Boltzmann factor for the kinetic energy of free particles:
Maxwell-Boltzmann distribution
Masatsugu Sei Suzuki
Department of Physics
(Date: September 06, 2018)

We now consider the Boltzmann factor for the kinetic energy of free particles with a mass *m*. We derive the form of Maxwell-Boltzmann distribution function/

The kinetic energy of the particle is given by

$$E_k = \frac{1}{2}mv^2$$

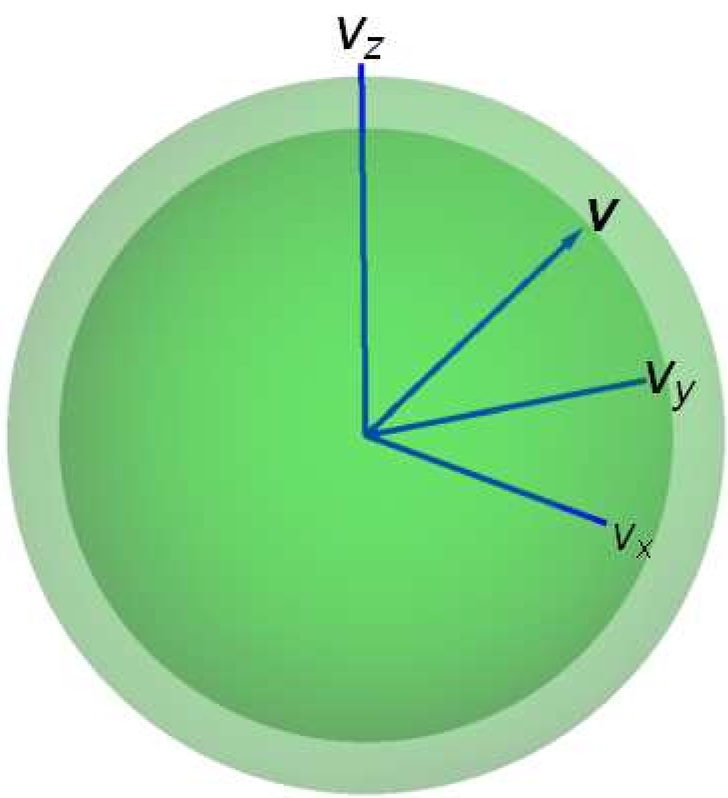
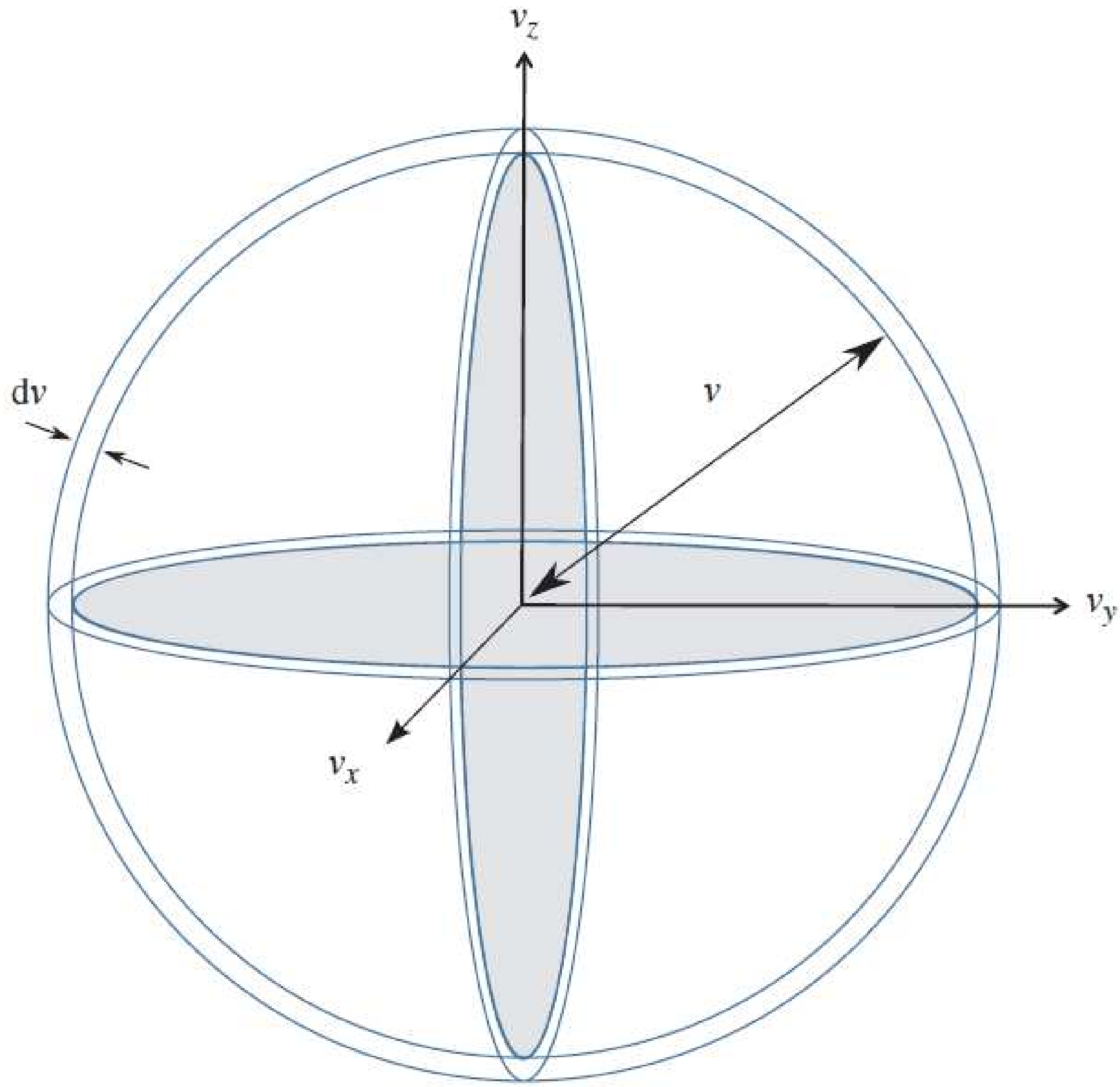


