Efficiency Measurement: Some old issues and new directions

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Abstract

This paper addresses directional issues related to efficiency measurement from an econometric point of view. It is argued that the estimated technical efficiency is not invariant to the choice of efficiency-orientation, viz., input- and output-oriented measure of technical efficiency. Some possible solutions to choose directions are explored. In particular, we discuss (i) input-specific inefficiency along with behavioral assumptions, and (ii) generalized distance function representation of the technology.

Prepared for presentation at the University of Georgia

Comments/suggestions are welcome.

Problems associated with econometric estimation of technical efficiency in stochastic frontier models

- The focus is on directional issues such as the input-oriented, output-oriented measure of technical efficiency.

- Some of the issues are at the idea stage and are being developed for full papers. So feel free to ask questions.

Two measures of technical efficiency are mostly used in the efficiency literature. These are:

(i) Input-oriented (I-O) technical efficiency,
(ii) Output oriented (O-O) technical efficiency.

Consider a single output production technology where Y is a scalar output and *X* is a vector of inputs. Then

I-O representation of the production technology $\Rightarrow Y = f(X \cdot \theta)$ where θ is a scalar.

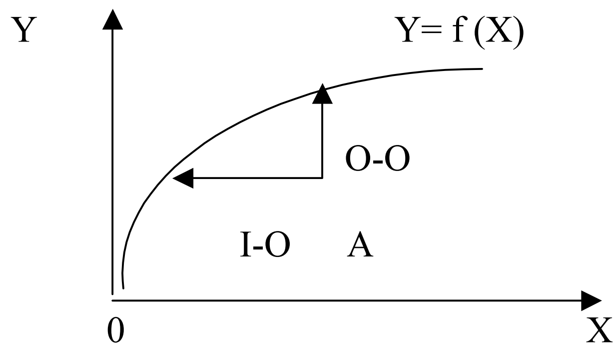
Technical efficiency ($TE = \theta \leq 1$). $(1-\theta)$ is the percent at which all inputs can be contracted without reducing output, and can be labeled as I-O technical inefficiency.

O-O representation of the production technology $\Rightarrow$ $Y = f(X)\lambda$.

Technical efficiency ($TE = \lambda \le 1$). $(1-\lambda)$ is the percent at which the output can be increased without increasing the inputs, and is labeled as O-O technical inefficiency.

Although we consider technologies with a single output, the I-O and O-O inefficiency can be discussed in the context of multiple output technologies as well.

Graphical representation:



Question: (i) Can one go from the I-O representation to the O-O representation and vice-versa? That is, if one knows I-O technical efficiency, can he compute O-O technical efficiency?

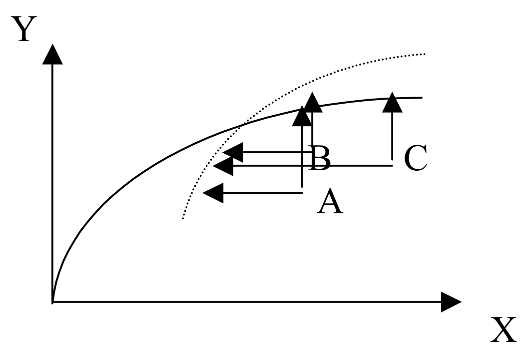
Answer: yes, it requires some algebraic manipulations. No need to assume a homogeneous production function. This point is illustrated with examples later.

Question: (ii) In measuring efficiency does it matter (theoretically and/or empirically) whether one uses the I-O or the O-O representation?

Answer: no if the frontier is known. One can measure λ from θ and vice-versa.

If the frontier is not known then in measuring efficiency, orientation matters. Why?

Econometric estimation of the frontier depends on the choice of directions. That means the vertical and horizontal projections don't give the same frontier empirically. This result is similar to regressing Y on X and X on Y, which does not give the same result so far as the slope and intercepts are concerned. See the graph below.



To examine this formally, consider the following:

(i) Estimate the O-O model (i.e., the ALS model and obtain λ using the Jondrow et al. formula) and solve for θ from $f(X)\lambda = f(X\theta)$ using a parametric form for *f*(.).

