Chapter 33
Electromagnetic waves
Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
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1. Introduction

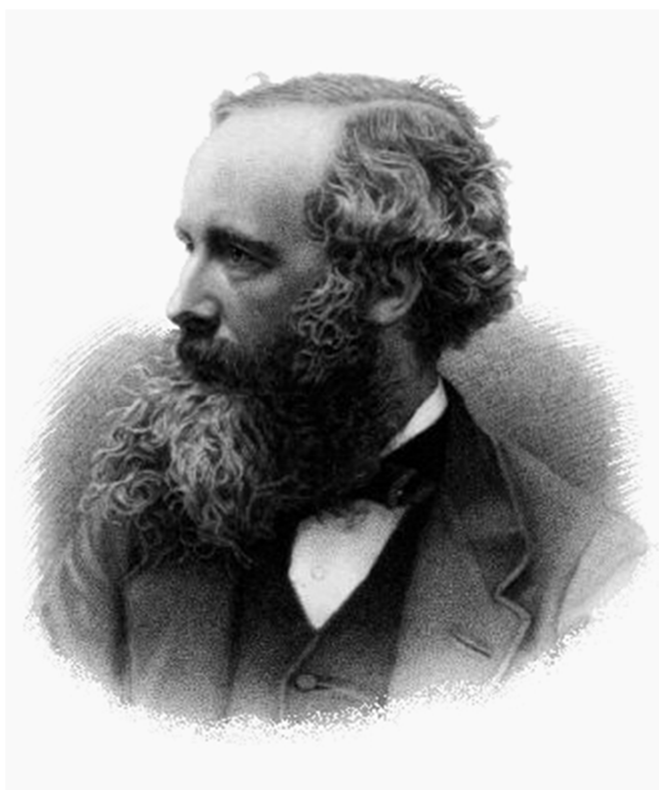
In 1865, James Clerk Maxwell (1831 – 1879) provided a mathematical theory that showed a close relationship between all electric and magnetic phenomena. Maxwell's equations also predicted the existence of electromagnetic waves that propagate through space. Einstein showed these equations are in agreement with the special theory of relativity.

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{B} = \mu_0 (\boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t})$$



James Clerk Maxwell
Born 13 June 1831
Edinburgh, Scotland, United Kingdom

Died 5 November 1879
Cambridge, England, United Kingdom

Nationality Scottish
Fields Mathematics, Science
Alma mater University of Edinburgh, University of Cambridge
Doctoral advisor William Hopkins
Known for Maxwell's Equations
The Maxwell Distribution
Maxwell's Demon
Notable awards Rumford Medal
Adams Prize

2 Maxwell's equations in vacuum (exact description)

Maxwell predicted the existence of electromagnetic waves. The electromagnetic waves consist of oscillating electric and magnetic fields. The changing fields induce each other, which maintain the propagation of the wave. A changing electric field induces a magnetic field. A changing magnetic field induces an electric field. We start with the Maxwell's equation ($\boldsymbol{J} = 0$, $\rho = 0$).

$$\nabla \cdot \boldsymbol{E} = 0$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t}$$

Since

$$\nabla \times (\nabla \times \boldsymbol{B}) = \nabla(\nabla \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B} = \mu_0 \varepsilon_0 \nabla \times \frac{\partial \boldsymbol{E}}{\partial t} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \nabla \times \boldsymbol{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2}$$

or

$$\boxed{\nabla^2 \boldsymbol{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2 \boldsymbol{B}}{\partial t^2}} \qquad \text{(wave equation)}$$

or

$$(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \boldsymbol{B} = 0$$

where c is the velocity of light,

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

Similarly,

$$\nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{B}) = -\mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \boldsymbol{E}$$

or

$$\nabla^2 \boldsymbol{E} = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \boldsymbol{E} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \boldsymbol{E} \qquad \text{(wave equation)}$$

We consider the special case when $\boldsymbol{E}$ or $\boldsymbol{B}$ depends only on x. In this case the equation for the field becomes

$$\frac{\partial^2}{\partial t^2} f = c^2 \frac{\partial^2}{\partial x^2} f$$

where f is understood any component of the vector $\boldsymbol{E}$ or $\boldsymbol{B}$.

$$(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}) f = 0$$

We introduce new variables

$$\xi = t - \frac{x}{c}$$

$$\eta = t + \frac{x}{c}$$

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = -\frac{1}{c}\frac{\partial}{\partial \xi} + \frac{1}{c}\frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

So that the equation for f becomes

$$\frac{\partial^2 f}{\partial \xi \partial \eta} = 0$$

The solution obviously has the form

$$f = f_1(\xi) + f_2(\eta)$$

where f_1 and f_2 are arbitrary function.

or

$$f = f_1(t - \frac{x}{c}) + f_2(t + \frac{x}{c})$$

The function f_1 represents a plane wave moving in the positive direction along the x axis. The function f_2 represents a plane wave moving in the negative direction along the x axis.

3. Plane-wave

3.1 Solutions for *E* and *B*

We are going to construct a rather simple electromagnetic field that will satisfy Maxwell's equation for empty space. We will assume that the vectors for the electric and magnetic fields in an EM wave have a specific space-time behavior that is consistent with Maxwell's equations. The components of the electric and magnetic fields of plane electromagnetic waves are perpendicular to each other and perpendicular to the direction of propagation. This can be summarized by saying that electromagnetic waves are *transverse waves*.

Suppose that ***E*** and ***B*** are described by a plane waves

$$\boldsymbol{E} = \boldsymbol{E}_0 \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)$$

$$\boldsymbol{B} = \boldsymbol{B}_0 \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)$$

where $\boldsymbol{k}$ is the wave number and ω is the angular frequency. The direction of $\boldsymbol{k}$ is the same as that of the propagation of the wave.

(i) Step-1

From the wave equation

$$\nabla^2 \boldsymbol{E} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \boldsymbol{E}$$

we have

$$-\boldsymbol{k}^2\boldsymbol{E}_0 = -\frac{\omega^2}{c^2}\boldsymbol{E}_0$$

The angular frequency ω satisfies the dispersion relation given by

$$\omega = ck = \frac{2\pi c}{\lambda} = 2\pi f$$

where

$$\omega = 2\pi f$$
$$k = \frac{2\pi}{\lambda}$$
$$\lambda f = c$$

(ii) Step-2

From

$$\nabla\cdot\boldsymbol{E} = 0, \qquad \text{and} \qquad \nabla\cdot\boldsymbol{B} = 0$$

we have

$$\boldsymbol{k}\cdot\boldsymbol{E}_0 = 0, \qquad \text{and} \qquad \boldsymbol{k}\cdot\boldsymbol{B}_0 = 0$$

The wave vector $\boldsymbol{k}$ is perpendicular to $\boldsymbol{E}$ and $\boldsymbol{B}$.

(iii) Step-3

From

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$$

$$(\boldsymbol{k}\times\boldsymbol{E}_0) = \omega\boldsymbol{B}_0$$

$$(\boldsymbol{k}\times\boldsymbol{E}_0) = ck\boldsymbol{B}_0$$

or

$$(\hat{\boldsymbol{k}}\times\boldsymbol{E}_0) = c\boldsymbol{B}_0$$

or

$$\boldsymbol{B}_0 = \frac{1}{c}(\hat{\boldsymbol{k}} \times \boldsymbol{E}_0)$$

Note that

$$E_0 = cB_0$$

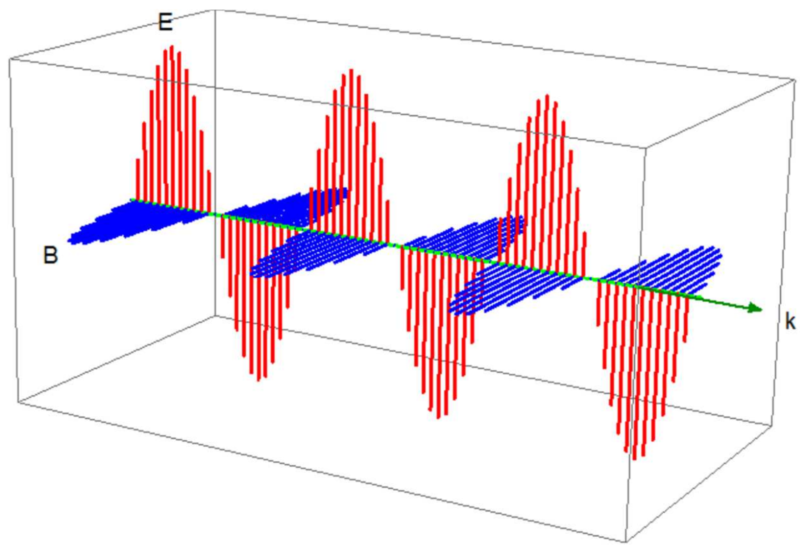


Fig. A linearly polarized, sinusoidally varying plane wave propagating in the positive x direction. The figure represents a snapshot at a particular time. This figure is made by using the ParametricPlot3D of the Mathematica.

((**Conclusion**))

The solutions of Maxwell's equation are wave-like, with both $\boldsymbol{E}$ and B satisfying a wave equation. Electromagnetic waves travel at the speed of light. This comes from the solution of Maxwell's equations. Waves in which the electric and magnetic fields are restricted to being parallel to a pair of perpendicular axes are said to be *linearly polarized transverse waves*. The direction of the wave's polarization coincides with that of the electric field.

3.2 Energy density and Poynting vector in electromagnetic wave (exact description)

Electromagnetic waves carry energy. As they propagate through space, they can transfer that energy to objects in their path. The rate of flow of energy in an electromagnetic (EM) wave is described by a vector called the Poynting vector.

The energy density u is given by

$$u = \frac{1}{2}(\varepsilon_0 \boldsymbol{E}^2 + \frac{1}{\mu_0}\boldsymbol{B}^2)$$

The Poynting vector $\boldsymbol{S}$ is given by

$$\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E}\times\boldsymbol{B})$$

(a) The time-averaged energy density <u>

First we calculate the time average of $\mathbf{E}^2$

$$\boldsymbol{E}^2 = \boldsymbol{E}_0^{\ 2}\cos^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)$$

The time average of $\boldsymbol{E}^2$

$$E_{rms}^{\ \ 2} = \frac{1}{T}\int_0^T \boldsymbol{E}^2 dt = \frac{1}{T}\int_0^T E_0^{\ 2}\cos^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)dt$$
$$= \frac{1}{2T}\int_0^T E_0^{\ 2}[1+\cos(2\boldsymbol{k}\cdot\boldsymbol{r}-2\omega t)]dt$$
$$= \frac{1}{2}E_0^{\ 2}$$

The root-mean square value of the electric field is given by

$$\boxed{E_{rms} = \frac{1}{\sqrt{2}}E_0 = \frac{1}{\sqrt{2}}E_{\max}}$$

where $E_0 = E_{\max}$, $T = \dfrac{2\pi}{\omega}$, and $\dfrac{1}{T}\displaystyle\int_0^T \cos(2\boldsymbol{k}\cdot\boldsymbol{r}-2\omega t)dt = 0$

Similarly, we have

$$B_{rms}^{\ \ 2} = \frac{1}{T}\int_0^T \boldsymbol{B}^2 dt = \frac{1}{2}B_0^{\ 2},$$

The root-mean square value of the magnetic field is

$$\boxed{B_{rms} = \frac{1}{\sqrt{2}}B_0 = \frac{1}{\sqrt{2}}B_{\max}}$$

where $B_0 = B_{\max}$,

$$\boxed{\frac{E_{rms}}{B_{rms}} = \frac{E_{\max}}{B_{\max}} = \frac{\omega}{k} = c}$$

Then the time-average of the energy density is given by

$$\langle u\rangle = \frac{1}{2}(\varepsilon_0 E_{rms}{}^2 + \frac{1}{\mu_0}B_{rms}{}^2) = \frac{1}{4}(\varepsilon_0 E_0{}^2 + \frac{1}{\mu_0}B_0{}^2)$$

Here we note

$$B_0{}^2 = \frac{1}{c^2}E_0{}^2$$

Then we have

$$\boxed{\langle u\rangle = \frac{1}{4}(\varepsilon_0 E_0{}^2 + \frac{1}{\mu_0 c^2}E_0{}^2) = \frac{1}{2}\varepsilon_0 E_0{}^2 = \varepsilon_0 E_{rms}{}^2}$$

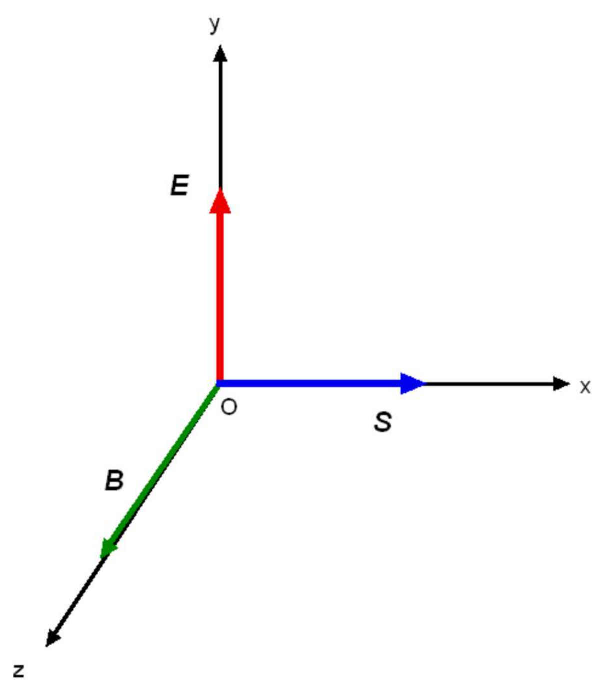
(b) The time-averaged Poynting vector <*S*>

Next we calculate the Poynting vector ***S***

$$\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E}\times\boldsymbol{B}) = \frac{1}{\mu_0}(\boldsymbol{E}_0\times\boldsymbol{B}_0)\cos^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)$$

The time-averaged Poynting vector <***S***> is obtained as

$$\langle \boldsymbol{S}\rangle = \frac{1}{2\mu_0}(\boldsymbol{E}_0\times\boldsymbol{B}_0)$$



Noting that

$$\boldsymbol{E}_0 \times \boldsymbol{B}_0 = \frac{1}{c}\boldsymbol{E}_0 \times (\hat{\boldsymbol{k}} \times \boldsymbol{E}_0) = \hat{\boldsymbol{k}}\frac{1}{c}E_0^{\ 2}$$

we have

$$\langle \boldsymbol{S} \rangle = \frac{1}{2\mu_0}\hat{\boldsymbol{k}}\frac{1}{c}E_0^{\ 2}$$

or

$$\langle \boldsymbol{S} \rangle = c\frac{1}{2\mu_0}\hat{\boldsymbol{k}}\frac{1}{c^2}E_0^{\ 2} = c\hat{\boldsymbol{k}}\frac{\varepsilon_0}{2}E_0^{\ 2} = c\hat{\boldsymbol{k}}\langle u \rangle$$

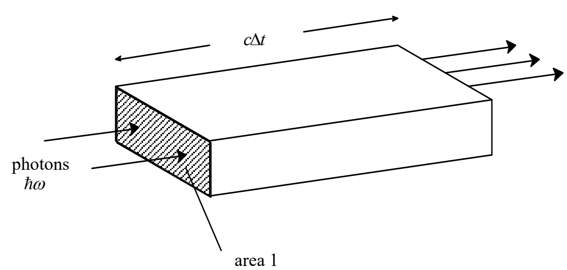
where $\hat{\boldsymbol{k}}$ is the unit vector of the wave vector $\boldsymbol{k}$, $\hat{\boldsymbol{k}} = \frac{\boldsymbol{k}}{k}$, and

$$\langle S \rangle = c\langle u \rangle$$

(c) The intensity *I* (= <*S*>)

Here we define the intensity I of the light. The intensity I is the energy flux (energy per unit area per unit time);

$I = \langle S \rangle$.



We now consider the photon flows (photon is the quantization of light with the velocity c) flows. During the time Δt, the total energy passing through the area A is

$$U = V\langle u \rangle = cA\Delta t\langle u \rangle$$

where the volume V is $c\Delta tA$ and the energy density is $\langle u \rangle$. From the definition of <*S*>, the total energy passing through the area A during the time Δt, is given by

$$U = A\Delta t < S >$$

Then we have

$$U = cA\Delta t\langle u\rangle = A\Delta t < S >,$$

leading to

$$I = \frac{U}{A\Delta t} = \langle S\rangle = c < u >= \frac{1}{2}c\varepsilon_0 E_0{}^2 = \frac{1}{2\mu_0 c}E_0{}^2 = \frac{1}{2\mu_0 c}E_{\max}{}^2 = \frac{1}{\mu_0 c}E_{rms}{}^2$$

where the unit of the intensity I is J/m^2 s = W/s.

