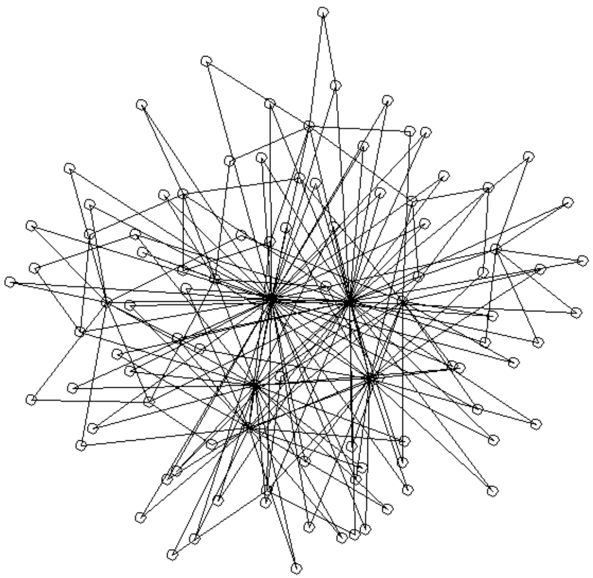
Review of Fundamentals of Graph Theory



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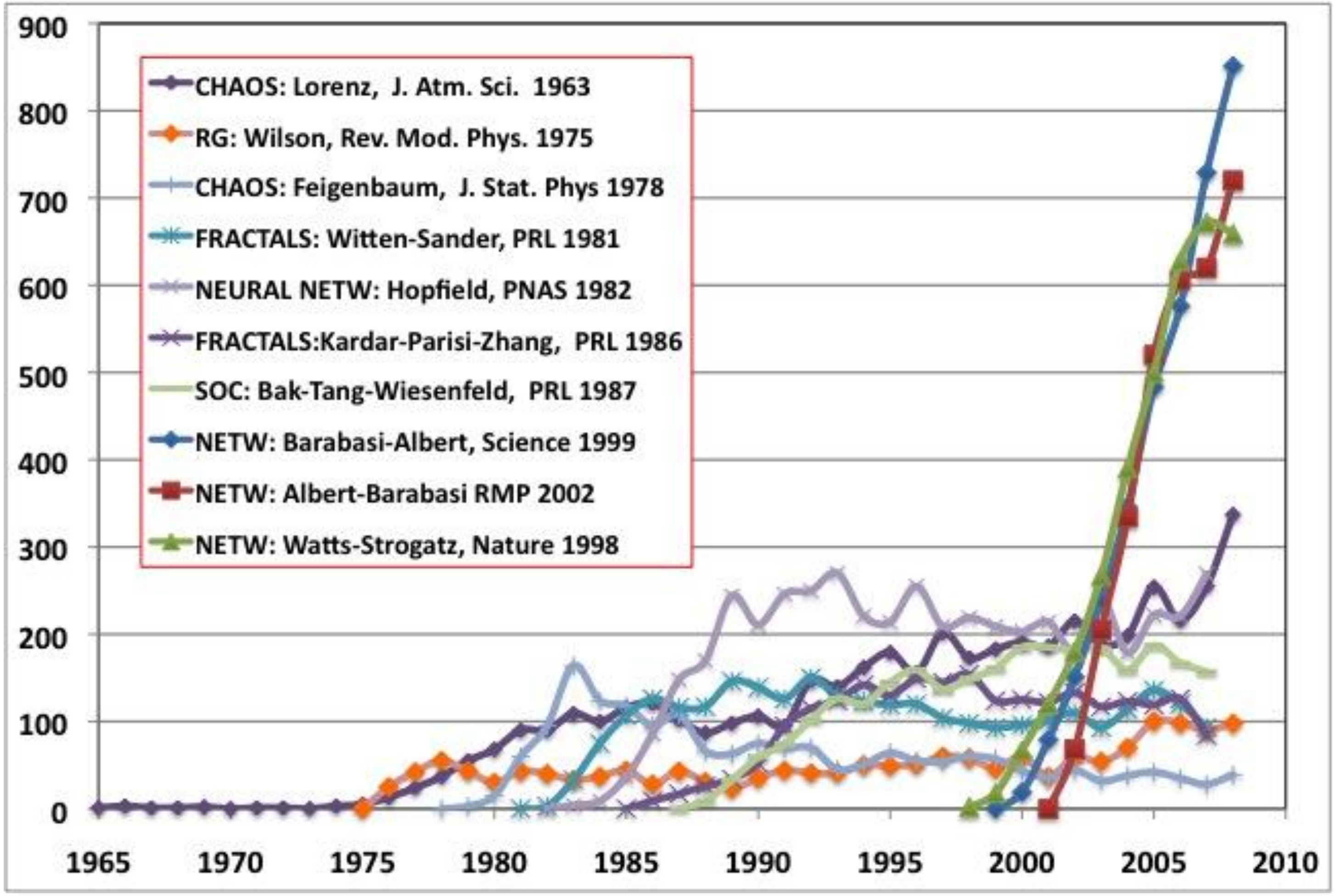
Rise of Network Science



(From Barabasi 2010)