(ii) Estimate the I-O model (not known in CS model) and get θ directly. How to estimate? Discuss later.

- Compare estimates of θ from (i) and (ii). These two θ's will be different numerically, if estimated econometrically (even for the CRS model because in the O-O model endogenous variable is output whereas in the I-O model inputs are endogenous variables. This is why a production function is estimated in the O-O model while a cost function is estimated in the I-O model). (Note that the comparison is <u>not</u> between θ and λ).

- Consequently, ranking of firms might also change.

- Instead of comparing estimates of θ one can also compare estimates of λ from these two models. For this, first, estimate θ from the I-O model. Then solve for λ from the equation $f(X)\lambda = f(X\theta)$. The next step is to estimate the O-O model and get λ directly. Finally, compare the two λ's. How to estimate? How to compare?

- One can also compare other features of the technology such as returns to scale (RTS), elasticities, etc., based on $f(X)$ and $f(X\theta)$. Since the "estimated technology" is likely to differ the "estimated" features are likely to be different?

- Inefficiency in the O-O formulation doesn't affect any parameters other than the intercept of the production function. That is, RTS, elasticities, etc., are estimated consistently without taking inefficiency into account (Schmidt 1984-5, Econometric Reviews). This makes selling the frontier stuff to non-efficiency people difficult!!

- Results are, however, dependent on the direction/orientation used. For example, opposite results are obtained if one uses an I-O model. Some of these issues are discussed in Alvarez, Arias and Kumbhakar (2002) in the context of panel data models.

- If a Cobb-Douglas or any other homogeneous function is used to represent the production technology, then $\lambda = \theta^r$ where r is the degree of homogeneity. Thus, it is not possible to separate I-O measure from the O-O measure econometrically (when one a single equation approach) in the sense that the I-O specification looks the same as the I-O specification. However, if estimated properly, estimated value of λ is likely to be different from θ even when r = 1. This is because in the I-O model inputs are endogenous whereas in the O-O model output is endogenous. Consequently, the estimation techniques will be different (e.g., a production function vs. a cost function).

- In any case, one cannot test econometrically whether inefficiency is I-O or O-O. Once a particular orientation is chosen (as is always the case), one can obtain both I-O and O-O measures, although these measures are not invariant to the choice of directions.

Practical/Economic Issues:

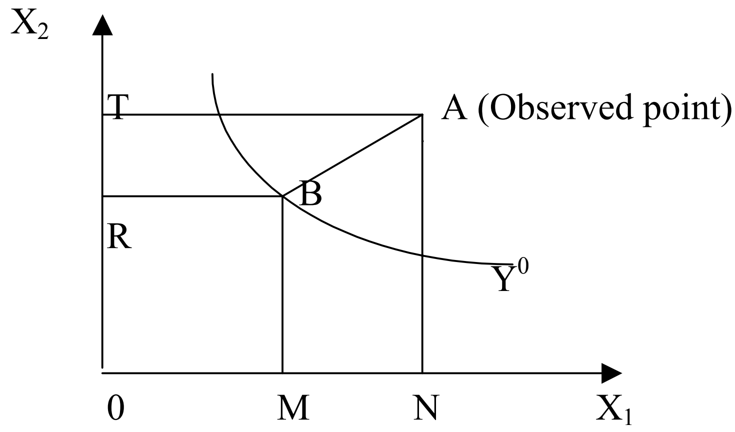
- Is it feasible to reduce all inputs by the same proportion? Feasibility.
- Is it desired, if feasible? That is, for example, does it lead to a least cost input allocation?
- When is it desirable to expand output(s), given the input quantities?
- Keep in mind that in reality producers decide the direction to choose!!

How to resolve the directional issue?

- Introduce behavioral assumption into the model explicitly.
- Specify more general models to address the directional issue.

Non-radial (input-specific) measure of technical efficiency

Y = f(θ_1X$_1$,..., θ_nX$_n$) , where $\theta_j \leq 1$ measures input-specific technical efficiency (ISTE).



