Micro Canonical Ensemble: Balls in bowls

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1. Four balls and two bowls

Given three balls labeled A, B, and C (distinguishable) and two bowls (a, b), what are the different ways in which we can apportion the balls to the two bowls? We solve this problem using Mathematica.

$$f = (A_1 a + A_2 b)(B_1 a + B_2 b)(C_1 a + C_2 b)(D_1 a + D_2 b)$$

Note that $A_1 a$ denotes the case when the ball A occupies the bowl a. $A_2 b$ denotes the case when the ball A occupies the bowl b. We expand f in terms of powers of a and b.

((**Mathematica**))

```
Clear["Global`*"];
f1 = (A1 a + A2 b) (B1 a + B2 b) (C1 a + C2 b) (D1 a + D2 b) // Expand;

G1[p_, q_] := Coefficient[f1, a^p b^q ];
g1 = Table[{a^p b^(4-p), G1[p, 4 - p]}, {p, 0, 4}] // TableForm
```

b^4	A2 B2 C2 D2
$a\ b^3$	A2 B2 C2 D1 + A2 B2 C1 D2 + A2 B1 C2 D2 + A1 B2 C2 D2
$a^2\ b^2$	A2 B2 C1 D1 + A2 B1 C2 D1 + A1 B2 C2 D1 + A2 B1 C1 D2 + A1 B2 C1 D2 + A1 B1 C2 D2
$a^3\ b$	A2 B1 C1 D1 + A1 B2 C1 D1 + A1 B1 C2 D1 + A1 B1 C1 D2
a^4	A1 B1 C1 D1

2. Ralph Baierlein (Thermal Physics) Problem 2-2

2-2

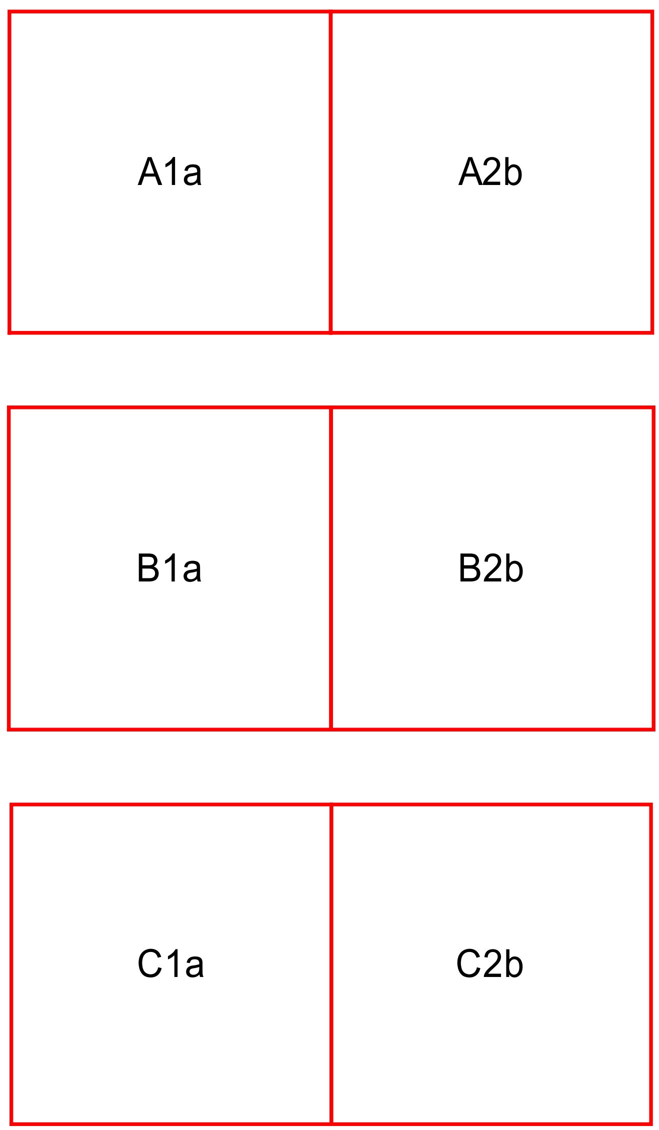
2. Work out the analog of table 2.2 but with $N = 6$ labeled balls. Draw a bar graph of multiplicity versus macrostate, the latter being specified by the number of balls in the right-hand bowl (together with the total number of balls, N). Using symmetry will expedite your work. Using the known total number of microstates (which you can reason to be 2^6) provides either a check on your arithmetic or a way to skip one multiplicity computation. Do you see the more-or-less even distribution growing in numerical significance?

