Intertemporal optimization: Value functions, Bellman equations

Review of micro stuff (Mas-Colell et. al.)

Static (one-period) problem:

Maximize $u(x)$ — vector of quantities of individual goods & services

subject to $p \cdot x \leq w$ — ("wealth")

p — vector of prices

$V(p, w)$ Indirect utility function (value function)

Result of utility maximization

p, w are what agent takes as given.

"Envelope Theorem":

If you give a little more wealth to the guy, how much does it increase his utility?

$$\frac{\partial V(p,w)}{\partial w} = \frac{\partial u(x)}{\partial x_1} \cdot \frac{1}{p_1} = \frac{\partial u(x)}{\partial x_2} \cdot \frac{1}{p_2} = \ldots$$

where $\frac{1}{p_1} = \frac{\partial x_1}{\partial w}$ if all of ∂w is spent on x_1

$\frac{1}{p_2} = \frac{\partial x_2}{\partial w}$ " " " " " " x_2

etc.

Intertemporal problem without uncertainty

Example similar to Romer ch-5

$$U_t = \sum_{\tau=0}^{\infty} \beta^{\tau} u(\overset{+}{c}_{t+\tau}, \overset{+}{1-l}_{t+\tau}) \qquad 0 < \beta < 1$$

discount factor

Labor time (fraction of time)

Leisure $= (1-l)$

$$\text{s.t.} \sum_{\tau=0}^{\infty} \left(\frac{1}{1+r_{t+\tau}}\right)^{\tau} c_{t+\tau} \le A_t + \sum_{\tau=0}^{\infty} \left(\frac{1}{1+r_{t+\tau}}\right)^{\tau} (w\, l)_{t+\tau}$$

real assets at time t

real wage

Value of U_t that results from maximization

is $V_t(A_t, \text{-----})$ Value function

A_t is one thing household takes as given as of time t (though it is result of past choices)

There may be other things household takes as given at time t. "State variables."

How can we solve this?

Turn it into a two-period problem,

and use fact that $V(\quad)$ is "recursive,"

Intertemporal...

Value function is recursive

Because lifetime is infinite, agent's problem will be the same at time $(t+1)$ as at t.

She's no closer to end of life.

Same number of periods (infinite) to deal with.

$V_t(A_t, \ldots)$ This function has some specific form.

$V_{t+1}(A_{t+1}, \ldots)$ Must have same form!
Same function, just different inputs

Turn it into a two-period problem

Utility function

Pull first period out of summation

$$U_t = u(c_t, 1-l_t) + \sum_{\tau=1}^{\infty} \beta^{\tau} u(c_{t+\tau}, 1-l_{t+\tau})$$

What will $\sum_{\tau=1}^{\infty} \beta^{\tau} u(\,.\,)$ turn out to be?

Tomorrow, agent will choose c_{t+1}, l_{t+1} to maximize utility looking forward from $t+1$.

She'll do it correctly, of course. So:

$$U_t = u(c_t, 1-l_t) + \beta V_{t+1}(A_{t+1}, \ldots)$$

$$V_t = \max_{c_t, l_t} u(c_t, 1-l_t) + \beta V_{t+1}(A_{t+1}, \ldots)$$

Bellman Equation

Intertemporal

Turn it into a two-period problem

Budget constraint

At time $t+1$, agent will take A_{t+1} as given.
How will c_t, l_t affect A_{t+1}?

$$A_{t+1} = (1+r_t)A_t + w_t l_t - c_t$$

What is effect of ∂c_t on V_{t+1}?

$$\frac{\partial V_{t+1}}{\partial c_t} = \frac{\partial V_{t+1}(A_{t+1}, \ldots)}{\partial A_{t+1}} \cdot \frac{\partial A_{t+1}}{\partial c_t}$$

$$= \ldots \ldots \cdot (-1)$$

Effect of ∂l_t?

$$\frac{\partial V_{t+1}}{\partial l_t} = \ldots \ldots \cdot \frac{\partial A_{t+1}}{\partial l_t}$$

$$= \ldots \ldots \cdot w_t$$

But what is $\dfrac{\partial V_{t+1}(\ )}{\partial A_{t+1}}$?