Fig. Velocity space. The kinetic energy is the same for the different state (v - $v + dv$).



The probability of finding the particles between v and $v + dv$ is

$$f(v)dv = \frac{4\pi v^2 dv \exp(-\frac{\beta m v^2}{2})}{\int_0^\infty 4\pi v^2 dv \exp(-\frac{\beta m v^2}{2})}.$$

Here we have

$$\int_0^\infty 4\pi v^2 dv \exp(-\frac{\beta m v^2}{2}) = 4\pi \sqrt{\frac{\pi}{2}} (m\beta)^{-3/2} = (\frac{m\beta}{2\pi})^{-3/2}$$

The Maxwell-Boltzmann distribution function is

$$\begin{aligned} f(v) &= (\frac{m\beta}{2\pi})^{3/2} 4\pi v^2 \exp(-\frac{\beta m v^2}{2}) \\ &= (\frac{m}{2\pi k_B T})^{3/2} 4\pi v^2 \exp(-\frac{m v^2}{2 k_B T}) \end{aligned}$$

where

$$\int_0^\infty f(v) dv = 1$$

The average velocity:

$$\langle v \rangle = \int_0^\infty v f(v) dv = 2\sqrt{\frac{2}{\pi}} (m\beta)^{-1/2} = \sqrt{\frac{8 k_B T}{\pi m}}$$

The variance:

$$\langle v^2 \rangle = \int_0^\infty v^2 f(v) dv = \frac{3}{m\beta} = \frac{3 k_B T}{m}$$

This can be rewritten as

$$\frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} k_B T \qquad \text{(equi-partition theorem)}$$

The root-mean square velocity is

$$v_{rms} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3 k_B T}{m}}.$$

What is the most probable velocity in which $f(v)$ takes a maximum.