Here $\theta_1 = 0M/0N$ and $\theta_2 = 0R/0T$ for the direction A→B.

- Individual θ_j cannot be identified especially if one uses the production function alone in estimation. For example, in the CD case,

$$\ln Y = \alpha_o + \sum \alpha_j \ln(\theta_j X_j) = \alpha_o + \sum \alpha_j \ln X_j + \sum_j \alpha_j \ln \theta_j$$

$$\equiv \alpha_o + \sum \alpha_j \ln X_j - u,\ u \geq 0,$$

which is not separable from the I-O and/or O-O measure of technical efficiency.

- It is possible to identify θ_j for some production functions with behavioral assumptions explicitly introduced.

- **Introduction of behavioral assumption**

Cost minimization with input-specific technical efficiency (ISTE)

<u>The Translog case:</u>

$$Y = f(\theta_1 X_1,\ \ldots,\ \theta_n X_n),$$

$$\Rightarrow C = C(\tilde{w}, Y) \text{ where } \tilde{w}_j = w_j/\theta_j.$$

$$\ln C = \alpha + \sum \alpha_j \ln \tilde{w}_j + \frac{1}{2}\sum\sum \alpha_{jk} \ln \tilde{w}_j \ln \tilde{w}_k + \alpha_y \ln Y + \frac{1}{2}\alpha_{yy} \ln Y^2 + \sum \gamma_{jy} \ln \tilde{w}_j \ln Y$$

$$S_j = \frac{\partial \ln C}{\partial \ln \tilde{w}_j} = \frac{W_j X_j}{C} = \alpha_j + \sum_k \alpha_{jk} \ln \tilde{w}_k + \gamma_{jy} \ln Y$$

$$= \alpha_j + \sum_k \alpha_{jk} \ln w_k + \gamma_{jy} \ln Y - \sum_k \alpha_{jk} \ln \theta_k$$

That is, $S_j = S_j^* - \sum \alpha_{jk} \ln \theta_k$

$\Rightarrow$ $S = S^* - A\varepsilon$

If linear homogeneity (in prices) is imposed by expressing all prices and cost in terms of $\tilde{w}_1$, then ε_j = ln (θ_j/θ_1), $j = 2, \ldots, n$ and A will be the $\{\alpha_{jk}\}$ matrix with the first row and first column deleted. Moreover, ε is likely to be two-sided. Thus, the cost share system can be written as

$$S = S^* - v,$$

where $v = A\varepsilon$ such that $E(v) = 0$. As before, ISTE is assumed to be random. This is likely to be the case unless for all the producers $\theta_j > (<) \theta_1$. Note that θ_j are both input and producer-specific as a result of which we can write ln (θ_j/θ_1) as an error term. No separate error terms are added to the cost share equations.

- The above system can be estimated using SUR, treating $v \sim (0, \Sigma)$. A non-zero mean on v can be accommodated.
- Since the elements of A are in S*, $\hat{\varepsilon} = A^{-1}\hat{v}$ $\Rightarrow est(\theta_j / \theta_1) = \exp(\hat{\varepsilon}_j)$ which is a non-radial contraction. |A|≠0 because rank of the $\{\alpha_{jk}\}$ matrix is (J-1).
- To estimate θ_1 we need to go back to the cost function and estimate it (conditional on the parameters obtained from the cost share equations) using the stochastic frontier approach (under the assumption that $\ln\theta_1 \le 0$). The individual θs can then be recovered from the estimates of $\hat{\varepsilon}_j$. The estimate of θ_j for each observation gives the direction.

Problems: (i) There no guarantee that all the θ_js will be ≤ 1. If θ_j are not one-sided they cannot be interpreted as ISTE. In that case how to estimate θ_1? The two-tier frontier of Polachek and Yoon?

(ii) On the other hand, if $\theta_j \leq 1$, can they be interpreted as ISTE?

(iii) Is it necessary for $\theta_j \leq 1$? No, if they are treated as a directional measure.