Intertemporal...

Turn it into...

Benveniste – Scheinkman equation

Recall that envelope theorem says effect on utility of giving agent a little more "wealth," assuming utility is being maximized, is equal to addition to utility from spending it all on x_1, or x_2, or whatever:

$$\frac{\partial V(p, w)}{\partial w} = \frac{\partial u(x)}{\partial x_1} \cdot \frac{1}{p_1} \quad \text{etc.}$$

So here

$\dfrac{\partial V_{t+1}(A_{t+1}, \ldots)}{\partial A_{t+1}}$ must be equal to effect on lifetime utility of spending all ∂A_{t+1} on c_{t+1}, not carrying any on into A_{t+2}.

So if I spend it all on c_{t+1}, what's $\dfrac{\partial c_{t+1}}{\partial A_{t+1}}$?

Recall $A_{t+2} = (1+r_{t+1}) A_{t+1} + w_{t+1} l_{t+1} - c_{t+1}$

$$\partial A_{t+2} = (1+r_{t+1}) \partial A_{t+1} + w_{t+1} \partial l_{t+1} - \partial c_{t+1}$$

Setting $\partial A_{t+2} = \partial l_{t+1} = 0$,

$$\partial c_{t+1} = (1+r_{t+1}) \partial A_{t+1}.$$

So....

Intertemporal....

Turn it into...

Benveniste-Scheinkman...

Applying envelope theorem to c_{t+1} gives B-S condition with respect to c_{t+1}:

$$\frac{\partial V_{t+1}(\ )}{\partial A_{t+1}} = \frac{\partial u(c_{t+1}, 1-l_{t+1})}{\partial c_{t+1}} \cdot (1+r_{t+1})$$

You can also apply envelope theorem to $(1-l_{t+1})$, using effect on utility of "spending" it all on leisure in t+1. Setting $\partial A_{t+2} = \partial c_{t+1} = 0$,

$$\partial l_{t+1} = (1+r_{t+1})\, \partial A_{t+1} \frac{1}{w_{t+1}} (-1)$$

$$\frac{\partial V_{t+1}(\ )}{\partial A_{t+1}} = \frac{\partial u(\qquad)}{\partial(1-l_{t+1})} \cdot (1+r_{t+1}) \frac{1}{w_{t+1}}$$

Intertemporal optimization...

First-order conditions from two-period problem without uncertainty

Choose c_t, l_t to maximize

$$U_t = u(c_t, 1-l_t) + \beta V_{t+1}(A_{t+1}, \ldots)$$

s.t. $A_{t+1} = (1+r_t)A_t + w_t l_t - c_t$

At optimal choices for c_t, l_t,

$$\frac{\partial U_t}{\partial c_t} = 0 = \frac{\partial u(c_t, 1-l_t)}{\partial c_t} + \beta \frac{\partial V_{t+1}}{\partial A_{t+1}} \cdot (-1)$$

$$\frac{\partial U_t}{\partial (1-l_t)} = 0 = \frac{\partial u(\quad)}{\partial (1-l_t)} + \beta \frac{\partial V_{t+1}}{\partial A_{t+1}} \cdot (-w_t)$$

where $\frac{\partial V_{t+1}}{\partial A_{t+1}} = \frac{\partial u(c_{t+1}, 1-l_{t+1})}{\partial c_{t+1}} \cdot (1+r_{t+1})$

or

$$= \frac{\partial u(\quad)}{\partial (1-l_{t+1})} \cdot (1+r_{t+1}) \frac{1}{w_{t+1}}$$

This can tell us a lot!

Intertemporal optimization ....