((**Note**))

Poynting, John Henry (1852-1914)

English physicist, mathematician, and inventor. He devised an equation by which the rate of flow of electromagnetic energy (now called the Poynting vector) can be determined. In 1891 he made an accurate measurement of Isaac Newton's gravitational constant. Poynting was born near Manchester and studied there at Owens College, and at Cambridge. From 1880 he was professor of physics at Mason College, Birmingham (which became Birmingham University in 1900). In On the Transfer of Energy in the Electromagnetic Field 1884, Poynting published the equation by which the magnitude and direction of the flow of electromagnetic energy can be determined. This equation is usually expressed as $\boldsymbol{S} = (1/\mu_0)\boldsymbol{E}\text{x}\boldsymbol{B}$ where $\boldsymbol{S}$ is the Poynting vector, is the permeability of the medium, E is the electric field strength, B is the magnetic field strength, and is the angle between the vectors representing the electric and magnetic fields. In 1903, he suggested the existence of an effect of the Sun's radiation that causes small particles orbiting the Sun to gradually approach it and eventually plunge in. This idea was later developed by US physicist Howard Percy Robertson (1903-1961) and is now known as the Poynting-Robertson effect. Poynting also devised a method for measuring the radiation pressure from a body; his method can be used to determine the absolute temperature of celestial objects. Poynting's other work included a statistical analysis of changes in commodity prices on the stock exchange 1884.

4. Physical meaning of Maxwell's equation

(a)

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}, \qquad \text{or} \qquad \oint \boldsymbol{E}\cdot d\boldsymbol{s} = -\frac{\partial}{\partial t}\Phi_B$$

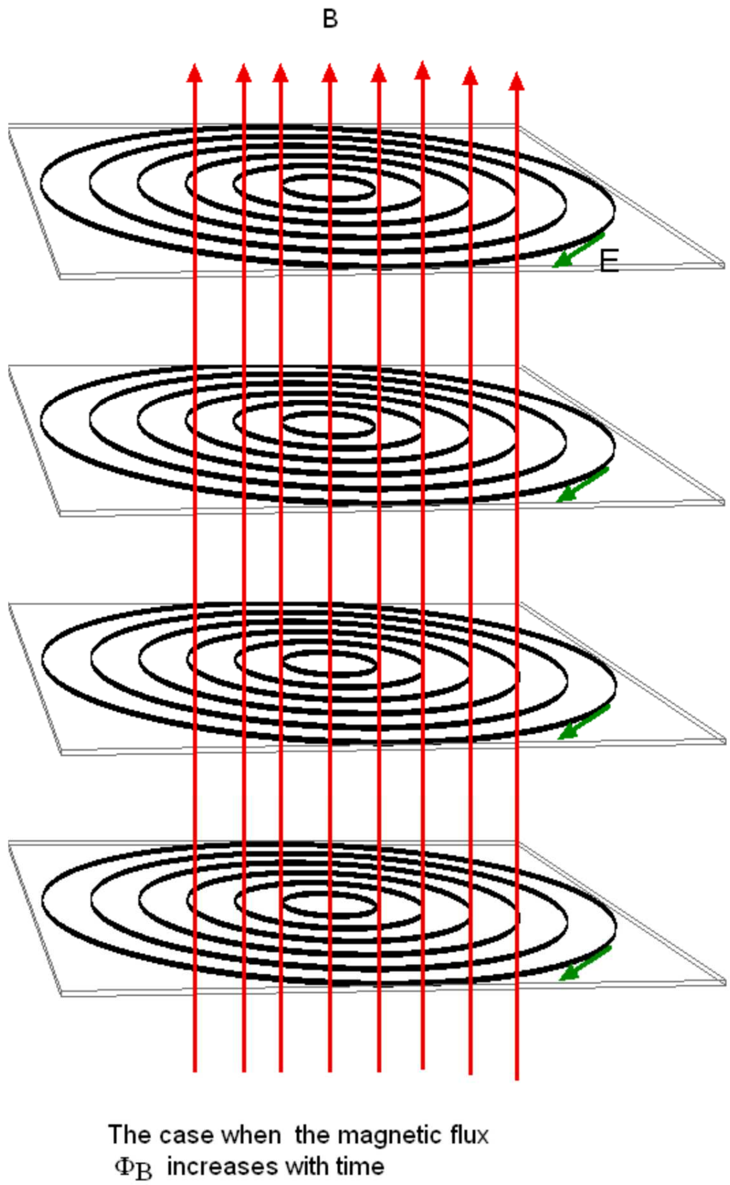


Fig. A time-varying ***B***-field. Surrounding each point where the magnetic flux Φ_B is changing the ***E***-field forms closed loops.

(b)

$$\nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}, \qquad \text{or} \qquad \oint \boldsymbol{B} \cdot d\boldsymbol{s} = \frac{1}{c^2}\frac{\partial}{\partial t}\Phi_E$$

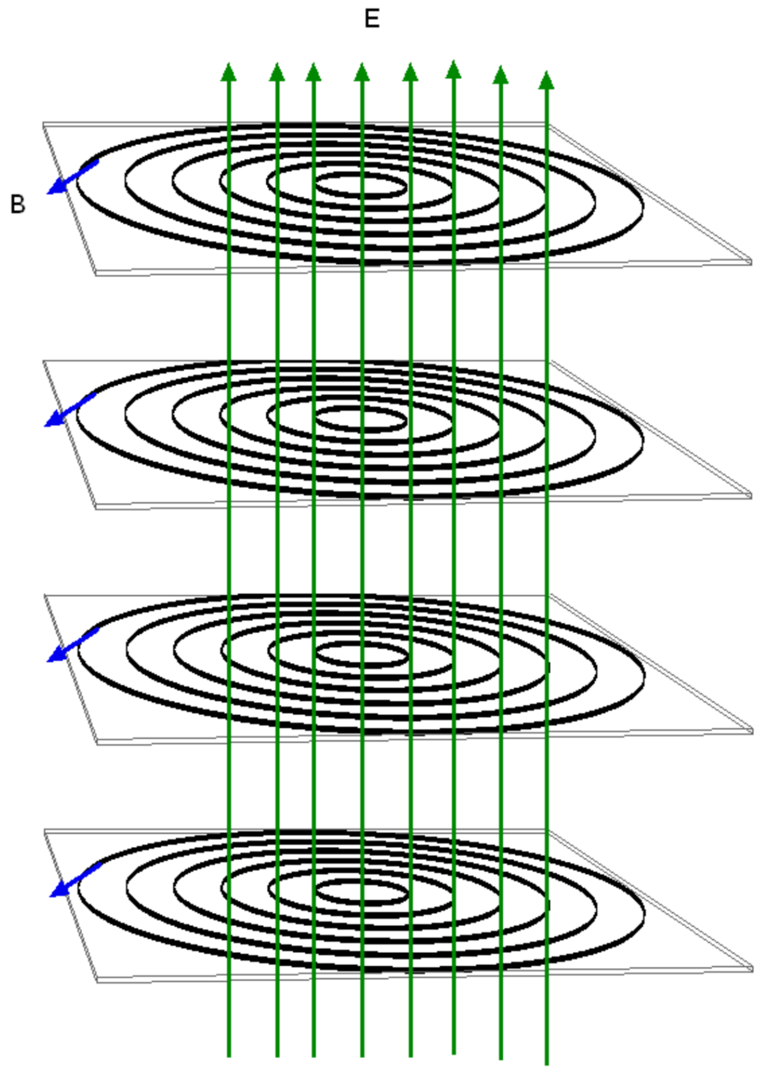
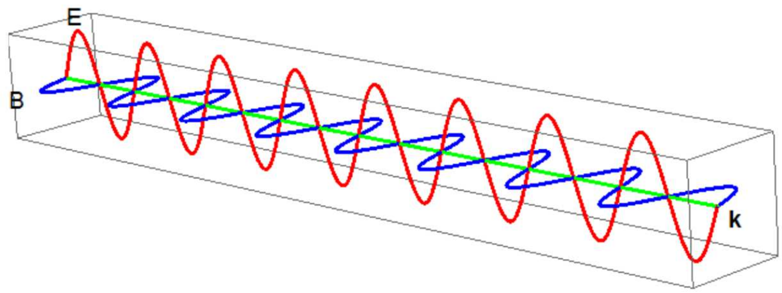
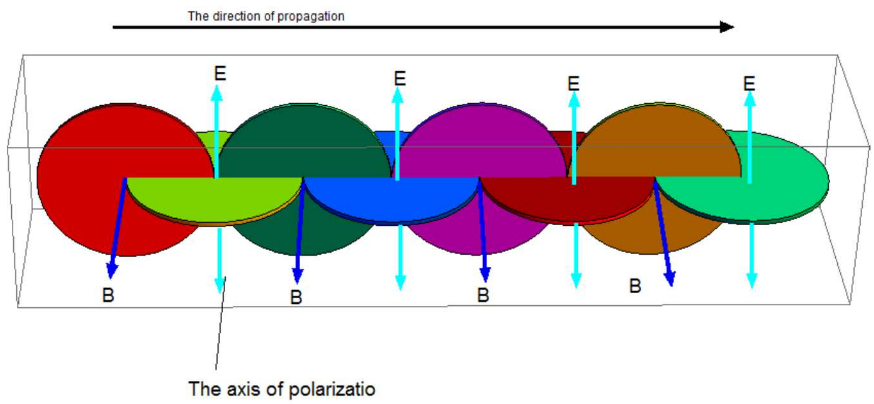


Fig. A time-varying ***E***-field. Surrounding each point where Φ_E is changing the ***B***-field forms closed loops.

A time-varying ***E***-field generates a ***B***-field which is everywhere perpendicular to the direction in which ***E***-field changes. In the same way, a time-varying ***B***-field generates an ***E***-field which is everywhere perpendicular to the direction in which ***B***-field changes. One can anticipate the general transverse nature of the ***E***- and ***B***-fields in an electromagnetic disturbance.





7. Energy conservation: Poynting theorem

We consider a general case where $\boldsymbol{J}$ and ρ are not zero. The system consists of charged particles and the fields $\boldsymbol{E}$ and $\boldsymbol{B}$.

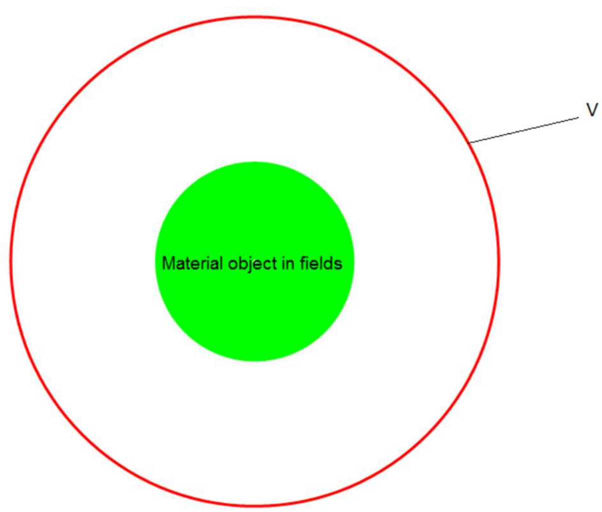


Fig. Combined system (particle and fields) inside volume V.

The work energy theorem:

$$\Delta K = \Delta W = \boldsymbol{F} \cdot \Delta \boldsymbol{r} = \boldsymbol{F} \cdot \boldsymbol{v} \Delta t$$

where K is the kinetic energy and $\boldsymbol{F}$ is the Lorentz force and is given by

$$\boldsymbol{F} = V\boldsymbol{f} = \rho V[\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B})]$$

where $\boldsymbol{f}$ is the force density,

$$\boldsymbol{f} = \rho[\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B})].$$

Then we have

$$\Delta W = \rho V[\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B})] \cdot \boldsymbol{v} \Delta t = \rho V(\boldsymbol{E} \cdot \boldsymbol{v}) \Delta t = V(\boldsymbol{E} \cdot \boldsymbol{J}) \Delta t$$

or

$$\frac{1}{V} \frac{\Delta W}{\Delta t} = \boldsymbol{E} \cdot \boldsymbol{J}$$

where

$$\boldsymbol{J} = \rho \boldsymbol{v}$$

More generally

$$\frac{dW}{dt} = \int \boldsymbol{E} \cdot \boldsymbol{J} d\tau$$

((**Poynting theorem**))

The work done on the changes by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy that flows out through the surface.

$$\frac{dW}{dt} = \int \boldsymbol{E} \cdot \boldsymbol{J} d\tau = -\frac{d}{dt} \int u d\tau - \int (\nabla \cdot \boldsymbol{S}) \cdot d\tau = -\frac{d}{dt} \int u d\tau - \int \boldsymbol{S} \cdot d\boldsymbol{a}$$

$$\frac{d}{dt} \int u d\tau + \int \boldsymbol{S} \cdot d\boldsymbol{a} + \int \boldsymbol{E} \cdot \boldsymbol{J} d\tau = 0 \qquad \text{(Energy conservation)}$$

The first term:	the rate of change of the total energy of the electromagnetic field in volume V.
The second term	the rate at which the electromagnetic field energy flows out through surface.

The third term the rate at which the field is doing work on the charges.

The above equation can be rewritten as

$$\int \frac{\partial u}{\partial t} d\tau + \int \nabla \cdot \boldsymbol{S} \cdot d\tau + \int \boldsymbol{E} \cdot \boldsymbol{J} d\tau = 0$$

or

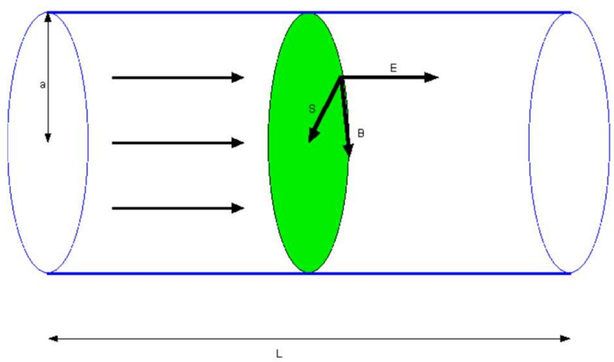
$$\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{S} + \boldsymbol{E} \cdot \boldsymbol{J} = 0$$

using the Gauss' law.

8. Example of the Poynting theorem

8.1 Energy flow in conduction

Here we show that the energy that ends up as joule heating is carried by the electromagnetic field outside the wire.



For simplicity we consider a DC current I flowing along a long straight wire of radius a and length L. The electric field E is given by

$$\boldsymbol{E} = \frac{V_0}{L} \hat{z}$$

Then we have

$$\int_V \boldsymbol{E} \cdot \boldsymbol{J} dV = \frac{V_0}{L} \frac{I}{\pi a^2} \pi a^2 L = V_0 I$$

The magnetic field on the surface of the wire is

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi a}\hat{\varphi}.$$

The Poyinting vector is

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} = \frac{1}{\mu_0}(\frac{V_0}{L}\hat{z})\times\frac{\mu_0 I}{2\pi a}\hat{\varphi} = -\frac{V_0 I}{2\pi aL}\hat{r}$$

The rate of transport energy through the lateral surface is

$$-\oint \boldsymbol{S}\cdot d\boldsymbol{a} = \frac{V_0 I}{2\pi aL}\oint \hat{r}\cdot d\boldsymbol{a} = \frac{V_0 I}{2\pi aL}2\pi aL = IV_0$$

Then we have

$$\int_V \boldsymbol{E}\cdot\boldsymbol{J}dV + \oint \boldsymbol{S}\cdot d\boldsymbol{a} = 0$$

Note that ***E*** and ***B*** are independent of *t*. It means that the energy density *u* is independent of *t*,

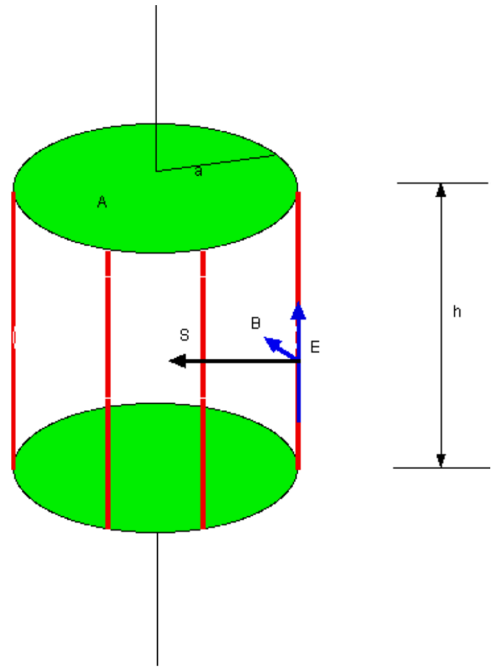
$$\int \frac{\partial u}{\partial t}d\tau = 0$$

Hence the Poyting's theorem is satisfied. The electromagnetic energy flow into the wire from its sides is converted into kinetic (heat) energy within the wire.

8.2 Energy flow in capacitance

We consider the second case where ***E*** and ***B*** are dependent on time.

$$\frac{dW}{dt} = \int \boldsymbol{E}\cdot\boldsymbol{J}d\tau = -\frac{d}{dt}\int u d\tau - \int(\nabla\cdot\boldsymbol{S})\cdot d\tau = -\frac{d}{dt}\int u d\tau - \int \boldsymbol{S}\cdot d\boldsymbol{a}$$



There is a displacement current in the space between two plates.

