Up and down under the gravity

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton

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Huang: Introduction to Statistical Mechanics
Problem 8-3

Problem 8-3 (Huang)

A chain made of N massless segments on equal length a hangs from a fixed point. A mass m is attached to the other end under gravity. Each segment can be in either of two states, up or down, as illustrated in the sketch. The segments have no mass, and the chain can go as far up as it can: there is no ceiling.

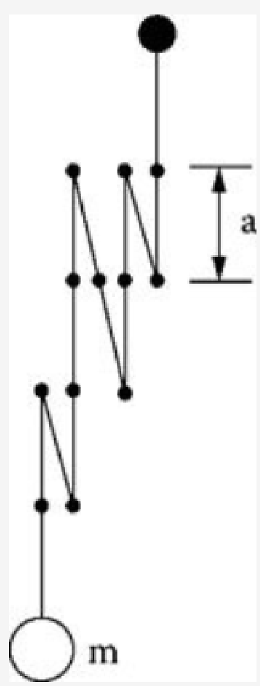
(a) Show that the partition function at temperature T is given by

$$Z_N = (1 + e^{-2\beta mga})^N .$$

(b) Find the entropy of the chain.

(c) Find the internal energy, and determine the length if the chain.

(d) Show that the chain obeys Hooke's law, namely, a small force pulling on the chain increases its length proportionately. Find the proportionality constant.



((**Solution**))

(a)

There are two kinds of segments shifting to upward and downward. Under the gravity, the downward segment contributes to the potential energy of $-mga$. The upward segment contributes to the potential energy of mga. The number of upward segments is N_1 and the number of downward segment is N_2. The total number N is

$$N_1 + N_2 = N .$$

The total length L is

$$L = |N_1 - N_2| a$$

The energy is

$$\begin{aligned} E(N_1, N_2) &= -mgaN_1 + mgaN_2 \\ &= -mga(N_1 - N_2) \quad . \\ &= -mga(2N_1 - N) \end{aligned}$$

Thus we have the partition function Z_{CN} for the N segment system is

$$\begin{aligned} Z_{CN} &= \sum_{N_1=0}^{N} \frac{N!}{N_1! N_2!} e^{-\beta E(N_1, N_2)} \\ &= \sum_{N_1=0}^{N} \frac{N!}{N_1!(N-N_1)!} e^{\beta mga(2N_1 - N)} \\ &= e^{-\beta mgaN} \sum_{N_1=0}^{N} \frac{N!}{N_1!(N-N_1)!} e^{2\beta mgaN_1} \\ &= e^{-\beta mga\beta} (1 + e^{2\beta mga})^N \\ &= (e^{\beta mga} + e^{-\beta mga} \_)^N \end{aligned}$$

or

$$Z_{CN} = [2\cosh(\beta mga)]^N$$

We note that the one segment partition function Z_{C1} is

$$Z_{C1} = e^{\beta mga} + e^{-\beta mga}$$

So the N-segment partition function Z_{CN} is obtained as

$$Z_{CN} = (Z_{C1})^N = (e^{\beta mga} + e^{-\beta mga})^N = [2\cosh(\beta mga)]^N$$

(b) and (c)

The Helmholtz free energy:

$$F = -k_B T \ln Z_{CN} = -Nk_B T \ln[2\cosh(\beta mga)]$$

The entropy S is

$$S = -\frac{\partial F}{\partial T} = Nk_B \ln[2\cosh(\beta mga)] - Nk_B mga\beta \tanh(\beta mga)$$

The internal energy U is

$$U = -\frac{\partial}{\partial \beta} \ln Z_{CN} = -Nmga \tanh(\beta mga)$$

The length of L is

$$\begin{aligned}\langle L \rangle &= a\langle |N_1 - N_2| \rangle \\ &= \frac{1}{mg}\langle |E(N_1, N_2)| \rangle \\ &= \frac{1}{mg}|U| \\ &= \frac{1}{mg} Nmga \tanh(\beta mga) \\ &= Na \tanh(\beta mga)\end{aligned}$$

or

$$\langle L \rangle = Na \tanh(\beta mga) = y$$

and

$$\beta mga = arc\tanh(\frac{y}{Na})$$

(d)
Since $dF = -PdV - SdT$, we have the force *f* defined by

$$f = PA = -A\left(\frac{\partial F}{\partial V}\right)_T = -\left(\frac{\partial F}{\partial y}\right)_T .$$

Since

$$F = -Nk_B T\ln[2\cosh(\beta mga)] = -Nk_B T\ln[2\cosh(arc\tanh(\frac{y}{Na})]$$

the force *f* is evaluated as

$$f = Nk_B T\left(\frac{\partial}{\partial y}\ln[2\cosh(arc\tanh(\frac{y}{Na})]\right)_T$$
$$= Nk_B T\frac{y}{N^2a^2 - y^2}$$

f can be approximated by

$$f \approx Nk_B T\frac{y}{N^2a^2} = \frac{1}{N\beta a^2}y \approx \frac{1}{N\beta a^2}Na(\beta mga) = mg$$

So the force *f* is equal to *mg*.

Note that the exact expression of *f* is given by

$$f = \frac{1}{2}Nk_B T\sinh(2\beta mga).$$

REFERENCES
K. Huang, Introduction to Statistical Physics, second edition (CRC Press, 2010).