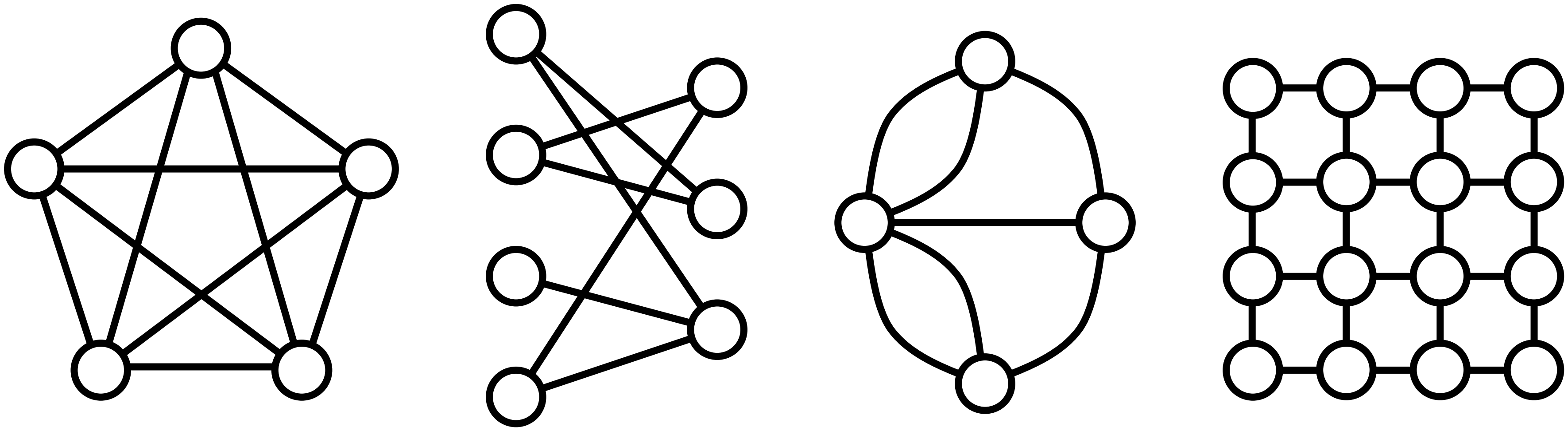
Network models

- Many discrete parts involved
  - Classic mean-field models not appropriate
- Parts are not uniformly connected
  - CA or cont. field models not appropriate
- Parts may dynamically increase / decrease in number
  - This deviates from typical dynamical systems assumptions

Fundamentals of Graph Theory

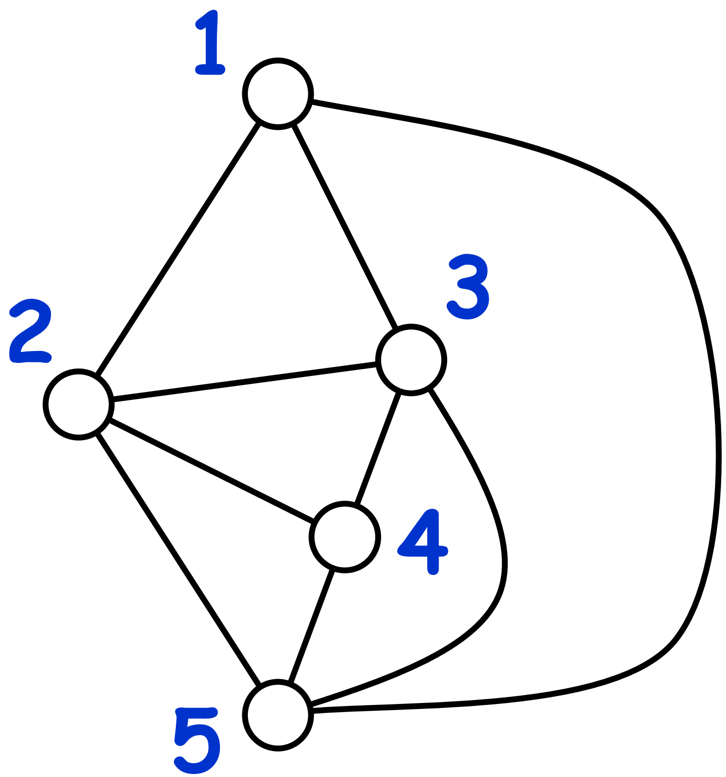
Graph?

- Mathematical structure that consists of "nodes" (or vertexes) and "edges" (or links) that make connection between nodes



Graph = Network

- G(V, E): graph (network)
  V: vertices (nodes), E: edges (links)



Nodes = 1, 2, 3, 4, 5

Links =
1<->2, 1<->3, 1<->5,
2<->3, 2<->4, 2<->5,
3<->4, 3<->5, 4<->5

(Nodes may have states; links may have directions and weights)

Representation of a network

- **Adjacency matrix:**

  A matrix with rows and columns labeled by nodes, where element a_{ij} shows the number of links going from node i to node j

  (becomes symmetric for undirected graph)

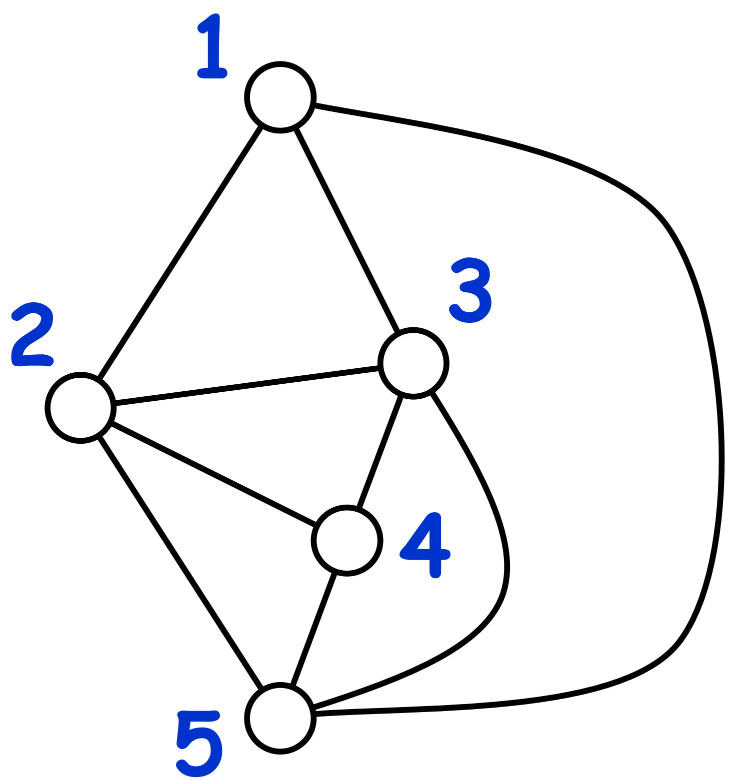
- **Adjacency list:**

  A list of links whose element "i->j" shows a link going from node i to node j

  (also represented as "i -> {j_1, j_2, j_3, …}")

Exercise

- Represent the following network in:



  - Adjacency matrix
  - Adjacency list

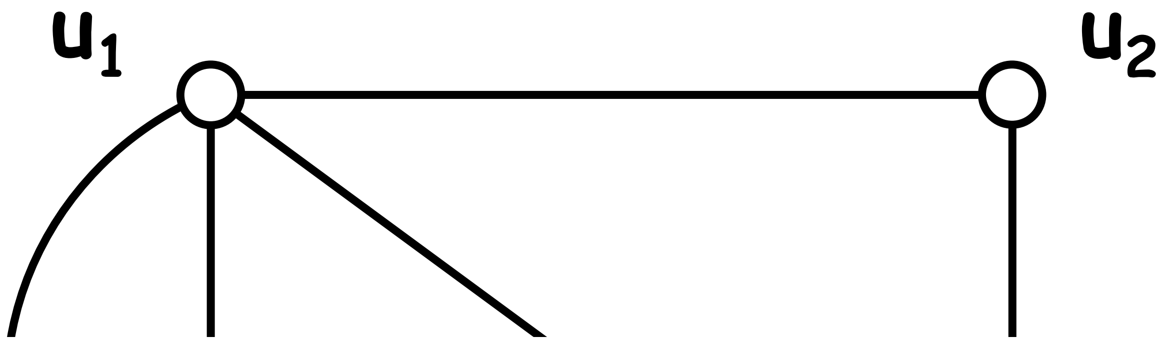
Properties of matrix description

- Labels of nodes and links are no more than a set of symbolic identifications that may be in any arbitrary order
  - → Re-labeling of nodes / links doesn't affect graph topology
  - → Permutation of rows / columns of the matrix doesn't affect graph topology

