

Feynman (Gibbs) Subscript Notations in Vector Analysis with the use of Mathematica

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1. Motivation

The electromagnetism is one of the most important courses in an undergraduate physics curriculum. It forms one of the core pillars of classical physics alongside mechanics and thermodynamics. Understanding the behavior of electric and magnetic fields is crucial for an comprehensive understanding of physical phenomena.

In 1986, I (M.S.) started to teach physics courses at SUNY at Binghamton (Binghamton University, BU). I have taught classical mechanics, statistical thermodynamics, and quantum mechanics in undergraduate physics courses. However, until 2023, surprisingly to me, I have no opportunity to teach electrodynamics in undergraduate physics courses. In Fall semester of 2023, I finally had an opportunity to teach Phys.332 (Electrodynamics II) and Phys.331 (Electrodynamics I, Spring, 2024), using the famous textbook, Griffiths, Introduction to Electrodynamics, 4th and 5th edition (Cambridge University Press).

It seems to me that many students of my class have difficulty in understanding the mathematics (in particular, vector analysis) such as Laplace's equation with spherical coordinates and cylindrical aa well as Cartesian coordinates. Fortunately, I am so familiar with the use of Mathematica (version 14) and enjoy in calculating vector analysis very much with Mathematica. SUNY system which our BU belongs to (about 50 Universities and Community colleges) has a license for the use of Mathematica from Wolfram. So, students at Binghamton University can get the latest version of Mathematica Program in free. So, I strongly suggest my students to use Mathematica program in calculating vector analyses, including Grad, Curl (Rot), Div, Laplacian, and so on.

What is the motivation for us to write this article? During my class, one of students asked me how to calculate a scalar differential operator which is defined by a scalar differential operator

$$P_c[\mathbf{u}, \mathbf{v}] = (\mathbf{u} \cdot \nabla) \mathbf{v},$$

where \mathbf{u} and \mathbf{v} are vectors. That expression appeared in the Maxwell stress tensor. As far as I know, there is no such an operator in standard Mathematica program. So, I wrote a Mathematica program (module) to calculate the scalar differential operator in Cartesian, cylindrical, and spherical coordinates for students, and made sure it works. After that, I think that students understand how to calculate the scalar differential operator in three coordinates (see later for detail).

Since then, I became familiar with the name of the scalar differential operator. In order to get more detail about this, I checked the Google web sites, and asked what is the scalar differential operator? How can it be used in physics, in particular electrodynamics? I realized that this operator is called a Feynman subscript notation in vector product. I found that Richard P. Feynman showed how to use this operator in his lecture (Feynman's lectures on Physics, vol. II, Chapter 27; Field energy and Field momentum). Fortunately, we can also listen to his lecture on the web site with his own voice (exciting). In fact, I listened to his lecture very carefully (originally presented at

January 31, 1963). He clearly mentioned “1881” when he explained the scalar differential equation (time around time 13 minutes 26 sec after the start).

So, I promptly asked Open AI ChatGPT;

((AI)) My question

Is it true that J.W. Gibbs gave an lecture on vector analysis at 1881? I found that Feynman referred to 1881 in Feynman's lectures on Physics vol 2 Chapter 27 audio (1963).

((AI)) Reply

Yes, J. Willard Gibbs did give lectures on vector analysis around 1881. Gibbs, a pioneering American scientist, significantly contributed to the development of vector analysis, which is a branch of mathematics that deals with vectors and vector-valued functions. The reference you mentioned, from Richard Feynman's "Lectures on Physics," Volume 2, Chapter 27, indeed notes that Gibbs was involved in the development of vector analysis during the late 19th century. While Feynman might have mentioned 1881 specifically, Gibbs's work on vector analysis spanned several years around that period. Gibbs began giving a course on vector analysis at Yale University in 1881. His notes from these lectures were later published as "Elements of Vector Analysis" by his student, Edwin Bidwell Wilson, in 1901. This work played a crucial role in popularizing and formalizing the use of vector methods in physics and engineering. Therefore, it is accurate to say that J.W. Gibbs was lecturing on vector analysis around 1881, and Feynman's mention in his lectures is consistent with historical accounts.

So I understand that the vector analysis which Feynman used in his book, was originally discussed by Prof. Gibbs at Yale University between 1881 and 1884.

2. Brief profiles of Feynman, Gibbs, Wilson; Comment from Joe Heafner

((Richard P. Feynman))

Between 1961 and 1963 **Richard Feynman** set out to record the lectures he gave to his freshman and sophomore introductory physics classes at Caltech. At the time, modern theories like those of relativity and quantum mechanics were exciting students and propelling them to pursue the study of our Universe’s fundamental nature.



Fig.1

Richard Feynman talking with a teaching assistant after the lecture on “The Dependence of Amplitudes on Time, Robert Leighton (left) and Matthew Sands (right) in background, April 20, 1963.

