From Canonical ensemble to Grand canonical ensemble
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We summarize the content of Chapter 8 of the textbook (R. Baierlein).

1. Approach from the canonical ensemble (occupation number representation)

The partition function of canonical ensemble (occupation number representation), is given by

$$Z_C(\bar{N}) = \sum_{n_1, n_2,..} e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 + ...)}\delta(n_1 + n_2 + ..., \bar{N})$$

where

$$E = n_1\varepsilon_1 + n_2\varepsilon_2 + ... \quad \text{(the energy of the system)}$$

$$N = n_1 + n_2 + ... \quad \text{(the number of particles in the system, fixed)}$$

2. Fermi-Dirac distribution function

We consider the fermion system obeying the Pauli exclusion principle

$$\begin{aligned}
\langle n_1 \rangle &= \frac{1}{Z_C(\bar{N})} \sum_{n_1, n_2,..} n_1 e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 + ...)}\delta(n_1 + n_2 + ..., \bar{N}) \\
&= \frac{1}{Z_C(\bar{N})} \sum_{n_2, n_3,..} e^{-\beta(\varepsilon_1 + n_2\varepsilon_2 + n_3\varepsilon_3 + ...)}\delta(1 + n_2 + ..., \bar{N}) \\
&\quad + \frac{1}{Z_C(\bar{N})} \sum_{n_2, n_3,..} 0\ e^{-\beta(n_2\varepsilon_2 + + n_3\varepsilon_3 + ...)}\delta(0 + n_2 + ..., \bar{N}) \\
&= \frac{1}{Z_C(\bar{N})} e^{-\beta\varepsilon_1} \sum_{n_2, n_3,..} e^{-\beta(n_2\varepsilon_2 + n_3\varepsilon_3 + ...)}\delta(n_2 + n_3 + ..., \bar{N} - 1)
\end{aligned}$$

where $n_i = 0,\ 1$ for fermions. We also Note that

$$\frac{1}{Z_C(\bar{N})}e^{-\beta\varepsilon_1}\sum_{n_1,n_2,..}(1-n_1)e^{-\beta(n_1\varepsilon_1+n_2\varepsilon_2+...)}\delta(n_1+n_2+...,\bar{N}-1)$$
$$=\frac{1}{Z_C(\bar{N})}e^{-\beta\varepsilon_1}\sum_{n_2,n_3,..}(1-0)e^{-\beta(0\varepsilon_1+n_2\varepsilon_2+...)}\delta(0+n_2+...,\bar{N}-1)$$
$$+\frac{1}{Z_C(\bar{N})}e^{-\beta\varepsilon_1}\sum_{n_2,n_3,..}(1-1)e^{-\beta(1\varepsilon_1+n_2\varepsilon_2+...)}\delta(1+n_2+...,\bar{N}-1)$$
$$=\frac{1}{Z_C(\bar{N})}e^{-\beta\varepsilon_1}\sum_{n_2,n_3,..}e^{-\beta(n_2\varepsilon_2+...)}\delta(n_2+n_3+...,\bar{N}-1)$$
$$=\langle n_1\rangle$$

From the definition of the partition function, we have

$$Z_C(\bar{N})=\sum_{n_1,n_2,..}e^{-\beta(n_1\varepsilon_1+n_2\varepsilon_2+...)}\delta(n_1+n_2+...,\bar{N})$$

and

$$Z_C(\bar{N}-1)=\sum_{n_1,n_2,..}e^{-\beta(n_1\varepsilon_1+n_2\varepsilon_2+...)}\delta(n_1+n_2+...,\bar{N}-1)$$

