B, M, and H for the magnetic materials
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Here **B**, **M**, and **H** for the uniformly magnetized sphere, cylinder, and disk are summarized. This chapter is related to the magnetic properties of ferromagnets n in Chapter 32 (Magnetism).

1.   Magnetic field at the hole of sphere: B_{plug} and B_{hole}
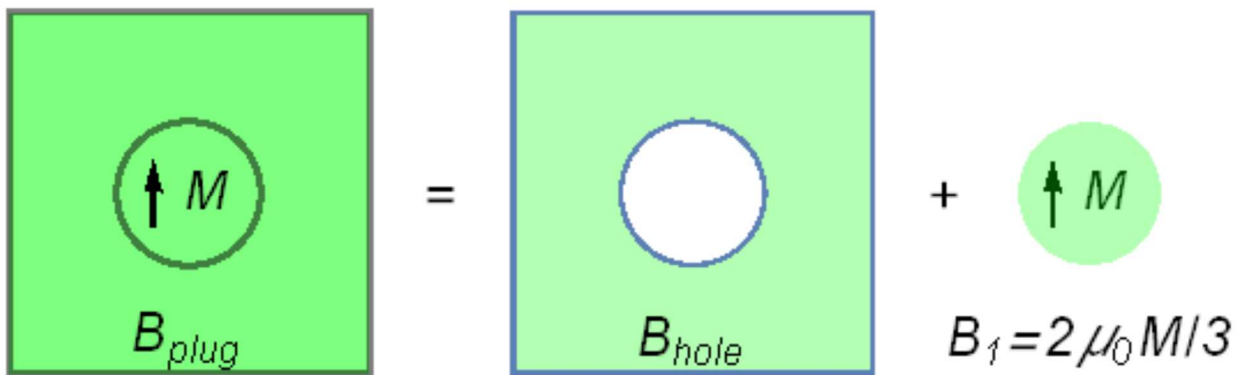 What is the magnetic field **B** inside the magnetic matter?

Fig.   The field at any point A in a magnetic system can be considered as the sum of the field in a spherical hole plus the field due to a spherical plug. $B_1 = B_{plug}$.

The magnetic field **B** inside the system (at any point A), before the sphere is removed, is the sum of the field from all magnetic moments outside the spherical volume plus the fields from magnetic moments within the sphere. That is, if we call the field in the uniform system we can write

$$\boldsymbol{B} = \boldsymbol{B}_{hole} + \boldsymbol{B}_{plug}$$

where $\boldsymbol{B}_{hole}$ is the magnetic field in the hole and $\boldsymbol{B}_{plug}$ is the magnetic field inside a sphere which is uniformly magnetized.

2.   B-field inside the uniformly magnetized sphere
 The magnetic field along the z axis at the point P $(0,0,z = r)$ which is generated from a circle centered at the origin (radius R) lying on the x-y plane.

$$B_z = \frac{\mu_0 I R^2}{2r^3}$$ (Biot-Savart law)

where r is the distance form any point on the circle to the point P on the z axis.
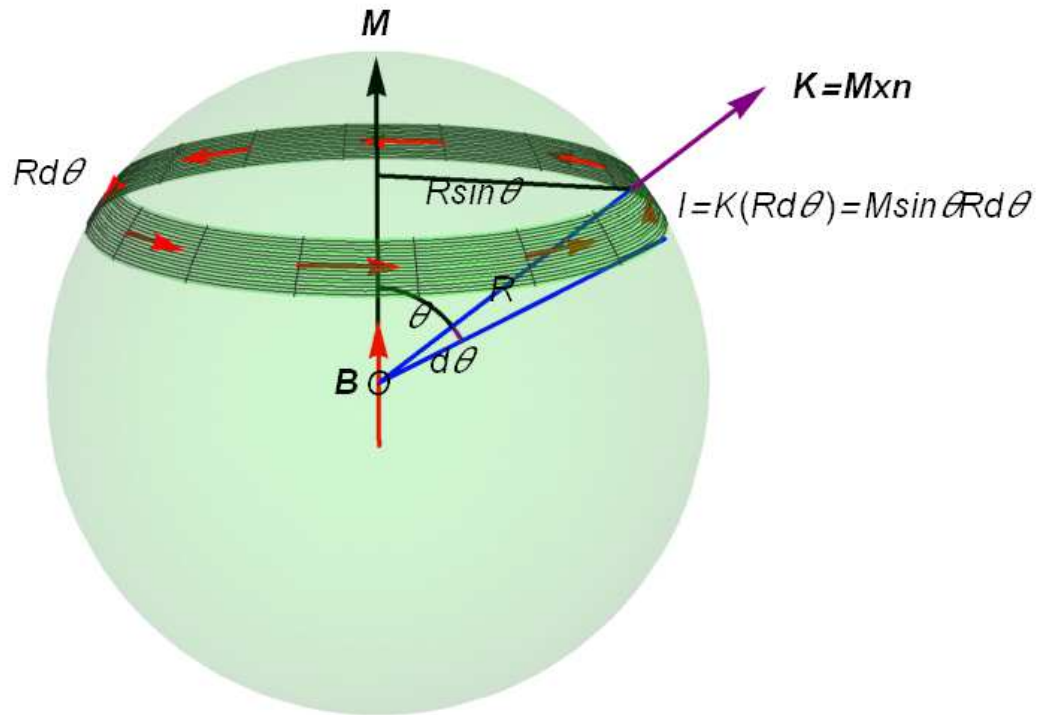
Using this formula, we calculate the magnetic field at the origin of a sphere (radius R) with a uniform magnetization M. The surface current vector on the surface of the sphere is

$$\boldsymbol{K} = \boldsymbol{M} \times \boldsymbol{n}$$

where $\boldsymbol{n}$ is the unit vector normal to the surface of sphere. The surface current is expressed by

$$I = K(Rd\theta) = M \sin\theta (Rd\theta) = MR \sin\theta d\theta$$

Thus the magnetic field at the origin (the direction is $+z$ axis) is obtained as

$$dB_z = \frac{\mu_0 (R\sin\theta)^2}{2R^3} MR \sin\theta a d\theta$$
$$= \frac{\mu_0 M}{2} \sin^3\theta d\theta$$

or

$$B_z = \frac{\mu_0 M}{2} \int_0^\pi \sin^3 \theta d\theta = \frac{2}{3}\mu_0 M$$