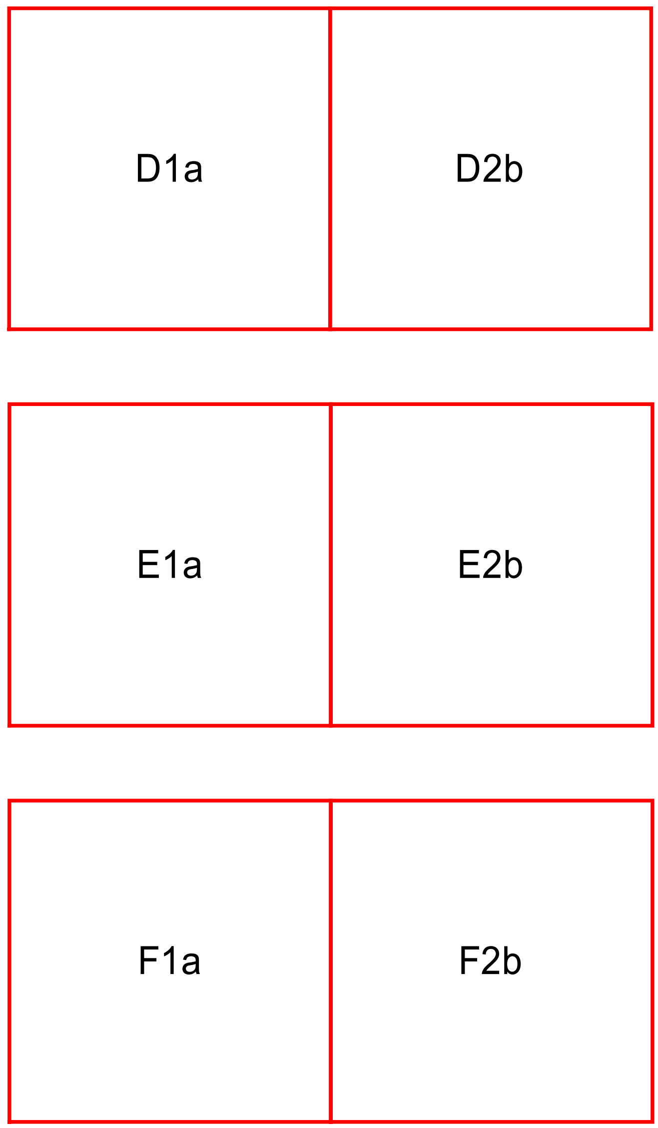
See the solution of HW-2 (home work).

Given three balls labeled A, B, C, D, E, and F (distinguishable) and two bowls (a, b), what are the different ways in which we can apportion the balls to the two bowls? We solve this problem using Mathematica.

$$f = (A_1 a + A_2 b)(B_1 a + B_2 b)(C_1 a + C_2 b)(D_1 a + D_2 b)(E_1 a + E_2 b)(F_1 a + F_2 b)$$

Note that $A_1 a$ denotes the case when the ball A occupies the bowl a. $A_2 b$ denotes the case when the ball A occupies the bowl b. We expand f in terms of powers of a and b.