Then we get

$$\begin{aligned}\langle n_1\rangle_{\bar{N}}&=\frac{1}{Z_C(\bar{N})}e^{-\beta\varepsilon_1}\sum_{n_1,n_2,..}(1-n_1)e^{-\beta(n_1\varepsilon_1+n_2\varepsilon_2+...)}\delta(n_1+n_2+...,\bar{N}-1)\\&=\frac{1}{Z_C(\bar{N})}e^{-\beta\varepsilon_1}[Z_C(\bar{N}-1)-Z_C(\bar{N}-1)\langle n_1\rangle_{\bar{N}-1}]\\&=\frac{Z_C(\bar{N}-1)}{Z_C(\bar{N})}e^{-\beta\varepsilon_1}[1-\langle n_1\rangle_{\bar{N}-1}]\end{aligned}$$

where

$$Z_C(\bar{N}-1)=\sum_{n_1,n_2,..}e^{-\beta(n_1\varepsilon_1+n_2\varepsilon_2+...)}\delta(n_1+n_2+...,\bar{N}-1)$$

$$\langle n_1\rangle_{\bar{N}_1-1}=\frac{1}{Z_C(\bar{N}-1)}\sum_{n_1,n_2,..}n_1e^{-\beta(n_1\varepsilon_1+n_2\varepsilon_2+...)}\delta(n_1+n_2+...,\bar{N}-1)$$

Here we use $N \to \bar{N}$. The average number $\bar{N} = \langle N \rangle_G$ is the same for both the canonical ensemble and the grand canonical ensemble. The chemical potential is defined by

$$\mu = \left(\frac{\partial F}{\partial \bar{N}}\right)_{T,V}$$
$$= F(\bar{N}) - F(\bar{N}-1)$$
$$= -\frac{1}{\beta}\ln[\frac{Z_C(\bar{N})}{Z_C(\bar{N}-1)}]$$

in the canonical ensemble, where $dF = -SdT - PdV + \mu dN$ and

$$F(\bar{N}) = -k_B T \ln Z_C(\bar{N}) = -\frac{1}{\beta}\ln Z_C(\bar{N})$$

From the above equation, we get

$$e^{\beta\mu} = \frac{Z_C(\bar{N}-1)}{Z_C(\bar{N})} = \frac{(z)^{-(\bar{N}-1)}Z_G(z)}{(z)^{-\bar{N}}Z_G(z)} = z$$

In the limit $\bar{N} \to \infty$, we have $\langle n_1 \rangle_{\bar{N}} \approx \langle n_1 \rangle_{\bar{N}-1}$

$$\langle n_1 \rangle_{\bar{N}} = \frac{Z_C(\bar{N}-1)}{Z_C(\bar{N})} e^{-\beta\varepsilon_1}[1 - \langle n_1 \rangle_{\bar{N}-1}]$$
$$\approx \frac{Z_C(\bar{N}-1)}{Z_C(\bar{N})} e^{-\beta\varepsilon_1}[1 - \langle n_1 \rangle_{\bar{N}}]$$
$$= e^{\beta(\mu-\varepsilon_1)}[1 - \langle n_1 \rangle_{\bar{N}}]$$

Thus we can derive the expression of the Fermi-Dirac distribution function,

$$\langle n_1 \rangle_{\bar{N}} \approx \langle n_1 \rangle_{\bar{N}-1} = \frac{1}{e^{\beta(\varepsilon_1-\mu)}+1} \qquad \text{(Fermi-Dirac distribution function)}$$

In general, we have

$$\langle n_a \rangle_{\bar{N}} = \frac{1}{e^{\beta(\varepsilon_a-\mu)}+1}$$

((**Note**))

Here we use the approximation that

$$Z_G(z) = \sum_{N=0}^{\infty} z^N Z_C(N) \approx z^{\bar{N}} Z_C(\bar{N})$$

where $z = e^{\beta\mu}$. We note that $z^N Z_C(N)$ has a very sharp peak at $N = \bar{N}$. In other words,

$$g(N) = \ln[z^N Z_C(N)] = N \ln z + \ln[Z_C(N)]$$

has a local maximum at $N = \bar{N}$.

$$g'(\bar{N}) = \ln z + \frac{\partial \ln[Z_C(\bar{N})]}{\partial \bar{N}} = 0$$

which means that

$$\mu = \left(\frac{\partial F(N)}{\partial N}\right)_{N=\bar{N}} = \left(\frac{\partial F(\bar{N})}{\partial \bar{N}}\right)$$