In the vector form, we have

$$\boxed{B = \frac{2}{3}\mu_0 M}$$

3. The surface current and the total magnetic moment for the uniformly magnetized sphere

Here we note that the magnetic moment M is obtained as follows.
Surface current:

$$K = M \times n$$

$$K = M \sin \theta e_\phi$$

The current I;

$$I = K(ad\theta) = M \sin \theta(ad\theta) = Ma \sin \theta d\theta$$

Magnetic moment:

$$d\mu = IA$$
$$= (Ma \sin \theta d\theta)\pi(a \sin \theta)^2$$
$$= M\pi a^3 \sin^3 \theta$$

$$\mu = M\pi a^3 \int_0^\pi \sin^3 \theta d\theta = M\frac{4}{3}\pi a^3$$

or

$$\mu = M\frac{4}{3}\pi a^3$$

which is expected from the definition of the uniformly magnetized sphere with the magnetization M, where

$$\int_0^\pi \sin^3 \theta d\theta = \frac{4}{3}$$

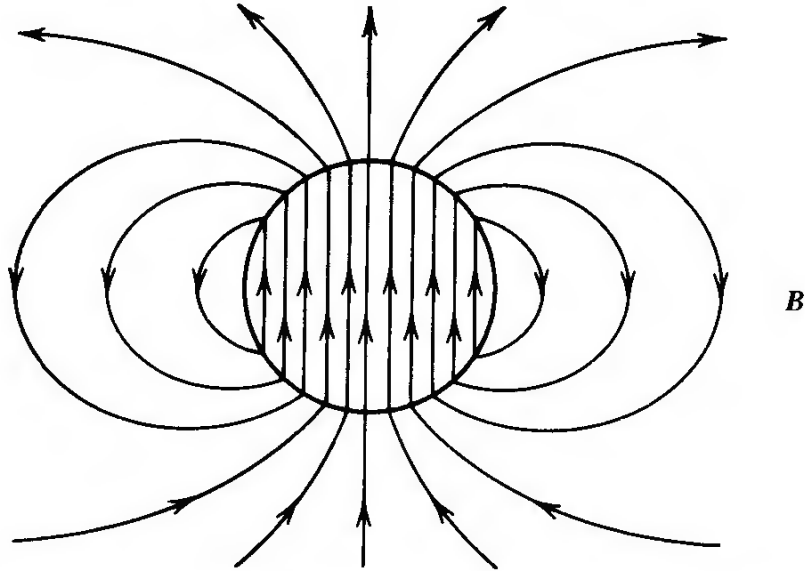## 4.   *B*-field inside and outside of the uniformly magnetized sphere

Fig.   *B*-field of the uniformly magnetized sphere with the magnetization *M*. The total magnetic moment is $\mu = \frac{4\pi}{3} MR^3$. $B_{in} = \frac{2}{3}\mu_0 M$

We consider a sphere which are uniformly magnetized, in the absence of an external magnetic field. The magnetic field **B** outside the sphere is expressed by a magnetic field due to the magnetic moment **m** along the *z* axis. At the north pole of the magnetic sphere, the magnetic field **B** can be expressed by

