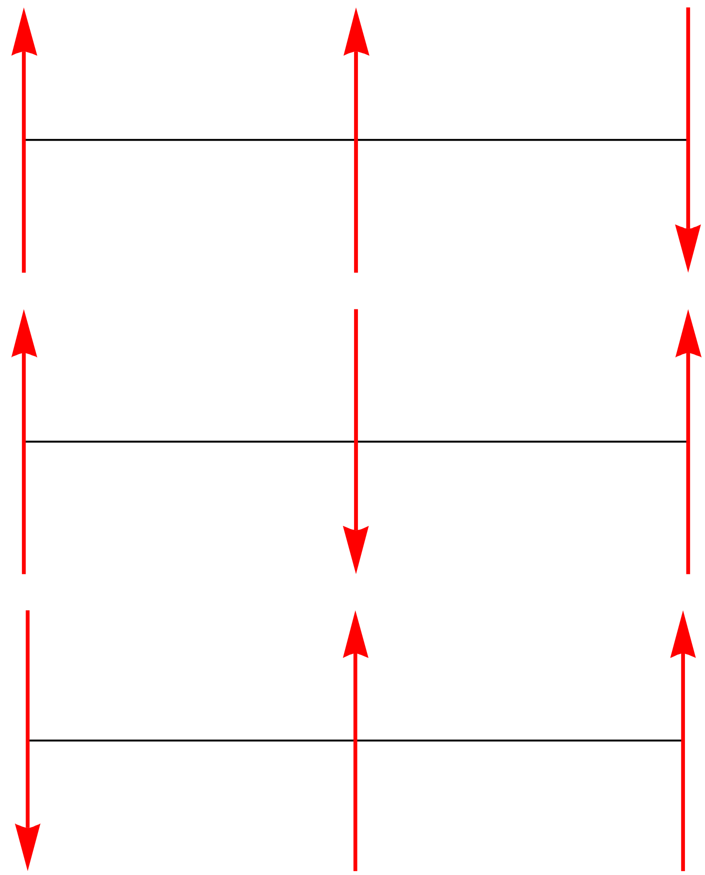
There is one state.



(ii) $N_{\uparrow} = 2$, $N_{\downarrow} = 1$, $2s = 2 - 1 = 1$ $(s = 1/2)$

$E = -mB$

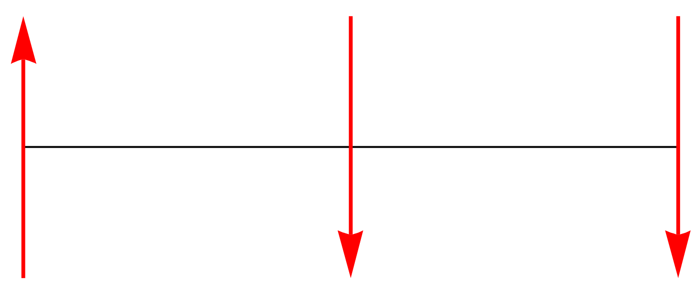
There are three states.

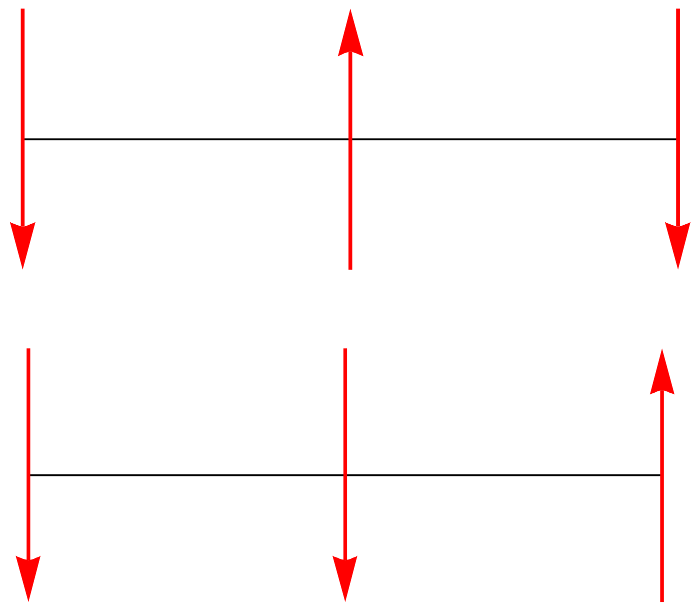


(iii) $N_{\uparrow} = 1$, $N_{\downarrow} = 2$, $2s = 1 - 2 = -1$ $(s = -1/2)$

$$E = mB$$

There are three states.