    (For adjacency matrices, the same permutation must be applied for both rows and columns)

Degree of a node

- A degree of node u, deg(u), is the number of links connected to u



$deg(u_1) = 4$ $deg(u_2) = 2$

Walk

- A list of links that are sequentially connected to form a continuous route
- In particular,

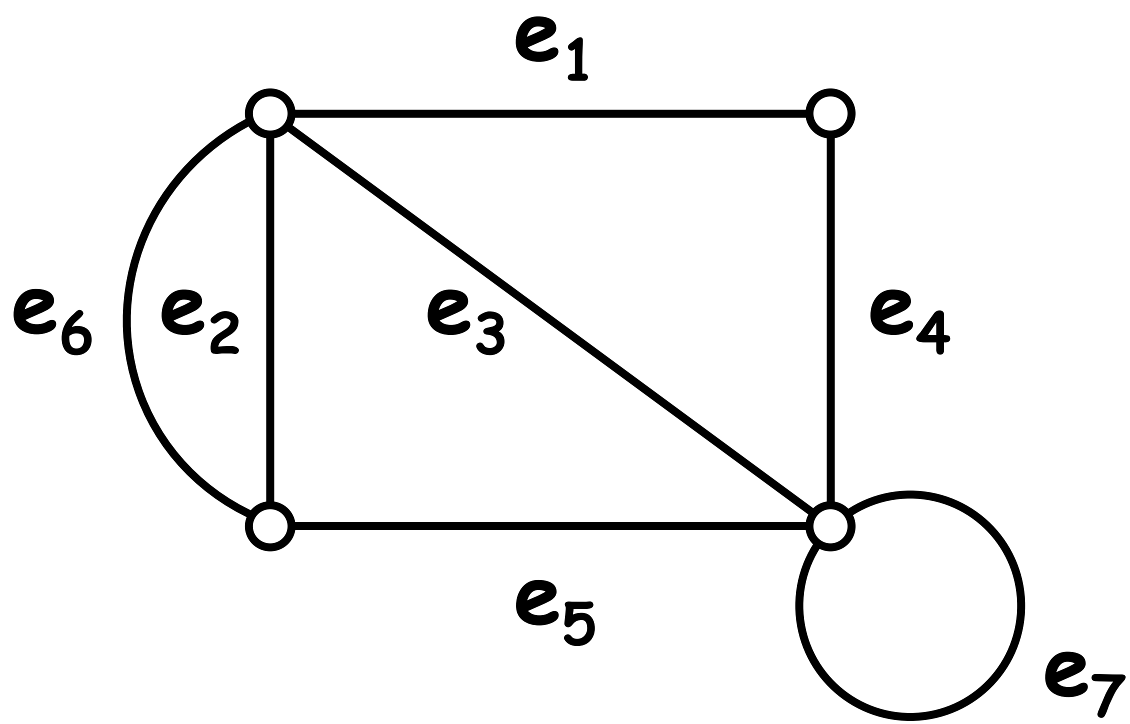
  Trail = a walk that does not go through the same link more than once

  Path = a walk that does not go through the same node more than once

  Cycle = a walk that starts and ends at the same node and that does not go through the same node on its way

Exercise

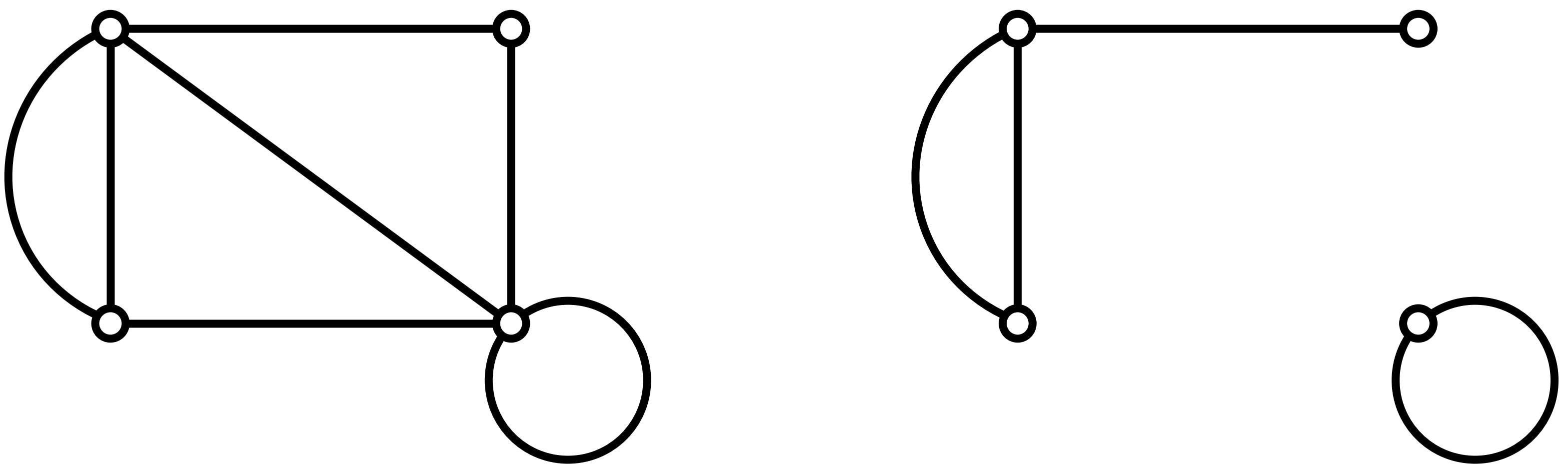
Classify the following walks into trail, path, cycle, or other