$$B_z = \frac{\mu_0}{2\pi R^3} \frac{4\pi R^3}{3} M = \frac{2\mu_0}{3} M$$

where *R* is the radius of sphere, the magnetic moment *m* is expressed by using the magnetization vector **M**
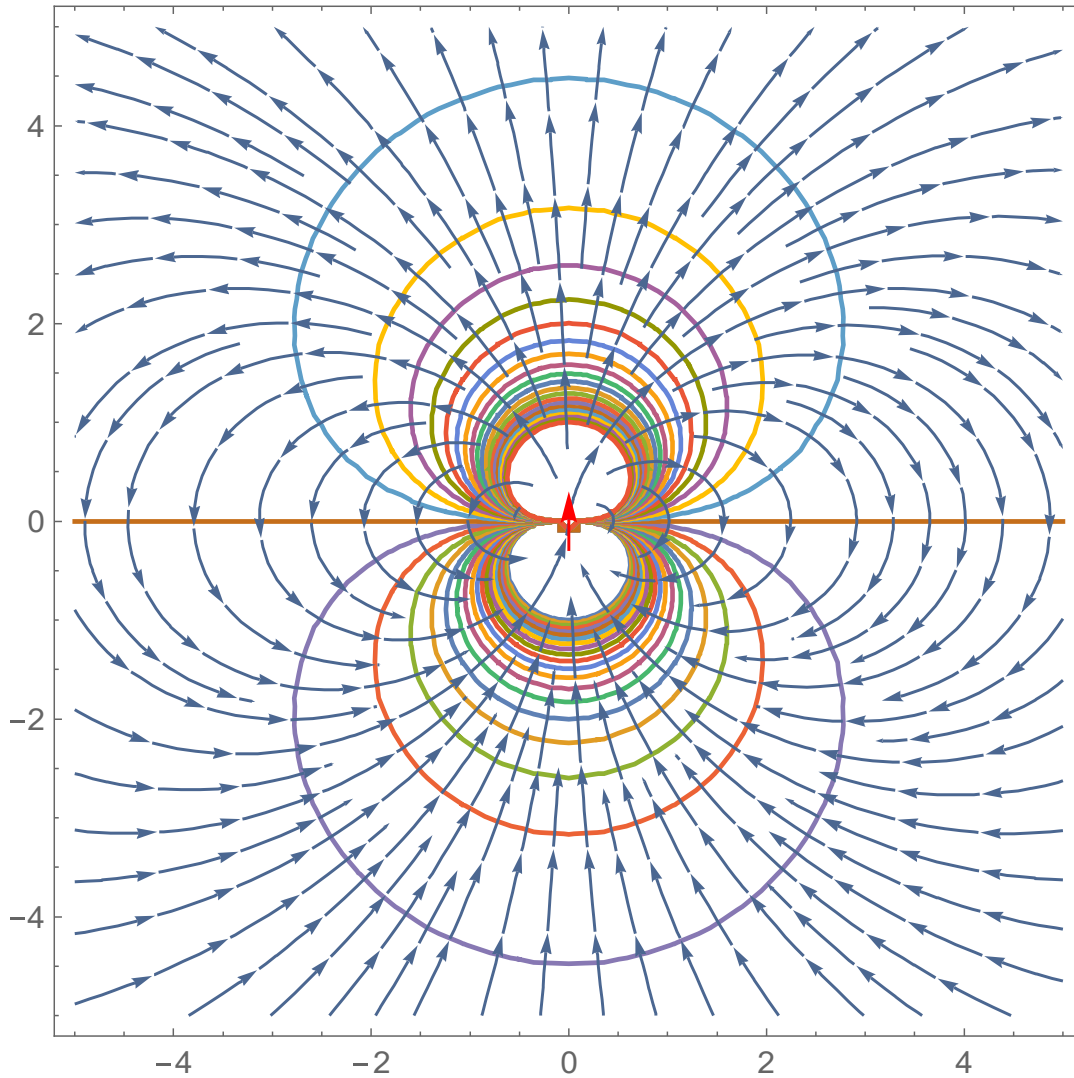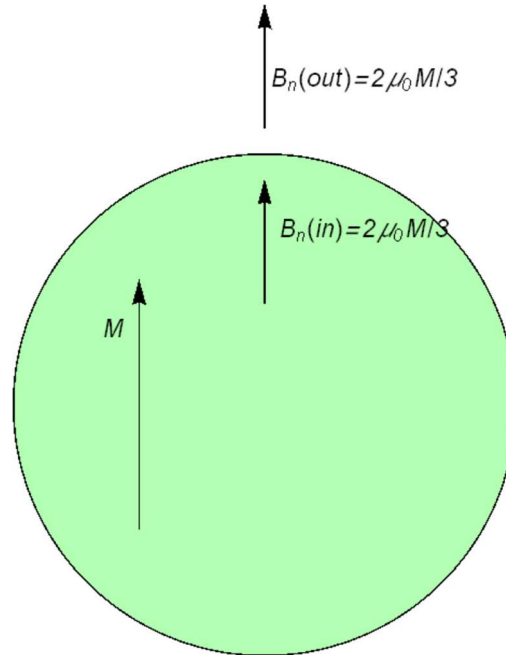
$$m = \frac{4\pi R^3}{3} M$$

Fig. B-field due to the magnetic moment (along the z axia). $B = \dfrac{\mu_0}{4\pi r^5}[3(\boldsymbol{m} \cdot \boldsymbol{r})\boldsymbol{r} - \boldsymbol{m}r^2]$.

We note that the normal component of the field $\boldsymbol{B}$ is continuous on the surface of the sphere. The normal component of $\boldsymbol{B}$ inside the sphere is the same as that of $\boldsymbol{B}$ outside the sphere.

$$B_{inside} = B_{outside} = \frac{2\pi}{3}\mu_0 M$$

((Note))

$$B_n^{in} = \frac{2}{3}\mu_0 M$$

$$\boldsymbol{B}_{out} = \frac{m\mu_0}{4\pi r^3}(2\cos\theta\boldsymbol{e}_r + \sin\theta\boldsymbol{e}_\theta) \qquad \text{(outside)}$$

When $\theta = 0$, $r = R$

$$B_{out} = \frac{2m\mu_0}{4\pi R^3} = \frac{2}{3}\mu_0 M$$

where

$$m = \frac{4\pi}{3}MR^3$$

5.   _H_-field inside and outside of the uniformly magnetized sphere
The **_H_**-field is defined by

$$H = \frac{B}{\mu_0} - M$$

So we have

$$H_n^{(in)} = \frac{2}{3}M - M = -\frac{1}{3}M \ , \qquad\qquad H_n^{(out)} = \frac{B}{\mu_0} = \frac{2}{3}M$$

Thus we have a discontinuity of the normal component of H as

$$H_n^{(in)} - H_n^{(out)} = M$$

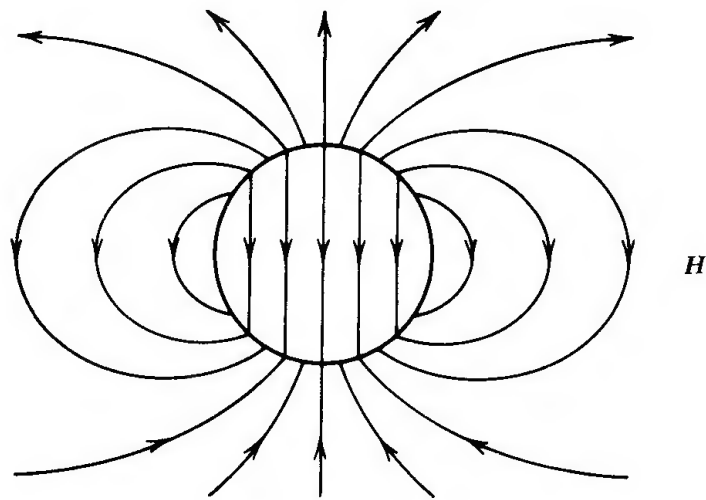((**Note**)) The tangential component of the H-field is continuous.