<https://www.feynmanlectures.caltech.edu/>

((Josiah Willard Gibbs))

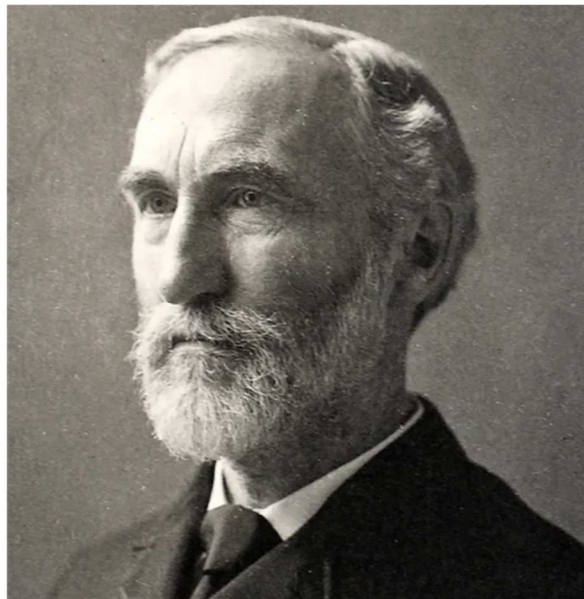


Fig.2

Josiah Willard Gibbs (February 11, 1839 – April 28, 1903) was an American scientist who made significant theoretical contributions to physics, chemistry, and mathematics. His work on the applications of thermodynamics was instrumental in transforming physical chemistry into a rigorous deductive science. Together with James Clerk Maxwell and Ludwig Boltzmann, he

created statistical mechanics (a term that he coined), explaining the laws of thermodynamics as consequences of the statistical properties of ensembles of the possible states of a physical system composed of many particles. Gibbs also worked on the application of Maxwell's equations to problems in physical optics. As a mathematician, he created **modern vector calculus** (independently of the British scientist Oliver Heaviside, who carried out similar work during the same period) and described the Gibbs phenomenon in the theory of Fourier analysis.
https://en.wikipedia.org/wiki/Josiah_Willard_Gibbs

Josiah Willard Gibbs and his contemporary, Edwin Bidwell Wilson, were both prominent figures in the fields of mathematics and theoretical physics during the late 19th and early 20th centuries. They collaborated on several works, and Wilson often contributed to the development and dissemination of Gibbs's ideas. The lecture notes of Gibbs was edited by Wilson in 1901 when he was 22 years old.

((Edwin Bidwell Wilson))

Edwin Bidwell Wilson (April 25, 1879 – December 28, 1964)

At Yale, Wilson worked under the supervision of Josiah Willard Gibbs and compiled an important textbook on vector analysis from Gibbs' lecture notes. Gibbs died when Wilson had just turned 24, but he exerted a strong influence on Wilson through his early supervision and through Wilson's experience compiling Gibbs' notes. Wilson became an Assistant Professor of Mathematics at Yale in 1906, then Associate Professor of Mathematics at Massachusetts Institute of Technology (MIT) in 1907, then Professor of Mathematics in 1911, then Head of the Department of Physics in 1917, and then Professor of Vital Statistics at the Harvard School of Public Health in 1922.

((Note from TensorTimes@2024 by Joe Heafner))

Did Feynman Invent Feynman subscript notation?

<https://tensortime.sticksandshadows.net/archives/718>

When we checked the web site about the Feynman's subscript notations, we found a very interesting article by Joe Heafner. A part of the article is reproduced here.

Feynman explains that the subscripted ∇ can be manipulated as though it were a vector (it is not) according to the rules of dot products (commutative), cross products (anticommutative), triple scalar products (cyclic permutation, swapping dots and crosses, etc.), and triple vector products (BAC-CAB, Jacobi identity, etc.) The strategy is to end up with only one vector (the one corresponding to a subscript) immediately to the right of each correspondingly subscripted ∇ . Then you drop the subscripts, and you should have a valid vector identity. In the audio version of this lecture, Feynman comments that he doesn't understand why this technique isn't taught. It was never shown to me as either an undergraduate or graduate student. I suspect it's treated as "one of those things" students are simply assumed to pick up at one point or another without it ever being explicitly addressed (much like critical thinking is treated).

The issue here, for me, is whether or not Feynman invented this way of manipulating vector expressions. After all, the notation carries his name so it might be reasonable to assume he invented the underlying method. My research shows that a very similar methodology is documented in the very first (as far as I know) textbook on vector analysis, Wilson's [Vector Analysis: A Text-Book for the use of Students of Mathematics and Physics](#). This is the famous work based on Gibbs' lecture notes and is the definitive work on contemporary vector analysis.

This brings to my mind the question of whether or not Feynman was aware of Wilson's textbook and this method documented therein and decided to change the nature of the subscript to show what is differentiated rather than what is not. I don't see how there is any way to know for sure, but it's an interesting question in my mind because I suspect many students are not aware of Wilson's textbook.

3. Overview: Feynman (Gibbs) subscript notations in vector products

Here we explain the detail of the Feynman (Gibbs) subscript notations in vector products and its application in vector analysis.

$$\begin{aligned}\nabla \cdot (\mathbf{u} \times \mathbf{v}) &= \nabla_{\mathbf{u}} \cdot (\mathbf{u} \times \mathbf{v}) + \nabla_{\mathbf{v}} \cdot (\mathbf{u} \times \mathbf{v}) \\ &= \mathbf{u} \cdot (\mathbf{v} \times \nabla_{\mathbf{u}}) + \mathbf{v} \cdot (\nabla_{\mathbf{v}} \times \mathbf{u}) \\ &= -\mathbf{u} \cdot (\nabla_{\mathbf{u}} \times \mathbf{v}) + \mathbf{v} \cdot (\nabla_{\mathbf{v}} \times \mathbf{u}) \\ &= -\mathbf{u} \cdot (\nabla \times \mathbf{v}) + \mathbf{v} \cdot (\nabla \times \mathbf{u})\end{aligned}$$

where $\nabla_{\mathbf{u}} \cdot (\mathbf{u} \times \mathbf{v})$ is the same as $\nabla \cdot (\mathbf{u} \times \mathbf{v})$ with \mathbf{u} is kept constant during calculation.