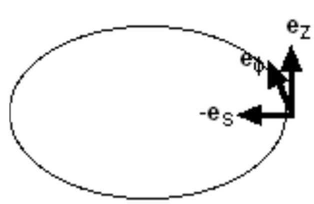
$$\nabla \times \boldsymbol{B} = \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

$$\oint (\nabla \times \boldsymbol{B}) \cdot d\boldsymbol{a} = \oint \boldsymbol{B} \cdot d\boldsymbol{l} = B 2\pi s = \mu_0 \varepsilon_0 \frac{dE}{dt} \pi s^2$$

$$B = \frac{1}{2\pi s} \mu_0 \varepsilon_0 \frac{dE}{dt} \pi s^2 = \frac{\mu_0 \varepsilon_0}{2} s \frac{dE}{dt}$$

The Poynting vector $\boldsymbol{S}$ on the cylinder surface is

$$\boldsymbol{S} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0} = \frac{1}{\mu_0} E \boldsymbol{e}_z \times \frac{\mu_0 \varepsilon_0}{2} a \frac{dE}{dt} \boldsymbol{e}_\varphi = -\boldsymbol{e}_s \frac{a}{2} \varepsilon_0 E \frac{dE}{dt}$$



The total amount of flow through the whole surface between the edges of the plate.

$$\int \boldsymbol{S} \cdot d\boldsymbol{a} = 2\pi a h S = -(2\pi a h \frac{a}{2} \varepsilon_0 E \frac{dE}{dt}) = -\pi a^2 h \frac{d}{dt} (\frac{1}{2} \varepsilon_0 E^2)$$

Since the current i is related to the charge Q by

$$i = \frac{dQ}{dt}$$

we have

$$\boldsymbol{E} = \frac{\sigma}{\varepsilon_0}\boldsymbol{e}_z = \frac{Q}{\pi a^2 \varepsilon_0}\boldsymbol{e}_z .$$

The potential difference between two plates is

$$V = Eh = \frac{Qh}{\pi a^2 \varepsilon_0} = \frac{Q}{C},$$

where the capacitance C is given by

$$\frac{\pi a^2 \varepsilon_0}{h} = C.$$

The energy density u is

$$u = \frac{1}{2}(\varepsilon_0 \boldsymbol{E}^2 + \frac{1}{\mu_0}\boldsymbol{B}^2)$$

The total energy U is

$$U = \int u d\tau = \int uh(2\pi s)ds = 2\pi h\int_0^a ds[s\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}(\frac{\mu_0{}^2\varepsilon_0{}^2}{4}s^3)\left(\frac{dE}{dt}\right)^2]$$

or

$$U = 2\pi h[\frac{a^2}{2}\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}(\frac{\mu_0{}^2\varepsilon_0{}^2}{4}\frac{a^4}{4})\left(\frac{dE}{dt}\right)^2]$$

or

$$U = \frac{\pi h a^2}{2}[\varepsilon_0 E^2 + \frac{1}{8}\mu_0\varepsilon_0{}^2 a^2\left(\frac{dE}{dt}\right)^2]$$

$$\frac{\partial U}{\partial t} = \pi h a^2 \varepsilon_0 E\frac{dE}{dt} + \frac{\pi h a^4}{8}\mu_0\varepsilon_0{}^2\frac{dE}{dt}\frac{d^2E}{dt^2}$$

Energy conservation (Poynting theorem)

$$\frac{dW}{dt} = \int \boldsymbol{E} \cdot \boldsymbol{J} d\tau = -\frac{d}{dt}U - \int \boldsymbol{S} \cdot d\boldsymbol{a}$$

The right-hand side of this equation is defined by K_1. K_1 is evaluated as follows.

$$K_1 = -\frac{d}{dt}U - \int \boldsymbol{S} \cdot d\boldsymbol{a} = -[\pi a^2 h \frac{d}{dt}(\frac{1}{2}\varepsilon_0 E^2) + \frac{\pi a^4 h}{8}\mu_0 \varepsilon_0{}^2 \frac{dE}{dt}\frac{d^2E}{dt^2}] + \pi a^2 h \frac{d}{dt}(\frac{1}{2}\varepsilon_0 E^2)$$

or

$$K_1 = -\frac{\pi a^4 h}{8}\mu_0 \varepsilon_0{}^2 \frac{dE}{dt}\frac{d^2E}{dt^2}$$

Since

$$E = \frac{Q}{\pi a^2 \varepsilon_0}$$

we have

$$\frac{dE}{dt} = \frac{1}{\pi a^2 \varepsilon_0}\frac{dQ}{dt} = \frac{I}{\pi a^2 \varepsilon_0}$$

$$\frac{d^2E}{dt^2} = \frac{1}{\pi a^2 \varepsilon_0}\frac{dI}{dt}.$$

Then $\frac{dW}{dt}$ can be rewritten as

$$\boxed{\frac{dW}{dt} = K_1 = -\frac{\pi a^4 h}{8}\mu_0 \varepsilon_0{}^2 \frac{dE}{dt}\frac{d^2E}{dt^2} = -\frac{h}{8\pi}\mu_0 I \frac{dI}{dt}}$$

When $I = I_0 \cos(\omega t)$,

$$\frac{dW}{dt} = \frac{h\mu_0}{16\pi} I_0{}^2 \omega \sin(2\omega t)$$

The time-averaged of dW/dt is

$$< \frac{dW}{dt} >= 0$$

over a period *T*.

9. Summary From Lecture Notes from Walter Lewin 8.02 Electricity and Magnetism

$$u_E = \frac{1}{2}\varepsilon_0 E^2 \qquad \text{(J/m}^3\text{)}$$

$$\begin{aligned} u_B &= \frac{1}{2\mu_0}B^2 \\ &= \frac{1}{2\mu_0}\left(\frac{E}{c}\right)^2 \qquad \text{(J/m}^3\text{)} \\ &= u_E \end{aligned}$$

where $E = cB$, $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

The total energy density is

$$u = u_E + u_B = \varepsilon_0 E^2 = \varepsilon_0 cEB.$$

The energy passing through unit area (1 m^2) per second is

$$cu = c^2 \varepsilon_0 EB = \frac{1}{\mu_0}EB = \frac{1}{\mu_0 c}E^2 \qquad \text{(J/m}^2\text{ s)}$$

The time average:

$$c\langle u \rangle = \frac{1}{2\mu_0}E_0 B_0 = \frac{1}{2c\mu_0}E_0^{\ 2} = \frac{1}{c\mu_0}E_{rms}^{\ \ 2}$$

where

$$E^2 \rightarrow \frac{1}{2}E_0^{\ 2} = E_{rms}^{\ \ 2} \qquad \text{(time average)}$$

The Poynting vector:

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} \qquad \text{(W/m}^2\text{)}$$

where W=J/s. The time average is

$$\langle S\rangle = \frac{1}{2\mu_0}E_0B_0 = \frac{1}{2\mu_0 c}E_0^{\ 2} = \frac{1}{\mu_0 c}E_{rms}^{\ 2}$$

or

$$\langle S\rangle = c\langle u\rangle$$

where $E_{rms} = \frac{1}{\sqrt{2}}E_0$

The average power per unit area transported by an electromagnetic wave is called the intensity,

$$I = \langle S\rangle$$

((**Example**))

(a) E_0 = 100 V/m

$$I = \langle S\rangle = \frac{1}{2\mu_0 c}E_0^{\ 2} = 13.2721\,\text{W/m}^2$$

(b) E_0 = 1000 V/m

$$I = \langle S\rangle = \frac{1}{2\mu_0 c}E_0^{\ 2} = 1327.21\,\text{W/m}^2$$

(c) Solar constant

$$I = \langle S\rangle = \frac{L_\odot}{4\pi A_u^{\ 2}} = 1361.17\ \text{W/m}^2 \qquad \text{(Solar constant)}$$

where

$A_u = 1.4959787\times10^{11}$ m, (distance between the sun and the earth)

$L_\odot = 3.828 \text{ x } 10^{26}$ W (Solar luminosity)

Note that

$$E_0 = \sqrt{2\mu_0 c\langle S\rangle} = 1012.71 \text{ V/m}$$

Poynting vector

$$\langle S\rangle = c\langle u\rangle$$

where p is the momentum of photon and P_{mom} is the momentum of the system

$$\varepsilon = cp\,, \qquad \text{(energy dispersion of photon)}$$

The momentum density is

$$G = \frac{\langle u\rangle}{c}.$$

The total energy is given by

$$\langle U\rangle = \langle u\rangle(\Delta A)(c\Delta t) = cP_{mom}$$

where P_{mom} is the total momentum of the system. Thus we get the relation between the radiation pressure P_{rad} and $< S >$ as

$$\langle S\rangle = c\langle u\rangle = \frac{cP_{mom}}{\Delta A\Delta t} = cP_{rad}$$

or

$$\boxed{P_{rad} = \langle u\rangle = \frac{\langle S\rangle}{c}} \qquad \text{(radiation pressure)}$$

((Note-1)) Definition of pressure and force

$$\frac{(P_{mom}/\Delta t)}{\Delta A} = P_{rad}\,, \qquad \text{(Pressure)}$$

$$P_{mom}/\Delta t \qquad \text{(Force)}$$

((**Note-2**))
The momentum density is

$$G = \frac{\langle u \rangle}{c} = \frac{\langle S \rangle}{c^2}.$$

The radiation pressure is

$$P_{rad} = \frac{\langle S \rangle}{c} = cG$$

Using the solar constant $G_{SC} = 1361.17$ W/m^2, $\langle S \rangle = G_{SC}$ and the radiation pressure on the surface of Earth is 4.5 μPa. For the radiation pressure of the Sun surface

$$\langle S \rangle = \sigma T_{sun}{}^4 = 6.32944 \text{ x } 10^7 \text{ W/m}^2.$$

and

$$P_{rad} = 0.211 \text{Pa}.$$

In general

$$P_{rad} = \alpha \frac{\langle S \rangle}{c}$$

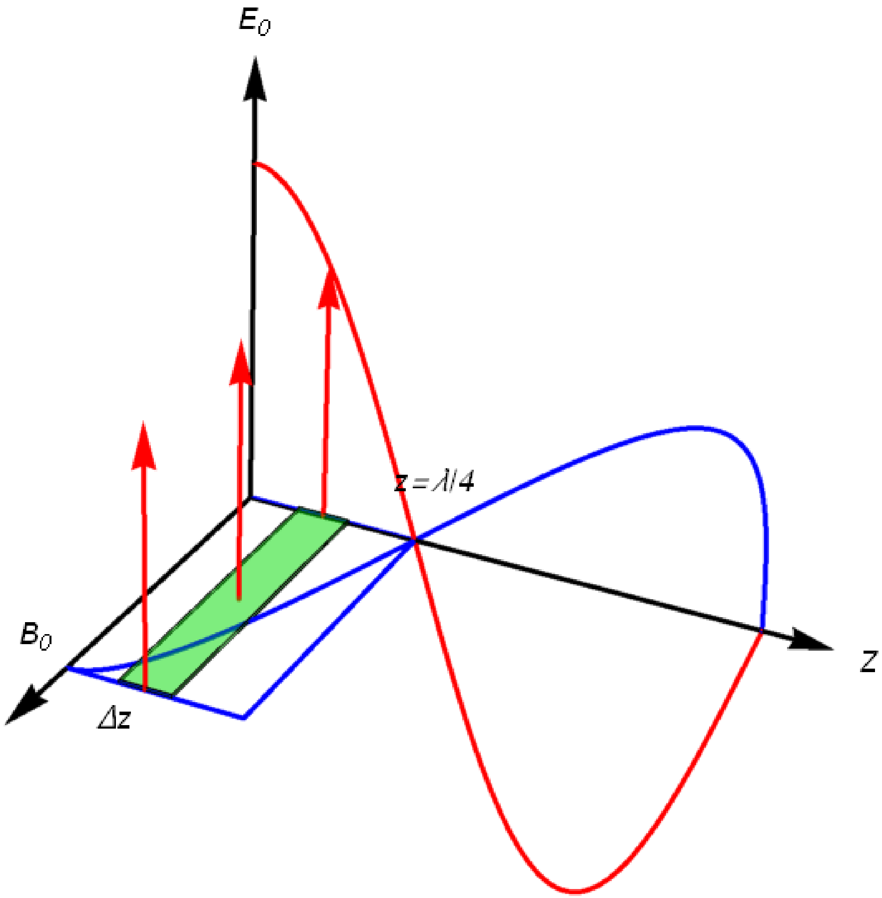
where $\alpha = 1$ for full absorption, 0 for the transparency, and 2 for the reflection (metal).

Radiation pressure is the pressure exerted upon any surface due to the exchange of momentum between the object and the electromagnetic field. This includes the momentum of light or electromagnetic radiation of any wavelength which is absorbed, reflected, or otherwise emitted (e.g. black body radiation) by matter on any scale (from macroscopic objects to dust particles to gas molecules).

10. Derivation of the relation $B_0 = \varepsilon_0 \mu_0 c E_0$

A part of this topics was discussed in the lecture of Prof. Walter Lewin (MIT 8.02 Electricity and Magnetism 2004).

(a) Method-I (from the Ampere-Maxwell law)



$$\oint (\nabla \times \boldsymbol{B}) \cdot d\boldsymbol{a} = \oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \oint \boldsymbol{E} \cdot d\boldsymbol{a} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \Phi_{\boldsymbol{E}}$$

$$\Phi_{\boldsymbol{E}} = \oint \boldsymbol{E} \cdot d\boldsymbol{a}$$

$$\Phi_{\boldsymbol{E}} = \int_0^{\lambda/4} l dz E_0 \cos(kz - \omega t)$$

$$\begin{aligned}
\frac{\partial}{\partial t} \Phi_{\boldsymbol{E}} &= \int_0^{\lambda/4} l dz E_0 \frac{\partial}{\partial t} \cos(kz - \omega t) \\
&= (-1)(-\omega) \int_0^{\lambda/4} l dz E_0 \sin(kz - \omega t) \\
&= l E_0 \omega \int_0^{\lambda/4} dz \sin(kz - \omega t)
\end{aligned}$$

$$\frac{\partial}{\partial t}\Phi_E\,|_{t=0} = lE_0\omega\int_0^{\lambda/4} dz\sin(kz) = l\frac{E_0\omega}{k}[-\cos(kz)]\,|_0^{\lambda/4} = lcE_0$$

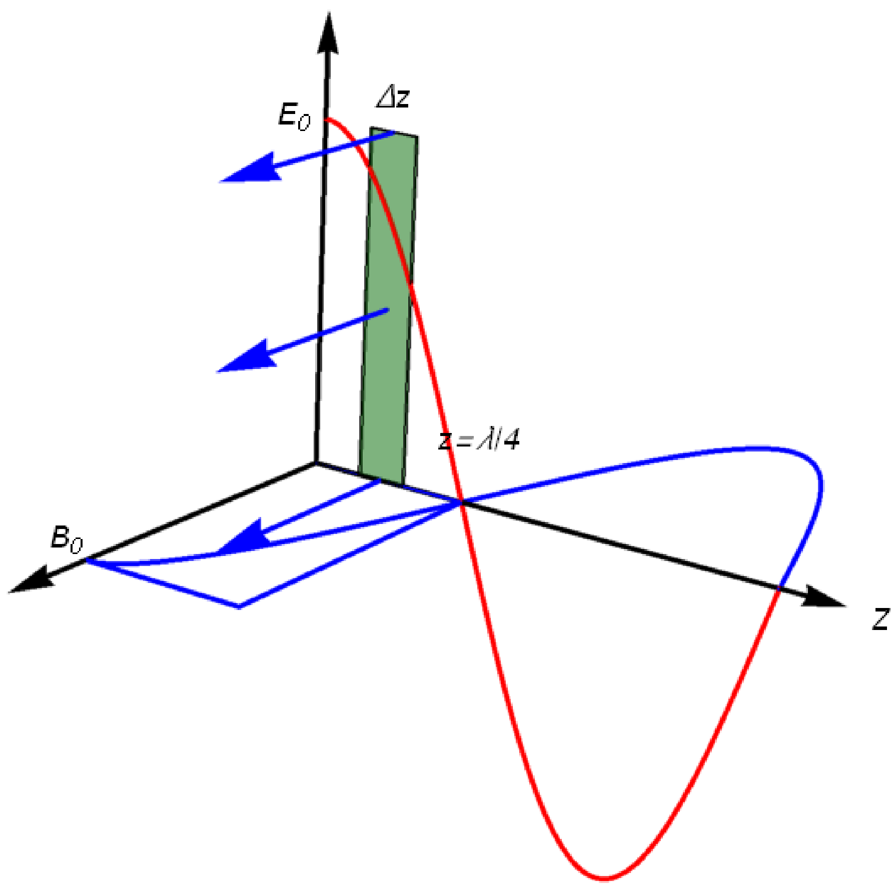
Ampere's law:

$$B_0 l = \varepsilon_0\mu_0 lcE_0$$

leading to

$$B_0 = \varepsilon_0\mu_0 cE_0 = \frac{1}{c}E_0$$

(b) Method II (from the Faraday's law)



$$\oint(\nabla\times\boldsymbol{E})\cdot d\boldsymbol{a}=\oint\boldsymbol{E}\cdot d\boldsymbol{l}=-\frac{\partial}{\partial t}\oint\boldsymbol{B}\cdot d\boldsymbol{a}=-\frac{\partial}{\partial t}\Phi_B$$

$$\Phi_{\boldsymbol{B}}=\oint\boldsymbol{B}\cdot d\boldsymbol{a}$$

$$\Phi_{\boldsymbol{B}}=\int_0^{\lambda/4} ldzB_0\cos(kz-\omega t)$$

$$\begin{aligned}\frac{\partial}{\partial t}\Phi_{\boldsymbol{B}}&=\int_0^{\lambda/4} ldzB_0\frac{\partial}{\partial t}\cos(kz-\omega t)\\&=(-1)(-\omega)\int_0^{\lambda/4} ldzB_0\sin(kz-\omega t)\\&=lB_0\omega\int_0^{\lambda/4} dz\sin(kz-\omega t)\end{aligned}$$

$$\frac{\partial}{\partial t}\Phi_{\boldsymbol{B}}\,|_{t=0}=lB_0\omega\int_0^{\lambda/4} dz\sin(kz)=l\frac{B_0\omega}{k}[-\cos(kz)]\,|_0^{\lambda/4}=lcB_0$$