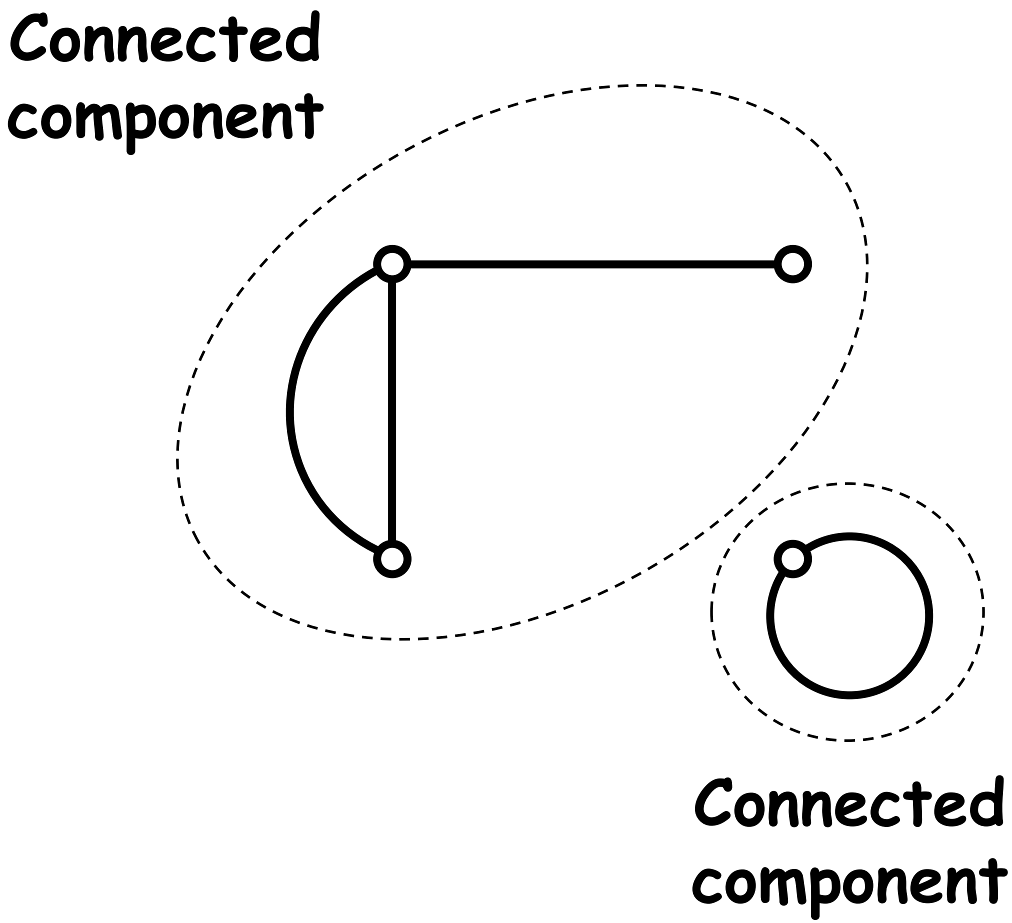
- $e_1, e_4, e_7, e_7, e_3, e_6$
- e_6, e_5, e_3, e_1, e_4
- e_1, e_6, e_5

Connected graph

- A graph in which there is a path between any pair of nodes



Connected components



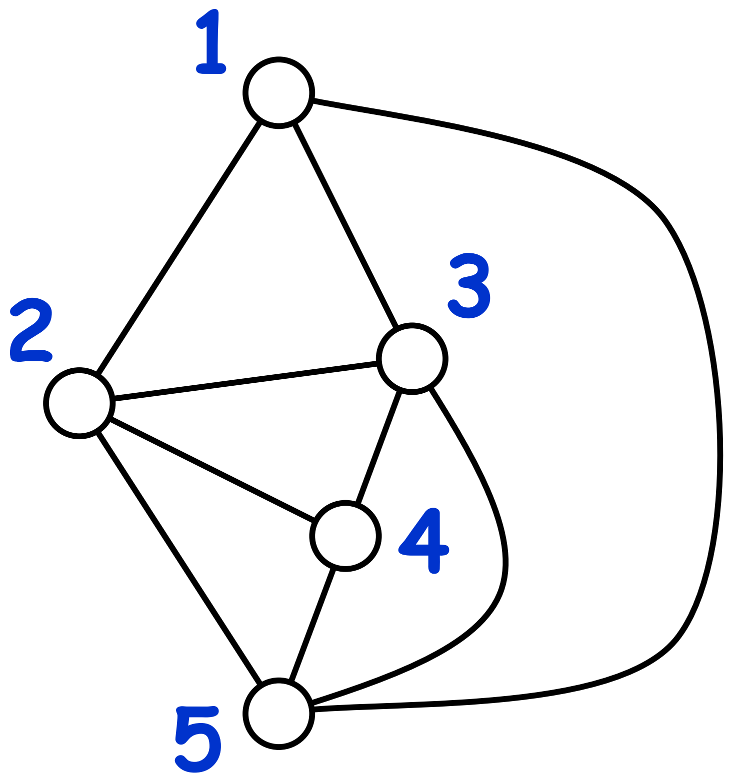
Number of connected components = 2

Subgraph, induced subgraph

- A graph G'(V', E') is a subgraph of another graph G(V, E) if V' and E' are subsets of V and E, respectively
- G'(V', E') is particularly called an induced subgraph of G(V, E) when E' is a set of all edges whose endpoints are both in V'

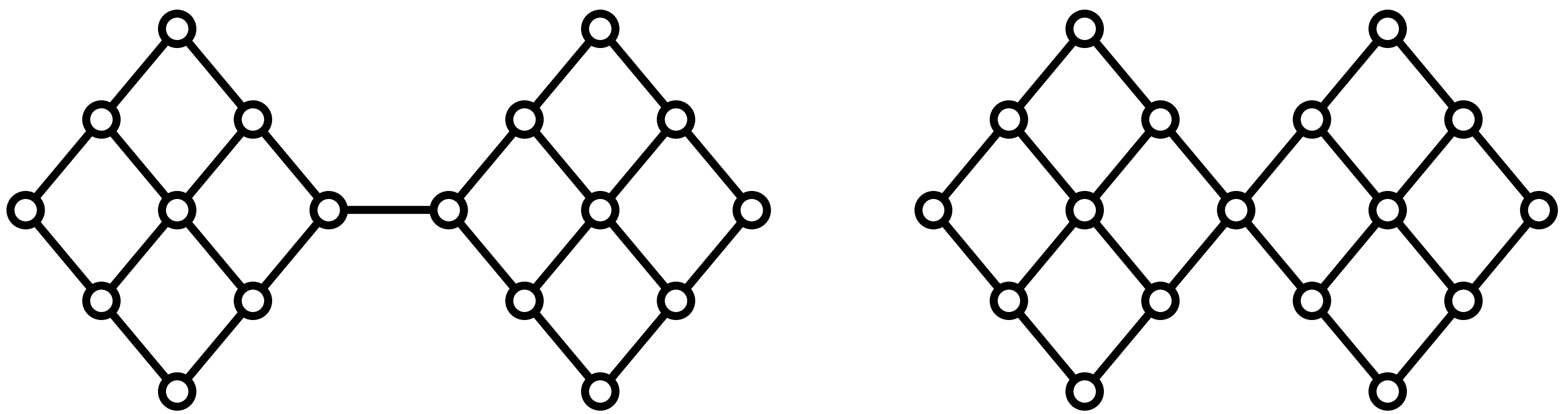
Exercise

- Find the subgraph of the following graph induced by nodes 1, 2 and 5



Bridge, cut point

- A bridge is an edge whose removal would increase the # of connected components
- A cut point is a node whose removal would increase the # of connected components