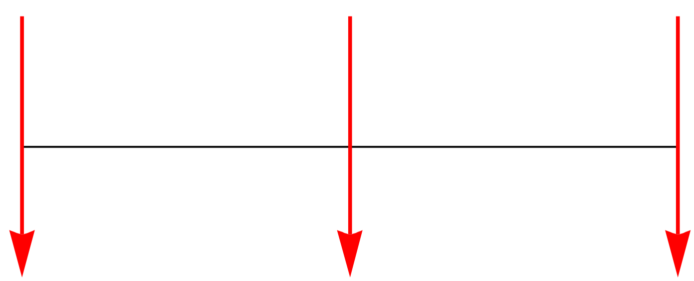


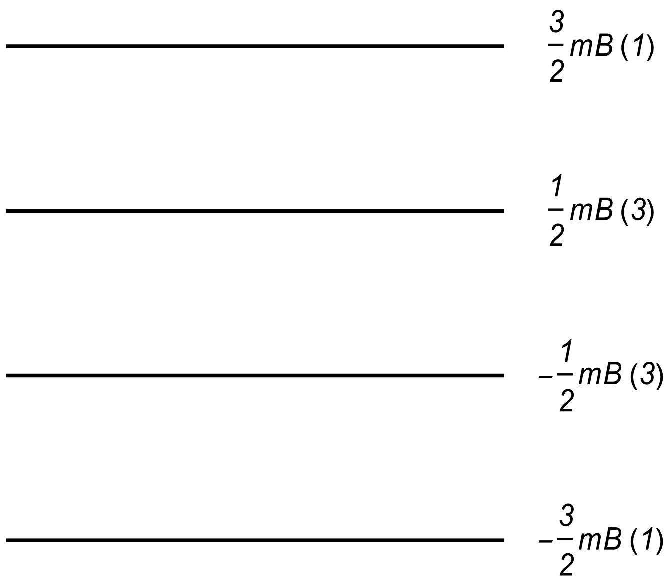


(iv) $N_{\uparrow} = 0$, $N_{\downarrow} = 3$, $2s = 0 - 3 = -3$ (s = -3/2)

$E = mB$

There is one state.





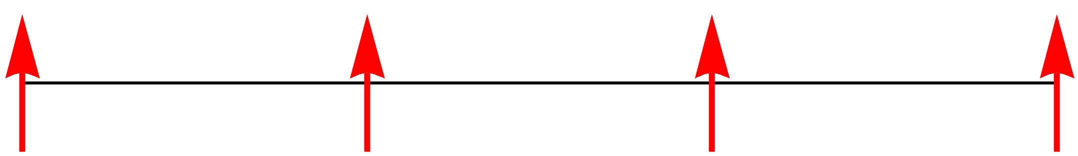
4. **Example-3 $N = 4$ case**

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

(i) $N_{\uparrow} = 4$, $N_{\downarrow} = 0$, $2s = 4 - 0 = 4$ $(s = 2)$

$$E = -4mB$$

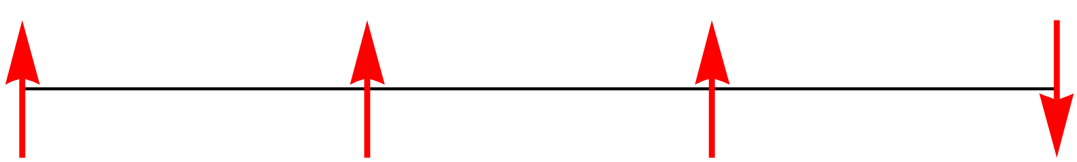
There is one state.