Fig. H-field of the uniformly magnetized sphere with the magnetization M. $B_{in} = \frac{2}{3}\mu_0 M$. Note the direction of H-field inside the sphere.
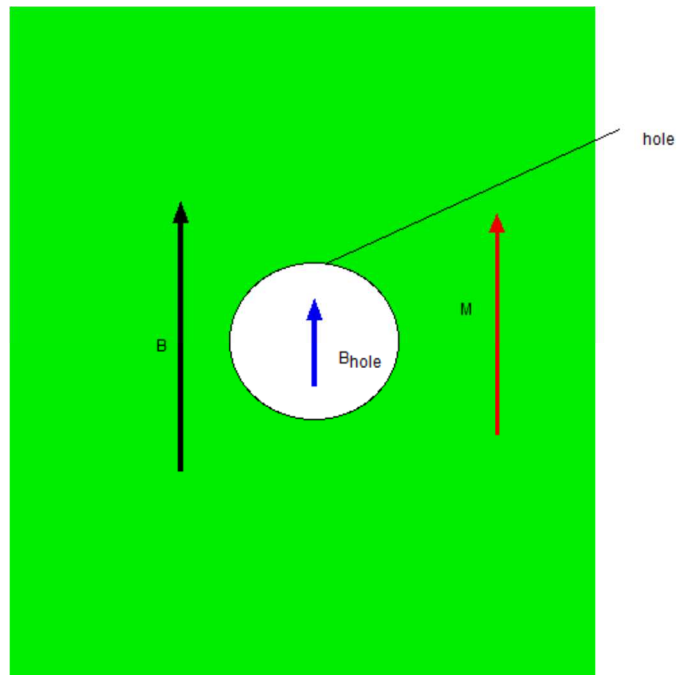
$$H_{in} = -\frac{1}{3}M$$

6.   Magnetic field at the spherical hole (B_{hole})
For the spherical hole, the magnetic field is given by

$$B_{plug} = \frac{2}{3}\mu_0 M$$

$$\boldsymbol{B}_{hole} = \boldsymbol{B} - \frac{2}{3}\mu_0\boldsymbol{M}$$

$$= \mu_0(\boldsymbol{H} + \boldsymbol{M}) - \frac{2}{3}\mu_0\boldsymbol{M}$$

$$= \mu_0(\boldsymbol{H} + \frac{1}{3}\boldsymbol{M})$$

We note that

$$B_{hole} = \mu_0 H_0, \qquad B_{plug} = B_{hole} + \frac{2}{3}\mu_0 M$$

Thus we have

$$B_{plug} = \mu_0(H_0 + \frac{2}{3}M)$$

Since $B_{plug} = \mu_0(H + M)$, we get

$$B_{plug} = \mu_0(H_0 + \frac{2}{3}M) = \mu_0(H + M)$$

or

$$H = H_0 - \frac{1}{3}M = H_0 + H_d$$

with $\quad H_d = -\frac{1}{3}M \quad$ for sphere demagnetization factor.

7. Magnetic field at the cylindrical hole (B_{hole})

We consider a uniformly magnetized system (the magnetization M). B is the magnetic field due to both the true current and the equivalent current from the magnetization. We now remove a small cylinder with M, leaving the hole. We know that the magnetic field in the center of the small cylinder removed, is equal to $\mu_0 M$, because of the uniformly magnetized cylinder.

$$B_{plug} = \mu_0 M$$

From the principle of the superposition, the magnetic field at the hole (empty space) is equal to

$$\begin{aligned} B_{hole} &= B - \mu_0 M \\ &= \mu_0(H + M) - \mu_0 M \\ &= \mu_0 H \end{aligned}$$

which is a magnetic field produced by the system without the small cylinder.
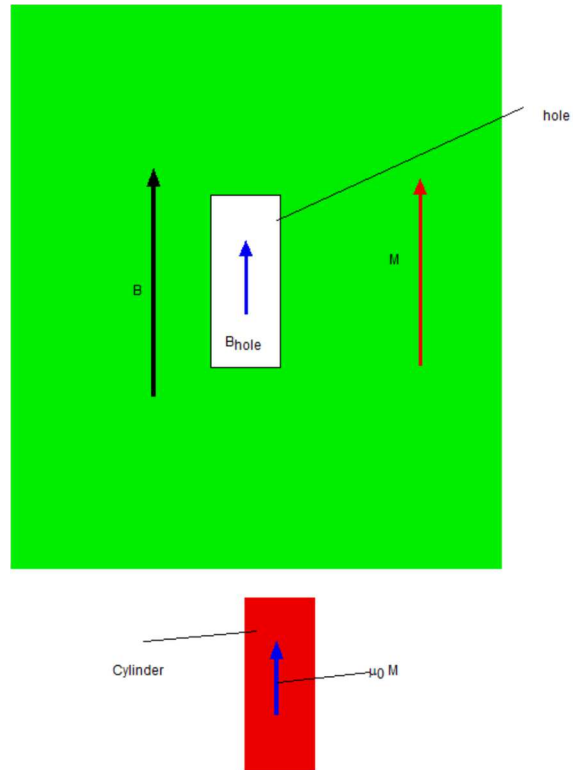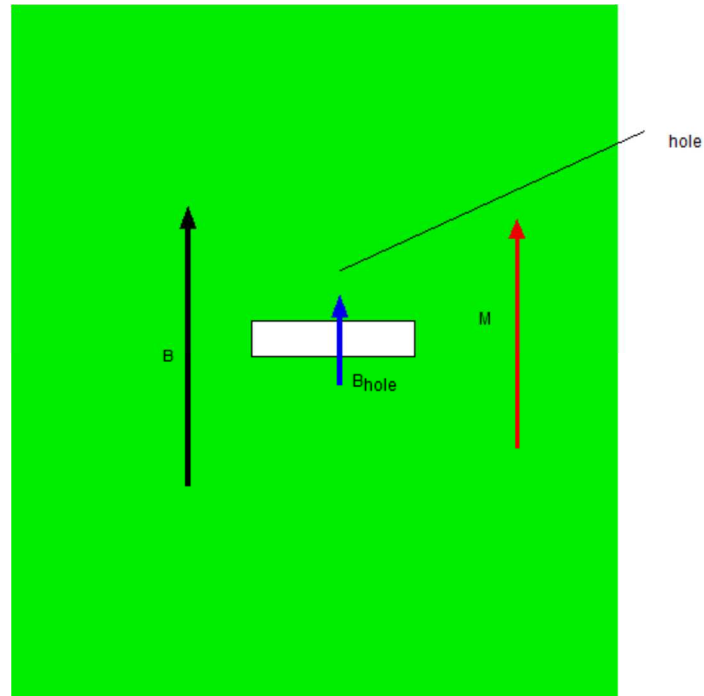
8. Magnetic field at the disk-like hole (B_{hole})