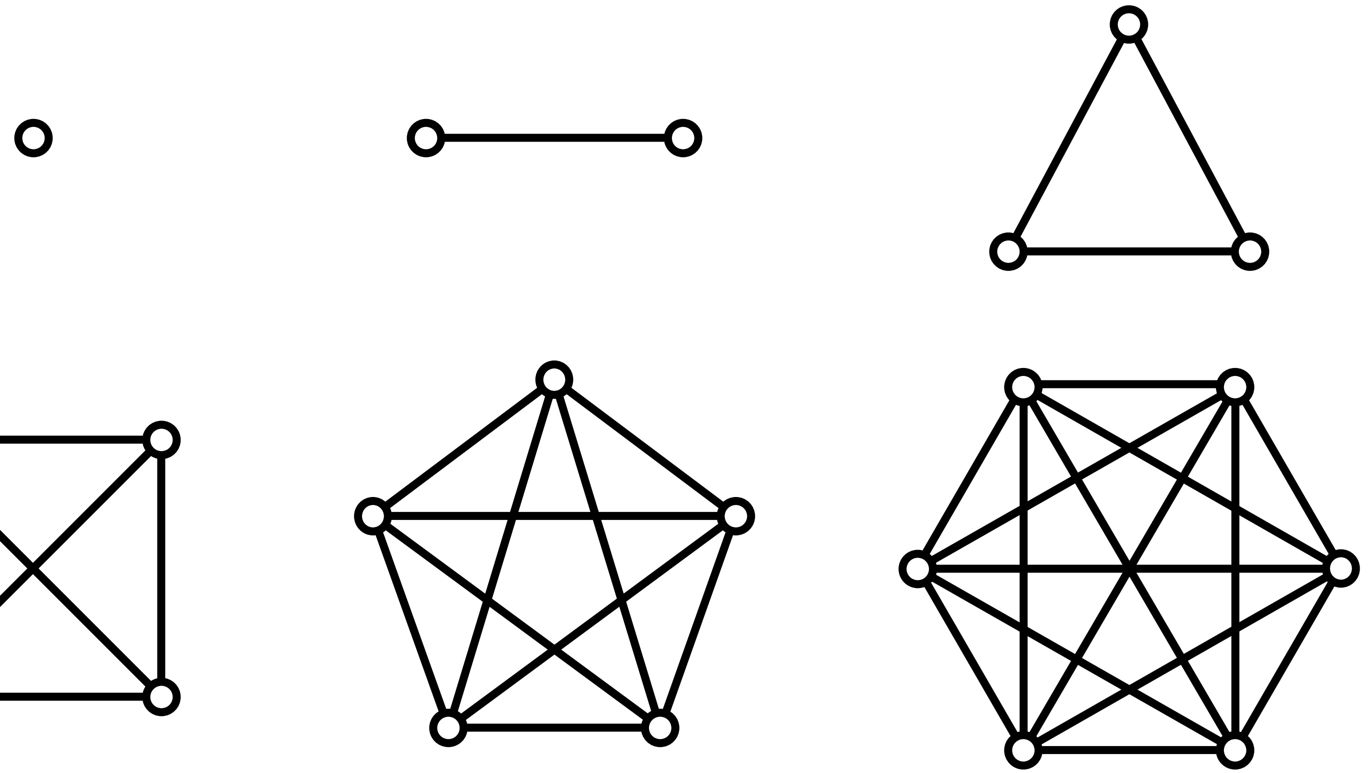
Graphs of characteristic shapes

- Complete graph
- Regular graph
- Bipartite graph, n-partite graph
- Complete n-partite graph
- Planar graph
- Dual graph

etc...