3. Bose-Einstein distribution function

We start with the canonical ensemble

$$\langle n_1 \rangle = \frac{1}{Z_C(\bar{N})} \sum_{n_1, n_2, ..} n_1 e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 + ...)} \delta(n_1 + n_2 + ..., \bar{N})$$

where $n_1 = 0, 1, 2, 3, 4, ..., \bar{N}$. We put $n_1 = n_1' + 1$.

$$\begin{aligned}
\langle n_1 \rangle_{\bar{N}} &= \frac{1}{Z_C(\bar{N})} e^{-\beta\varepsilon_1} \sum_{n_1', n_2, ..} (n_1' + 1) e^{-\beta(n_1'\varepsilon_1 + n_2\varepsilon_2 + ...)} \delta(n_1' + n_2 + ..., \bar{N} - 1) \\
&= \frac{1}{Z_C(\bar{N})} e^{-\beta\varepsilon_1} \sum_{n_1', n_2, ..} (n_1' + 1) e^{-\beta(n_1'\varepsilon_1 + n_2\varepsilon_2 + ...)} \delta(n_1' + n_2 + ..., \bar{N} - 1) \\
&= \frac{1}{Z_C(\bar{N})} e^{-\beta\varepsilon_1} [Z_C(\bar{N} - 1) + Z_C(\bar{N} - 1) \langle n_1 \rangle_{\bar{N}-1}] \\
&= \frac{Z_G(\bar{N} - 1)}{Z_C(\bar{N})} e^{-\beta\varepsilon_1} [1 + \langle n_1 \rangle_{\bar{N}-1}]
\end{aligned}$$

We use the relation

$$e^{\beta\mu} = \frac{Z_C(\bar{N}-1)}{Z_C(\bar{N})},$$

We also assume that

$$\langle n_1 \rangle_{\bar{N}-1} \approx \langle n_1 \rangle_{\bar{N}}$$

in the limit of $\bar{N} \to \infty$. Thus we get

$$\langle n_1 \rangle_{\bar{N}} = e^{\beta(\mu-\varepsilon_1)}[1 + \langle n_1 \rangle_{\bar{N}}]$$

or

$$\langle n_1 \rangle_{\bar{N}} [1 - e^{\beta(\mu-\varepsilon_1)}] = e^{\beta(\mu-\varepsilon_1)}$$

or

$$\langle n_1 \rangle_{\bar{N}} = \frac{e^{\beta(\mu-\varepsilon_1)}}{1 - e^{\beta(\mu-\varepsilon_1)}} = \frac{1}{e^{\beta(\varepsilon_1-\mu)} - 1} \qquad \text{(Bose-Einstein distribution)}$$

In general, we have

$$\boxed{\langle n_a \rangle_{\bar{N}} = \frac{1}{e^{\beta(\varepsilon_a-\mu)} - 1}}$$

4. Limits: classical and semi-classical

$$\langle n_a \rangle = e^{-\beta(\varepsilon_a-\mu)} = e^{\beta\mu} e^{-\beta\varepsilon_a}$$

$$\bar{N} = \sum_a \langle n_a \rangle = e^{\beta\mu} \sum_a e^{-\beta\varepsilon_a} = e^{\beta\mu} Z_C(1)$$

or

$$e^{\beta\mu} = \frac{\bar{N}}{Z_C(1)}$$

or

$$\beta\mu = -\ln\left(\frac{Z_C(1)}{\bar{N}}\right)$$

or

$$\mu = -k_B T \ln\left(\frac{Z_C(1)}{\bar{N}}\right)$$

Using this expression for the chemical potential, the above expression of $\langle n_a \rangle$ can be rewritten as

$$\langle n_a \rangle = \frac{\bar{N}}{Z_C(1)} e^{-\beta\varepsilon_a}$$ (Maxwell-Boltzmann distribution function)