For the disk-like cavity, the magnetic field inside the hole is

$$\boldsymbol{B}_{plug} = 0$$

or

$$\boldsymbol{B}_{hole} = \boldsymbol{B}$$
$$= \mu_0 (\boldsymbol{H} + \boldsymbol{M})$$

9. Magnetic susceptibility

$$B_{Plug} = B = \mu_0(H + M)$$

We define the magnetic susceptibility as

$$M = \chi_m H$$

Thus we have

$$B = \mu_0(H + \chi_m H) = \mu_0(1 + \chi_m)H = \mu H$$

where μ is the permeability of the margetic material and χ_m is the magnetic susceptibility.

$$B_{plug} = B = \mu_0(H + M) = \mu_0(H_0 + \frac{2}{3}M)$$

or

$$H = H_0 - \frac{1}{3}M$$
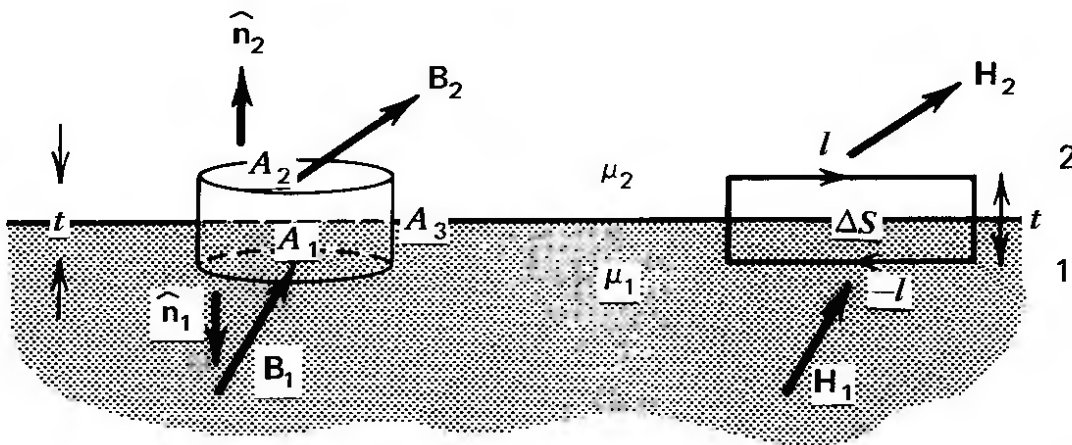
$$M = \chi_m H = \chi_m (H_0 - \frac{1}{3} M)$$

$$M = \chi_m H = \frac{\chi_m}{1 + \frac{1}{3} \chi_m} H_0$$

$$B = \mu_0 (1 + \frac{2}{3} \frac{\chi_m}{1 + \frac{1}{3} \chi_m}) H_0 = \mu_0 (\frac{1 + \chi_m}{1 + \frac{1}{3} \chi_m}) H_0$$

and

$$H = \frac{1}{1 + \frac{1}{3} \chi_m} H_0$$

10. Boundary condition (I)

Obtaining the boundary conditions on the fields **B** and **H** at the interface between two magnetic materials by applying Gauss' law to the pillbox and Ampere's law to the rectangle.

(a)

$$\oint (\nabla \cdot \boldsymbol{B}) dV = \oint \boldsymbol{B} \cdot d\boldsymbol{a} = 0$$

leading to

$$B_{n1} = B_{n2} \qquad \text{(normal component)}$$

Since

$$B_{n1} = \mu_0(H_{n1} + M_{n1}), \qquad B_{n2} = \mu_0(H_{n2} + M_{n2})$$

we have

$$H_{n1} + M_{n1} = H_{n2} + M_{n2}$$

or

$$H_n^{in} + M\cos\theta = H_n^{out}, \qquad \text{or} \qquad H_n^{in} - H_n^{out} = -M\cos\theta$$