((**Mathematica**)) We solve this problem using the mathematica.

```
Clear["Global`*"];

f1 = (A1 a + A2 b) (B1 a + B2 b) (C1 a + C2 b) (D1 a + D2 b) (E1 a  + E2 b) (F1 a  +  F2 b) // Expand
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$$
\begin{aligned}
&a^6 A1\ B1\ C1\ D1\ E1\ F1 + a^5 A2\ b\ B1\ C1\ D1\ E1\ F1 + a^5 A1\ b\ B2\ C1\ D1\ E1\ F1 + a^4 A2\ b^2 B2\ C1\ D1\ E1\ F1 + \\
&a^5 A1\ b\ B1\ C2\ D1\ E1\ F1 + a^4 A2\ b^2 B1\ C2\ D1\ E1\ F1 + a^4 A1\ b^2 B2\ C2\ D1\ E1\ F1 + a^3 A2\ b^3 B2\ C2\ D1\ E1\ F1 + \\
&a^5 A1\ b\ B1\ C1\ D2\ E1\ F1 + a^4 A2\ b^2 B1\ C1\ D2\ E1\ F1 + a^4 A1\ b^2 B2\ C1\ D2\ E1\ F1 + a^3 A2\ b^3 B2\ C1\ D2\ E1\ F1 + \\
&a^4 A1\ b^2 B1\ C2\ D2\ E1\ F1 + a^3 A2\ b^3 B1\ C2\ D2\ E1\ F1 + a^3 A1\ b^3 B2\ C2\ D2\ E1\ F1 + a^2 A2\ b^4 B2\ C2\ D2\ E1\ F1 + \\
&a^5 A1\ b\ B1\ C1\ D1\ E2\ F1 + a^4 A2\ b^2 B1\ C1\ D1\ E2\ F1 + a^4 A1\ b^2 B2\ C1\ D1\ E2\ F1 + a^3 A2\ b^3 B2\ C1\ D1\ E2\ F1 + \\
&a^4 A1\ b^2 B1\ C2\ D1\ E2\ F1 + a^3 A2\ b^3 B1\ C2\ D1\ E2\ F1 + a^3 A1\ b^3 B2\ C2\ D1\ E2\ F1 + a^2 A2\ b^4 B2\ C2\ D1\ E2\ F1 + \\
&a^4 A1\ b^2 B1\ C1\ D2\ E2\ F1 + a^3 A2\ b^3 B1\ C1\ D2\ E2\ F1 + a^3 A1\ b^3 B2\ C1\ D2\ E2\ F1 + a^2 A2\ b^4 B2\ C1\ D2\ E2\ F1 + \\
&a^3 A1\ b^3 B1\ C2\ D2\ E2\ F1 + a^2 A2\ b^4 B1\ C2\ D2\ E2\ F1 + a^2 A1\ b^4 B2\ C2\ D2\ E2\ F1 + a\ A2\ b^5 B2\ C2\ D2\ E2\ F1 + \\
&a^5 A1\ b\ B1\ C1\ D1\ E1\ F2 + a^4 A2\ b^2 B1\ C1\ D1\ E1\ F2 + a^4 A1\ b^2 B2\ C1\ D1\ E1\ F2 + a^3 A2\ b^3 B2\ C1\ D1\ E1\ F2 + \\
&a^4 A1\ b^2 B1\ C2\ D1\ E1\ F2 + a^3 A2\ b^3 B1\ C2\ D1\ E1\ F2 + a^3 A1\ b^3 B2\ C2\ D1\ E1\ F2 + a^2 A2\ b^4 B2\ C2\ D1\ E1\ F2 + \\
&a^4 A1\ b^2 B1\ C1\ D2\ E1\ F2 + a^3 A2\ b^3 B1\ C1\ D2\ E1\ F2 + a^3 A1\ b^3 B2\ C1\ D2\ E1\ F2 + a^2 A2\ b^4 B2\ C1\ D2\ E1\ F2 + \\
&a^3 A1\ b^3 B1\ C2\ D2\ E1\ F2 + a^2 A2\ b^4 B1\ C2\ D2\ E1\ F2 + a^2 A1\ b^4 B2\ C2\ D2\ E1\ F2 + a\ A2\ b^5 B2\ C2\ D2\ E1\ F2 + \\
&a^4 A1\ b^2 B1\ C1\ D1\ E2\ F2 + a^3 A2\ b^3 B1\ C1\ D1\ E2\ F2 + a^3 A1\ b^3 B2\ C1\ D1\ E2\ F2 + a^2 A2\ b^4 B2\ C1\ D1\ E2\ F2 + \\
&a^3 A1\ b^3 B1\ C2\ D1\ E2\ F2 + a^2 A2\ b^4 B1\ C2\ D1\ E2\ F2 + a^2 A1\ b^4 B2\ C2\ D1\ E2\ F2 + a\ A2\ b^5 B2\ C2\ D1\ E2\ F2 + \\
&a^3 A1\ b^3 B1\ C1\ D2\ E2\ F2 + a^2 A2\ b^4 B1\ C1\ D2\ E2\ F2 + a^2 A1\ b^4 B2\ C1\ D2\ E2\ F2 + a\ A2\ b^5 B2\ C1\ D2\ E2\ F2 + \\
&a^2 A1\ b^4 B1\ C2\ D2\ E2\ F2 + a\ A2\ b^5 B1\ C2\ D2\ E2\ F2 + a\ A1\ b^5 B2\ C2\ D2\ E2\ F2 + A2\ b^6 B2\ C2\ D2\ E2\ F2
\end{aligned}
$$

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Coefficient[f1, a^6]
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A1 B1 C1 D1 E1 F1