4. The semi-classical limit of the partition function

We start with the above relations

$$e^{\beta\mu} = \frac{\bar{N}}{Z_C(1)}$$

and

$$e^{\beta\mu} = \frac{Z_C(\bar{N}-1)}{Z_C(\bar{N})}$$

Form these equation we get the recursion formula for $Z_C(\bar{N})$

$$e^{\beta\mu} = \frac{Z_C(\bar{N}-1)}{Z_C(\bar{N})} = \frac{\bar{N}}{Z_C(1)}$$

or

$$Z_C(\bar{N}) = \frac{Z_C(1)}{\bar{N}} Z_C(\bar{N}-1)$$

Using this relation, we get

$$\begin{aligned}
Z_C(\bar{N}) &= \frac{Z_C(1)}{\bar{N}} \frac{Z_C(1)}{\bar{N}-1} Z_C(\bar{N}-2) \\
&= \frac{Z_C(1)}{\bar{N}} \frac{Z_C(1)}{\bar{N}-1} \frac{Z_C(1)}{\bar{N}-2} \cdots\cdots \frac{Z_C(1)}{2} Z_C(1) \\
&= \frac{Z_{C1}{}^{\bar{N}}}{\bar{N}!}
\end{aligned}$$

or

$$\boxed{Z_C(\bar{N}) = \frac{Z_{C1}{}^{\bar{N}}}{\bar{N}!}}$$

which is the partition function of identical N particles. Note that

$$Z_{C1} = Z_C(1)$$

5. Pressure

From the thermodynamics property

$$dU = TdS - PdV$$

the pressure can be obtained as

$$P = -\frac{\partial U}{\partial V} = -\frac{1}{Z} \sum_j \left(\frac{\partial E_j}{\partial V} \right) e^{-\beta E_j} = -\sum_j \left(\frac{\partial E_j}{\partial V} \right) \frac{e^{-\beta E_j}}{Z}$$

For the energy eigenvalue for the one-particle energy

$$\varepsilon_a = \frac{\hbar^2}{2m} k^2 = \frac{\hbar^2}{2m} \left(\frac{2\pi}{L} n \right)^2 = \frac{\hbar^2 (2\pi n)^2}{2m} L^{-2} = \frac{\hbar^2 (2\pi n)^2}{2m} V^{-\frac{2}{3}} \qquad \text{with} \qquad k = \frac{2\pi}{L} n$$

$$\frac{\partial \varepsilon_a}{\partial V} = -\frac{2}{3} \frac{\varepsilon_a}{V}$$

For the energy eigenvalue for the *N*-particle energy,

$$\frac{\partial E_j}{\partial V} = -\frac{2}{3}\frac{E_j}{V}$$

Thus we have the pressure as

$$P = \frac{2}{3V}\sum_j E_j \frac{e^{-\beta E_j}}{Z} = \frac{2U}{3V}$$

or

$$PV = \frac{2}{3}U\,.$$

We now calculate the internal energy

$$\begin{aligned} U &= \frac{1}{Z(N)}\sum_{n_1,n_2,n_3,\ldots}(n_1\varepsilon_1 + n_2\varepsilon_2 + \ldots)e^{-(\beta n_1\varepsilon_1 + \beta n_2\varepsilon_2 + \ldots)}\delta(n_1 + n_2 + \ldots, N) \\ &= \varepsilon_1\langle n_1\rangle + \varepsilon_2\langle n_2\rangle + \ldots. \\ &= \sum_a \varepsilon_a\langle n_a\rangle \end{aligned}$$

or

$$U = \sum_a \varepsilon_a\langle n_a\rangle = \sum_a \frac{\varepsilon_a}{e^{\beta(\varepsilon_a-\mu)} \pm 1}$$

Note that

$$\langle n_a\rangle = \frac{1}{e^{\beta(\varepsilon_a-\mu)} \pm 1} = e^{\beta(\varepsilon_a-\mu)} \mp e^{2\beta(\varepsilon_a-\mu)}$$

The chemical potential is determined by the condition that

$$N = \sum_a \langle n_a\rangle = \sum_a e^{\beta(\varepsilon_a-\mu)} \mp \sum_a e^{2\beta(\varepsilon_a-\mu)}$$

For the one-particle state with the free energy:

$$\sum_a = \sum_{\boldsymbol{k}}$$

$$D(\varepsilon)d\varepsilon = \frac{(2s+1)V}{(2\pi)^3} 4\pi k^2 dk$$
$$= \frac{(2s+1)V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon}d\varepsilon$$