Faraday's law:

$$-E_0l=-lcB_0,\qquad \text{or}\qquad E_0=cB_0$$

leading to

$$B_0=\varepsilon_0\mu_0cE_0=\frac{1}{c}E_0$$

11. Momentum conservation

We start with the expression of the pointing vector,

$$\frac{\partial}{\partial t}\boldsymbol{S}=\frac{1}{\mu_0}\frac{\partial}{\partial t}(\boldsymbol{E}\times\boldsymbol{B})=\frac{1}{\mu_0}(\frac{\partial\boldsymbol{E}}{\partial t}\times\boldsymbol{B}+\boldsymbol{E}\times\frac{\partial}{\partial t}\boldsymbol{B})$$

Here we only use the Maxwell's equation.

$$\begin{aligned}\frac{\partial\boldsymbol{B}}{\partial t}&=-\nabla\times\boldsymbol{E}\\\frac{\partial\boldsymbol{E}}{\partial t}&=\frac{1}{\mu_0\varepsilon_0}(\nabla\times\boldsymbol{B}-\mu_0\boldsymbol{J})\end{aligned}$$

Then we have

$$\mu_0 \frac{\partial}{\partial t}\boldsymbol{S} = \frac{\partial \boldsymbol{E}}{\partial t}\times\boldsymbol{B} + \boldsymbol{E}\times\frac{\partial}{\partial t}\boldsymbol{B} = \frac{1}{\mu_0\varepsilon_0}(\nabla\times\boldsymbol{B} - \mu_0\boldsymbol{J})\times\boldsymbol{B} + \boldsymbol{E}\times(-\nabla\times\boldsymbol{E})$$

leading to the momentum conservation,

$$\varepsilon_0\mu_0\frac{\partial}{\partial t}\boldsymbol{S} + \boldsymbol{f} = \nabla\cdot\vec{\vec{T}} \qquad \text{(momentum conservation)}$$

or

$$\frac{1}{c^2}\frac{\partial}{\partial t}\boldsymbol{S} + \boldsymbol{f} = \nabla\cdot\vec{\vec{T}}$$

where T_{ij} is called the Maxwell stress tensor,

$$T_{ij} = -\delta_{ij}\left(\frac{1}{2}\varepsilon_0\boldsymbol{E}^2 + \frac{1}{2\mu_0}\boldsymbol{B}^2\right) + \varepsilon_0 E_i E_j + \frac{1}{\mu_0}B_i B_j$$

$$\begin{aligned}\nabla\cdot\vec{\vec{T}} &= -\sum_{i,j}\boldsymbol{e}_i\delta_{ij}\frac{\partial}{\partial x_j}\left(\frac{1}{2}\varepsilon_0\boldsymbol{E}^2 + \frac{1}{2\mu_0}\boldsymbol{B}^2\right) + \varepsilon_0\sum_{i,j}\left(E_j\frac{\partial E_i}{\partial x_j} + \frac{\partial E_j}{\partial x_j}E_i\right)\boldsymbol{e}_i + \frac{1}{\mu_0}\sum_{i,j}\left(B_j\frac{\partial B_i}{\partial x_j} + \frac{\partial B_j}{\partial x_j}B_i\right)\boldsymbol{e}_i \\ &= \sum_{i,j}\frac{\partial}{\partial x_j}T_{ji}\boldsymbol{e}_i\end{aligned}$$

12. Momentum of the field, momentum density

12.1 Definition

The pointing vector $\boldsymbol{S}$ gives not only the energy flow but, if it is divided by c^2, also the momentum density.

For particles, we have

$$\Delta\boldsymbol{P}_{mech} = \boldsymbol{F}\Delta t = \boldsymbol{f}V\Delta t$$

($\mathbf{P}_{mec}$: the total momentum of all the particles in a volume V)

or

$$\frac{1}{V}\frac{\Delta\boldsymbol{P}_{mech}}{\Delta t} = \boldsymbol{f} = \rho\boldsymbol{E} + \boldsymbol{J}\times\boldsymbol{B}$$

or

$$\frac{d\boldsymbol{P}_{mec}}{dt} = \int\boldsymbol{f}\cdot d\tau \qquad \text{(in general)}$$

Here we define the momentum of the field (per unit volume), the momentum density as

$$\boldsymbol{G} = \varepsilon_0 \mu_0 \boldsymbol{S} = \frac{1}{c^2}\boldsymbol{S} \qquad \text{(momentum density)}$$

$$\boldsymbol{P}_{em} = \int \boldsymbol{G} d\tau$$

where $\boldsymbol{P}_{em}$ is the total electromagnetic momentum stored in the electromagnetic field.

We now consider the physical meaning.

$$\varepsilon_0 \mu_0 \frac{\partial}{\partial t}\boldsymbol{S} + \boldsymbol{f} = \nabla \cdot \vec{\vec{T}} \quad \text{(momentum conservation)}$$

or

$$\int (\frac{\partial}{\partial t}\boldsymbol{G} + \boldsymbol{f}) d\tau = \int (\nabla \cdot \vec{\vec{T}}) d\tau = \int \vec{\vec{T}} \cdot d\boldsymbol{a}$$

or

$$\frac{\partial}{\partial t}(\boldsymbol{P}_{em} + \boldsymbol{P}_{mech}) = \int \vec{\vec{T}} \cdot d\boldsymbol{a}$$

where

$$\frac{\partial \boldsymbol{P}_{mech}}{\partial t} = \int \boldsymbol{f} d\tau = \boldsymbol{F}$$

$$\frac{\partial \boldsymbol{P}_{em}}{\partial t} = \frac{\partial}{\partial t}\int \boldsymbol{G} d\tau = \boldsymbol{F}_{em}$$

The impulse $\boldsymbol{I}_{\mathrm{em}}$ is defined by

$$\boldsymbol{I}_{em} = \boldsymbol{F}_{em}\Delta t = \Delta \boldsymbol{P}_{em}$$

((Note)) Definition of mass density μ

$$\langle G \rangle = \varepsilon_0 \mu_0 \langle S \rangle = \frac{1}{c^2}\langle S \rangle = \frac{c\langle u \rangle}{c^2}$$

The mass density μ is defined as

$$\mu = \frac{\langle u \rangle}{c^2}, \qquad \text{or} \qquad \langle u \rangle = \mu c^2$$

which is the main result of relativistic theory (Einstein).

12.2 Correspondence

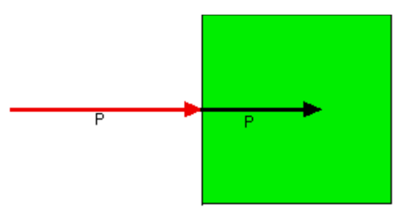
The following table shows the correspondence between the particles and fields.

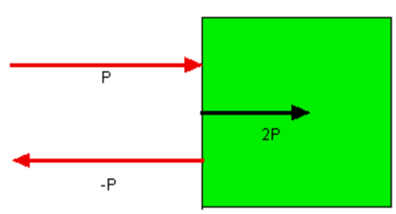
Particles	*E*-*B* fields	
	$\boldsymbol{G} = \frac{1}{c^2}\boldsymbol{S}$	(momentum density)
$\boldsymbol{P}_{\text{mech}}$	$\boldsymbol{P}_{em} = \int \boldsymbol{G} d\tau$	(momentum)
$\boldsymbol{f}$	$\frac{\partial \boldsymbol{G}}{\partial t} = \frac{1}{c^2}\frac{\partial \boldsymbol{S}}{\partial t}$	(force density)
$\boldsymbol{F}_{mech} = \int \boldsymbol{f} d\tau = \frac{d\boldsymbol{P}_{mech}}{dt}$	$\boldsymbol{F}_{em} = \frac{\partial}{\partial t}\int \boldsymbol{G} d\tau = \frac{d\boldsymbol{P}_{em}}{dt}$	(force)
$\boldsymbol{I}_{mech} = \Delta \boldsymbol{P}_{mech} = \boldsymbol{F}_{mech}\Delta t$	$\boldsymbol{I}_{em} = \Delta \boldsymbol{P}_{em} = \boldsymbol{F}_{em}\Delta t$	(impulse)

13. Radiation pressure

Radiation pressure is the pressure exerted upon any surface exposed to electromagnetic radiation. For example, the radiation of the Sun at the Earth has an energy flux density of 1370 W/m^2, so the radiation pressure is 4.6 μPa (absorbed).

Here we consider the two cases: the absorption and reflection of light waves at a surface of the object.

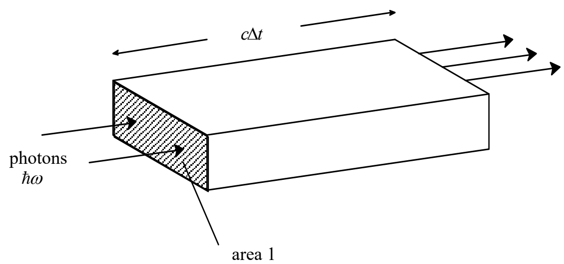




(1) Total absorption

The momentum $\boldsymbol{P}_{em}$ is delivered to the object. $\boldsymbol{I}$ is the impulse.

$$\boldsymbol{F}_{em} = \frac{\Delta \boldsymbol{P}_{em}}{\Delta t}$$
$$\boldsymbol{I} = \boldsymbol{F}_{em}\Delta t = \Delta \boldsymbol{P}_{em} = \boldsymbol{P}_{em}$$



where

$$\boldsymbol{P}_{em} = \langle \boldsymbol{G} \rangle Ac\Delta t = \frac{\langle \boldsymbol{S} \rangle}{c^2} Ac\Delta t$$

where A is the area.

((Note)) G[(J s/m)/ m^3], A[m^2], $GAc\Delta t$ [(J s/m)/ m^3][m^2][m/s][s]=[J.s/m]

Then we have the impulse and the force given by

$$\boldsymbol{I}_{em} = P_{em} = \frac{\langle \boldsymbol{S} \rangle}{c^2} Ac\Delta t$$
$$\boldsymbol{F}_{em} = \frac{P_{em}}{\Delta t} = \frac{\langle \boldsymbol{S} \rangle}{c} A = \frac{I_0}{c} A$$

since $\langle \boldsymbol{S} \rangle = I_0 = c\langle u \rangle$ (the intensity) and I_0 is the intensity of the light wave.

The radiation pressure is given by

$$\frac{F_{em}}{A} = \frac{I_0}{c}$$

(2) Total reflection

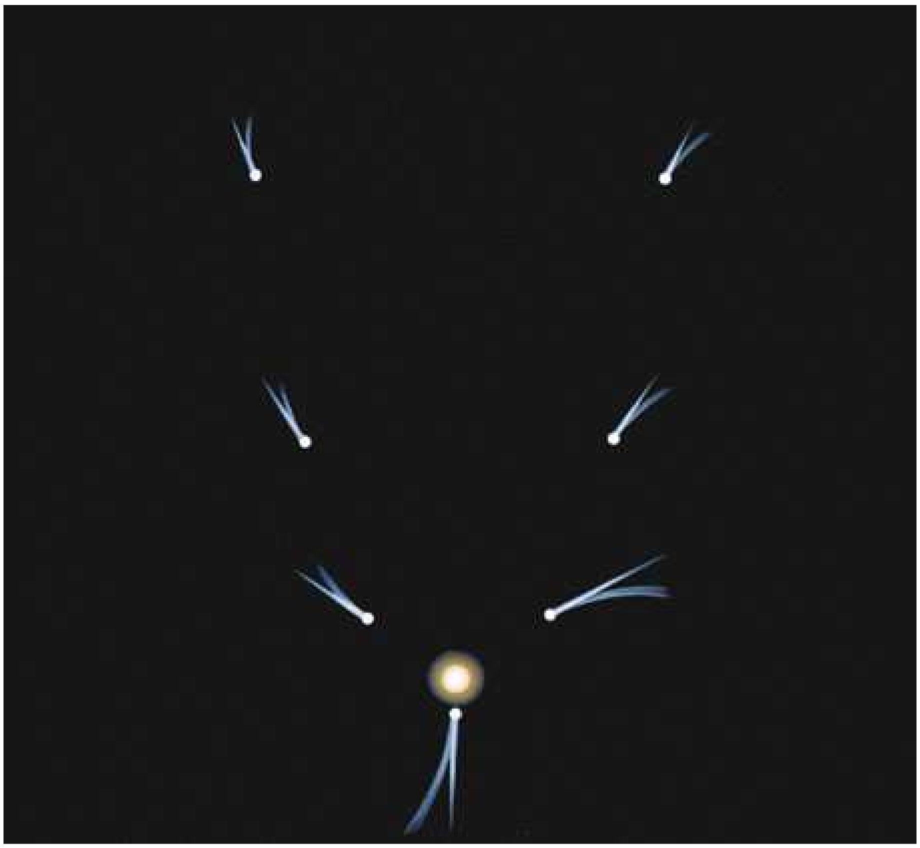
$$\boldsymbol{I} = \boldsymbol{F}_{em}\Delta t = \Delta \boldsymbol{P} = 2\boldsymbol{P}_{em} = 2\frac{\langle \boldsymbol{S} \rangle}{c^2} Ac\Delta t$$
$$\boldsymbol{F}_{em} = 2\frac{\langle \boldsymbol{S} \rangle}{c} A = 2\langle u \rangle A = \frac{2I_0}{c} A$$

since $\langle \boldsymbol{S} \rangle = I_0 = c\langle u \rangle$ (the intensity)

The radiation pressure is

$$\frac{F_{em}}{A} = \frac{2I_0}{c}$$

((**Radiation pressure on the tails of comets**))



The minute pressure exerted on a surface at right-angles to the direction of travel of the incident electromagnetic radiation. Its existence was first predicted by James Maxwell in 1899 and demonstrated experimentally by Peter Lebedev. In quantum mechanics, radiation pressure can be interpreted as the transfer of momentum from photons as they strike a surface. Radiation pressure on dust grains in space can dominate over gravity and explains why the tail of a comet always points away from the Sun.

(3) Partial reflection and absorption
Problem 33-23; partial absorption and partial reflection

Prove, for a plane electromagnetic wave that is normally incident on a flat surface, that the radiation pressure on the surface is equal to the energy density in the incident beam. (This relation between pressure and energy density holds no matter what fraction of the incident energy is reflected.