$$\begin{aligned}\nabla(\mathbf{u} \cdot \mathbf{v}) &= \nabla_{\mathbf{u}}(\mathbf{u} \cdot \mathbf{v}) + \nabla_{\mathbf{v}}(\mathbf{u} \cdot \mathbf{v}) \\ &= [(\mathbf{u} \cdot \nabla_{\mathbf{u}})\mathbf{v} + \mathbf{u} \times (\nabla_{\mathbf{u}} \times \mathbf{v})] + [(\mathbf{v} \cdot \nabla_{\mathbf{v}})\mathbf{u} + \mathbf{v} \times (\nabla_{\mathbf{v}} \times \mathbf{u})] \\ &= (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u})\end{aligned}$$

where $(\mathbf{u} \cdot \nabla)\mathbf{v}$ is called a scalar differential operator.

$$\begin{aligned}\nabla \times (\mathbf{u} \times \mathbf{v}) &= \nabla_{\mathbf{u}} \times (\mathbf{u} \times \mathbf{v}) + \nabla_{\mathbf{v}} \times (\mathbf{u} \times \mathbf{v}) \\ &= (\nabla_{\mathbf{u}} \cdot \mathbf{v})\mathbf{u} - (\nabla_{\mathbf{u}} \cdot \mathbf{u})\mathbf{v} + (\nabla_{\mathbf{v}} \cdot \mathbf{v})\mathbf{u} - (\nabla_{\mathbf{v}} \cdot \mathbf{u})\mathbf{v} \\ &= (\mathbf{v} \cdot \nabla_{\mathbf{v}})\mathbf{u} - (\mathbf{u} \cdot \nabla_{\mathbf{u}})\mathbf{v} + (\nabla_{\mathbf{u}} \cdot \mathbf{v})\mathbf{u} - (\nabla_{\mathbf{v}} \cdot \mathbf{u})\mathbf{v} \\ &= (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + (\nabla \cdot \mathbf{v})\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{v}\end{aligned}$$

In this article, we have discussed the following things

- (i) We will briefly summarize the discussions given by Gibbs (1881-1884) and Feynman (1963).
- (ii) Construction of Mathematica Programs for the scalar differential operators in terms of three coordinate systems

(iii) We will make proof that Feynman (Gibbs) subscript notations are mathematically valid using the Mathematica program based on Cartesian coordinates.

(iv) **Application-1**

We show that the magnetic field due to constant magnetic moment \mathbf{m} and the electric field due to the constant electric dipole moment \mathbf{p} can be expressed by

$$\begin{aligned}\mathbf{B}_{dipole} &= \nabla \times \mathbf{A}_{dipole} \\ &= \frac{\mu_0}{4\pi} \nabla (\mathbf{m} \times \frac{\mathbf{r}}{r^3}) \\ &= -\frac{\mu_0}{4\pi} (\mathbf{m} \cdot \nabla) \frac{\mathbf{r}}{r^3}, \\ &= -\frac{\mu_0}{4\pi} P_c[\mathbf{m}, \frac{\mathbf{r}}{r^3}]\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}_{dipole} &= -\nabla V_{dipole} \\ &= -\frac{1}{4\pi\epsilon_0} \nabla (\mathbf{p} \cdot \frac{\mathbf{r}}{r^3}) \\ &= -\frac{1}{4\pi\epsilon_0} (\mathbf{p} \cdot \nabla) \frac{\mathbf{r}}{r^3} \\ &= -\frac{1}{4\pi\epsilon_0} P_c[\mathbf{p}, \frac{\mathbf{r}}{r^3}]\end{aligned}$$

Clearly, \mathbf{B}_{dipole} and \mathbf{E}_{dipole} can be expressed by the scalar differential operator.

(v) **Application II: Magnetic and electric force (Purcell)**

The force on the constant magnetic moment \mathbf{m} under the presence of an external magnetic field \mathbf{B} is given by

$$\mathbf{F}_B = (\mathbf{m} \cdot \nabla)\mathbf{B} = P_c[\mathbf{m}, \mathbf{B}].$$

The force on the constant electric dipole moment \mathbf{p} under the presence of an external magnetic field \mathbf{E} is given by

$$\mathbf{F}_E = (\mathbf{p} \cdot \nabla)\mathbf{E} = P_c[\mathbf{p}, \mathbf{E}].$$

(vi) Using the scalar differential operators in terms of three coordinate systems, we solve typical problems in Arfken's book,

4. From the lecture of Gibbs on vector analysis (1881-1884): Summary

Here we just show how the subscript notations in vector products were discussed in Gibbs in his lectures on vector analysis. A part of Gibb's book is reproduced here, although some revisions are added by us.