- Interpretation of θ_j when they are not necessarily less than unity: It is possible to give an alternative interpretation of θ_j by expressing them in terms of technical (radial) and allocative inefficiency. This can be done as follows:

Equate the two cost functions (one from the ISTE and the other from the standard radial technical inefficiency and allocative inefficiency), and the cost share equations derived from each cost functions. These will generate J equations to solve for u (radial cost inefficiency) and $\xi_j,\ j = 2,\ldots,J$ where ξ_j represents allocative inefficiency for the input pair (j,1) (see, e.g., Kumbhakar, 1997). These equations are, however, highly non-linear.

Summary: Does this solve the directional problem?

Distance function Approach:

I-O: Input distance from D (X,Y) is homogeneous of degree 1 in X
O-O: Output distance from D(X,Y) is homogeneous of degree 1 in Y

- Using D (X,Y) one can estimate the technology (frontier) and technical efficiency parametically/nonparametically
- Again one has to choose between I-O and O-O measures.

Hyperbolic distance function:

$$\mathrm{D}_{\mathrm{H}}(X,Y) = \inf\{\theta > 0 : (X\theta, Y/\theta) \in \mathrm{T}(X,Y)\}$$

Almost-homogeneous of degree -1, 1, 1 in X and Y

- Hyperbolic measure decreases inputs by θ and increases output by $1/\theta$.

Generalized distance function [Cox and Chavas 1999]

$$\mathrm{D_G}(X,Y,\alpha) = \inf\{\theta > 0 : (X\theta^{1-\alpha}, Y\theta^{-\alpha}) \in \mathrm{T}(X,Y)\}$$

Almost homogeneous of degree α-1, α, 1 in X and Y and 0 $\leq \alpha \leq$ 1. Also $D_G(X,Y,\alpha) \leq 1$.

Some special cases of generalized distance function.

α=0 input distance function
α=1 output distance function
$\alpha = \frac{1}{2}$ hyperbolic distance function.

- The generalized distance function might have some potential in solving the orientation.

Use modified Euler's lemma to impose almost homogeneous restrictions.

$$(\alpha-1)\sum_j \frac{\partial D}{\partial X_j} X_j + \alpha \sum_m \frac{\partial D}{\partial Y_m} Y_m = D$$

$$\Rightarrow \qquad (\alpha-1)\sum_j \frac{\partial \ln D}{\partial \ln X_j} + \alpha \sum_m \frac{\partial \ln D}{\partial \ln Y_m} = 1$$

- One-input, one output case: the <u>Translog model</u>
  $\ln D = \alpha_o + \alpha_x \ln X + \alpha_y \ln Y + \frac{1}{2}[\alpha_{xx} \ln X^2 + \alpha_{yy} \ln Y^2] + \alpha_{xy} \ln X \ln Y.$
  Using modified Euler's lemmas gives
  $(\alpha-1)\ \{\alpha_x + \alpha_{xx}\ln X + \alpha_{xy}\ln Y\} + \alpha\{\alpha_y + \alpha_{yy}\ln Y + \alpha_{xy}\ln X\}$
  $= [\alpha(\alpha_x + \alpha_y) - \alpha_x] + \ln X\,[(\alpha-1)\,\alpha_{xx} + \alpha . \alpha_{xy}] + \ln Y\,[(\alpha-1)\cdot \alpha_{xy} + \alpha . \alpha_{yy}] = 1$

  $\Rightarrow$ (i) $\alpha\ (\alpha_x + \alpha_y) - \alpha_x = 1 \qquad \Rightarrow \alpha = \frac{1+\alpha_x}{\alpha_x + \alpha_y}$

  (ii) $(\alpha-1)\ \alpha_{xx} + \alpha \cdot \alpha_{xy} = 0$
  $\Rightarrow \alpha_{xy} = \frac{1-\alpha}{\alpha} \cdot \alpha_{xx} = \frac{\alpha_y - 1}{1+\alpha_x} \cdot \alpha_{xx}$

  (iii) $(\alpha-1) \cdot \alpha_{xy} + \alpha \cdot \alpha_{yy} = 0$

  $\Rightarrow \alpha_{yy} = \frac{1-\alpha}{\alpha}\alpha_{xy} = \left(\frac{\alpha_y - 1}{1+\alpha_x}\right)^2 \alpha_{xx}$

- Free parameters are: α_o, α_x, α_y and α_{xx}
- Has one extra parameter compared to the standard input
  and output distance function

Extension to multiple inputs and multiple outputs is straightforward. Work in progress (Cuesta, Kumbhakar and Zofio (2002)).