Let f be the fraction of the incident beam intensity that is reflected. The fraction absorbed is $1 - f$. The reflected portion exerts a radiation pressure of

$$p_r = \frac{2fI_0}{c}$$

and the absorbed portion exerts a radiation pressure of

$$p_a = \frac{(1-f)I_0}{c}$$

where I_0 is the incident intensity. The factor 2 enters the first expression because the momentum of the reflected portion is reversed. The total radiation pressure is the sum of the two contributions:

$$p_{total} = p_r + p_a = \frac{2fI_0 + (1-f)I_0}{c} = \frac{(1+f)I_0}{c}$$

To relate the intensity and energy density, we consider a tube with length ℓ and cross-sectional area A, lying with its axis along the propagation direction of an electromagnetic wave. The electromagnetic energy inside is $U = uA\ell$, where u is the energy density. All this energy passes through the end in time $t = \ell / c$, so the intensity is

$$I = \frac{\langle U \rangle}{At} = \frac{\langle u \rangle Alc}{Al} = \langle u \rangle c$$

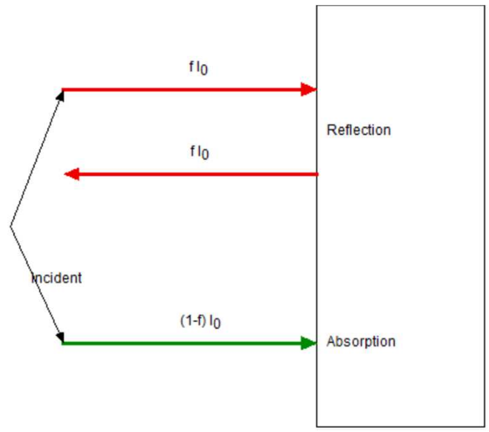
Thus $u = I/c$. The intensity and energy density are positive, regardless of the propagation direction. For the partially reflected and partially absorbed wave, the intensity just outside the surface is

$$I = I_0 + fI_0 = (1+f)I_0$$

where the first term is associated with the incident beam and the second is associated with the reflected beam. Consequently, the energy density is

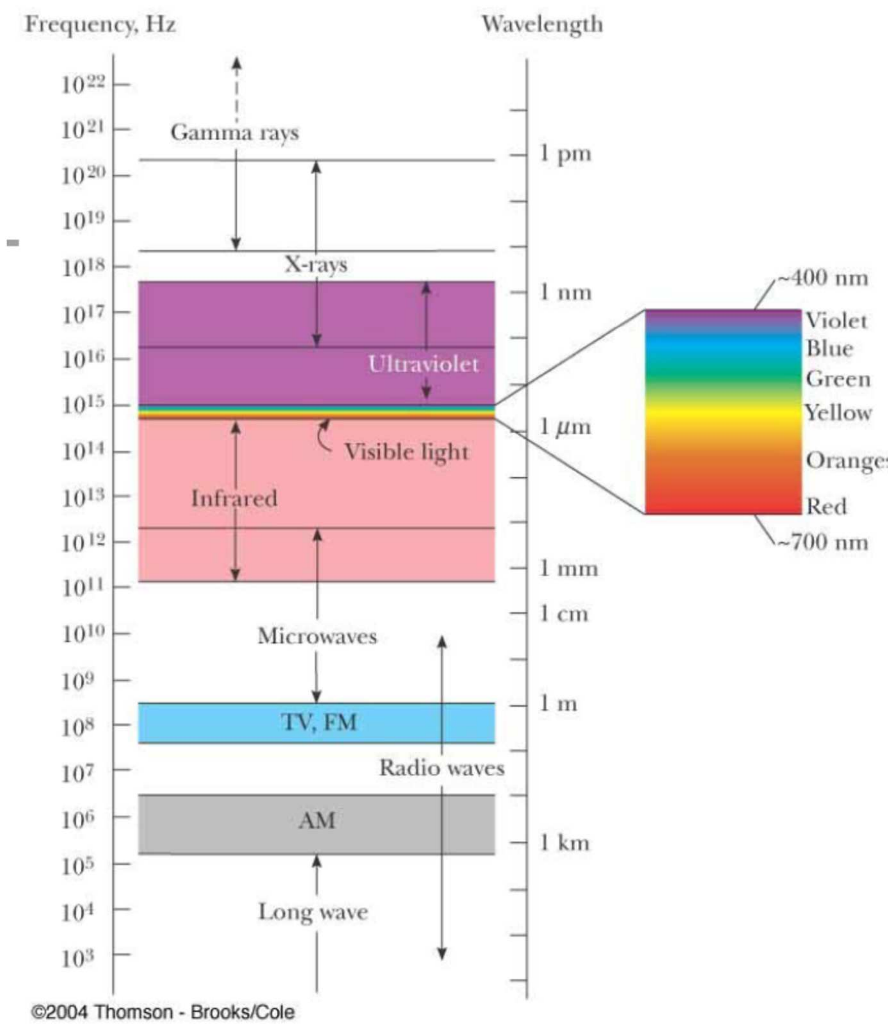
$$\langle u \rangle = \frac{I}{c} = \frac{(1+f)I_0}{c}$$

the same as radiation pressure.



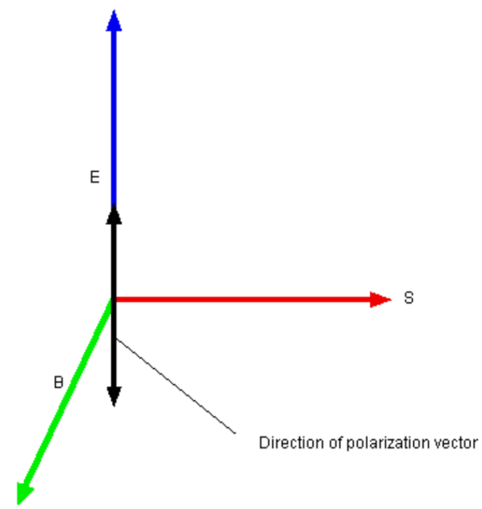
14. Electromagnetic waves

Table shows the frequency and wavelength of the light ranging from the gamma ray to the AM wave.

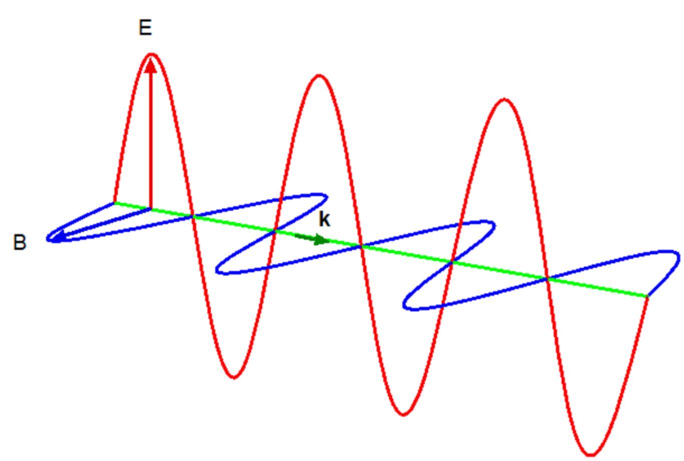


15. Polarization

The electric and magnetic vectors associated with an electromagnetic wave are perpendicular to each other and to the direction of wave propagation. Polarization is a property that specifies the directions of the electric and magnetic fields associated with an EM wave. The direction of polarization is defined to be the direction in which the *electric field is vibrating.*

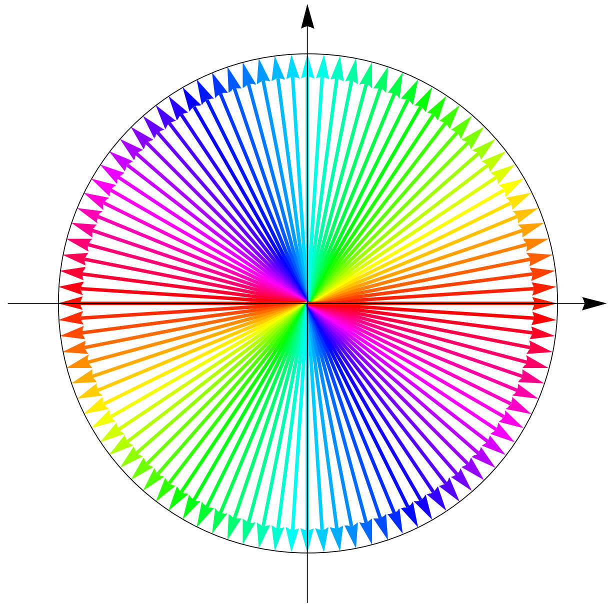


The plane containing the ***E***-vector is called the plane of oscillation of the wave. Hence the wave is said to be plane polarized in the y direction. We can represent the wave's polarization by showing the direction of electric field oscillations in a head-on view of the plane of oscillation.



16 Unpolarized light

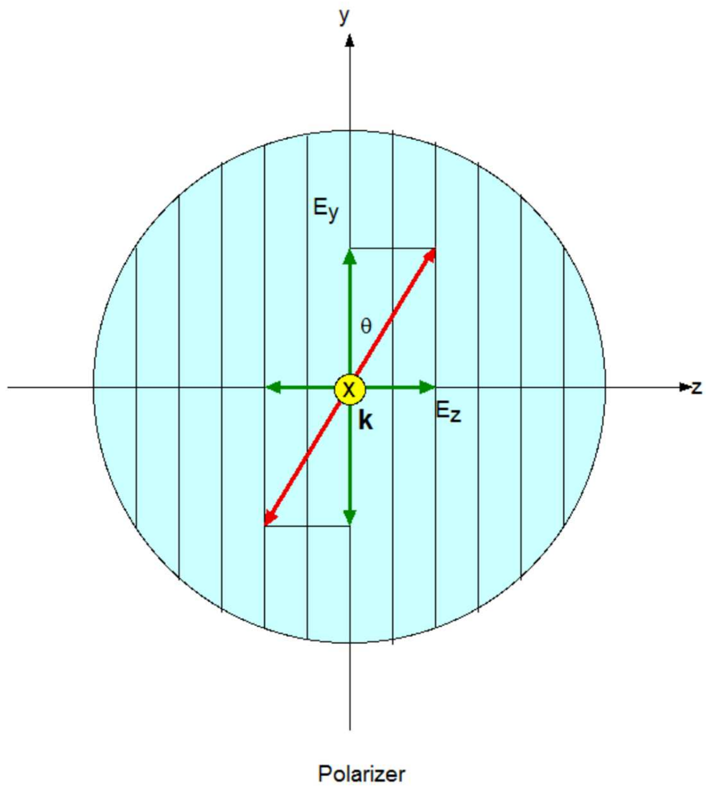
All directions of vibration from a wave source are possible. The resultant EM wave is a superposition of waves vibrating in many different directions. This is an unpolarized wave. The arrows show a few possible directions of the waves in the beam. The representing unpolarized light is the superposition of two polarized waves (E_x and E_y) whose planes of oscillation are perpendicular to each other.



17. Intensity of transmitted polarized light

(1) Malus' law

An electric field component parallel to the polarization direction is passed (transmitted) by a polarizing sheet. A component perpendicular to it is absorbed.



The electric field along the direction of the polarizing sheet is given by

$$E_y = E\cos\theta$$

Then the intensity I of the polarized light with the polarization vector parallel to the y axis is given by

$$I = I_0 \cos^2\theta \quad \text{(Malus' law)}$$

where

$$I = S_{avg} = \frac{E_{rms}^{\ \ 2}}{c\mu_0} = I_0 \cos^2\theta$$

((Note)) Etienne Louis Malus (1775 – 1812).

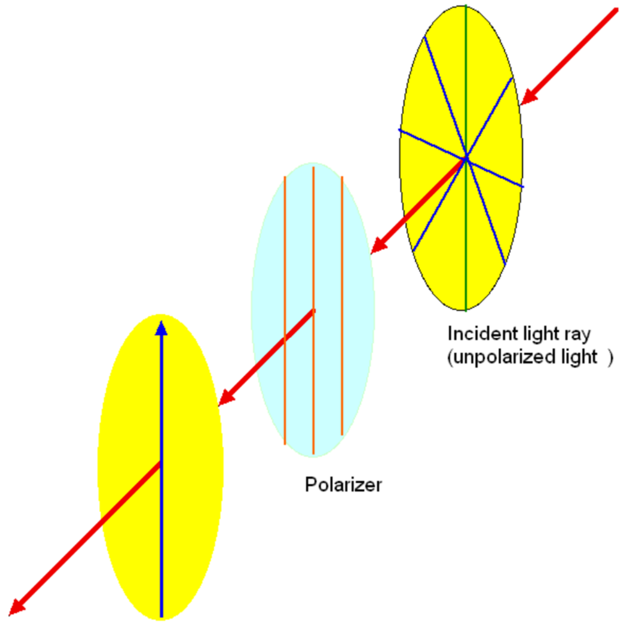
(2) One-half rule for unpolarized light

When the light reaching a polarization sheet is unpolarized, we get a polarized light with the intensity

$$I = \frac{I_0}{2} \qquad \text{(one-half rule)}$$

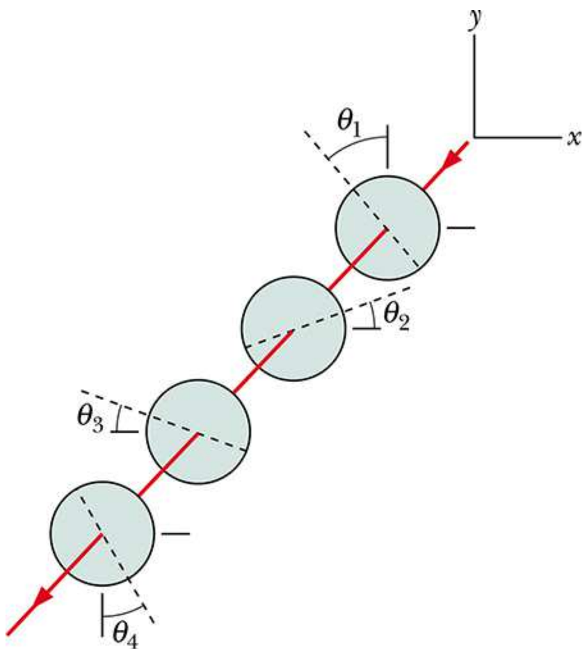
since

$$I = \frac{1}{2\pi}\int_0^{2\pi} I_0 \cos^2 \theta d\theta = \frac{I_0}{2\pi}\frac{1}{2}\int_0^{2\pi} [1 - \cos(2\theta)] d\theta = \frac{I_0}{2}$$



((Example)) **Problem 33-74**

In Fig., unpolarized light with an intensity I_0 (25 W/m^2) is sent into a system of four polarising sheet with polarizing directions at angles θ_1 = 40°, θ_2 = 20°, θ_3 = 20°, and θ_4 = 30°. What is the intensity of the light emerges from the system?



((**Solution**))
$\theta_1 = 40°$
$\theta_2 = 20°$
$\theta_3 = 20°$
$\theta_4 = 30°$
$I_0 = 25$ W/m^2

The intensity is given by

$$\begin{aligned}
I &= \frac{I_0}{2}\cos^2(\theta_1 + 90° - \theta_2)\cos^2(180° - \theta_2 - \theta_3)\cos^2(90° + \theta_3 + \theta_4) \\
&= \frac{I_0}{2}\sin^2(\theta_1 - \theta_2)\cos^2(\theta_2 + \theta_3)\sin^2(\theta_3 + \theta_4) \\
&= \frac{25}{2}\sin^2(20°)\cos^2(40°)\sin^2(50°) \\
&= 0.504 W/m^2
\end{aligned}$$

18. Index of refraction

The velocity of the light in the material with ε and μ

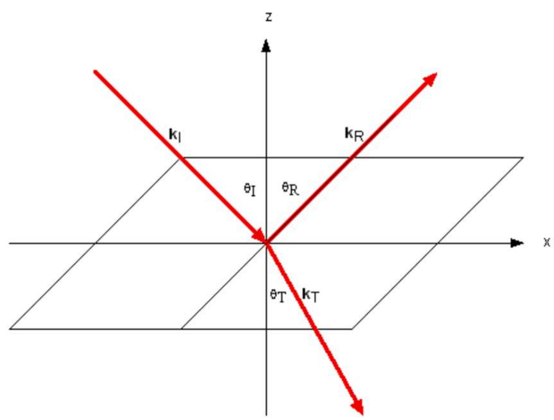
$$v = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{c}{n}$$

Here n is the index of refraction;

$$n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}$$

For most materials, $\mu = \mu_0$

19. Reflection and transmission



((First law))

The incident, reflected, and transmitted wave vectors form a plane (plane of incidence), which also includes the normal to the surface.

((Second law))

$$\theta_i = \theta_R$$

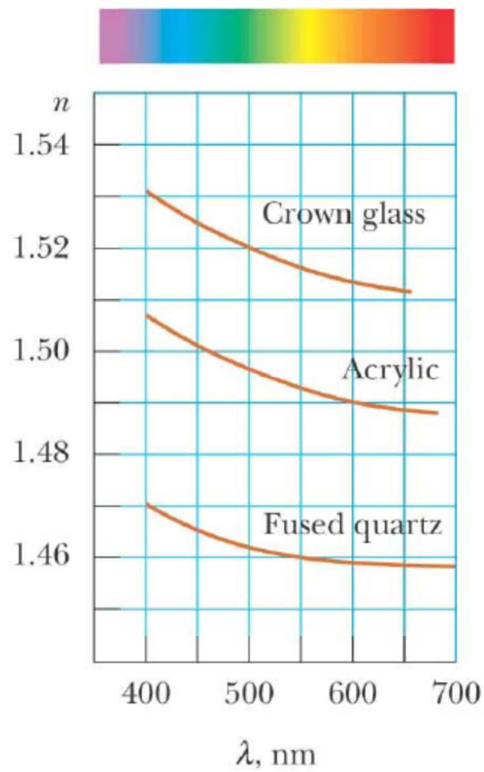
((Third law))

$$\frac{\sin\theta_i}{\sin\theta_T} = \frac{n_2}{n_1} \qquad \text{(Snell's law)}$$

Willebrod Snell (1591 – 1626)

20. Dispersion

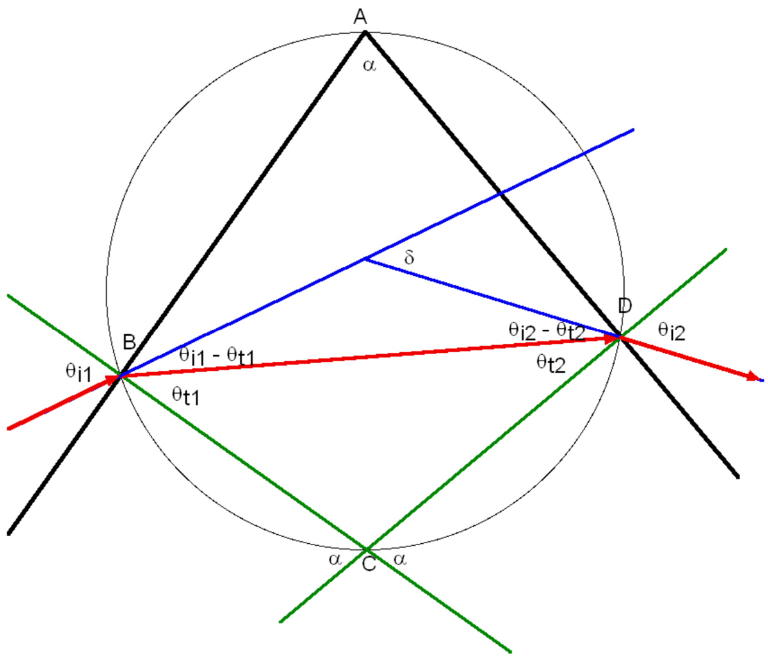
For a given material, the index of refraction varies with the wavelength of the light passing through the material. This dependence of n on λ is called dispersion. Snell's Law indicates light of different wavelengths is bent at different angles when incident on a refracting material



21. Prism

The ray emerges refracted from its original direction of travel by an angle δ, called the angle of deviation. δ depends on the apex angle α of the prism and the index of refraction n of the material. Since all the colors have different angles of deviation, white light will spread out into a spectrum.