The density of states:

$$D(\varepsilon) = \frac{(2s+1)V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon}$$

where $s = \frac{1}{2}$ for electron

$$\varepsilon = \frac{\hbar^2 k^2}{2m}$$

((**Note**)) Classical limit

$$\langle n_a \rangle = e^{-\beta(\varepsilon_a - \mu)} = e^{\beta\mu} e^{-\beta\varepsilon_a}$$

$$N = \sum_a \langle n_a \rangle = e^{\beta\mu} \sum_a e^{-\beta\varepsilon_a} = e^{\beta\mu} Z_{C1}$$

$$e^{\beta\mu} = \frac{N}{Z_{C1}}$$

or

$$\beta\mu = -\ln\left(\frac{Z_{C1}}{N}\right)$$

or

$$\mu = -k_B T \ln\left(\frac{Z_{C1}}{N}\right)$$

Note that

$$U = -N\frac{\partial}{\partial\beta}\ln Z_{C1} = \sum_a \varepsilon_a \langle n_a\rangle = \frac{N}{Z_{C1}}\sum_a \varepsilon_a e^{-\beta\varepsilon_a} = N\sum_a \varepsilon_a P_a$$

and

$$\langle n_a\rangle = \frac{N}{Z_{C1}} e^{-\beta\varepsilon_a}$$

We now calculate the partition function Z_{C1} as

$$\begin{aligned}
Z_{C1} &= \frac{V(2s+1)}{(2\pi)^3}\int_0^\infty 4\pi k^2 dk e^{-\frac{\beta\hbar^2 k^2}{2m}} \\
&= \frac{V(2s+1)}{2\pi^2}\left(\frac{2m}{\beta\hbar^2}\right)^{3/2}\int_0^\infty dx x^2 e^{-x^2} \\
&= \frac{V(2s+1)}{\pi^2}\left(\frac{2m}{\beta\hbar^2}\right)^{3/2}\frac{\sqrt{\pi}}{2^3} \\
&= V(2s+1)\left(\frac{mk_B T}{2\pi\hbar^2}\right)^{3/2} \\
&= V(2s+1)\lambda_{th}{}^3
\end{aligned}$$

or

$$Z_{C1} = \frac{V(2s+1)}{n_Q}$$

where the thermal wavelength is

$$\lambda_{th} = \left(\frac{2\pi\hbar^2}{mk_B T}\right)^{1/2}. \qquad n_Q = \frac{1}{\lambda_{th}{}^3}$$

(i) Internal energy

$$U = -N\frac{\partial}{\partial\beta}\ln Z_{C1} = \frac{3}{2}Nk_BT$$

(ii) Helmholtz free energy

$$F = -Nk_BT\ln Z_{C1} = -Nk_BT\ln[\frac{V(2s+1)}{n_Q}]$$

(iii) Pressure

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{Nk_BT}{V}$$

(iv) Chemical potential

$$\mu = k_BT\ln[(2s+1)\frac{n}{n_Q}]$$

(v) Helmholtz free energy and chemical potential depend on the number of spin *s*.

6. Probability

The probability is defined as

$$P[E_s(N),N] = \frac{1}{Z_G}z^N e^{-\beta E_s(N)}$$

$E_s(N)$ is the energy eigenvalue of *N*-particle eigenfunction.

$$\sum_{N,s}P[E_s(N),N] = \frac{1}{Z_G}\sum_{N,s}z^N e^{-\beta E_s(N)} = 1$$

where the partition function of the grand canonical ensemble,

$$Z_G = \sum_{N,s}z^N e^{-\beta E_s(N)} = \sum_{N,s}e^{\mu N}e^{-\beta E_s(N)}$$

with

$$z = e^{\beta\mu}$$

$E_s(N)$ is the energy eigenvalue for the N particles. The average number of particle number:

$$\langle N \rangle = \sum_{N,s} N z^N e^{-\beta E_s(N)}$$

$$\langle N \rangle = \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln Z_G = \frac{1}{\beta Z_G}\frac{\partial Z_G}{\partial\mu} = \frac{1}{Z_G}\sum_{N,s} N z^N e^{-\beta E_s(N)}$$