.... without uncertainty

Euler equation

Recall the Euler equation from RCK model

$$\underset{\text{Change in MU}}{\dot{u}'(c_t)} \Big/ \underset{\text{MU}}{u'(c_t)} = \rho + n - \underbrace{f'(k)}_{r}$$

where $\underbrace{\dot{u}'(c_t)}_{\dot{c}} = \frac{\partial c_t}{\partial t} \cdot \frac{\partial u'(c_t)}{\partial c_t} = \frac{\partial c_t}{\partial t} u''(c_t)$

Implicitly defines path for c over time.

with CRRA $u'(c) = c^{-\theta}$ $\quad u''(c) = -\theta c^{-\theta-1}$

so Euler equation becomes

$$\frac{\dot{c}_t(-\theta c_t^{-\theta-1})}{c_t^{-\theta}} = \rho + n - f'(k)$$

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left[\underbrace{f'(k_t)}_{r} - \rho - n\right]$$

defines path of c over time given either

— starting point

— ending point (work backwards)

Intertemporal ....
.... without uncertainty
Euler equation (cont.)

Using B-S equation, we can define an Euler eqn. from Bellman equation

$$U_t = u(c_t, 1-l_t) + \beta V_{t+1}[A_{t+1}, \dots)$$

where $A_{t+1} = (1+r_t)A_t + w_t l_t - c_t$

$$0 = \frac{\partial U_t}{\partial c_t} = u'(c_t, \dots) + \beta \frac{\partial V_{t+1}(\ )}{\partial A_{t+1}} \frac{\partial A_{t+1}}{\partial c_t}$$

$$B-S: \frac{\partial V_{t+1}(\ )}{\partial A_{t+1}} = u'(c_{t+1}, \dots) \cdot (1+r_{t+1})$$

$$0 = u'(c_t, \dots) + \beta u'(c_{t+1}, \dots)(1+r_{t+1})(-1)$$

$$\frac{u'(c_{t+1}, \dots)}{u'(c_t, \dots)} = (\beta(1+r_{t+1}))^{-1}$$

Specific felicity function: $\frac{1}{1-\theta} c_t^{1-\theta} + v(1-l_t)$

$$\frac{c_{t+1}^{-\theta}}{c_t^{-\theta}} = (\beta(1+r_{t+1}))^{-1}$$

$$\frac{c_{t+1}}{c_t} = (\beta(1+r_{t+1}))^{1/\theta}$$

(Recall high discounting means small β)

Intertemporal optimization

With uncertainty

I don't know what V_{t+1} will be

$$V_t = \max_{c_t, l_t} u(c_t, 1-l_t) + \beta E_t V_{t+1}(A_{t+1} \cdots)$$

F.O.C.'s become:

$$0 = \frac{\partial u(c_t, 1-l_t)}{\partial c_t} + \beta \frac{\partial E_t V_{t+1}(A_{t+1})}{\partial c_t}$$

etc.

plus a handy thing:

$$\frac{\partial E_t V_{t+1}(\;)}{\partial c_t} = E_t\left[\frac{\partial V_{t+1}(\;)}{\partial A_{t+1}} \frac{\partial A_{t+1}}{\partial c_t}\right]$$

How do you know?

probability of state j

$$E_t V_{t+1}(\;) = \sum_{j=1}^{n} \nu_j \; V_{t+1,j}$$

where there are n possible futures, each with probability ν_j, and a value of V_{t+1} in that future,

$$\frac{\partial E_t V_{t+1}(\;)}{\partial X_t} = \sum_{j=1}^{n} \nu_j \frac{\partial V_{t+1,j}}{\partial X_t} = E_t\left[\frac{\partial V_{t+1}(\;)}{X_t}\right]$$

anything

Intertemporal optimization ....

With uncertainty (cont.)

B-S condition with uncertainty

Because of envelope theorem, we know that in each possible future state j

$$\frac{\partial V_{t+1,j}}{\partial c_t} = \frac{\partial u(\quad)}{\partial c_{t+1,j}} \frac{\partial c_{t+1,j}}{\partial A_{t+1}} \frac{\partial A_{t+1}}{\partial c_t}$$

so

$$\sum_{j=1}^{n} \nu_j \frac{\partial V_{t+1,j}}{\partial A_{t+1}} = \sum_{j=1}^{n} \nu_j \frac{\partial u(\,)}{\partial c_{t+1,j}} \frac{\partial c_{t+1,j}}{\partial A_{t+1}} \frac{\partial A_{t+1}}{\partial c_t}$$

so

$$E_t\left[\frac{\partial V_{t+1}(\,)}{\partial A_{t+1}}\right] = E_t\left[\frac{\partial u(\,)}{\partial c_{t+1,j}} \cdots\cdots\right]$$

Now we can define Euler equation.