4-1 Discussion by Gibbs (I): 73

73: It frequently happens, as in the case of the application just cited, that the operators ∇ , $\nabla \cdot$, $\nabla \times$, have to be applied to combinations of scalar functions, vector functions, or both. The following rules of operation will be found useful. Let u , v be scalar functions and \mathbf{u} , \mathbf{v} vector functions of position in space. Then

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{u}) + \mathbf{u} \times (\nabla \times \mathbf{v}) \quad (41)$$

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v}) \quad (42)$$

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} - \mathbf{v}(\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{u}(\nabla \cdot \mathbf{v}) \quad (43)$$

The proofs of the formula may be given most naturally by expanding the expressions in terms of three unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ (Cartesian coordinates)

4-2 Discussion by Gibbs (II): 74

74. The notation $\nabla_{\mathbf{u}}(\mathbf{u} \cdot \mathbf{v})$ will be used to denote that in applying the operator ∇ to the product $(\mathbf{u} \cdot \mathbf{v})$, the quantity \mathbf{u} is to be regarded as constant, That is, the operation ∇ is carried out partially upon the product $(\mathbf{u} \cdot \mathbf{v})$. In general, if ∇ is to be carried out partially upon any number of functions which occur after it in a parenthesis, those functions which are constant for the differentiations are written after the parenthesis as subscripts. Gibbs proved that

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = \nabla_{\mathbf{u}}(\mathbf{u} \cdot \mathbf{v}) + \nabla_{\mathbf{v}}(\mathbf{u} \cdot \mathbf{v}). \quad (41)'$$

The formula given above may be written in the following manner, as is obvious from analogy with the corresponding formula in differentials;

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \nabla_{\mathbf{u}} \cdot (\mathbf{u} \times \mathbf{v}) + \nabla_{\mathbf{v}} \cdot (\mathbf{u} \times \mathbf{v}). \quad (42)'$$

and

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = \nabla_{\mathbf{u}} \times (\mathbf{u} \times \mathbf{v}) + \nabla_{\mathbf{v}} \times (\mathbf{u} \times \mathbf{v}). \quad (43)'$$

Take for instance (43)'. Expand $\nabla_{\mathbf{u}} \times (\mathbf{u} \times \mathbf{v})$ formally.

$$\nabla_{\mathbf{u}} \times (\mathbf{u} \times \mathbf{v}) = (\nabla_{\mathbf{u}} \cdot \mathbf{v})\mathbf{u} - (\nabla_{\mathbf{u}} \cdot \mathbf{u})\mathbf{v},$$

where it must be understood that \mathbf{u} is constant for the differentiations which occur in $\nabla_{\mathbf{u}}$. Then in the last term, the factor \mathbf{u} may be placed before the sign $\nabla_{\mathbf{u}}$. Hence

$$\begin{aligned}\nabla_{\mathbf{u}} \times (\mathbf{u} \times \mathbf{v}) &= \mathbf{u}(\nabla_{\mathbf{u}} \cdot \mathbf{v}) - (\mathbf{u} \cdot \nabla_{\mathbf{u}})\mathbf{v} \\ &= \mathbf{u}(\nabla \cdot \mathbf{v}) - (\mathbf{u} \cdot \nabla)\mathbf{v} \\ &= \mathbf{u}(\nabla \cdot \mathbf{v}) - P_c[\mathbf{u}, \mathbf{v}]\end{aligned}$$

In like manner

$$\begin{aligned}\nabla_{\mathbf{v}} \times (\mathbf{u} \times \mathbf{v}) &= (\mathbf{v} \cdot \nabla_{\mathbf{v}})\mathbf{u} - (\nabla_{\mathbf{v}} \cdot \mathbf{u})\mathbf{v} \\ &= (\mathbf{v} \cdot \nabla)\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{v} \\ &= P_c[\mathbf{v}, \mathbf{u}] - \mathbf{v}(\nabla \cdot \mathbf{u})\end{aligned}$$

Hence,

$$\begin{aligned}\nabla \times (\mathbf{u} \times \mathbf{v}) &= \nabla_{\mathbf{u}} \times (\mathbf{u} \times \mathbf{v}) + \nabla_{\mathbf{v}} \times (\mathbf{u} \times \mathbf{v}) \\ &= (\mathbf{v} \cdot \nabla_{\mathbf{v}})\mathbf{u} - (\mathbf{u} \cdot \nabla_{\mathbf{u}})\mathbf{v} - \mathbf{v}(\nabla_{\mathbf{v}} \cdot \mathbf{u}) + \mathbf{u}(\nabla_{\mathbf{u}} \cdot \mathbf{v}) \\ &= (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} - \mathbf{v}(\nabla \cdot \mathbf{u}) + \mathbf{u}(\nabla \cdot \mathbf{v}) \\ &= P_c[\mathbf{v}, \mathbf{u}] - P_c[\mathbf{u}, \mathbf{v}] - \mathbf{v}(\nabla \cdot \mathbf{u}) + \mathbf{u}(\nabla \cdot \mathbf{v})\end{aligned}$$

4-3 Discussion by Gibbs (1II):75

75. There are number of important relations in which the partial operation $\nabla_{\mathbf{u}}(\mathbf{u} \cdot \mathbf{v})$ figures.

$$\mathbf{u} \times (\nabla_{\mathbf{u}} \times \mathbf{v}) = \nabla_{\mathbf{u}}(\mathbf{u} \cdot \mathbf{v}) - (\mathbf{u} \cdot \nabla_{\mathbf{u}})\mathbf{v}, \quad (46)$$

or

$$\nabla_{\mathbf{u}}(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{u} \times (\nabla \times \mathbf{v}), \quad (46)'$$

or

$$(\mathbf{u} \cdot \nabla)\mathbf{v} = \nabla_{\mathbf{u}}(\mathbf{u} \cdot \mathbf{v}) + (\nabla \times \mathbf{v}) \times \mathbf{u}, \quad (46)''$$