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Coefficient[f1, a^5 b]
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A2 B1 C1 D1 E1 F1 + A1 B2 C1 D1 E1 F1 + A1 B1 C2 D1 E1 F1 + A1 B1 C1 D2 E1 F1 + A1 B1 C1 D1 E2 F1 + A1 B1 C1 D1 E1 F2

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Coefficient[f1, a^4 b^2]
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A2 B2 C1 D1 E1 F1 + A2 B1 C2 D1 E1 F1 + A1 B2 C2 D1 E1 F1 + A2 B1 C1 D2 E1 F1 + A1 B2 C1 D2 E1 F1 +
A1 B1 C2 D2 E1 F1 + A2 B1 C1 D1 E2 F1 + A1 B2 C1 D1 E2 F1 + A1 B1 C2 D1 E2 F1 + A1 B1 C1 D2 E2 F1 +
A2 B1 C1 D1 E1 F2 + A1 B2 C1 D1 E1 F2 + A1 B1 C2 D1 E1 F2 + A1 B1 C1 D2 E1 F2 + A1 B1 C1 D1 E2 F2

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Coefficient[f1, a^3 b^3]
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A2 B2 C2 D1 E1 F1 + A2 B2 C1 D2 E1 F1 + A2 B1 C2 D2 E1 F1 + A1 B2 C2 D2 E1 F1 + A2 B2 C1 D1 E2 F1 +
A2 B1 C2 D1 E2 F1 + A1 B2 C2 D1 E2 F1 + A2 B1 C1 D2 E2 F1 + A1 B2 C1 D2 E2 F1 + A1 B1 C2 D2 E2 F1 +
A2 B2 C1 D1 E1 F2 + A2 B1 C2 D1 E1 F2 + A1 B2 C2 D1 E1 F2 + A2 B1 C1 D2 E1 F2 + A1 B2 C1 D2 E1 F2 +
A1 B1 C2 D2 E1 F2 + A2 B1 C1 D1 E2 F2 + A1 B2 C1 D1 E2 F2 + A1 B1 C2 D1 E2 F2 + A1 B1 C1 D2 E2 F2

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Coefficient[f1, a^2 b^4]
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A2 B2 C2 D2 E1 F1 + A2 B2 C2 D1 E2 F1 + A2 B2 C1 D2 E2 F1 + A2 B1 C2 D2 E2 F1 + A1 B2 C2 D2 E2 F1 +
A2 B2 C2 D1 E1 F2 + A2 B2 C1 D2 E1 F2 + A2 B1 C2 D2 E1 F2 + A1 B2 C2 D2 E1 F2 + A2 B2 C1 D1 E2 F2 +
A2 B1 C2 D1 E2 F2 + A1 B2 C2 D1 E2 F2 + A2 B1 C1 D2 E2 F2 + A1 B2 C1 D2 E2 F2 + A1 B1 C2 D2 E2 F2

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Coefficient[f1, a^1 b^5]
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A2 B2 C2 D2 E2 F1 + A2 B2 C2 D2 E1 F2 + A2 B2 C2 D1 E2 F2 + A2 B2 C1 D2 E2 F2 + A2 B1 C2 D2 E2 F2 + A1 B2 C2 D2 E2 F2

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Coefficient[f1, b^6]
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A2 B2 C2 D2 E2 F2

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f1 = (a + b)^6 // Expand
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$$a^6 + 6 a^5 b + 15 a^4 b^2 + 20 a^3 b^3 + 15 a^2 b^4 + 6 a b^5 + b^6$$

3. The case of two balls (distinguishable) in the three bowls

Given three balls labeled A and B (distinguishable) and three bowls (a, b, c), what are the different ways in which we can apportion the balls to the two bowls? We solve this problem using Mathematica. There are 3x3 = 9 possible arrangements.

$$f = (A_1 a + A_2 b + A_3 c)(B_1 a + B_2 b + B_3 c)$$

((**Mathematica**))