(Note: you can do same for z_t.)

Intertemporal...

With uncertainty (cont.)

Euler equation

$$0 = u'(c_t, \dots) + \beta\, E_t\left[u'(c_{t+1}, \dots) \cdot (1 + r_{t+1}) \right]$$

But here it gets tricky. What is $E_t[\dots]$?

You *might* think

$$\_\_\_\_\_ = u'(E_t[c_{t+1}, \dots]) \cdot (1 + E_t[r_{t+1}])$$

But no! Two reasons. Math things.

1) c_{t+1} & r_{t+1} might both be uncertain (random vars.) at time t. For two random variables X & Z,

$$E_t[X_{t+1} \cdot Z_{t+1}] \neq E_t[X_{t+1}] \cdot E_t[Z_{t+1}]!$$

$$E_t[X_{t+1} \cdot Z_{t+1}] = E_t[X_{t+1}] \cdot E_t[Z_{t+1}] + \mathrm{Cov}(X_{t+1}, Z_{t+1})$$

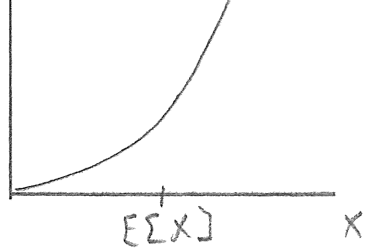
2) Jensen's inequality

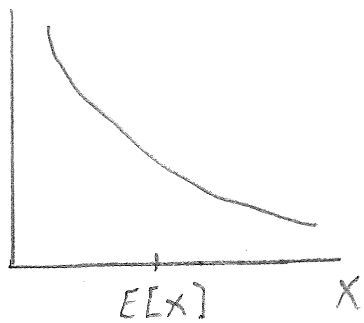
Intertemporal...
With uncertainty (cont.)
Jensen's inequality

Random variable X, $E[X]$, Function $F(X)$

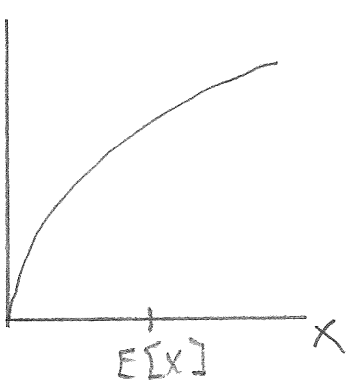
$E\left[F(X)\right] = F(E[X])$ if $F(X)$ linear in X, $F''(X) = 0$

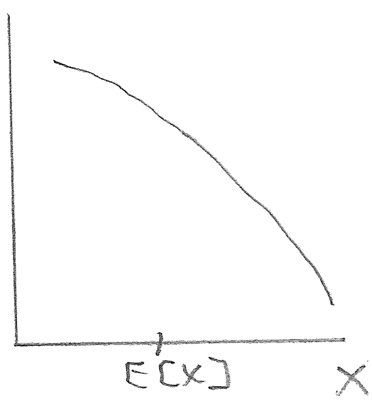
$E\left[F(X)\right] > F(E[X])$ if $F(X)$ convex $F''(X) > 0$





$E\left[F(X)\right] < F(E[X])$ if $F(X)$ concave $F''(X) < 0$





So here, even if only c_{t+1} is uncertain, we need to know if $u'(c_{t+1}, \dots)$ is convex or concave. Diminishing MU means concave.

Intertemporal ...

With uncertainty (cont.)

Another approximation: "certainty equivalence"

It's often hard to get a usable expression for $E_t[u'(c_{t+1}, \ldots) \cdot \text{----}]$

so we often approximate with "certainty equivalence":

$$E_t[F(X)] \approx F(E[X])$$

We can use Jensen's inequality to figure out the direction of the resulting error if we can at least figure out $F''(X)$.