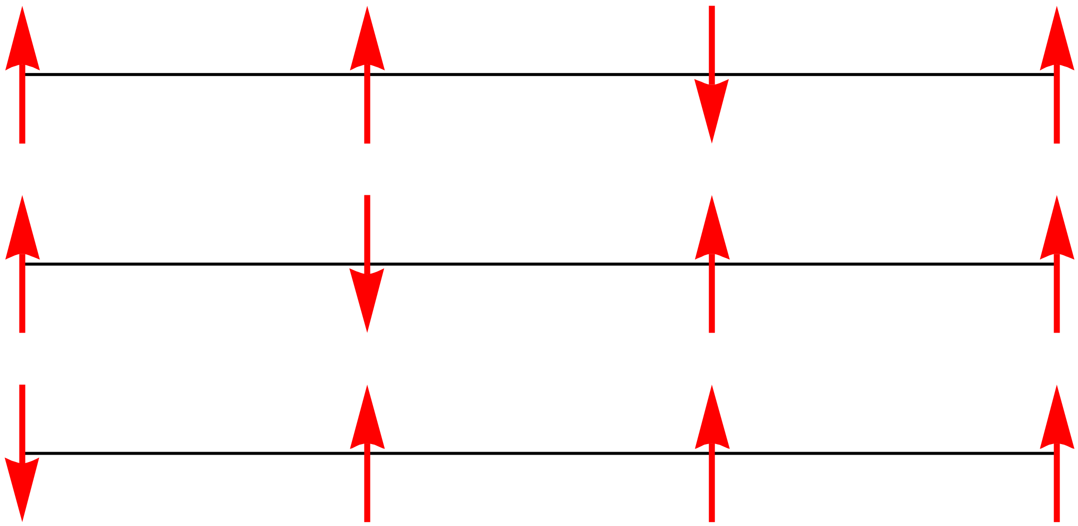


(ii) $N_{\uparrow} = 3$, $N_{\downarrow} = 1$, $2s = 3 - 1 = 2$ $(s = 1)$

$$E = -2mB$$

There are 4 states.