(a) Violet deviates the most.
(b) Red deviates the least.
(c) The remaining colors are in between.



AB and AD are the surface of the prism. α is the vertex angle of the prism and δ is the deviation angle. From the geometrical consideration, the points A, B, C, and D are on the same circle. Then we have the following relations,

$$\begin{aligned}
\alpha &= \theta_{t1} + \theta_{t2} \\
\delta &= (\theta_{i1} - \theta_{t1}) + (\theta_{i2} - \theta_{t2}). \\
&= \theta_{i1} + \theta_{i2} - \alpha
\end{aligned}$$

Snell's law:

$$\sin\theta_{i1} = n\sin\theta_{t1}, \qquad \sin\theta_{i2} = n\sin\theta_{t2}$$

where n is the index of refraction of the prism.

((**Angle of minimum deviation**))

Here we discuss the angle δ as a function of the incident angle θ_{i1}. From the Snell's law,

$$\sin\theta_{i2} = n\sin\theta_{t2} = n\sin(\alpha - \theta_{t1}) = n(\sin\alpha\cos\theta_{t1} - \cos\alpha\sin\theta_{t1}).$$

We note that

$$\cos\theta_{t1} = \sqrt{1-\sin^2\theta_{t1}} = \sqrt{1-\frac{1}{n^2}\sin^2\theta_{i1}} \quad .$$

$$\sin\theta_{t1} = \frac{\sin\theta_{i1}}{n}$$

Then we get

$$\begin{aligned}\sin\theta_{i2} &= n\sin\alpha\sqrt{1-\frac{1}{n^2}\sin^2\theta_{i1}} - n\cos\alpha\frac{\sin\theta_{i1}}{n} \\ &= \sin\alpha\sqrt{n^2-\sin^2\theta_{i1}} - \cos\alpha\sin\theta_{i1}\end{aligned} \quad .$$

The angle of deviation is obtained as

$$\delta = \theta_{i1} - \alpha + \arcsin(\sin\alpha\sqrt{n^2-\sin^2\theta_{i1}} - \cos\alpha\sin\theta_{i1}) \, .$$

Here we assume that $\alpha = 60°$. We make a plot of the angle δ as a function of θ_{i1}, where the index of refraction n is changed as a parameter. It is found from the figure that δ takes minimum at a characteristic angle (the angle of minimum deviation),

$$\theta_{i1} = \arcsin[n\sin(\frac{\alpha}{2})] \, .$$

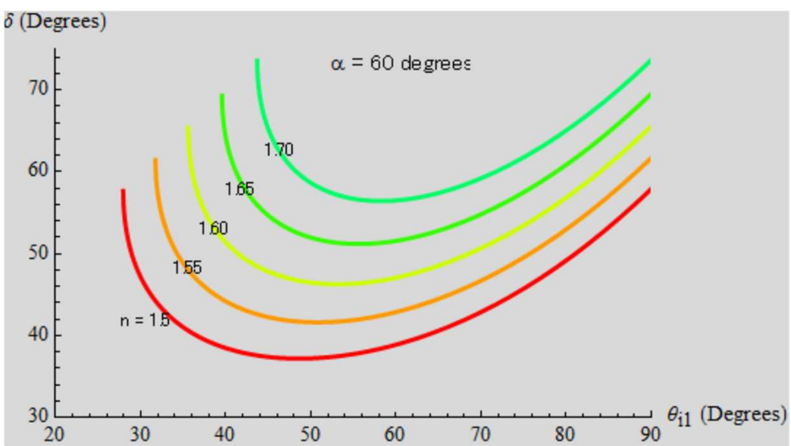


Fig. Deviation vs incident angle, where n is changed as a parameter.

What is the condition for the angle of minimum deviation? The condition is derived as

$$\begin{aligned}\theta_{i1} &= \theta_{i2} \\ \theta_{t1} &= \theta_{t2}\end{aligned} \qquad \text{(symmetric configuration)}$$

In other words, the ray BD should be parallel to the base of the prism (the isosceles triangle with the apex angle α) in the case of the angle of minimum deviation.

((**Proof**))

The angle δ has a minimum at the angle of minimum deviation,

$$\frac{d\delta}{d\theta_{i1}} = 0,$$

or

$$d\theta_{i2} = -d\theta_{i1}, \tag{1}$$

From $\alpha = \theta_{t1} + \theta_{t2}$, we have

$$d\theta_{t1} = -d\theta_{t2}. \tag{2}$$

From the Snell's law,

$$\begin{aligned} \cos\theta_{i1} d\theta_{i1} &= n\cos\theta_{t1} d\theta_{t1} \\ \cos\theta_{i2} d\theta_{i2} &= n\cos\theta_{t2} d\theta_{t2} \end{aligned}. \tag{3}$$

From Eqs.(1), (2), and (3), we have

$$\frac{\cos\theta_{i1}}{\cos\theta_{i2}} = \frac{\cos\theta_{t1}}{\cos\theta_{t2}} = \frac{\sqrt{1 - \frac{1}{n^2}\sin^2\theta_{i1}}}{\sqrt{1 - \frac{1}{n^2}\sin^2\theta_{i2}}},$$

$$\frac{\sqrt{1-\sin^2\theta_{i1}}}{\sqrt{1-\sin^2\theta_{i2}}} = \frac{\sqrt{n^2-\sin^2\theta_{i1}}}{\sqrt{n^2-\sin^2\theta_{i2}}}$$

which leads to the condition given by

$$\begin{aligned} \theta_{i1} &= \theta_{i2} \\ \theta_{t1} &= \theta_{t2} = \frac{\alpha}{2} \end{aligned}.$$

Using this condition, the incident angle can be calculated as

$$\theta_{i1} = \arcsin[n\sin(\frac{\alpha}{2})].$$

When $\alpha = 60°$, we have

$$\theta_{i1} = 48.59° \text{ for } n = 1.50$$

$$\theta_{i1} = 53.13° \text{ for } n = 1.60$$

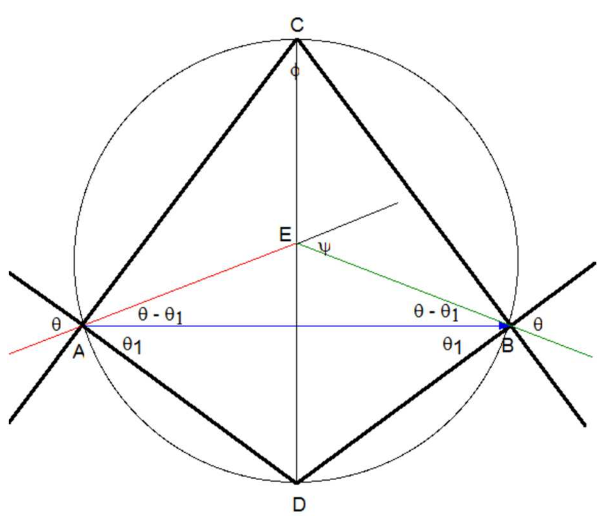
$$\theta_{i1} = 58.21° \text{ for } n = 1.70$$

((**Example**)) **Problem 33-55**
Index of refraction *n*, the angle of minimum deviation

In Fig., a ray is incident on one surface of a triangular glass prism in air. The angle of incidence θ is chosen so that the emerging ray also makes the same angle θ with the normal to the other face. Show that the index of refraction n of the glass prism is given by

$$n = \frac{\sin\left(\dfrac{\psi + \phi}{2}\right)}{\sin\dfrac{\phi}{2}}$$

where ϕ is the vertex angle of the prism and ψ is the deviation angle, the total angle through which the beam is turned in passing through the prism. (Under these conditions, the deviation angle ψ has the smallest possible value, which is called the angle of minimum deviation).



((**Solution**))
From the symmetry, the points A, B, C and D are on the same circle. The line CD is the diameter of the circle. Note that an angle inscribed in a semicircle is a right angle.

Snell's law:

$$\sin\,\theta = n\sin\,\theta_1$$

We also have the relations

$$\psi = 2(\theta - \theta_1)$$

and

$$2\theta_1 = \phi$$

which leads to the expression given by

$$\theta = \frac{\psi + \phi}{2}$$

Then the index of refraction n is derived as

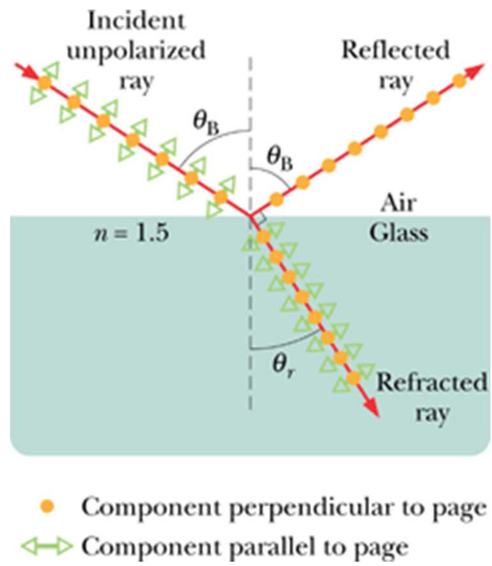
$$\boxed{n = \frac{\sin\left(\dfrac{\psi+\phi}{2}\right)}{\sin\left(\dfrac{\phi}{2}\right)}}$$

22. Brewster's angle: Polarization without polarizer

Figure shows a ray of unpolarized light incident on a glass surface. Let us resolve the electric field vectors of the light into two components. The perpendicular components are perpendicular to the plane of incidence and thus also to the page in Fig.; these components are represented with dots (as if we see the tips of the vectors). The parallel components are parallel to the plane of incidence and the page; they are represented with double-headed arrows. Because the light is unpolarized, these two components are of equal magnitude.

A ray of unpolarized light in air is incident on a glass surface at the Brewster angle θ_B. The electric fields along that ray have been resolved into components perpendicular to the page (the plane of incidence, reflection, and refraction) and components parallel to the page. The reflected light consists only of components perpendicular to the page and is thus polarized in that direction.

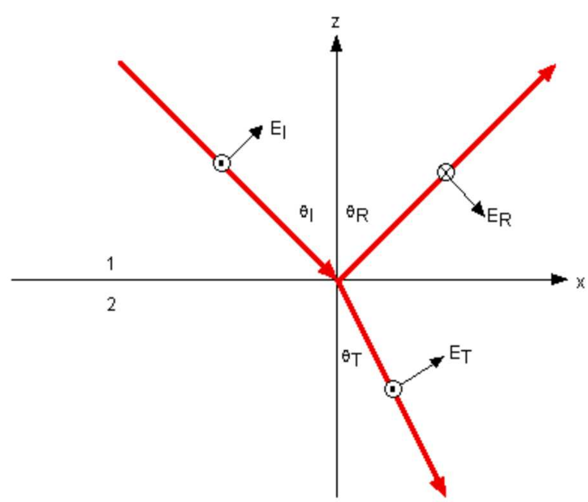
The refracted light consists of the original components parallel to the page and weaker components perpendicular to the page; this light is partially polarized



Scottish physicist, Sir David Brewster (1781-1868)

(A) Reflection and transmission for the polarization vector in the plane of incidence (application of the Fresnel's equations)

The polarization of the incident wave is parallel to the plane of incidence. The reflected and transmitted waves are also polarized in this plane.



Then we have two independent equations. Here we define

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I}$$

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2}$$

Reflection coefficient

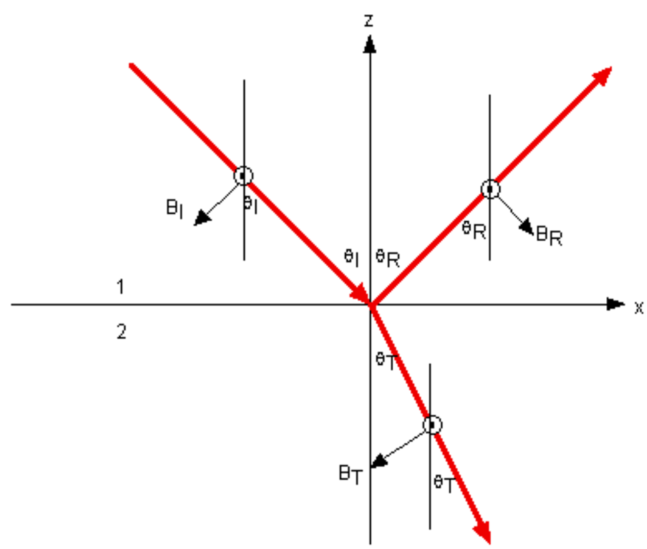
$$R_{//} = \frac{I_R}{I_I} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2$$

Transmission coefficient

$$T_{//} = \frac{I_T}{I_I} = \alpha\beta\left(\frac{2}{\alpha + \beta}\right)^2 = \frac{4\alpha\beta}{(\alpha + \beta)^2}$$

where // means that the polarization vector is in the plane of incidence.

(B) Reflection and transmission for the polarization vector perpendicular to the plane of incidence (application of the Fresnel's equations)



Reflection coefficient

$$R_{\perp} = \frac{I_R}{I_I} = \left(\frac{1 - \alpha\beta}{1 + \alpha\beta}\right)^2$$

Transmission coefficient

$$T_{\perp} = \frac{I_T}{I_I} = \alpha = \frac{4\alpha\beta}{(1 + \alpha\beta)^2}$$

where $\perp$ means that the polarization vector perpendicular to the plane of incidence.

We now consider the case when $\mu_1 = \mu_2$

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I}$$

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{v_1}{v_2} = \frac{n_2}{n_1} = \frac{\sin\theta_I}{\sin\theta_T}$$

Then we have

$$R_{//} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 = \left(\frac{\sin(2\theta_T)-\sin(2\theta_I)}{\sin(2\theta_T)+\sin(2\theta_I)}\right)^2$$

$$T_{//} = 1 - R_{//}$$

$$R_{\perp} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2 = \frac{\sin^2(\theta_T-\theta_I)}{\sin^2(\theta_T+\theta_I)}$$

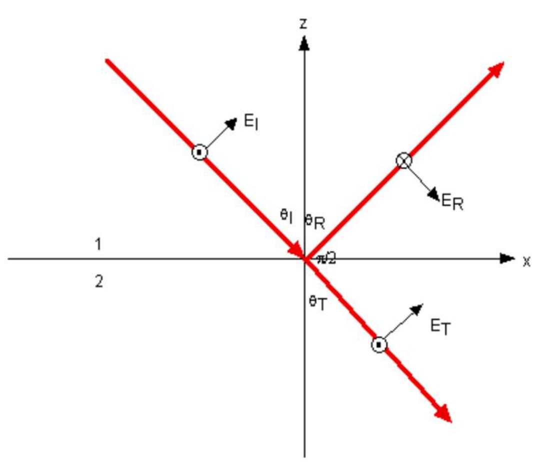
$$T_{\perp} = 1 - R_{\perp}$$

Note that $R_{//} = 0$ for $\sin(2\theta_T) = \sin(2\theta_I)$. This means that

$2\theta_T = 2\theta_I$ or $2\theta_T = \pi - 2\theta_I$

or

$\theta_T + \theta_I = \frac{\pi}{2}$ (Brewster angle).



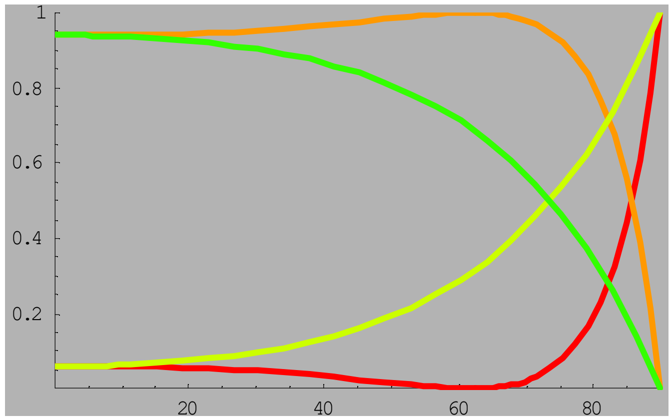
Brewsterís angle at which R// becomes zero

Using the Snell's law, we have

$$n_1 \sin\theta_I = n_2 \sin(\frac{\pi}{2} - \theta_I) = n_2 \cos\theta_I$$

then we have a Brewster's angle (θ_I), which is defined by

$$\tan\theta_I = \frac{n_2}{n_1}$$



$R_{//}$ (red), $T_{//}$(orange), $R_\perp$ (yellow), $T_\perp$(green)
$R+T=1$

((**Experiment**)) **Brewster angle**

You need only a polarizer for this experiment. Suppose that sun light enters from outside through a window and is reflected on the floor of your class room. When you look at the reflected light using the polarizer and slowly rotates the polarizer in one direction, you may easily find that the unpolarized light is polarized at some angle (that is a Brewster angle).