(b)
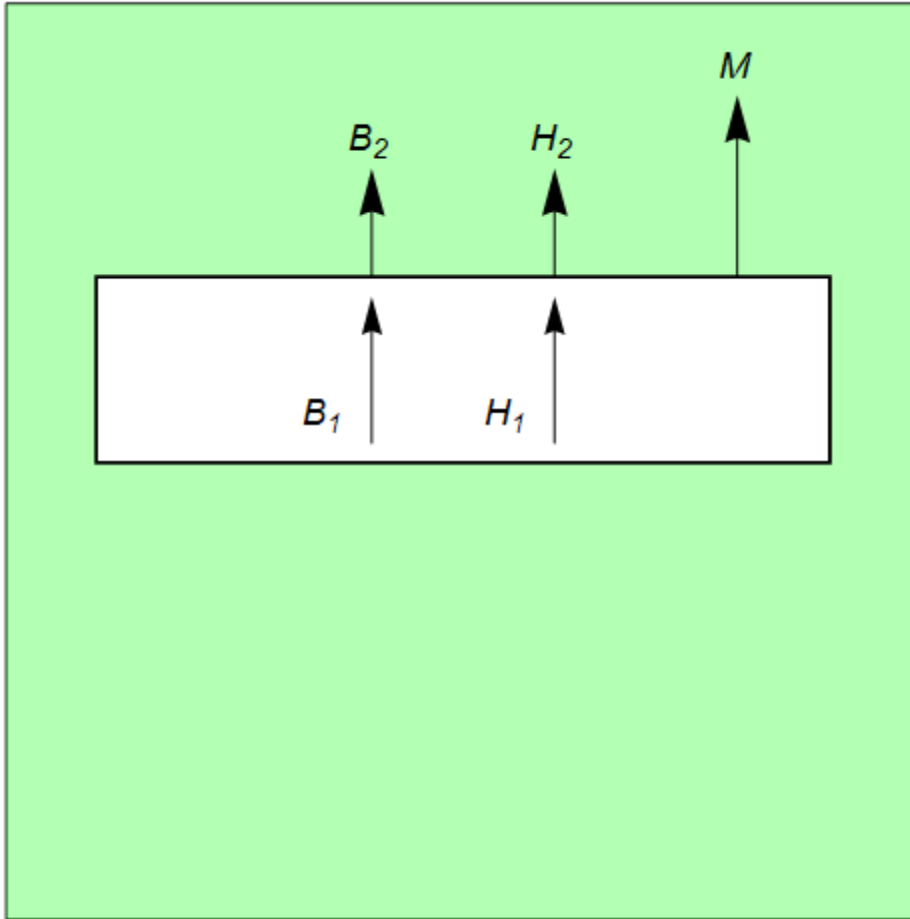
$$\oint (\nabla \times \boldsymbol{B}) \cdot d\boldsymbol{a} = \oint \boldsymbol{B} \cdot d\boldsymbol{l} = 0$$

leading to

$$H_{t1} = H_{t2} \qquad \text{(tangential component).}$$

11   Example for the boundary condition
Example-1

$B_1 = B_2$ (continuity of the normal component)

$B_2 = \mu_0 (H_2 + M)$, $B_1 = \mu_0 H_1$

leading to

$H_1 = H_2 + M$

which means that the H-field inside the rectangle cavity is higher than the H-field outside the cavity.

Example-2

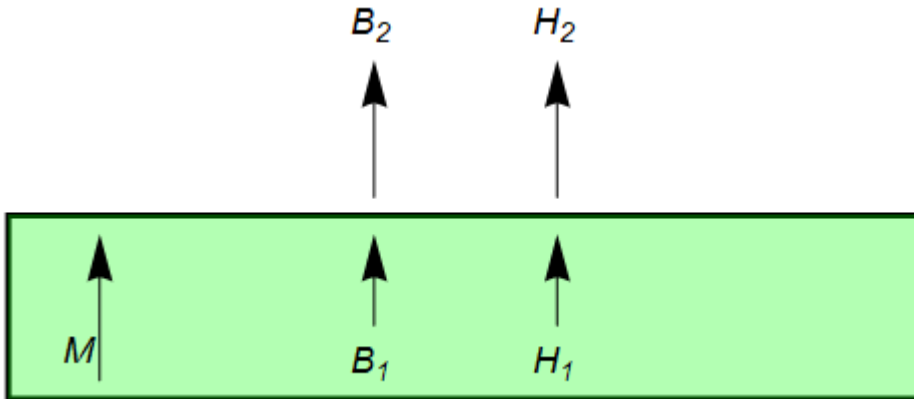$$B_1 = B_2 \qquad \text{(continuity of the normal component)}$$