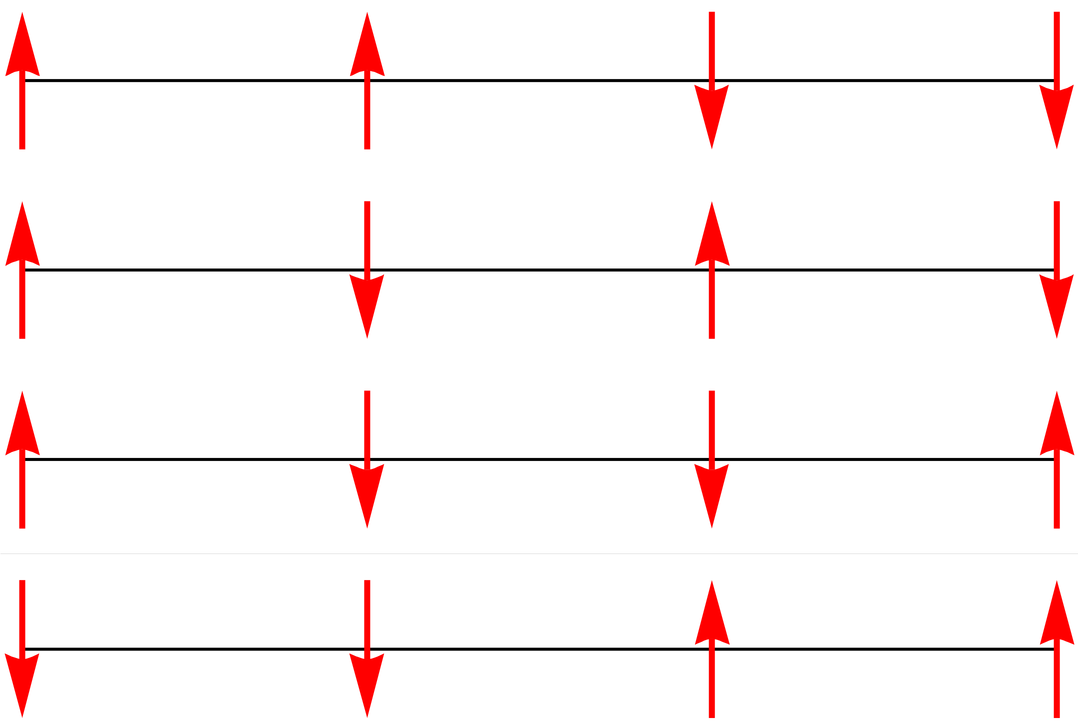


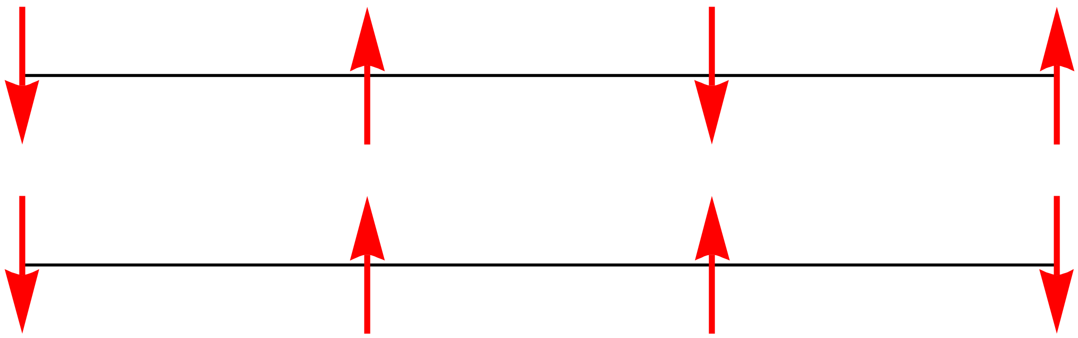


(iii) $N_{\uparrow} = 2$, $N_{\downarrow} = 2$, $2s = 2 - 2 = 0$ $(s = 0)$

There are 6 states.

$E = 0$

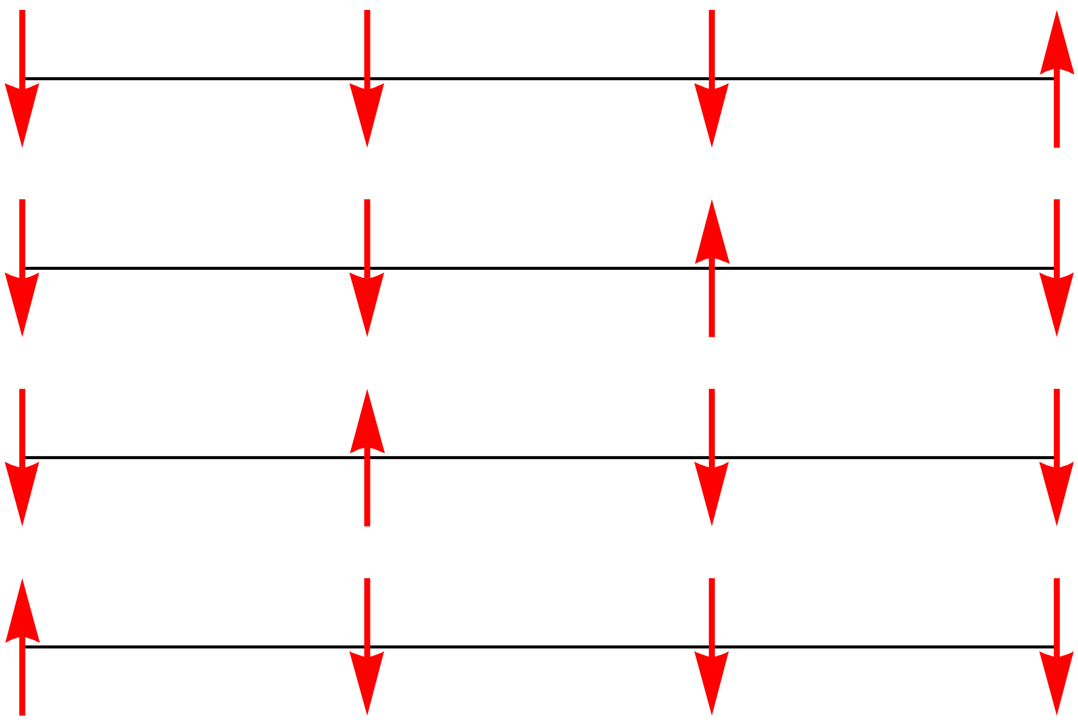




(iv) $N_{\uparrow} = 1$, $N_{\downarrow} = 3$, $2s = 1 - 3 = -2$ (s = -1)