We note that the partition function of the N particle (canonical ensemble) is defined by

$$Z_C(N) = \sum_s e^{-\beta E_s(N)}$$

Thus Z_G is expressed by

$$Z_G = \sum_N e^{\beta\mu N} Z_C(N)$$

7. Occupation number representation:

$$E_s(N) = n_1\varepsilon_1 + n_2\varepsilon_2 + ...$$

$$N = n_1 + n_2 + ...$$

where ε_1, ε_2,… are the energy eigenvalues of one-particle system.

$$\begin{aligned}
Z_G &= \sum_N \sum_{\{n_1,n_2,..\}} e^{\mu(n_1+n_2+...)} e^{-\beta(\varepsilon_1 n_1+\varepsilon_2 n_2+...)} \delta[n_1+n_2+...-N] \\
&= \sum_N \sum_{\{n_1,n_2,..\}} e^{\mu(n_1+n_2+...)} e^{-\beta(\varepsilon_1 n_1+\varepsilon_2 n_2+...)} \delta[n_1+n_2+...-N] \\
&= \sum_{\{n_1,n_2,..\}} e^{\beta(\mu-\varepsilon_1)n_1} e^{\beta(\mu-\varepsilon_1)n_2} .... \\
&= \sum_{n_1} e^{\beta(\mu-\varepsilon_1)n_1} \sum_{n_2} e^{\beta(\mu-\varepsilon_1)n_2} ....
\end{aligned} \tag{1}$$

It is difficult to perform the summation in Eq.(1) in the canonical ensemble, because of the restriction $N = n_1 + n_2 + ...$ In the grand partition function with the summation $\sum_N$ the calculation can be made much more easier. The problem is reduced to one-particle problem.

For fermions; n = 0 and 1

$$Z_G = [1 + e^{\beta(\mu-\varepsilon_1)}][1 + e^{\beta(\mu-\varepsilon_2)}]........$$

$$\begin{aligned}\ln Z_G &= \ln[1 + e^{\beta(\mu-\varepsilon_1)}] + \ln[1 + e^{\beta(\mu-\varepsilon_2)}] + ...\\ &= \sum_{\boldsymbol{k}} \ln[1 + e^{\beta(\mu-\varepsilon_{\boldsymbol{k}})}]\end{aligned}$$

$$\begin{aligned}\frac{1}{\beta}\frac{\partial}{\partial\mu}\ln Z_G &= z\frac{\partial}{\partial z}\ln Z_G\\ &= \sum_{\boldsymbol{k}} \frac{e^{\beta(\mu-\varepsilon_{\boldsymbol{k}})}}{1 + e^{\beta(\mu-\varepsilon_{\boldsymbol{k}})}}\\ &= \sum_{\boldsymbol{k}} \frac{1}{e^{\beta(\varepsilon_{\boldsymbol{k}}-\mu)} + 1}\end{aligned}$$

The Fermi-Dirac distribution function;

$$f_{\boldsymbol{k}} = \frac{1}{e^{\beta(\varepsilon_{\boldsymbol{k}}-\mu)} + 1}$$

For bosons; n = 0, 1, 2, …….

$$Z_G = [1 - e^{\beta(\mu-\varepsilon_1)}]^{-1}[1 - e^{\beta(\mu-\varepsilon_2)}]^{-1}........$$

$$\begin{aligned}\ln Z_G &= -\ln[1 - e^{\beta(\mu-\varepsilon_1)}] - \ln[1 - e^{\beta(\mu-\varepsilon_2)}] + ...\\ &= -\sum_{\boldsymbol{k}} \ln[1 - e^{\beta(\mu-\varepsilon_{\boldsymbol{k}})}]\end{aligned}$$

$$\frac{1}{\beta}\frac{\partial}{\partial\mu}\ln Z_G = z\frac{\partial}{\partial z}\ln Z_G$$
$$= -\sum_{\boldsymbol{k}}\frac{-e^{\beta(\mu-\varepsilon_{\boldsymbol{k}})}}{1-e^{\beta(\mu-\varepsilon_{\boldsymbol{k}})}}$$
$$= \sum_{\boldsymbol{k}}\frac{1}{e^{\beta(\varepsilon_{\boldsymbol{k}}-\mu)}-1}$$