This proof of this relation may be given by expanding in terms of i, j, k (Cartesian coordinates). A method of remembering the result easily is as follows. Expand the product

$$\mathbf{u} \times (\nabla \times \mathbf{v}).$$

Formally as if ∇ , \mathbf{u} , \mathbf{v} were all real vectors. Then

$$\mathbf{u} \times (\nabla_{\mathbf{u}} \times \mathbf{v}) = (\mathbf{u} \cdot \nabla) \mathbf{v} - (\mathbf{u} \cdot \nabla_{\mathbf{u}}) \mathbf{v}$$

The second term is capable of interpretation as it stands. The first term, however, is not. The operator ∇ has nothing upon which to operate. It therefore must be transposed so that it shall have $\mathbf{u} \cdot \mathbf{v}$ as an operand. But \mathbf{u} being outside of the parenthesis in $\mathbf{u} \times (\nabla_{\mathbf{u}} \times \mathbf{v})$ is constant for the differentiations. Hence

$$(\mathbf{u} \cdot \nabla) \mathbf{v} = \nabla_{\mathbf{u}} (\mathbf{u} \cdot \mathbf{v}),$$

and

$$\mathbf{u} \times (\nabla_{\mathbf{u}} \times \mathbf{v}) = \nabla_{\mathbf{u}} (\mathbf{u} \cdot \mathbf{v}) - (\mathbf{u} \cdot \nabla_{\mathbf{u}}) \mathbf{v}. \quad (46)$$

If \mathbf{u} be a unit vector, say $\hat{\mathbf{a}}$, the formula

$$(\hat{\mathbf{a}} \cdot \nabla) \mathbf{v} = \nabla_{\hat{\mathbf{a}}} (\hat{\mathbf{a}} \cdot \mathbf{v}) + (\nabla \times \mathbf{v}) \times \hat{\mathbf{a}}, \quad (47)$$

expresses the fact that the directional derivative $(\hat{\mathbf{a}} \cdot \nabla) \mathbf{v}$ of a vector function \mathbf{v} in the direction $\hat{\mathbf{a}}$ is equal to the derivative of the projection of the vector \mathbf{v} in that direction plus the vector product of the curl of \mathbf{v} into the direction $\hat{\mathbf{a}}$.

5. From the book of Feynman on Feynman's lectures on Physics vol.2 Chapter 27.

Here we just show how the subscript notations in vector products were discussed in Feynman's lectures on Physics. A part of Feynman's book (vol.2 Chapter 27) is reproduced here, although some revisions are added by us.

5.1 Detail of Formula-1

In his book, Feynman explained his subscript notations using the formula 1 given by

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \nabla_{\mathbf{u}} \cdot (\mathbf{u} \times \mathbf{v}) + \nabla_{\mathbf{v}} \cdot (\mathbf{u} \times \mathbf{v}). \quad (27.10)$$

The moment we do that, we do not have to keep the order straight any more. We always know that $\nabla_{\mathbf{v}}$ operates on \mathbf{u} only (constant \mathbf{v}), and $\nabla_{\mathbf{u}}$ operates on \mathbf{v} only (constant \mathbf{u}). In these circumstances, we can use ∇ as though it were an ordinary vector. (Of course, when we are

finished, we will want to return to the “standard” notation that everybody usually uses.). So now, we can do the various things like interchanging dots and crosses and making other kinds of rearrangements of the terms. For instance, the last term of Eq. (27.10) can be rewritten as

$$\begin{aligned}\nabla_{\mathbf{v}} \cdot (\mathbf{u} \times \mathbf{v}) &= \mathbf{v} \cdot (\nabla_{\mathbf{v}} \times \mathbf{u}) \\ &= (\nabla_{\mathbf{v}} \times \mathbf{u}) \cdot \mathbf{v}\end{aligned}$$

[You remember that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$]. And the first term of Eq. (27.10) is the same as

$$\begin{aligned}\nabla_{\mathbf{u}} \cdot (\mathbf{u} \times \mathbf{v}) &= \mathbf{u} \cdot (\mathbf{v} \times \nabla_{\mathbf{u}}) \\ &= -\mathbf{u} \cdot (\nabla_{\mathbf{u}} \times \mathbf{v}) \\ &= -(\nabla_{\mathbf{u}} \times \mathbf{v}) \cdot \mathbf{u}\end{aligned}$$

It looks freakish, but it is all right. Now if we try to go back to the ordinary convention, we have to arrange that the ∇ operates only on its “own” variable. The first one is already that way, so we can just leave off the subscript. The second one needs some rearranging to put the ∇ in front of the \mathbf{v} , which we can do by reversing the cross product and changing sign

Now it is in a conventional order, so we can return to the usual notation. Equation (27.10) is equivalent

$$\begin{aligned}\nabla \cdot (\mathbf{u} \times \mathbf{v}) &= +\nabla_{\mathbf{u}} \cdot (\mathbf{u} \times \mathbf{v}) + \nabla_{\mathbf{v}} \cdot (\mathbf{u} \times \mathbf{v}) \\ &= -\mathbf{u} \cdot (\nabla_{\mathbf{u}} \times \mathbf{v}) + \mathbf{v} \cdot (\nabla_{\mathbf{v}} \times \mathbf{u}) \\ &= -\mathbf{u} \cdot (\nabla \times \mathbf{v}) + \mathbf{v} \cdot (\nabla \times \mathbf{u})\end{aligned}\tag{27.11}$$

(A quicker way would have been to use components in this special case, but it was worth taking the time to show you the mathematical trick. You probably won’t see it anywhere else, and it is very good for unlocking vector algebra from the rules about the order of terms with derivatives).