```
Clear["Global`*"];

G1[p_, q_, r_] := Module[{g1, f1, f2},
   f1 = (A1 a + A2 b + A3 c) (B1 a + B2 b + B3 c) // Expand;
   f2 = If[p + q + r == 2, Coefficient[f1, a^p b^q c^r], 0]];

g1 = Table[{a^p b^q c^r, G1[p, q, r]}, {p, 0, 2},
   {q, 0, 2}, {r, 0, 2}] // TableForm
```

```
1     0              b       0              b^2       A2 B2
c     0              b c     A3 B2 + A2 B3  b^2 c     0
c^2   A3 B3          b c^2   0              b^2 c^2   0

a     0              a b     A2 B1 + A1 B2  a b^2     0
a c   A3 B1 + A1 B3  a b c   0              a b^2 c   0
a c^2 0              a b c^2 0              a b^2 c^2 0

a^2     A1 B1        a^2 b     0            a^2 b^2     0
a^2 c   0            a^2 b c   0            a^2 b^2 c   0
a^2 c^2 0            a^2 b c^2 0            a^2 b^2 c^2 0
```

From this table, we have the following conclusion.

(a) Coefficient of a^2;

A1B1 {2,0,0} $\frac{2!}{2!0!0!} = 1$

(b) Coefficient of b^2;

A2B2 {0,2,0} $\frac{2!}{0!2!0!} = 1$

(c) Coefficient of c^2;

A3B3 {0,0,2} $\frac{2!}{0!0!2!} = 1$

(d) Coefficient of ab;

A1B2 A2B1 {1,1,0} $\frac{2!}{1!1!0!} = 2$

(e) Coefficient of bc;

A2B3 A3B2 {0,1,1} $\frac{2!}{0!1!1!} = 2$

(f) Coefficient of ca;

A1B3 A3B1 {1,0,1} $\frac{2!}{1!0!1!} = 2$

(ii) If the balls are indistinguishable,

(a) Coefficient of a^2;