$$B_2 = \mu_0 H_2, \qquad B_1 = \mu_0 (H_1 + M)$$

leading to

$$H_1 = H_2 - M$$

which means that the H-field inside the rectangle cavity is lower than that outside the rectangle cavity.
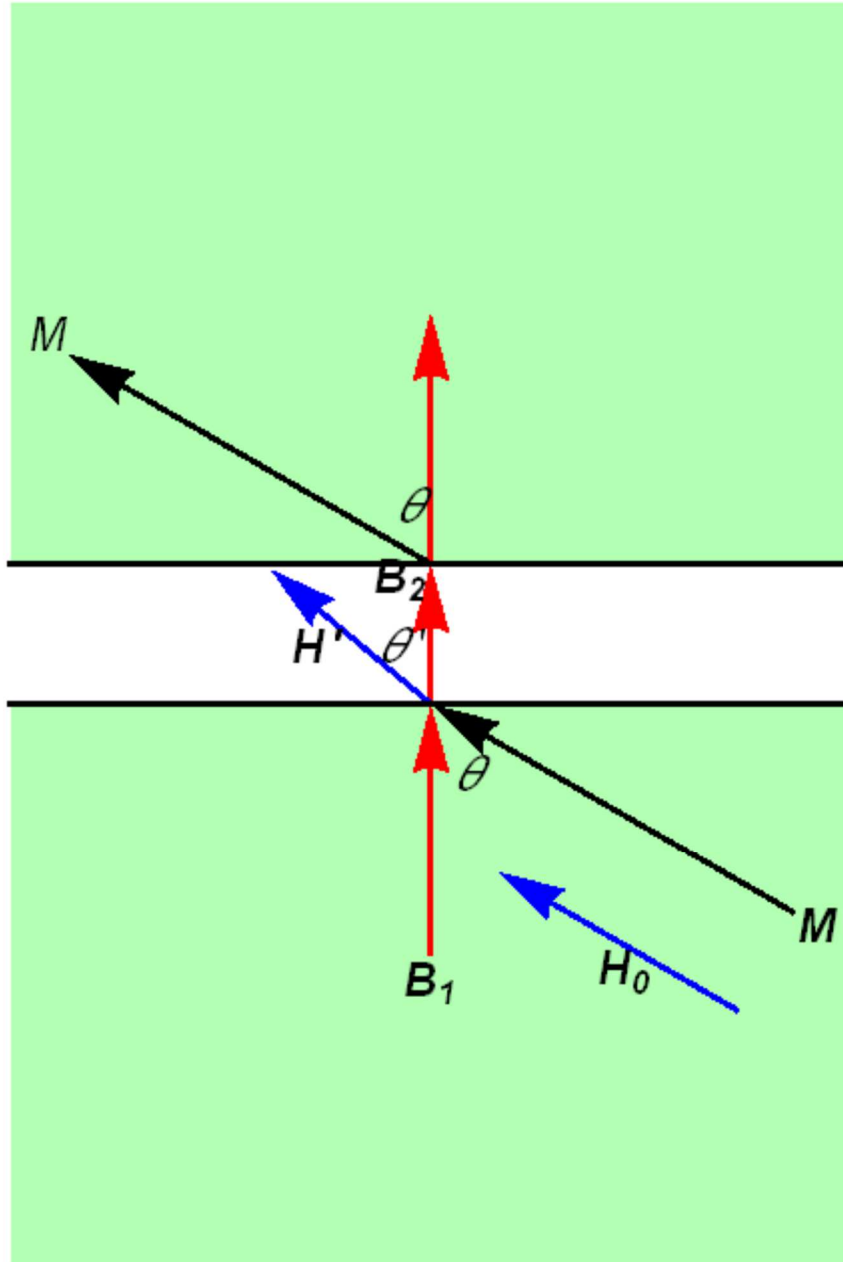
We assume that

$$M = \chi_m H_1 = \chi_m (H_2 - M)$$

where $H_1 = H_2 - M$ and χ_m is the magnetic susceptibility. Then we get

$$M = \frac{\chi_m}{1 + \chi_m} H_2$$

Example-3 Magnetic field H in a small gap in a magnetized system (infinite size)
We consider a gap inside the magnetized system (infinite size) with the magnetization M. We find H' inside the gap.

$B_1 = B_2$ (continuity of the normal component of $\boldsymbol{B}$)

$B_1 = \mu_0(H_0 \cos\theta + M \cos\theta) = \mu_0(H_0 + M)\cos\theta$

$B_2 = \mu_0 H' \cos\theta'$

So we have

$$H'\cos\theta' = (H_0 + M)\cos\theta \qquad (1)$$

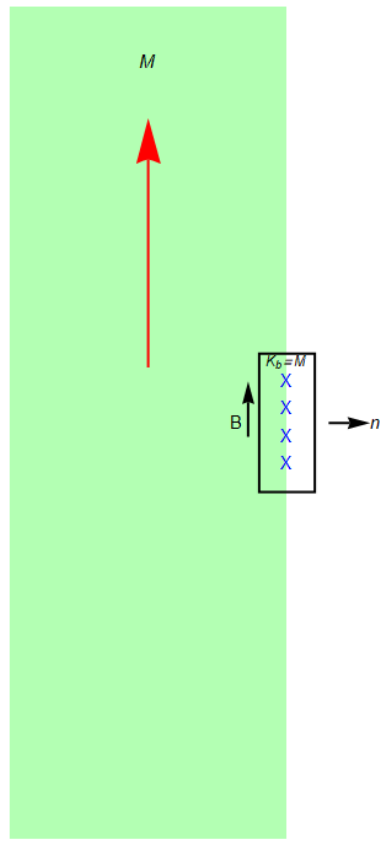The boundary condition for the tangential component of **H**;

$$H'\sin\theta' = H_0\sin\theta \qquad (2)$$

Using Eqs.(1) and (2), we get

$$\tan\theta' = \frac{H_0}{H_0 + M}\tan\theta$$

$$H' = \sqrt{(H_0\sin\theta) + (H_0 + M)^2\sin^2\theta}$$

12. The case of the long cylinder

$$\mathbf{K}_b = \mathbf{M}\times\mathbf{n} \qquad\qquad I_b = Mz$$

Ampere's law for the B field:

$$B_t^{(in)} z = \mu_0 M z$$

or

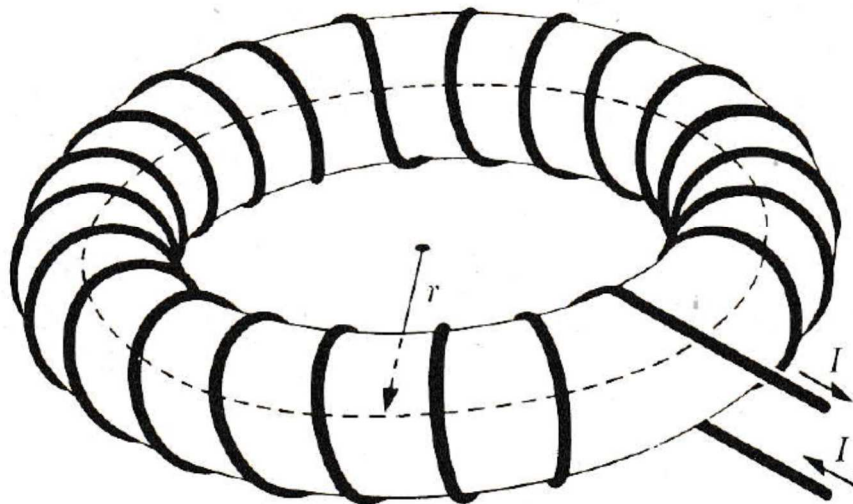$$B_t^{(in)} = \mu_0 M$$

Using the relation $H = \dfrac{B}{\mu_0} - M$, the H-field can be obtained as

$$H_t^{(in)} = \frac{B_t^{(in)}}{\mu_0} - M = 0$$

13. Toroidal coil

A. Kip, Fundamentals of Electricity and Magnetism, 2nd edition, McGraw-Hill (1969).

We apply the Ampere's law to the toroidal coil where the magnetic specimen is inserted.