5.2 Detail of Formula-2

We start with

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = \nabla_{\mathbf{u}}(\mathbf{u} \cdot \mathbf{v}) + \nabla_{\mathbf{v}}(\mathbf{u} \cdot \mathbf{v})$$

$\nabla_{\mathbf{u}}(\mathbf{u} \cdot \mathbf{v})$ will be used to denote that in applying the operator ∇ to the product $(\mathbf{u} \cdot \mathbf{v})$, the quantity \mathbf{u} is to be regarded as constant. That is, the operation ∇ is carried out only partially upon the product $(\mathbf{u} \cdot \mathbf{v})$. In general, if ∇ is to be carried out partially upon any number of functions which occur after it in a parenthesis, those functions which are constant for the differentiations are written after the parenthesis as subscripts.

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = \nabla_{\mathbf{u}}(\mathbf{u} \cdot \mathbf{v}) + \nabla_{\mathbf{v}}(\mathbf{u} \cdot \mathbf{v}),$$

with

$$\mathbf{u} \times (\nabla_{\mathbf{u}} \times \mathbf{v}) = \nabla_{\mathbf{u}}(\mathbf{u} \cdot \mathbf{v}) - (\mathbf{u} \cdot \nabla_{\mathbf{u}})\mathbf{v},$$

$$\mathbf{v} \times (\nabla_{\mathbf{v}} \times \mathbf{u}) = \nabla_{\mathbf{v}}(\mathbf{u} \cdot \mathbf{v}) - (\mathbf{v} \cdot \nabla_{\mathbf{v}})\mathbf{u},$$

using the usual vector formula

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \\ &= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \end{aligned}$$

Thus, we have

$$\begin{aligned} \nabla(\mathbf{u} \cdot \mathbf{v}) &= \nabla_{\mathbf{u}}(\mathbf{u} \cdot \mathbf{v}) + \nabla_{\mathbf{v}}(\mathbf{u} \cdot \mathbf{v}) \\ &= [(\mathbf{u} \cdot \nabla_{\mathbf{u}})\mathbf{v} + \mathbf{u} \times (\nabla_{\mathbf{u}} \times \mathbf{v})] + [(\mathbf{v} \cdot \nabla_{\mathbf{v}})\mathbf{u} + \mathbf{v} \times (\nabla_{\mathbf{v}} \times \mathbf{u})] \\ &= (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{u} \times (\nabla \times \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{u} + \mathbf{v} \times (\nabla \times \mathbf{u}) \\ &= (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u}) \\ &= P_c[\mathbf{u}, \mathbf{v}] + P_c[\mathbf{v}, \mathbf{u}] + \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u}) \end{aligned}$$

Note that one can find the scalar differential operators $P_c[\mathbf{u}, \mathbf{v}]$ and $P_c[\mathbf{v}, \mathbf{u}]$ in $\nabla(\mathbf{u} \cdot \mathbf{v})$.

$$P_c[\mathbf{u}, \mathbf{v}] = (\mathbf{u} \cdot \nabla)\mathbf{v}, \quad P_c[\mathbf{v}, \mathbf{u}] = (\mathbf{v} \cdot \nabla)\mathbf{u}.$$

5.3 Detail of Formula-3

We start with

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = \nabla_{\mathbf{u}} \times (\mathbf{u} \times \mathbf{v}) + \nabla_{\mathbf{v}} \times (\mathbf{u} \times \mathbf{v}).$$

Using the formula in vector analysis,

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \\ &= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \end{aligned}$$

we get

$$\begin{aligned} \nabla_{\mathbf{u}} \times (\mathbf{u} \times \mathbf{v}) &= (\nabla_{\mathbf{u}} \cdot \mathbf{v})\mathbf{u} - (\nabla_{\mathbf{u}} \cdot \mathbf{u})\mathbf{v} \\ &= -(\mathbf{u} \cdot \nabla_{\mathbf{u}})\mathbf{v} + (\nabla_{\mathbf{u}} \cdot \mathbf{v})\mathbf{u} \end{aligned}$$

and

$$\begin{aligned}\nabla_{\mathbf{v}} \times (\mathbf{u} \times \mathbf{v}) &= (\nabla_{\mathbf{v}} \cdot \mathbf{v})\mathbf{u} - (\nabla_{\mathbf{v}} \cdot \mathbf{u})\mathbf{v} \\ &= (\mathbf{v} \cdot \nabla_{\mathbf{v}})\mathbf{u} - (\nabla_{\mathbf{v}} \cdot \mathbf{u})\mathbf{v}\end{aligned}$$

where we use

$$(\nabla_{\mathbf{v}} \cdot \mathbf{v})\mathbf{u} = (\mathbf{v} \cdot \nabla_{\mathbf{v}})\mathbf{u} = (\mathbf{v} \cdot \nabla)\mathbf{u} = P_c[\mathbf{v}, \mathbf{u}],$$

$$(\nabla_{\mathbf{u}} \cdot \mathbf{u})\mathbf{v} = (\mathbf{u} \cdot \nabla_{\mathbf{u}})\mathbf{v} = (\mathbf{u} \cdot \nabla)\mathbf{v} = P_c[\mathbf{u}, \mathbf{v}].$$