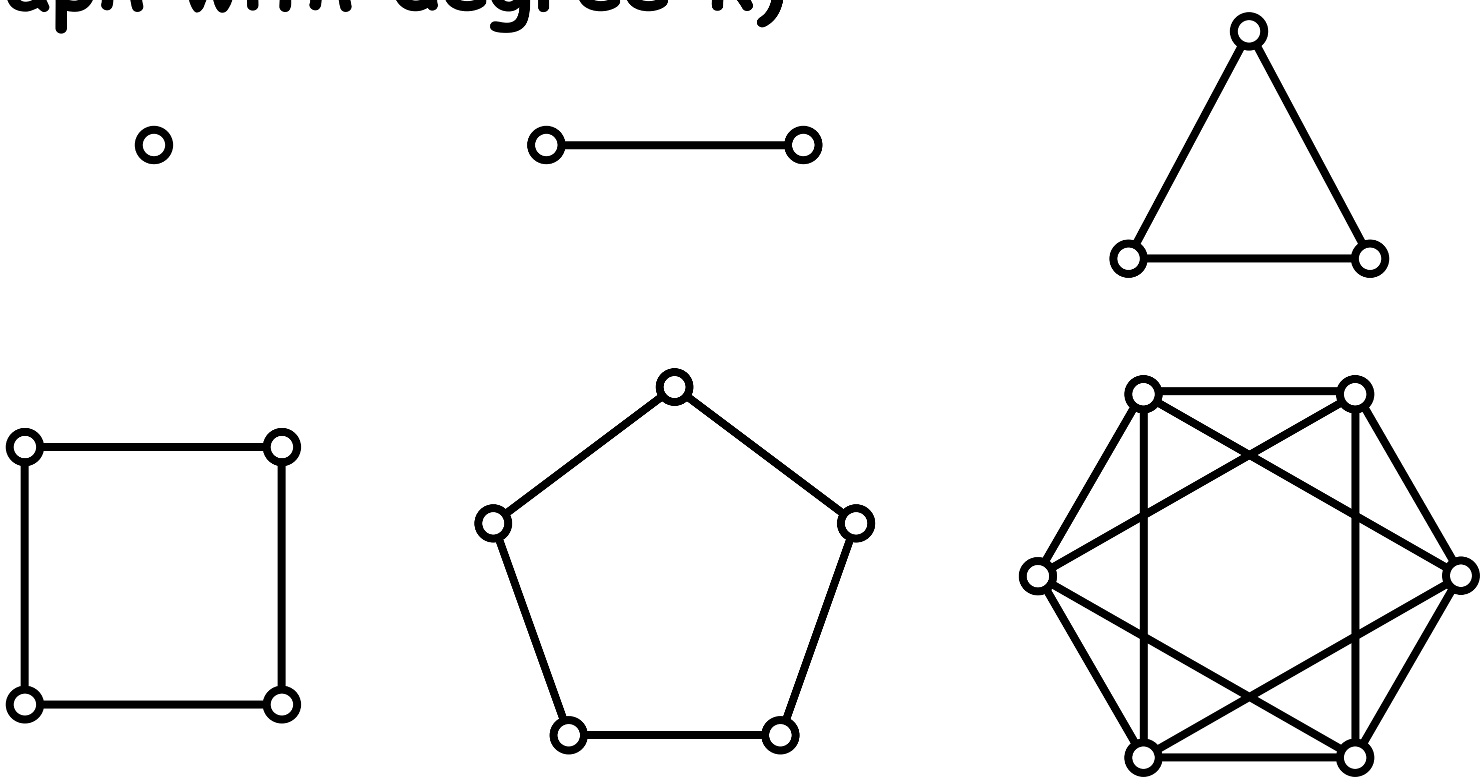
Complete graph

- A graph in which any pair of nodes are connected (often written as K_1, K_2, ...)



Regular graph

- A graph in which all nodes have the same degree (often called k-regular graph with degree k)



Exercise

- How many links does a complete graph with n nodes have?

- How many links does a k-regular graph with n nodes have?

Exercise

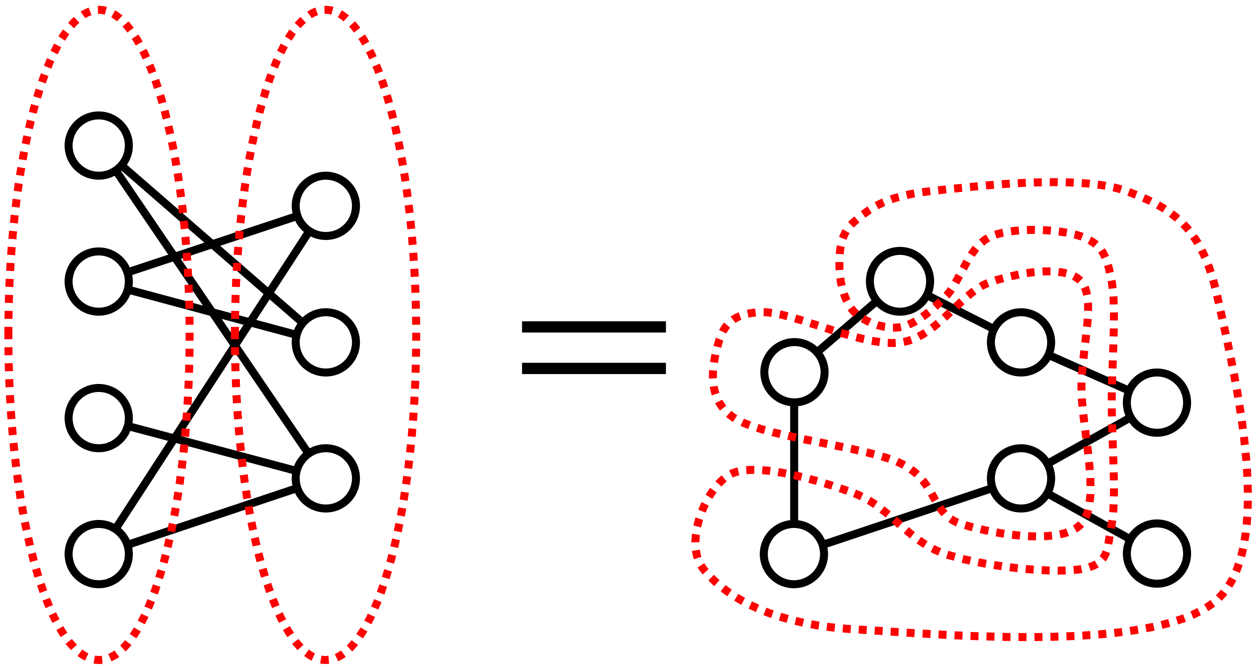
- What value can k take when a k-regular graph has 5 nodes?

- What value can k take when a k-regular graph has 6 nodes?

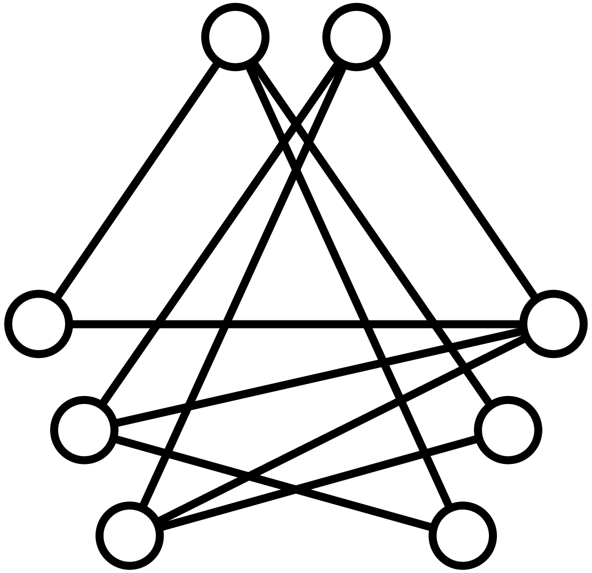
(Assume there is no multiple links or loops)

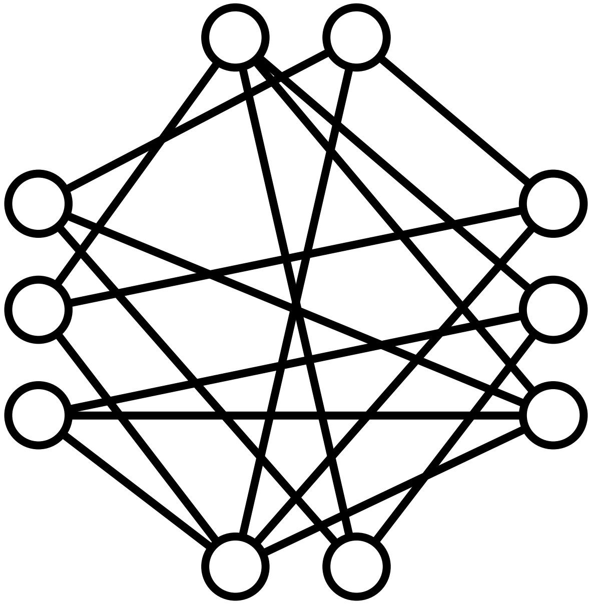
Bipartite graph

- A graph whose nodes can be divided into two subsets so that no link connects nodes within the same subset