$$\frac{df}{dv} = 0$$

$$2v\exp(-\frac{mv^2}{2k_BT})+v^2(-\frac{mv}{k_BT})\exp(-\frac{mv^2}{2k_BT})=0$$

or

$$v_{probable}=\sqrt{\frac{2k_BT}{m}}$$

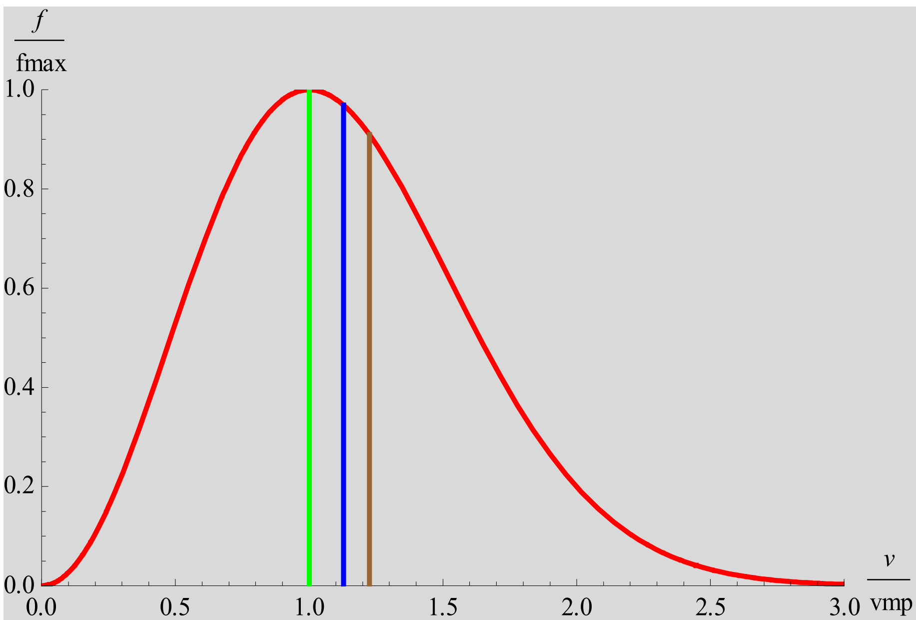


Fig. Plot of the normalized $f(v)/f_{\max}$ as a function of a normalized v/v_{mp}.

Green:	most probable speed	(v_{mp})
Blue:	averaged speed	(v_{avg})
Brown:	root-mean squared speed	(v_{rms})

$$1<\frac{v_{avg}}{v_{mp}}(=1.128)<\frac{v_{rms}}{v_{mp}}(=\frac{\sqrt{6}}{2}=1.225)$$

((**Example**)) Neutron (obeying the Maxwell Boltzmann distribution)

(a) $T = 300$ K.

$$E = \frac{1}{2} m_n v_p{}^2 = 25.852 \text{ meV}, \qquad \lambda = \frac{h}{m_n v_p} = 1.77885 \text{ Å}$$

(b) $T = 20$ K (liquid hydrogen or deuterium)

$$E = \frac{1}{2} m_n v_p{}^2 = 1.723472 \text{ meV}, \qquad \lambda = \frac{h}{m_n v_p} = 6.8895 \text{ Å}$$

((**Note-1**)) The degeneracy of the state

The Boltzmann factor is given by

$$\exp(-\frac{\beta}{2} m v^2)$$

Each states with the velocity having v and $v + dv$, has the same Boltzmann factor. In other words, there is a degeneracy of states $4\pi v^2 dv$.

((**Note-2**)) Definition of $f(v)$

$$N = \int N(\mathbf{v}) d\mathbf{v} = \int 4\pi v^2 N(v)$$

or

$$1 = \int 4\pi v^2 dv \frac{N(v)}{N} = \int 4\pi v^2 dv \overline{n}(v) = \int f(v) dv$$

where

$$f(v) = 4\pi v^2 \overline{n}(v), \qquad \overline{n}(v) = \frac{N(v)}{N}$$

N is the total number of particles.

((**Mathematica**))

```
Clear["Global`*"];
```

$$A1 = \int_0^\infty 4\pi v^2 \, \mathrm{Exp}\left[-\frac{\beta}{2} m\, v^2\right] dv \;//\; \mathrm{Simplify}[\#, \{m > 0, \beta > 0\}]\,\&$$

$$\frac{2\sqrt{2}\,\pi^{3/2}}{(m\,\beta)^{3/2}}$$

$$f1 = \frac{1}{A1} 4\pi v^2 \, \mathrm{Exp}\left[-\frac{\beta}{2} m\, v^2\right]$$

$$e^{-\frac{1}{2} m v^2 \beta} \sqrt{\frac{2}{\pi}}\, v^2\, (m\,\beta)^{3/2}$$

$$\int_0^\infty f1\, dv \;//\; \mathrm{Simplify}[\#, \{m > 0, \beta > 0\}]\,\&$$

1

```
vav = ∫_0^∞ v f1 ⅆv // Simplify[#, {m > 0, β > 0}] &
```

$$\frac{2\sqrt{\frac{2}{\pi}}}{\sqrt{m\,\beta}}$$

```
vsq = ∫_0^∞ v^2 f1 ⅆv // Simplify[#, {m > 0, β > 0}] &
```

$$\frac{3}{m\,\beta}$$

```
δv = √vsq // Simplify
```

$$\sqrt{3}\,\sqrt{\frac{1}{m\,\beta}}$$

APPENDIX Gauss integrals

$$\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

We take a derivative of this equation with respect to a

$$\int_0^\infty (-x^2)e^{-ax^2}dx = -\frac{\sqrt{\pi}}{4}a^{-\frac{3}{2}}$$

leading to the formula given by

$$\int_0^\infty x^2 e^{-ax^2}dx = \frac{\sqrt{\pi}}{4a^{3/2}}.$$