$E = 2mB$

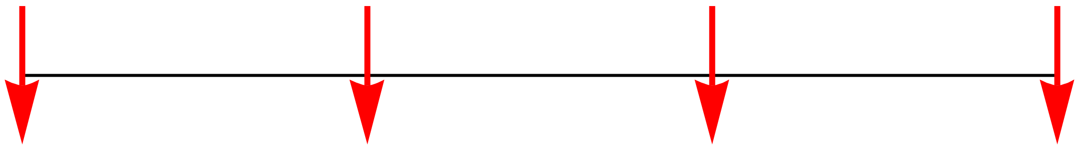
There are 4 states.



(v) $N_{\uparrow} = 0$, $N_{\downarrow} = 4$, $2s = 0 - 4 = -4$ (s = -2)

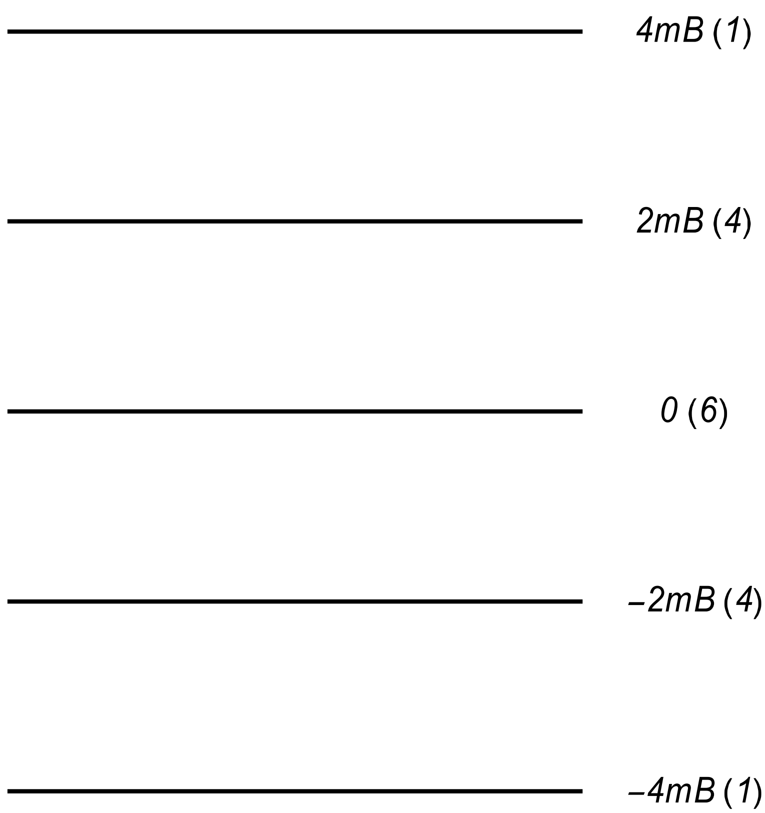
$E = 4mB$

There is one states.



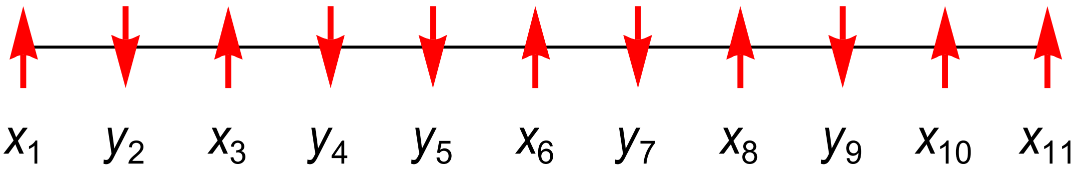
((**Energy and multiplicity**))

total states = 2^4 = 16 states
1+4+6+4+1 = 16 (the same as 2^4)



5. State of model system

We use the following simple notation for a single state of the system of N sites. The magnetic-moment up-state at the site i is denoted by x_i, and the magnetic moment down-state at the site j is denoted by y_j.