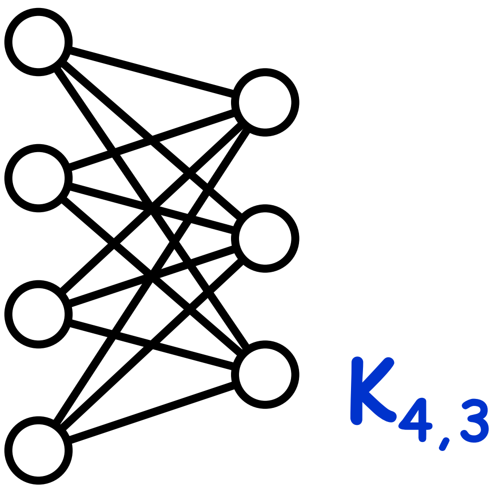
n-partite graph

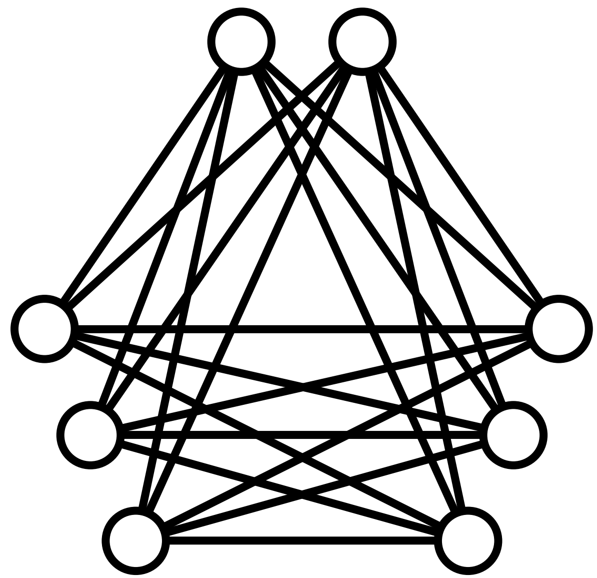




Complete n-partite graph

- An n-partite graph in which each node is connected to all the other nodes that belong to different subsets

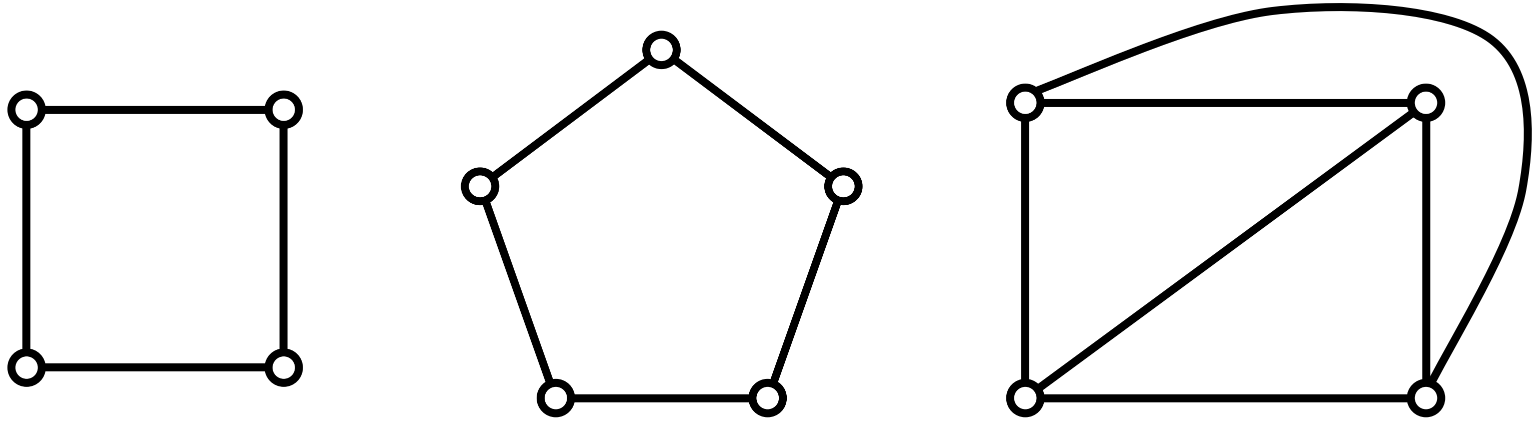




- $K_{m,n}$ represents a complete bipartite graph

Planar graph

- A graph that can be graphically drawn in a two-dimensional plane with no link crossing



Exercise

- Check if K_4, K_5, $K_{2,2}$, $K_{2,3}$, and $K_{3,3}$ are planar or not

Planar graph and Euler's formula

- Euler's formula for any planar graph:

  $|V| - |E| + R = s + 1$

  - R: # of regions
  - s: # of connected components
- For a simple planar graph (no multiple edges, loops, or edge crossings), each region must be encircled by at least three edges, i.e.

  $|E| >= 3R / 2$

Exercise

- By using the previous two formulae, show that K_5 cannot be a planar graph

Exercise

- Prove Euler's formula
  - Hint: use mathematical induction for the number of edges $|E|$

A (rough) proof

- With $|E|=0$, the formula is true ($|V|-0+1 = |V|+1$)
- Assume that the formula is true with $|E|=p$; When a new edge is added to the graph ($|E| \to p+1$):
  - If the added edge divides a region into two, it must be part of a cycle and thus included in a connected graph (i.e. # of connected components does not change):

    $|V|-(p+1)+R+1 = s+1$ (the formula still holds)
  - If the added edge does not divide any region, it must be a bridge between two originally disconnected components (i.e. # of connected components decreases by 1):