Problem: How to estimate the model? Which variable/variables is/are endogenous?

- Mechanically one can use the stochastic frontier approach.
- Use instrumental variable approach and make inefficiency deterministic functions of exogenous variables?
- Use programming techniques (deterministic)?

Does this generalized distance function solve the orientation problem? Not really. Why?

Note that α is a parameter that is same for all observations. That is, all firms follow the same direction!!

Assuming that the model can be estimated, one can test whether $\alpha = 1, 0, .5, etc.$

<u>Mixing model</u>:

Traditional mixing models:

- Allow technology to differ among producers classified into groups. Number of groups is usually chosen exogenously but the producers to be included in each group are chosen by the model. For example, consider two groups (with O-O technical efficiency) in which

  (i) p is proportion of firms that use the technology $Y = f_1(X)\lambda_1$, and
  (ii) $(1\text{-}p)$ proportion of firms that use the technology $Y = f_2(X)\lambda_2$.

This framework will allow one to estimate p, parameters of $f_1(.)$ and $f_2(.)$, as well as in λ_1 and λ_2. But one has to first decide whether the I-O or the O-O models are to be used.

- Instead of using a production function, one can use a cost or profit function and use the mixing models.

See Caudill (2002, other years), Tsionas (2002).

Here we propose using the mixing models to solve the directional problem.

Mixing model using distance functions (Orea and Kumbhakar, 2002 work in progress)

Let the technology be represented by a stochastic distance function which can be expressed in general terms as:

$$D = f(y, x, \beta) \cdot \exp(v) \tag{1}$$

or, in logs

$$\ln D = \ln f(y, x, \beta) + v \tag{2}$$

The output-oriented distance function can be written as

$$-\ln y_M = \ln f(y / y_M, x, \beta) + v + u^{+} \tag{3}$$

Under the usual assumptions the likelihood function for the output-oriented model can then be denoted as $g_O(y,x,\theta_O)$, where θ_O is the vector of parameters associated with the output-oriented model.
The input-oriented distance function can be written as

$$-\ln x_N = \ln f(y, x / x_N, \beta_I) + v - u^{+} \tag{4}$$

Under the usual assumptions the likelihood function for the input-oriented model can be denoted as $g_I(y,x,\theta_I)$, where θ_I is the vector of parameters associated with the input-oriented model.

Efficiency orientation for each firm is addressed by adopting a latent class structure. Here, the likelihood function for a particular firm is the weighted sum of both output-oriented and input-oriented likelihood functions, where the weights are the probabilities of choosing output and input orientation. That is,

$$g(y,x,\theta,\delta) = g_O(y,x,\theta_O) \cdot P_O(\delta_O) + g_I(y,x,\theta_I) \cdot P_I(\delta_I) \tag{5}$$

where $0 \le P_j \le 1$ (j = O,I), and $P_O + P_I = 1$, $\theta = (\theta_O, \theta_I)$, $\delta = (\delta_O, \delta_I)$ and the probabilities of output and input orientation are specified as

$$P_j(\delta_j) = \frac{\exp(\delta_j{}'q)}{\sum_j \exp(\delta_j{}'q)} \quad , \quad j = O, I \tag{6}$$

where q is a vector of firm-specific variables.