Combining with these equations, we have

$$\begin{aligned}\nabla \times (\mathbf{u} \times \mathbf{v}) &= \nabla_{\mathbf{u}} \times (\mathbf{u} \times \mathbf{v}) + \nabla_{\mathbf{v}} \times (\mathbf{u} \times \mathbf{v}) \\ &= (\nabla_{\mathbf{u}} \cdot \mathbf{v})\mathbf{u} - (\nabla_{\mathbf{u}} \cdot \mathbf{u})\mathbf{v} + (\nabla_{\mathbf{v}} \cdot \mathbf{v})\mathbf{u} - (\nabla_{\mathbf{v}} \cdot \mathbf{u})\mathbf{v} \\ &= (\nabla_{\mathbf{u}} \cdot \mathbf{v})\mathbf{u} - (\mathbf{u} \cdot \nabla_{\mathbf{u}})\mathbf{v} + (\mathbf{v} \cdot \nabla_{\mathbf{v}})\mathbf{u} - (\nabla_{\mathbf{v}} \cdot \mathbf{u})\mathbf{v} \\ &= (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + (\nabla \cdot \mathbf{v})\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{v} \\ &= -P_c[\mathbf{u}, \mathbf{v}] + P_c[\mathbf{v}, \mathbf{u}] + (\nabla \cdot \mathbf{v})\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{v}\end{aligned}$$

Note that one can find the scalar differential operators $P_c[\mathbf{u}, \mathbf{v}]$ and $P_c[\mathbf{v}, \mathbf{u}]$ in $\nabla \times (\mathbf{u} \times \mathbf{v})$.

6. Overview on Proof of Formula and Gibbs with the use of Mathematica

Here we prove three formula related to the Feynman (Gibbs) subscript notations using Mathematica. For the sake of simplicity, we use the vector analysis based on the Cartesian coordinates.

We introduce \mathbf{u} and \mathbf{v} in cartesian coordinates as follows

(a) Space-independent vectors:

$$\mathbf{u}_{011} = (u_{01x}, u_{01y}, u_{01z}), \quad \mathbf{v}_{011} = (v_{01x}, v_{01y}, v_{01z})$$

(b) Space-dependent vectors:

$$\mathbf{u}_{11} = (u_{1x}(x, y, z), u_{1y}(x, y, z), u_{1z}(x, y, z))$$

and

$$\mathbf{v}_{11} = (v_{1x}(x, y, z), v_{1y}(x, y, z), v_{1z}(x, y, z))$$

When we calculate, for example,

$$\nabla_{\mathbf{u}} \times (\mathbf{u} \times \mathbf{v}),$$

Instead we calculate $\nabla \times (\mathbf{u}_{011} \times \mathbf{v}_{11})$, since \mathbf{u} is constant vector [independent of $\mathbf{r} = (x, y, z)$]. After calculating this, the space independent vector \mathbf{u}_{011} is replaced by the space dependent vector \mathbf{u}_{11} .

(c) Definition of a scalar differential operator (Cartesian coordinates) in Mathematica

The scalar differential operator $Pc[\mathbf{u}, \mathbf{v}] = (\mathbf{u} \cdot \nabla) \mathbf{v}$ which is defined in this article in the Cartesian coordinates.

$$\begin{aligned} Pc[\mathbf{u}, \mathbf{v}] &= (u_x \nabla_x + u_y \nabla_y + u_z \nabla_z) \mathbf{v} \\ &= u_x \nabla_x \mathbf{v} + u_y \nabla_y \mathbf{v} + u_z \nabla_z \mathbf{v} \\ &= u_x D[\mathbf{v}, x] + u_y D[\mathbf{v}, y] + u_z D[\mathbf{v}, z] \end{aligned}$$

7. Proof using the Mathematica

In this note, we will present Mathematica programs how to calculate $(\mathbf{u} \cdot \nabla) \mathbf{v}$ in three kinds of coordinates, cartesian coordinates, cylindrical coordinates, and spherical coordinates. Note that at this moment we do not know that the useful program for the calculation of $(\mathbf{u} \cdot \nabla) \mathbf{v}$ may be available from any possible program sources from Wolfram. If it is available, this note is not so significant, we think. Anyway, we will show our Mathematica program to evaluate the Feynman (Gibbs) subscript notations in vector products for there coordinate systems.

((**Mathematica**))

We use the Mathematica program, in order to give proof of formula obtained by Gibbs and Feynman.

```

Clear["Global`"];
SetCoordinates[Cartesian[x, y, z]];
ux = {1, 0, 0};
uy = {0, 1, 0};
uz = {0, 0, 1};
r = {x, y, z};
R =  $\sqrt{(r.r)}$ ;
Lap := Laplacian[#, {x, y, z}, "Cartesian"] &;
Gra := Grad[#, {x, y, z}, "Cartesian"] &;
Curla := Curl[#, {x, y, z}, "Cartesian"] &;
Diva := Div[#, {x, y, z}, "Cartesian"] &;
Pc[u_, v_] := (u[[1]]  $\times$  D[v, x] + u[[2]]  $\times$  D[v, y] + u[[3]]  $\times$  D[v, z]);
u011 = {u01x, u01y, u01z};
v011 = {v01x, v01y, v01z};
u11 = {u1x[x, y, z], u1y[x, y, z], u1z[x, y, z]};
v11 = {v1x[x, y, z], v1y[x, y, z], v1z[x, y, z]};
rule1 = {u01x  $\rightarrow$  u1x[x, y, z], u01y  $\rightarrow$  u1y[x, y, z],
  u01z  $\rightarrow$  u1z[x, y, z]};
rule2 = {v01x  $\rightarrow$  v1x[x, y, z], v01y  $\rightarrow$  v1y[x, y, z],
  v01z  $\rightarrow$  v1z[x, y, z]};