A1A1 {2,0,0} $\frac{2!}{2!1!0!} = 1$

(b) Coefficient of b^2;

A2A2 {0,2,0} $\frac{2!}{0!2!0!} = 1$

(c) Coefficient of c^2;

A3A3 {0,0,2} $\frac{2!}{0!0!2!}=1$

(d) Coefficient of ab;

A1A2 1

(e) Coefficient of bc;

A2A3 1

(f) Coefficient of ca;

A1A3 1

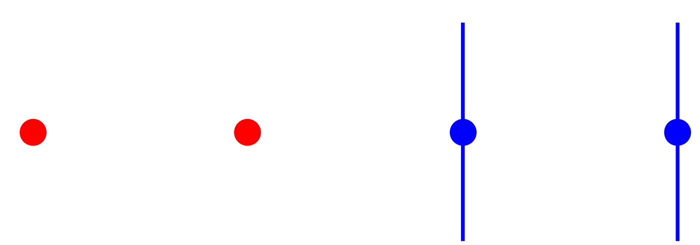
So there are 6 arrangements.

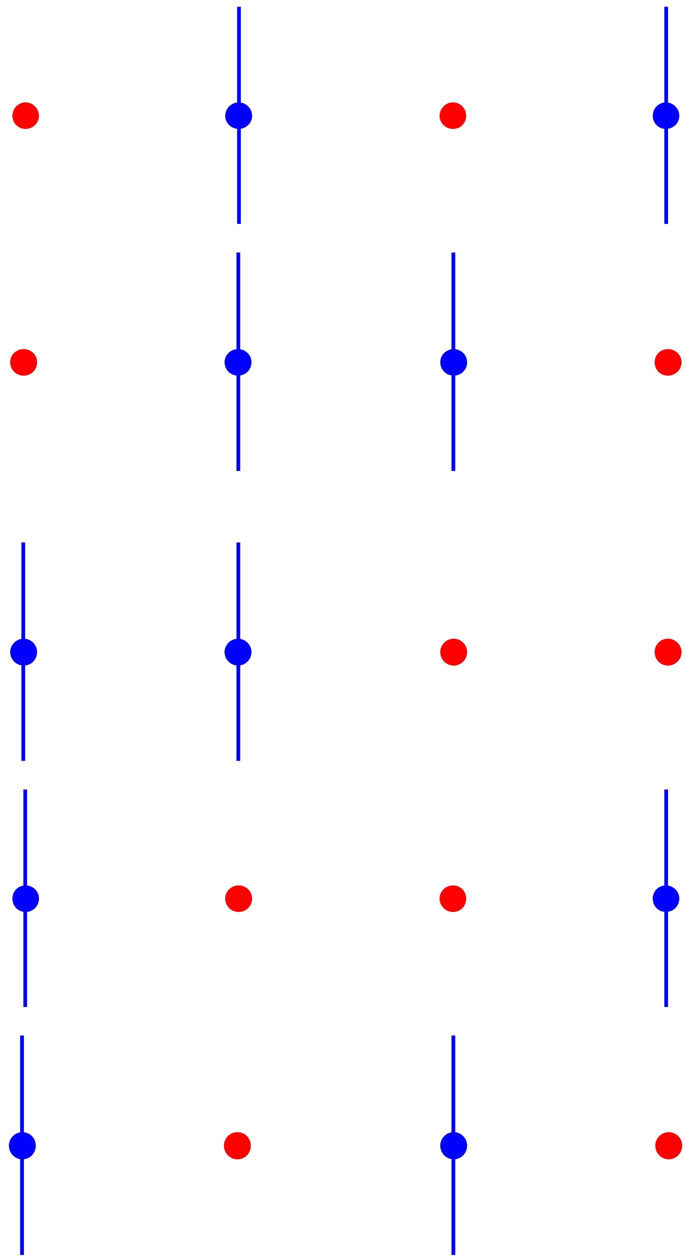
$$\frac{4!}{2!2!}=6$$

This problem is equivalent to the following problem,

$$x_1 + x_2 + x_3 = 2$$

The number of ways of distributing 2 red balls (instead of using A) among 3 labelled bars (|) (instead of using bowls). As is evident from **Fig**. one can get this number by finding the number of permutations of placing in a row all the red balls together with (3-1) = 2 that designate the dividing walls.





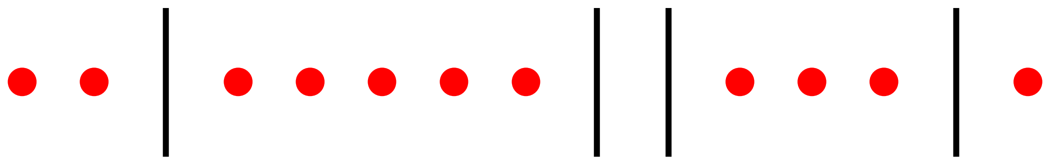
So there are 6 arrangements;

$$\frac{4!}{2!2!} = 6$$

((**Note**))

For convenience, we use the black balls instead of the bar. If one labels all the balls with the running numbers, 1, 2, 3, …, $n+f-1$, the number of permutations is $(n+f-1)!$. Note that the numbers of permutations among the balls with the same color are given by $n!$ for red balls and $(f-1)!$ for black balls. Thus we have

$$W_n = \frac{(f+n-1)!}{n!(f-1)!}$$



which is equivalent to

by replacing the bars by the black dots.