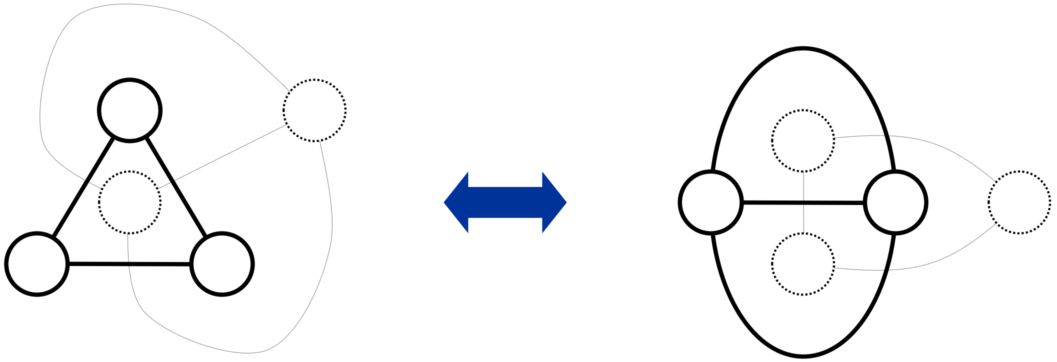
    $|V|-(p+1)+R = (s-1)+1$ (the formula still holds)

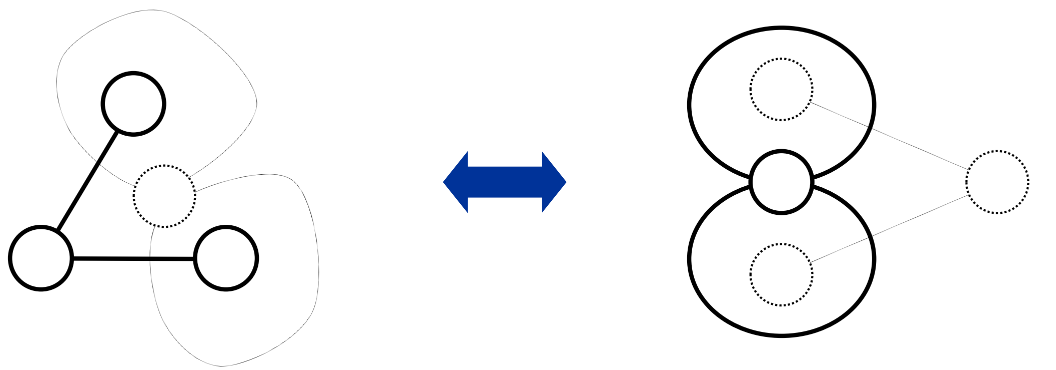
Dual graph

- For any planar graph, one can obtain a new planar graph (dual graph) by
  - assigning a node to each separate region (including background), and
  - connect nodes if the corresponding regions are adjacent across one edge

(A dual of a dual will be the original planar graph)

Examples of dual graphs

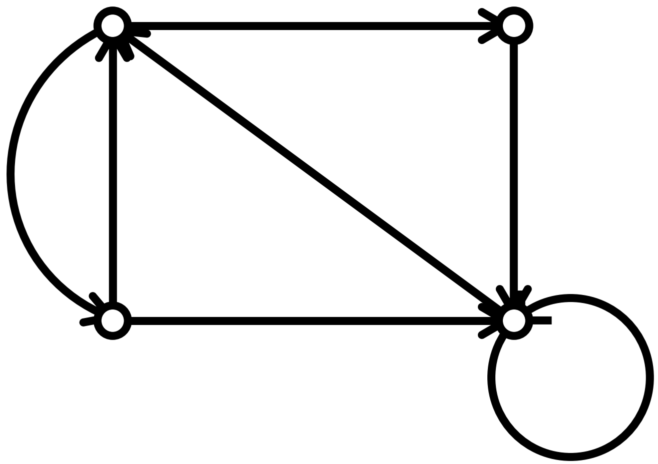




Exercise

- Represent the numbers of nodes and edges of a dual graph of G using the numbers of nodes and edges of G
  (Assume that G is a connected graph)
- By using the result above, obtain all the complete graphs whose dual graphs are identical to themselves

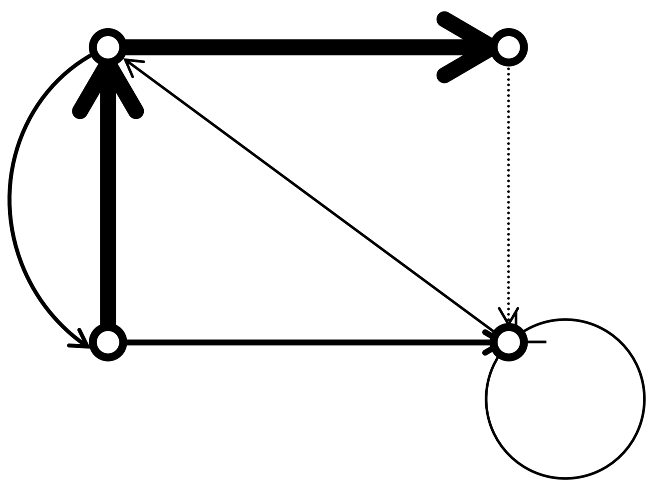
Directed graph



- Each link is directed
- Direction represents either order of relationship or accessibility between nodes

E.g. genealogy

Weighted directed graph



- Most general version of graphs
- Both weight and direction is assigned to each link

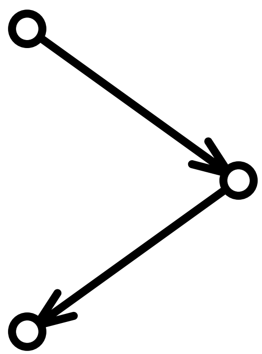
E.g. traffic network

Levels of connectivity

- For every pair of nodes in a directed graph:
  - if there is a semi-path (path ignoring directions) between them, the graph is weakly connected
  - if there is a path from one to the other, the graph is unilaterally connected
  - if there are paths from either direction between them, the graph is strongly connected

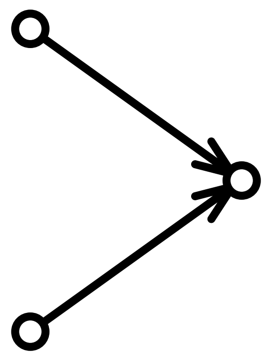
Exercise

- Determine the connectivity of the graphs below



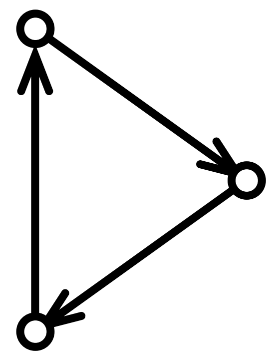
Unilateral

There is always some causal relationship between any pair



Weak

Some pairs are completely independent from each other



Strong

There is always bidirectional influences between any pair

General structure of large directed networks