((**Feynman**)) From "Surely you are joking, Mr Feynman"
Surely you are joking Mr. Feynman
Brewster angle for the polarization of light

In regard to education in Brazil, I had a very interesting experience. I was teaching a group of students who would ultimately become teachers, since at that time there were not many opportunities in Brazil for a highly trained person in science. These students had already had many courses, and this was to be their most advanced course in electricity and magnetism - Maxwell's equations, and so on.

The university was located in various office buildings throughout the city, and the course I taught met in a building which overlooked the bay. I discovered a very strange phenomenon: I could ask a question, which the students would answer immediately. But the next time I would ask the question - the same subject, and the same question, as far as I could tell - they couldn't answer it at all!

For instance, one time I was talking about polarized light, and I gave them all some strips of polaroid. Polaroid passes only light whose electric vector is in a certain direction, so I explained how you could tell which way the light is polarized from whether the polaroid is dark or light. We first took two strips of polaroid and rotated them until they let the most light through. From doing that we could tell that the two strips were now admitting light polarized in the same direction - what passed through one piece of polaroid could also pass through the other. But then I asked them how one could tell the absolute direction of polarization, for a single piece of polaroid. They hadn't any idea. I knew this took a certain amount of ingenuity, so I gave them a hint: "Look at the light reflected from the bay outside." Nobody said anything. Then I said, "Have you ever heard of Brewster's Angle?". "Yes, sir! Brewster's Angle is the angle at which light reflected from a medium with an index of refraction is completely polarized." "And which way is the light polarized when it's reflected?" "The light is polarized perpendicular to the plane of reflection, sir." Even now, I have to think about it; they knew it cold! They even knew the tangent of the angle equals the index! I said, "Well?" Still nothing. They had just told me that light reflected from a medium with an index, such as the bay outside, was polarized; they had even told me which way it was polarized. I said, "Look at the bay outside, through the polaroid. Now turn the polaroid." "Ooh, it's polarized!" they said.

After a lot of investigation, I finally figured out that the students had memorized everything, but they didn't know what anything meant. When they heard "light that is reflected from a medium with an index," they didn't know that it meant a material such as water. They didn't know that the "direction of the light" is the direction in which you see something when you're looking at it, and so on. Everything was entirely memorized, yet nothing had been translated into meaningful words. So if I asked, "What is Brewster's Angle?" I'm going into the computer with the right keywords. But if I say, "Look at the water," nothing happens they don't have anything under "Look at the water"!

23. Total internal reflection

A phenomenon called total internal reflection can occur when light is directed from a medium having a given index of refraction toward one having a *lower* index of refraction

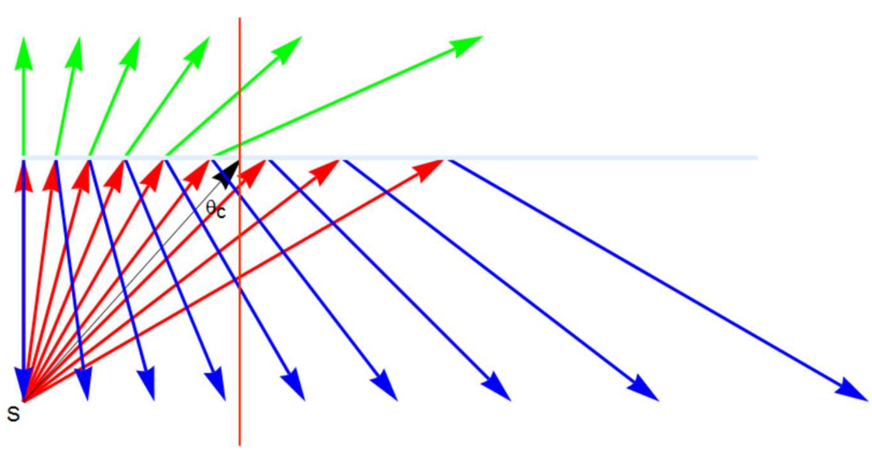
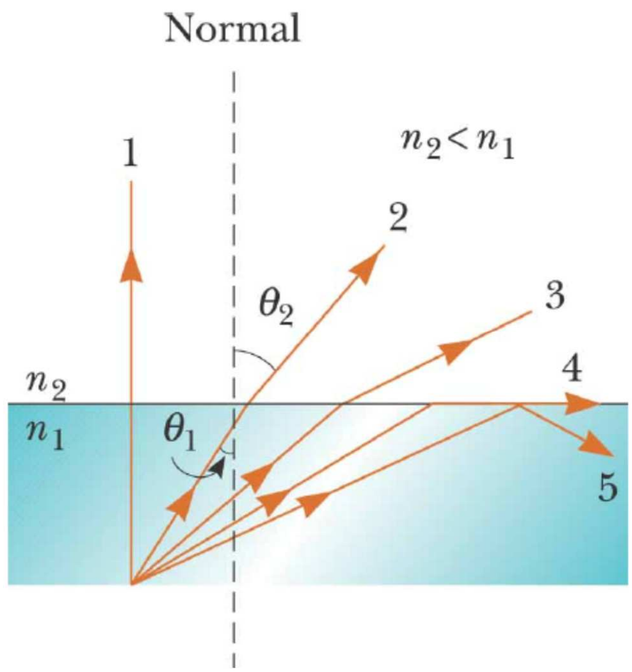
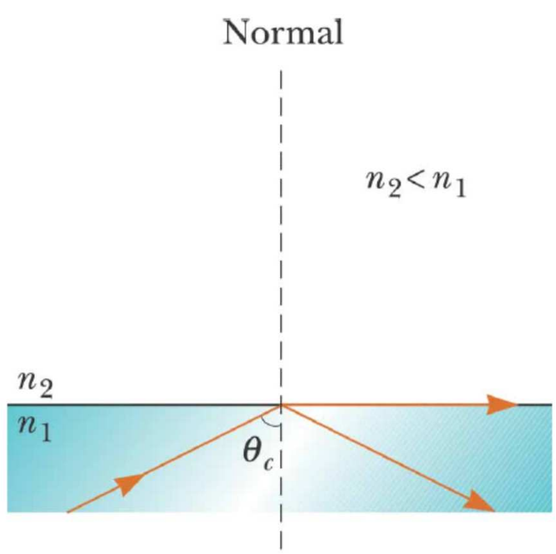


Fig. The total internal reflection of light from a point S, occurs for all angles of incidence greater than θ_c. In this case $\theta_c = \arcsin(1/n) = 41.8°$ for $n = 1.50$. At θ_c, the refracted ray points along the air-glass interface.

Possible directions of the beam are indicated by rays numbered 1 through 5. The refracted rays are bent away from the normal since $n_1 > n_2$.



Critical angle, θ_c

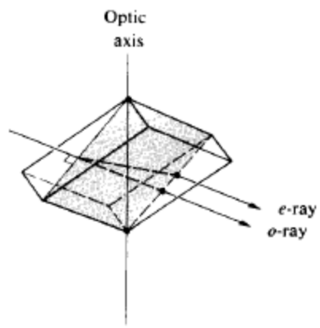


There is a particular angle of incidence that will result in an angle of refraction of 90°. This angle of incidence is called the critical angle, θ_c.

$$\sin\theta_c = \frac{n_2}{n_1}$$

24. Birefringence

We observe a birefringence in calcite.



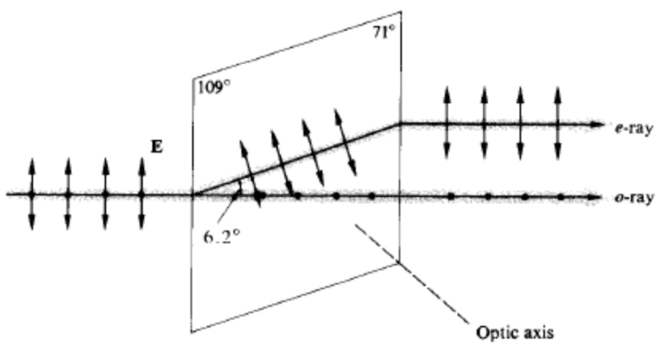


Fig. 8.22 A light beam with two orthogonal field components traversing a calcite principal section.

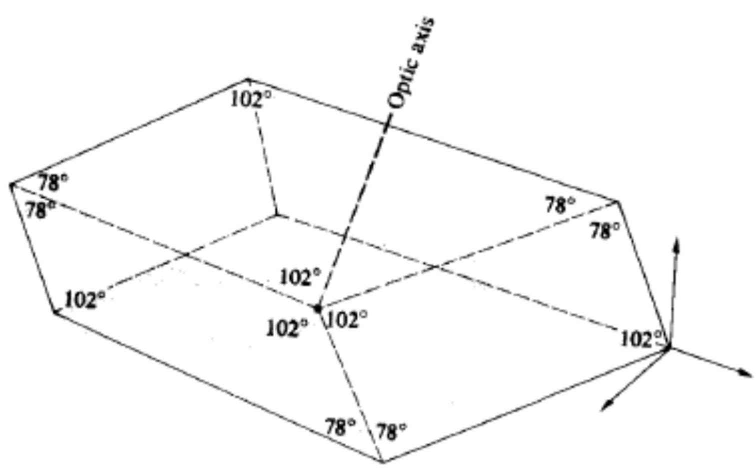
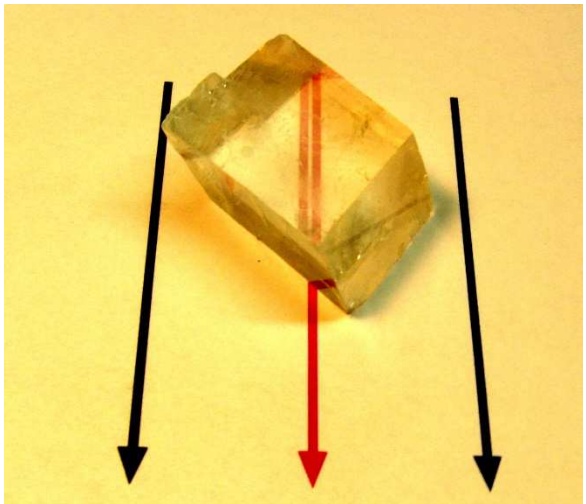
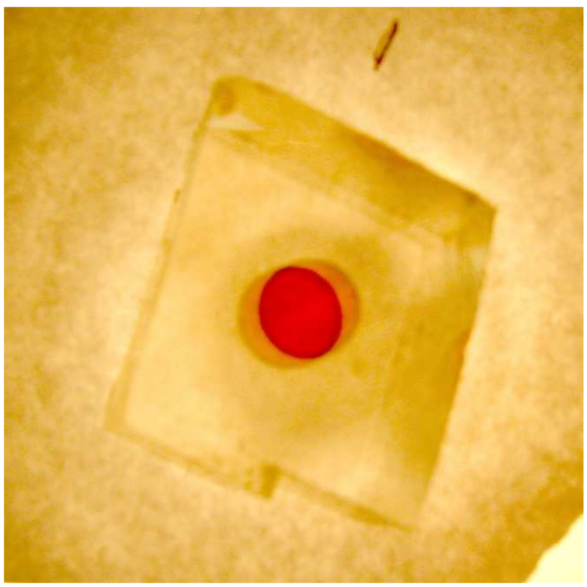


Fig. 8.19 Calcite cleavage form.





25. Typical problems

Formula

$$I = \frac{E_m^{\ 2}}{2\mu_0 c} = \frac{cB_m^{\ 2}}{2\mu_0}$$

c = 2.99792458 x 10^8 m/s
μ_0 = 12.566370614 x 10^{-7} (H/m)
ε_0 = 8.854187817 x 10^{-12} (F/m)

$E_m = c\ B_m$

$$E_{rms} = \frac{1}{\sqrt{2}} E_m$$

$$B_{rms} = \frac{1}{\sqrt{2}} B_m$$

$$\frac{E_{rms}}{B_{rms}} = \frac{E_{\max}}{B_{\max}} = \frac{\omega}{k} = c$$

$\langle u \rangle = I / c$ energy density

$P_r = I/c$ radiation pressure for the total absorption

$P_r = 2I/c$ radiation pressure for the totale reflection

25.1 Problem 33-15

Sunlight just outside Earth's atmosphere has an intensity of 1.40 kW/m^2. Calculate (a) E_m and (b) B_m for sunlight there, assuming it to be a plane wave.

((**Solution**))

I = 1.40 kW/m^2

$$I = \frac{E_m^{\ 2}}{2\mu_0 c}$$

or

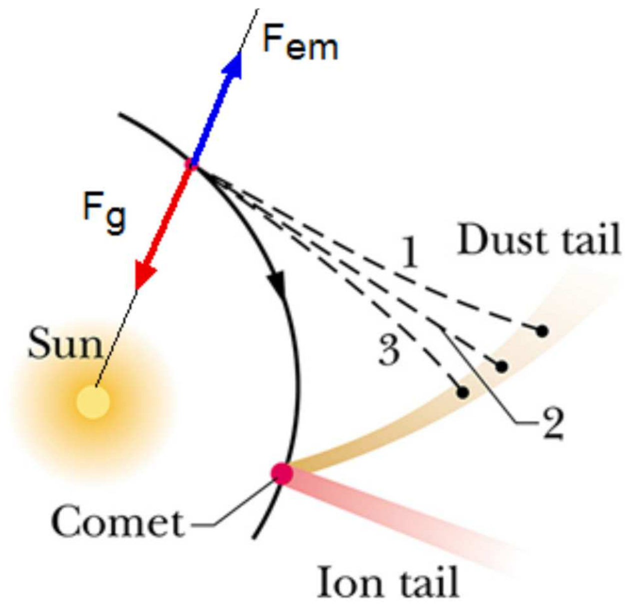
$$E_m = \sqrt{2\mu_0 cI} = 1.03 \text{ x } 10^3 \text{ V/m}$$

$$B_m = \frac{E_m}{c} = 3.43 \text{ x } 10^{-6} \text{ T}$$

25.2 Problem 33-31(Sp-33) radiation pressure

As a comet swings around the Sun, ice on the comet's surface vaporized, releasing trapped dust particles and ions. The ions, because they are electrically charged, are forced by the electrically charged solar wind into a straight ion tail that points radially away from the Sun (Fig). The (electrically neutral) dust particles are pushed radially outward from the Sun by the radiation force on them from sunlight. Assume that the dust particles are spherical, have density 3.5 x 10^3 kg/m^3, and are totally absorbing. (a) What radius must a particle have in order to follow a straight path, like path 2 in the figure? (b) If its radius is larger, does its path curve away from the Sun (like path 1) or toward the Sun (like path 3)?

((**Solution**))



dust particle
spherical
$\rho = 3.5 \times 10^3$ kg/m^3
Totally absorbing

(a)

The radiation pressure:

$$P_{em} = \frac{I}{c}$$

where

$$I = \frac{P}{4\pi r^2}$$

Then the force due to the radiation pressure,

$$F_{em} = P_{em}A = \frac{I}{c}(\pi R^2) = \frac{1}{c}(\pi R^2)\frac{P}{4\pi r^2} = \frac{PR^2}{4cr^2}$$

The central force due to the gravitation,

$$F_g = \frac{GmM_{sun}}{r^2} = \frac{GM_{sun}}{r^2}(\frac{4\pi}{3}R^3\rho)$$

From the condition that $F_g = F_{em}$

$$\frac{PR^2}{4cr^2} = \frac{4\pi GM_{sun}}{3r^2} R^3 \rho$$

$$R = R_c = \frac{3P}{16\pi c\rho GM_{sun}} = 1.7\times 10^{-7} m$$

(path-2)

(b)

$$F_{em} = \frac{PR^2}{4cr^2}$$

$$F_g = \frac{GM_{sun}}{r^2}\left(\frac{4\pi}{3}R^3\rho\right)$$

For $R>R_c$, $F_g>F_{em}$. (path-3)

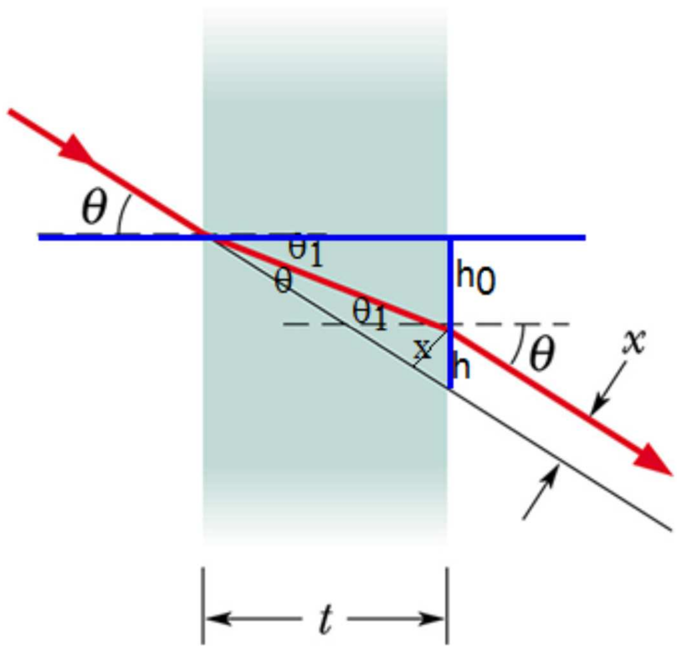
For $R<R_c$, $F_g<F_{em}$. (path-1)

25.3 Problem 33-75 Snell's law

(a) Prove that a ray of light incident on the surface of a sheet of plate glass of thickness t emerges from the opposite face parallel to its initial direction but displaced sideway, as in Fig. (b) Show that, for small angles of incidence θ, this displacement is given by

$$x = t\theta\left(\frac{n-1}{n}\right),$$

where n is the index of refraction of the glass and q is measured in radians.