The estimated parameters can then be used to compute posterior probabilites of output and input orientation:

$$P(j \mid y,x) = \frac{g_j(y,x,\theta_j) \cdot P_j(\delta_j)}{\sum_j g_j(y,x,\theta_j) \cdot P_j(\delta_j)} \quad , \quad j = O, I \tag{7}$$

The model proposes that a firm may be, with some probability, maximizing revenue *and* minimizing cost. A measure of technical efficiency in the mixing model can be obtained from:

$$\ln M\hat{T}E = P(j = O \mid y,x) \cdot \ln O\hat{T}E + P(j = I \mid y,x) \cdot \ln I\hat{T}E \tag{8}$$

where $O\hat{T}E = E\left[\exp(-u^+ \mid e)\right]$ and $I\hat{T}E = 1/E\left[\exp(u^+ \mid e)\right]$

Alternatively, if P(O|y,x) > P(I|y,x) for a firm, it can be assumed to belong in the O-O firm and ite TE can be computed using OTE.

Mixing model with behavioral assumptions taken into account explicitly.

Assume that firms in O-O are maximizing output, while those in I-O are minimizing cost. So the technology for the O-O farms can be expressed as

$$Y = f(X) \cdot \lambda, \lambda \leq 1.$$

Similarly, the technology for the I-O farms can be expressed as

$$C = C(w, y) \cdot \frac{1}{\theta}$$

As before assume that the probabilities of a firm belonging in the O-O and I-O groups are given by

$$P_j(\delta_j) = \frac{\exp(\delta_j' q)}{\sum_j \exp(\delta_j' q)} \quad , \quad j = O, I$$

Use these probabilities to define the likelihood function for a sample of firms to obtain the ML estimate of the parameters. Then use to posterior probabilities to classify whether a firm belongs to the O-O or I-O class, and estimate TE.

Features of the mixing model

(i) endogeneity problem is taken into account explicitly.

(ii) can test for simpler functional forms, such as CD vs. Translog.

Example:
The CD case: Cost function parameters are functions of the production function parameters. So one can test/restrict cost function parameter to match with the production function parameters.

<u>Productivity Implications</u>:

O-O Model: $Y = f(X, t)\lambda$

$$T\dot{F}P = \dot{Y} - \sum_j S_j \dot{X}_j = (RTS - 1)\sum_j S_j \dot{X}_j + \dot{f}_t + \dot{\lambda}_t,$$

where $S_j = w_j X_j / C$ and RTS = $\sum \partial \ln f / \partial \ln X_j$. The first-order conditions of cost minimization are used to get the above formula. The components of TFP growth are:

Scale $\rightarrow$ (RTS – 1) $\sum_j S_j \dot{X}_j$,

$\dot{f}_t$ $\rightarrow$ technical change = $\frac{\partial \ln f}{\partial t}$,

$\dot{\lambda}_t \rightarrow$ change in technical efficiency = $\frac{\partial \ln \lambda}{\partial t}$.

$\Rightarrow T\dot{F}P$ is decomposed into scale, technical change, and change in technical inefficiency.

I-O Model: $Y = f(\theta X, t)$

$$\Rightarrow T\dot{F}P = [RTS(\theta X, t) - 1]\sum_j S_j \dot{X}_j + \dot{f}_t(\theta X, t) + RTS(\theta X, t)\dot{\theta}$$

where $RTS(\theta X, t) = \sum \partial \ln f(\theta x, t) / \partial \ln X_j$.

Note that RTS and TC above are defined at $f(\theta X, t)$ NOT at the frontier $f(X, t)$. If they are defined at the frontier, the decomposition formula becomes

$$\Rightarrow T\dot{F}P = [RTS(X, t) - 1]\sum_j S_j \dot{X}_j + \dot{f}_t(X, t) + RTS(X, t)\dot{\theta}$$

$$+ [RTS(\theta X, t) - RTS(X, t)]\sum_j S_j \dot{X}_j + [\dot{f}_t(\theta X, t) - \dot{f}_t(X, t)] + (RTS(\theta X, t) - RTS(X, t))\dot{\theta}$$

$\Rightarrow T\dot{F}P$ is decomposed into scale, technical change, and technical inefficiency (the last part) components. Note that the technical inefficiency component depends on $\dot{\theta}$, θ and data.

The BIG question is how to estimate the I-O model using the production function in a SF framework? This is still an unanswered question.