$$Bl = \mu_0 (nlI + K_b l)$$

or

$$B = \mu_0(nI + K_b)$$

where

$$\mathbf{K}_b = \mathbf{M} \times \mathbf{n} \qquad \text{or} \qquad K_b = M$$

Thus we have

$$B = \mu_0(nI + M)$$

leading to

$$H = H_0 = nI .$$

There is no demagnetization effect

In the above figure, we show an experimental arrangement using a toroidal specimen and coil for measuring magnetic effects of matter, uninfluenced by end effect. An extra secondary winding is employed which connects to a ballistic galvanometer. In order to measure the magnetic effect of matter, a fixed current is passed through the toroidal coil. The magnetic flux in the empty coil is then compared with the flux when the coil is filled with matter. The difference in these two measurements gives the magnetic contribution of the specimen. The flux Φ in the coil is determined by measuring the deflection of the ballistic galvanometer when the current is suddenly turned off.

In the vacuum, the magnetic induction in the solenoid is given by

$$B_0 = \mu_0 nI$$

The corresponding magnetic flux is

$$\Phi_0 = B_0 A = \mu_0 nIA$$

When the experiment is repeated with a specimen filling the toroid the field B is modified by the matter.

$$B = \mu_0(nI + M)$$

The corresponding magnetic flux is

$$\Phi = BA = \mu_0(nI + M)A$$

Thus the difference in the magnetic flux.

((**Note**)) The physics of the toroidal coil is the same as that of the infinitely long cylinder.

$$B = \mu_0(nI + M).$$

There is no demagnetization effect in this case.

APPENDIX-I

$$H_d^x = -N_x M_x, \qquad H_d^y = -N_y M_y, \qquad H_d^z = -N_z M_z$$

where

$$N_x + N_y + N_z = 1$$

For sphere $\qquad N_x = N_y = N_z = \dfrac{1}{3}$

For disk, $\qquad N_z = 1. \qquad N_x = N_y = 0.$

For cylinder, $\qquad N_z = 0, \qquad N_x = N_y = \dfrac{1}{2}$

B.   Magnetic scalar potential   (This topics will not be taught in the lecture)
   We notice that

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} = 0,$$

   A large circuit C can be divided into many small circuits by means of a mesh.
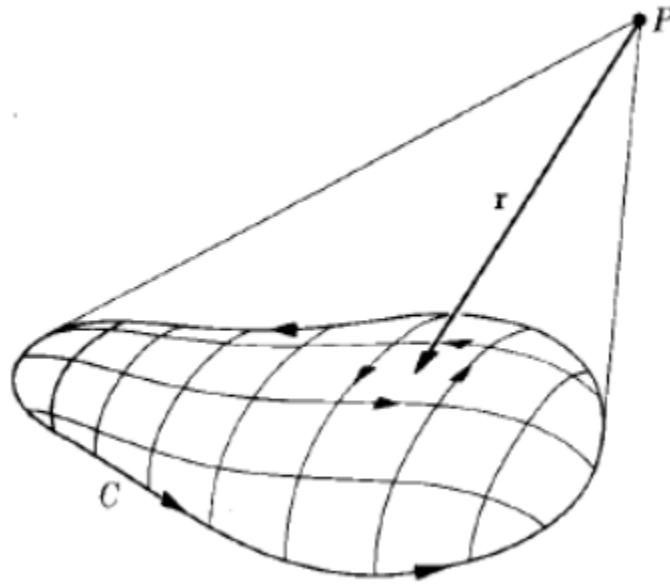
Fig.   A macroscopic current circuit constructed from elemental magnetic dipoles.

If each small loop formed by the mesh carries the same current as originally was carried by the circuit C, then, because of the cancellation of currents in the common branch of adjacent loops, the net effect is the same as if the charge flowed only in the circuit C. For any one of the small loops, the magnetic moment may be written as

$$d\boldsymbol{m} = I d\boldsymbol{a} = I\boldsymbol{n} da$$

since each of the loops is sufficiently small to be regarded as planar. Using this expression, we have

$$d\varphi_m = \frac{\boldsymbol{r}\cdot d\boldsymbol{m}}{4\pi r^3} = \frac{I}{4\pi}\frac{\boldsymbol{r}\cdot\boldsymbol{n} da}{r^3}$$

or

$$\varphi_m = \frac{I}{4\pi}\int_S \frac{\boldsymbol{r}\cdot\boldsymbol{n} da}{r^3}$$

where S is the surface bounded by C. In this equation, $\boldsymbol{r}$ is the vector from da to the point P, that is, -r as shown in Fig. Making the change $(\boldsymbol{r} \rightarrow \boldsymbol{-r})$ results in

$$\phi_m = -\frac{I}{4\pi}\int_S \frac{\mathbf{r}\cdot\mathbf{n}da}{r^3} = -\frac{I}{4\pi}\Omega$$

where $\dfrac{\mathbf{r}\cdot\mathbf{n}da}{r^3}$ is the solid angle ($d\Omega$) subtended by da at P.

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