((**Solution**))
Snell's law

$$\sin\theta = n\sin\theta_1$$
$$\theta \approx n\theta_1$$

From the geometry,

$$x = t\cdot\tan(\theta) - t\cdot\tan\theta_1$$
$$\approx t(\theta - \theta_1)$$
$$= t\theta(1-\frac{1}{n})$$

25.4 Problem 33-89 Plane wave

The magnetic component of a polarized wave of light is

$$B_x = (4.0\times 10^{-6}T)\sin[(1.57\times 10^{7}m^{-1})y + \omega t]$$

(a) Parallel to which axis is the light polarized? What are the (b) frequency and (c) intensity of the light?

((**Solution**))

$$B_x = B_0\sin(ky+\omega t)$$

Maxwell's equation

$$\nabla \times \boldsymbol{B} = \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

or

$$\nabla \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & 0 & 0 \end{vmatrix} = (0, 0, -\frac{\partial}{\partial y} B_x)$$

$$= \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \boldsymbol{E} = \mu_0 \varepsilon_0 (\frac{\partial}{\partial t} E_x, \frac{\partial}{\partial t} E_y, \frac{\partial}{\partial t} E_z)$$

Then we have

$$-\frac{\partial B_x}{\partial y} = \mu_0 \varepsilon_0 \frac{\partial E_z}{\partial t}, \qquad E_x = E_y = 0$$

The electric field E_z has the form

$$E_z = E_0 \sin(ky + \omega t)$$

$$-\frac{\partial B_x}{\partial y} = -B_0 k \cos(ky + \omega t)$$

$$\mu_0 \varepsilon_0 \frac{\partial E_z}{\partial t} = \mu_0 \varepsilon_0 \omega E_0 \cos(ky + \omega t)$$

From this relation, we get

$$-B_0 k = \mu_0 \varepsilon_0 \omega E_0$$

$$E_0 = -cB_0$$

where $c = \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}}$ and $\omega = ck$

(a) $E_z = E_0 \sin(ky + \omega t)$

with

$$E_0 = -cB_0$$

The polarization axis is the z axis.

(b) $\omega = 2\pi f = ck$ = (2.99792458 x 10^8) (1.57 x 10^7)

or

f = 7.491 x 10^{14} Hz

where

c = 2.99792458 x 10^8 m/s

(c) $I = \dfrac{E_0^{\ 2}}{2\mu_0 c} = \dfrac{cB_0^{\ 2}}{2\mu_0}$ = 1.91 kW/m^2

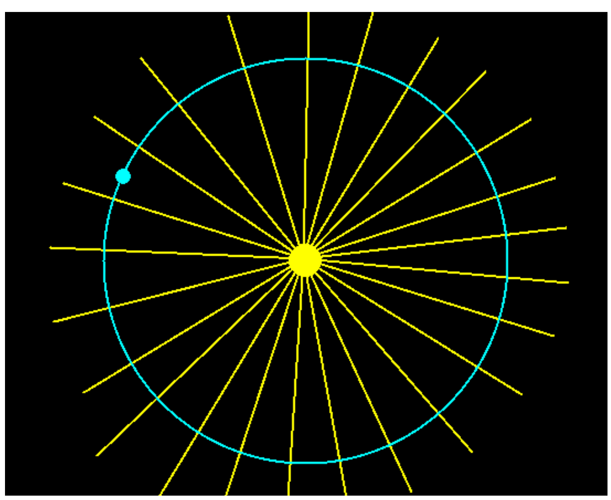
where

μ_0 = 12.566370614 x 10^{-7} (H/m), B_0 = 4.0 x 10^{-6} T

APPENDIX-I

A. Solar luminosity and surface temperature of the sun

The solar luminosity is a unit of luminosity (power emitted in the form of photons) conventionally used by astronomers to give the luminosities of stars. It is equal to the current luminosity of the Sun, which is 3.827 × 10^{26} W, or 3.827 × 10^{33} erg/s. (Wikipedia)



Imagine a huge sphere with a radius 1 AU with the Sun at its center. Each square meter of that sphere receives 1370 W of power from the Sun. So we can calculate the total energy output of the Sun (the Sun's Luminosity L_{sun}) from

$$G_{SC} = \frac{L_{sun}}{4\pi (AU)^2} = 1370 \text{ W/m}^2$$

where AU is an astronomical unit = average distance between the Earth and the Sun;

AU = 1.49597870 x 10^{11} m.

((**Definition**)) **Solar constant**: G_{SC}

The solar constant (G_{SC}) is a flux density measuring mean solar electromagnetic radiation (solar irradiance) per unit area. It is measured on a surface perpendicular to the rays, one astronomical unit (AU) from the Sun (roughly the distance from the Sun to the Earth).

Then the solar luminosity is obtained as

L_{sun}=3.85284 x 10^{26} W.

The radius of the Sun is

R_{sun} = 6.9599 x 10^8 m.

The surface temperature of the Sun is evaluated as follows,

$$F_{sun} = \sigma_{SB} T_{sun}^{\ 4} = \frac{L_{sun}}{4\pi R_{sun}^{\ 2}} = 6.32944 \times 10^7 \text{ W/m}^2.$$

where σ_{SB} is the Stefan-Boltzmann constant, σ_{SB} = 5.670400 x 10^{-8} W/m^2 K^4. The Sun's surface temperature is

T_{sun} = 5780 K.

APPENDIX-II Temperature of the Earth

A. Temperature of the Earth

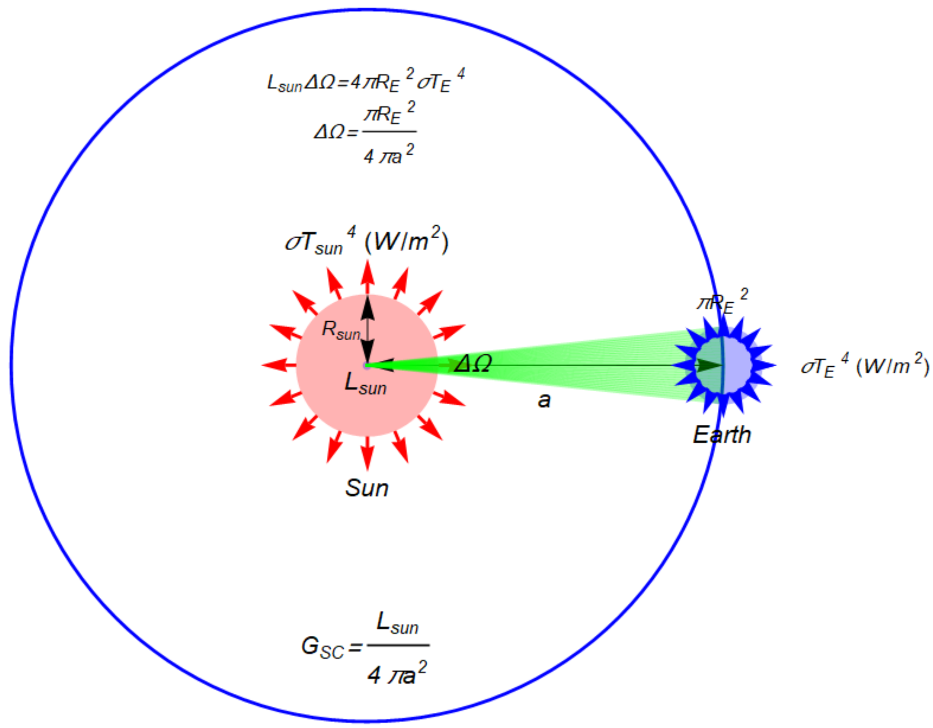


Fig. Solar luminosity L_{sun} =3.828x10^{26} W. Solar constant on the Earth is G_{SC}=1.3608 kW/m^2. a =AU = 1.49597870 x 10^{11} m. σ is the Stefan-Boltzmann constant.

We can use the Stefan-Boltzmann law to estimate the temperature of the Earth from first principles. The Sun is a ball of glowing gas of radius $R_{Sun} \approx 6.9599\times10^5$ km and surface temperature $T_{Sun} \approx 5780$ K. Its luminosity is

$$L_{Sun} = 4\pi R_{Sun}{}^2 \sigma T_{Sun}{}^4 \qquad \text{(W)}$$

according to the Stefan-Boltzmann law. The Earth is a globe of radius $R_E \approx 6372$ km located an average distance a = 1.49597870 x 10^8 km (= 1 AU) from the Sun. The Earth intercepts an amount of energy

$$\Delta P_{\mathrm{int}ercept} = \pi R_E{}^2 G_{SC} = \frac{\pi R_E{}^2}{4\pi a^2} L_{sun} = \Delta\Omega L_{sun} \qquad \text{(W)}$$

or

$$\Delta P_{\text{intercept}} = L_{Sun}\frac{\Delta\Omega}{4\pi} = L_{Sun}\frac{\pi R_E{}^2}{4\pi a^2} = \sigma T_{Sun}{}^4\frac{\pi R_{Sun}{}^2 R_E{}^2}{a^2} \qquad \text{(W)}$$

per second from the Sun's radiative output;

$$\pi R_E{}^2 = a^2\Delta\Omega = \frac{\pi R_E{}^2}{4\pi a^2} = \frac{R_E{}^2}{4a^2},$$

The Earth absorbs this energy, and then re-radiates it at longer wavelengths (the Kirchhoff's law). The luminosity of the Earth is

$$L_E = \pi R_E{}^2 G_{SC} = 4\pi R_E{}^2\sigma T_E{}^4, \qquad \text{(W)}$$

or

$$G_{SC} = 4\sigma T_E^4$$

according to the Stefan-Boltzmann law, where T_E is the average temperature of the Earth's surface. Here, we are ignoring any surface temperature variations between polar and equatorial regions, or between day and night. In steady-state, the luminosity of the Earth must balance the radiative power input from the Sun, we arrive at

$$\Delta P_{\text{int ercept}} = L_E,$$

or

$$L_{Sun}\frac{\pi R_E{}^2}{4\pi a^2} = 4\pi R_E{}^2\sigma T_E{}^4$$

$$\frac{T_{Sun}}{T_E} = \sqrt{2}\frac{\sqrt{a}}{\sqrt{R_{Sun}}}$$

or

$$T_E = \frac{1}{\sqrt{2}} \frac{\sqrt{R_{Sun}}}{\sqrt{a}} T_{Sun}$$

Remarkably, the ratio of the Earth's surface temperature to that of the Sun depends only on the Earth-Sun distance and the radius of the Sun. The above expression yields $T_E = 278.78$ K. This is slightly on the cold side, by a few degrees, because of the greenhouse action of the Earth's atmosphere, which was neglected in our calculation. Nevertheless, it is quite encouraging that such a crude calculation comes so close to the correct answer.

B. Evaluation of the average surface temperature of our solar system

The average surface temperature of the planet may be expressed by

$$T_{av} = \frac{1}{\sqrt{2}} \frac{\sqrt{R_{Sun}}}{\sqrt{d}} T_{Sun} = \frac{278.774}{\sqrt{d(AU)}} \text{ [K]}$$

where d is the mean distance from the Sun and d(AU) is the same distance in units of AU. The values of d, the calculated surface temperature T_{av}, and the reported surface temperature T_{obs} for each planet are listed in Table.

Planet	d (AU)	T_{av} [K]	T_{obs} [K]
Mercury	0.24	569.0	700
Venus	0.61	356.9	740
Earth	1	278.8	287.2
Mars	1.52	226.1	227
Jupitor	5.20	122.3	165 (1 bar level) 112 (0.1 bar level)
Saturn	9.53	90.3	134 (1 bar level) 84 (0.1 bar level)

Uranus	19.19	63.6	76 (1 bar level) 53 (0.1 bar level)
Neptune	30.06	50.8	72 (1 bar level) 55 (0.1 bar level)
Pluto	39.53	44.3 K	44

APPENDIX-III. Polarizers (some interesting experiment)

B.1 Simulation using Mathematica

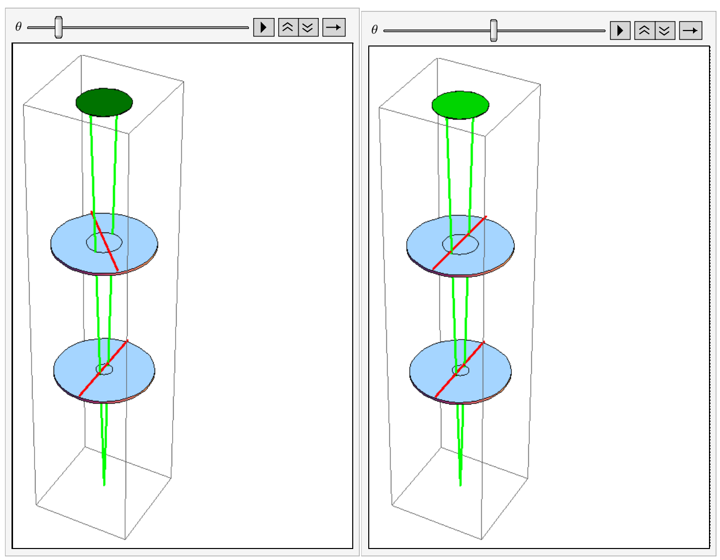


Fig.1 Demonstration for the role of two polarizers. The light passes when the directions of the two polarizers are the same. The light does not pass when the directions of two polarizers are perpendicular to each other.

B.2 Polarized light coming from the computer monitor

(i)

For convenience, I type a word "Computer monitor" in the computer monitor of the lap top computer. One can see clearly the word through the polarizer.

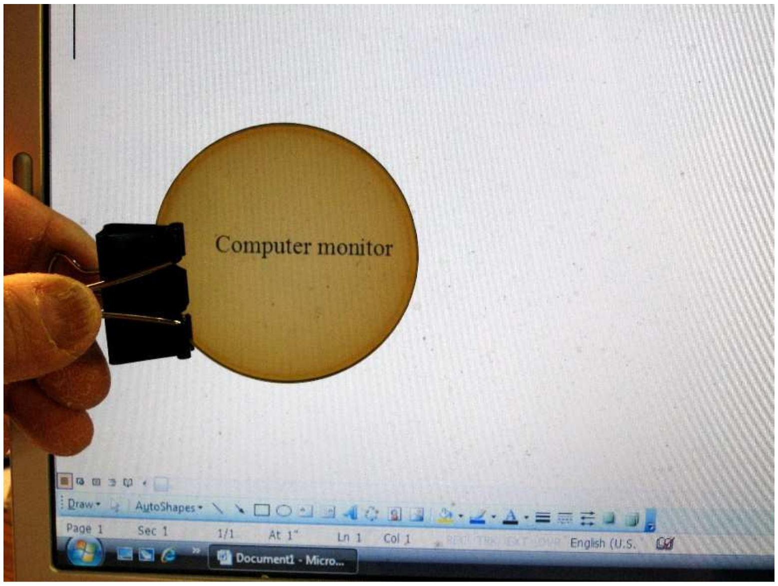


Fig.3 Polarizes in front of the computer monitor, where the direction of the polarization for rays from the computer monitor is the same as that of the polarizer.

(ii) The rotation of the polarizer by 90 degrees from the case (i).

When the polarizer is rotated 90° from the case (i), we find that the word has disappeared. This means that the light does not pass the polarizer.

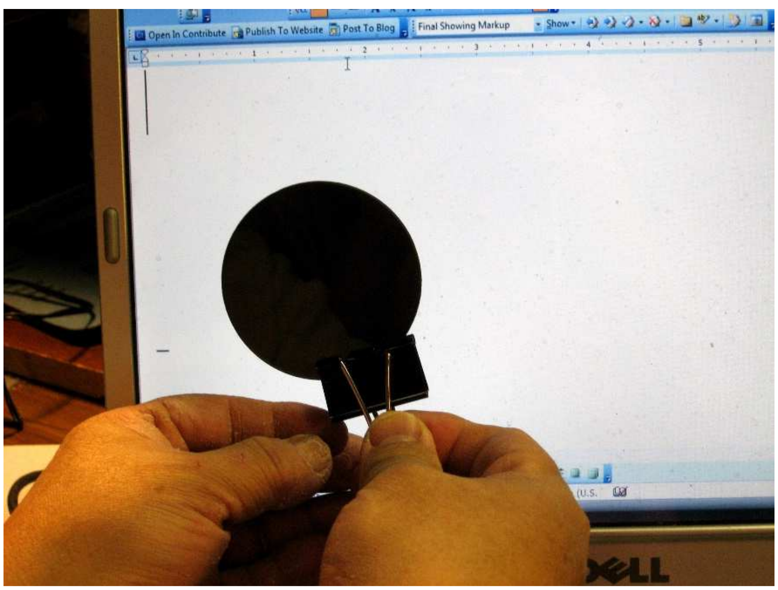


Fig.4 Polarizes in front of the computer monitor, where the direction of the polarization for rays from the computer monitor is perpendicular to that of the polarizer.