Every distinct state of the system is contained in a symbolic product of N factors,

$$(x_1 + y_1)(x_2 + y_2)(x_3 + y_3)(x_4 + y_4).......(x_{N-1} + y_{N-1})(x_N + y_N)$$

which is called a generating function. It generates the states of the system. The generating function for the states of a system of $N = 2$ is given by

$$(x_1 + y_1)(x_2 + y_2) = x_1 x_2 + x_1 y_2 + y_1 x_2 + y_1 y_2 .$$

for the $N = 2$ system. From this we have

$x_1 x_2$ (state) with the multiplicity (1)

$x_1 y_2$ (state) with the multiplicity (1)

$y_1 x_2$ (state) with the multiplicity (1)

$y_1 y_2$ (state) with the multiplicity (1)

For $N = 3$ we have

$$
\begin{aligned}
(x_1 + y_1)(x_2 + y_2)(x_3 + y_3) &= (x_1 x_2 + x_1 y_2 + y_1 x_2 + y_1 y_2)(x_3 + y_3) \\
&= x_1 x_2 x_3 \\
&\quad + x_1 x_2 y_3 + x_1 y_2 x_3 + y_1 x_2 x_3 \\
&\quad + x_1 y_2 y_3 + y_1 x_2 y_3 + y_1 y_2 x_3 \\
&\quad + y_1 y_2 y_3
\end{aligned}
$$

In summary we have

$x_1 x_2 x_3$	$M = 3m$ state	(multiplicity 1)
$x_1 x_2 y_3 + x_1 y_2 x_3 + y_1 x_2 x_3$	$M = m$ state	(multiplicity 3)
$x_1 y_2 y_3 + y_1 x_2 y_3 + y_1 y_2 x_3$	$M = -m$ state	(multiplicity 3)
$y_1 y_2 y_3$	$M = -3m$ state	(multiplicity 1)

For $N = 4$ we have

$$
\begin{aligned}
(x_1 + y_1)(x_2 + y_2)(x_3 + y_3)(x_4 + y_4) &= (x_1 x_2 x_3 + x_1 x_2 y_3 + x_1 y_2 x_3 + y_1 x_2 x_3 \\
&\quad + x_1 y_2 y_3 + y_1 x_2 y_3 + y_1 y_2 x_3 + y_1 y_2 y_3)(x_4 + y_4) \\
&= x_1 x_2 x_3 x_4 \\
&\quad + x_1 x_2 x_3 y_4 + x_1 x_2 y_3 x_4 + x_1 y_2 x_3 x_4 + y_1 x_2 x_3 x_4 \\
&\quad + x_1 x_2 y_3 y_4 + x_1 y_2 x_3 y_4 + y_1 x_2 x_3 y_4 + x_1 y_2 y_3 x_4 + y_1 x_2 y_3 x_4 + y_1 y_2 x_3 x_4 \\
&\quad + x_1 y_2 y_3 y_4 + y_1 x_2 y_3 y_4 + y_1 y_2 x_3 y_4 + y_1 y_2 y_3 x_4 \\
&\quad + y_1 y_2 y_3 y_4
\end{aligned}
$$

In summary we have

$x_1 x_2 x_3 x_4$	$M = 4m$ state (multiplicity 1)
$x_1 x_2 x_3 y_4 + x_1 x_2 y_3 x_4 + x_1 y_2 x_3 x_4 + y_1 x_2 x_3 x_4$	$M = 2m$ state (multiplicity 4)
$x_1 x_2 y_3 y_4 + x_1 y_2 x_3 y_4 + y_1 x_2 x_3 y_4 + x_1 y_2 y_3 x_4 + y_1 x_2 y_3 x_4 + y_1 y_2 x_3 x_4$	