Bose-Einstein distribution function;

$$n_{\boldsymbol{k}} = \frac{1}{e^{\beta(\varepsilon_{\boldsymbol{k}}-\mu)}-1}.$$

REFERENCES

R. Baierlein, Thermal Physics (Cambridge, 1999).

APPENDIX-I

$$z\frac{\partial}{\partial z} = \frac{\partial}{\partial \ln z} = \frac{\partial}{\partial(\beta\mu)} = \frac{1}{\beta}\frac{\partial}{\partial\mu} = k_B T\frac{\partial}{\partial\mu}$$

APPENDIX

(a) Grand canonical ensemble:

$$\langle N\rangle_G = z\frac{\partial}{\partial z}\ln Z_G$$

The partition function of the canonical ensemble:

$$Z_{CN} = \frac{(Z_{C1})^N}{N!}$$

The partition function of the grand canonical ensemble;

$$
\begin{aligned}
Z_G &= \sum_N z^N Z_{CN} \\
&= \sum_N z^N \frac{(Z_{C1})^N}{N!} \\
&= \sum_N \frac{(zZ_{C1})^N}{N!} \\
&= \exp(zZ_{C1})
\end{aligned}
$$

or

$$\ln Z_G = zZ_{C1}$$

Thus we have

$$\boxed{\langle N \rangle_G = z\frac{\partial}{\partial z}\ln Z_G = z\frac{\partial}{\partial z}(zZ_{C1}) = zZ_{C1}}$$

or

$$\langle N \rangle_G = zZ_{C1} = e^{\beta\mu} Z_{C1}$$

The chemical potential is thus obtained as

$$\mu = k_B T \ln\left(\frac{\langle N \rangle_G}{Z_{C1}}\right)$$

where

$$Z_{C1} = Vn_Q(T)$$

$$\mu = k_B T \ln\left(\frac{N}{Z_{C1}}\right) = k_B T \ln\left(\frac{N}{Vn_Q}\right) = k_B T \ln\left(\frac{n}{n_Q}\right)$$

with

$$n = \frac{N}{V}$$

(b) Canonical ensemble:

Approximation:

$$Z_G = \sum_N z^N Z_{CN} \approx z^{\bar{N}} Z_{C\bar{N}}$$

where

$$Z_{C\bar{N}} = \frac{\left(Z_{C1}\right)^{\bar{N}}}{\bar{N}!}$$

$$Z_G \approx z^{\bar{N}} Z_{C\bar{N}} = z^{\bar{N}} \frac{\left(Z_{C1}\right)^{\bar{N}}}{\bar{N}!}$$

$$\begin{aligned}\ln Z_G &\approx \ln[z^{\bar{N}} \frac{\left(Z_{C1}\right)^{\bar{N}}}{\bar{N}!}] \\ &= \bar{N}\ln z + \bar{N}\ln Z_{C1} - \ln \bar{N}\end{aligned}$$

$$\begin{aligned}\langle N \rangle_G &= z\frac{\partial}{\partial z}\ln Z_G \\ &= z\frac{\partial}{\partial z}[\bar{N}\ln z + \bar{N}\ln Z_{C1} - \ln \bar{N}] \\ &= \bar{N}\end{aligned}$$

(c) Occupation number representation

$$\begin{aligned}\langle N \rangle_G &= \sum_a \langle n_a \rangle_G \\ &= \sum_a \frac{1}{e^{\beta(\varepsilon_a - \mu)} \pm 1} \\ &\approx \sum_a e^{-\beta(\varepsilon_a - \mu)} \\ &= z\sum_a e^{-\beta\varepsilon_a} \\ &= zZ_{C1}\end{aligned}$$

since

$$\frac{1}{e^{\beta(\varepsilon-\mu)}-1} \approx$$