$M = 0$ state (multiplicity 6)

$x_1 y_2 y_3 y_4 + y_1 x_2 y_3 y_4 + y_1 y_2 x_3 y_4 + y_1 y_2 y_3 x_4$ — $M = -2m$ state (multiplicity 4)

$y_1 y_2 y_3 y_4$ — $M = -4m$ state (multiplicity 1)

In order to find the expression for the multiplicity, we may drop the site label.

$$(x+y)^N = \sum_{t=0}^{N} \frac{N!}{(N-t)!t!} x^{N-t} y^t \qquad (1)$$

where

$$N_\uparrow = N - t, \qquad N_\downarrow = t$$

$$N_\uparrow - N_\downarrow = N - 2t = 2s$$

Thus we get

$$t = N_\downarrow = \frac{1}{2}N - s, \qquad N - t = N_\uparrow = \frac{1}{2}N + s$$

Equation (1) can be rewritten as

$$(x+y)^N = \sum_{s=-N/2}^{N/2} \frac{N!}{(\frac{1}{2}N+s)!(\frac{1}{2}N-s)!} x^{\frac{1}{2}N+s} y^{\frac{1}{2}N-s}$$

The coefficient of the term in $x^{\frac{1}{2}N+s} y^{\frac{1}{2}N-s}$ is the number of states where the net magnetic moment is having $2sm\,[= m(N_\uparrow - N_\downarrow)]$. We denote the number of states in this class by

$$g(N,s) = \frac{N!}{(\frac{1}{2}N+s)!(\frac{1}{2}N-s)!} = \frac{N!}{N_\uparrow! N_\downarrow!}$$

which is called the multiplicity function. In the expression of

$$(x+y)^N = \sum_{s=-N/2}^{N/2} g(N,s) x^{\frac{1}{2}N+s} y^{\frac{1}{2}N-s}$$

we put $x = y = 1$. Thus we have the total number states

$$2^N = \sum_{s=-N/2}^{N/2} g(N,s)$$

6. Sharpness of the multiplicity function

Using the Stirling formula

$$n! = n(\ln n - 1) + \frac{1}{2}\ln(2\pi n)$$

for $n >> 1$, we find the approximation for the multiplicity function,

$$\begin{aligned}\ln g(N,s) &= \ln N! - \ln(\frac{1}{2}N+s)! - (\frac{1}{2}N-s)! \\ &= \frac{1}{2}\ln(\frac{1}{2\pi N}) + N\ln 2 - \frac{2s^2}{N} + \frac{1}{2}\ln 2^2 \\ &= \ln[2^N\left(\frac{2}{\pi N}\right)^{1/2}] - \frac{2s^2}{N}\end{aligned}$$

or

$$g(N,s) = g(N,0)\exp[-\frac{2s^2}{N}]$$

with

$$g(N,0) = 2^N\left(\frac{2}{\pi N}\right)^{1/2}$$

$g(N,s)$ can be expressed by the Gaussian distribution function as

$$2^N \frac{1}{\sqrt{2\pi}\sigma}\exp[-\frac{s^2}{2\sigma^2}]$$

where

$$\sigma = \frac{\sqrt{N}}{2}$$

$$g(N,0) = 2^N\left(\frac{2}{\pi N}\right)^{1/2} = 2^N \frac{1}{\sqrt{2\pi}\sigma}\ .$$

We note that

$$P(s) = \frac{1}{\sqrt{2\pi}\sigma}\exp[-\frac{s^2}{2\sigma^2}]$$

is the Gaussian distribution function and

$$\int P(s)ds = \int ds \frac{1}{\sqrt{2\pi}\sigma}\exp[-\frac{s^2}{2\sigma^2}] = 1$$

The average:

$$\langle s \rangle = \int sP(s) = 0$$

$$\langle s^2 \rangle = \int s^2 P(s) = \sigma^2$$

The fluctuation

$$\sqrt{\langle (\Delta s)^2 \rangle} = \sqrt{\langle s^2 \rangle - \langle s \rangle^2} = \sqrt{\sigma^2} = \frac{\sqrt{N}}{2}$$