```

(1-1) Proof

$$(\mathbf{v} \cdot \nabla_{\mathbf{v}}) \mathbf{u} = (\mathbf{v} \cdot \nabla) \mathbf{u} = P_c[\mathbf{v}, \mathbf{u}]$$

t1 = Pc[v011, u11] // . rule2 // Simplify

$$\{v1z[x, y, z] u1x^{(0,0,1)}[x, y, z] + v1y[x, y, z] u1x^{(0,1,0)}[x, y, z] + v1x[x, y, z] u1x^{(1,0,0)}[x, y, z], v1z[x, y, z] u1y^{(0,0,1)}[x, y, z] + v1y[x, y, z] u1y^{(0,1,0)}[x, y, z] + v1x[x, y, z] u1y^{(1,0,0)}[x, y, z], v1z[x, y, z] u1z^{(0,0,1)}[x, y, z] + v1y[x, y, z] u1z^{(0,1,0)}[x, y, z] + v1x[x, y, z] u1z^{(1,0,0)}[x, y, z]\}$$

t2 = Pc[v11, u11] // Simplify

$$\{v1z[x, y, z] u1x^{(0,0,1)}[x, y, z] + v1y[x, y, z] u1x^{(0,1,0)}[x, y, z] + v1x[x, y, z] u1x^{(1,0,0)}[x, y, z], v1z[x, y, z] u1y^{(0,0,1)}[x, y, z] + v1y[x, y, z] u1y^{(0,1,0)}[x, y, z] + v1x[x, y, z] u1y^{(1,0,0)}[x, y, z], v1z[x, y, z] u1z^{(0,0,1)}[x, y, z] + v1y[x, y, z] u1z^{(0,1,0)}[x, y, z] + v1x[x, y, z] u1z^{(1,0,0)}[x, y, z]\}$$

t12 = t1 - t2 // Simplify

$$\{0, 0, 0\}$$

(1-2) Proof

$$(\mathbf{u} \cdot \nabla_{\mathbf{u}}) \mathbf{v} = (\mathbf{u} \cdot \nabla) \mathbf{v} = P_c[\mathbf{u}, \mathbf{v}]$$

t1 = Pc[u011, v11] // . rule1 // Simplify

$$\{u1z[x, y, z] v1x^{(0,0,1)}[x, y, z] + u1y[x, y, z] v1x^{(0,1,0)}[x, y, z] + u1x[x, y, z] v1x^{(1,0,0)}[x, y, z], u1z[x, y, z] v1y^{(0,0,1)}[x, y, z] + u1y[x, y, z] v1y^{(0,1,0)}[x, y, z] + u1x[x, y, z] v1y^{(1,0,0)}[x, y, z], u1z[x, y, z] v1z^{(0,0,1)}[x, y, z] + u1y[x, y, z] v1z^{(0,1,0)}[x, y, z] + u1x[x, y, z] v1z^{(1,0,0)}[x, y, z]\}$$

t2 = Pc[u11, v11] // Simplify

$$\{u1z[x, y, z] v1x^{(0,0,1)}[x, y, z] + u1y[x, y, z] v1x^{(0,1,0)}[x, y, z] + u1x[x, y, z] v1x^{(1,0,0)}[x, y, z], u1z[x, y, z] v1y^{(0,0,1)}[x, y, z] + u1y[x, y, z] v1y^{(0,1,0)}[x, y, z] + u1x[x, y, z] v1y^{(1,0,0)}[x, y, z], u1z[x, y, z] v1z^{(0,0,1)}[x, y, z] + u1y[x, y, z] v1z^{(0,1,0)}[x, y, z] + u1x[x, y, z] v1z^{(1,0,0)}[x, y, z]\}$$

t12 = t1 - t2 // Simplify

$$\{0, 0, 0\}$$

(ii)

$$\begin{aligned}\nabla \cdot (\mathbf{u} \times \mathbf{v}) &= \nabla_{\mathbf{u}} \cdot (\mathbf{u} \times \mathbf{v}) + \nabla_{\mathbf{v}} \cdot (\mathbf{u} \times \mathbf{v}) \\ &= \mathbf{u} \cdot (\mathbf{v} \times \nabla_{\mathbf{u}}) + \mathbf{v} \cdot (\nabla_{\mathbf{v}} \times \mathbf{u}) \\ &= -\mathbf{u} \cdot (\nabla_{\mathbf{u}} \times \mathbf{v}) + \mathbf{v} \cdot (\nabla_{\mathbf{v}} \times \mathbf{u}) \\ &= -\mathbf{u} \cdot (\nabla \times \mathbf{v}) + \mathbf{v} \cdot (\nabla \times \mathbf{u})\end{aligned}$$

(2-1) Proof

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \nabla_{\mathbf{u}} \cdot (\mathbf{u} \times \mathbf{v}) + \nabla_{\mathbf{v}} \cdot (\mathbf{u} \times \mathbf{v})$$

t1 = Diva[Cross[u011, v11]] // . rule1 // Simplify

$$\begin{aligned}u1x[x, y, z] (v1y^{(\theta, \theta, 1)}[x, y, z] - v1z^{(\theta, 1, \theta)}[x, y, z]) + \\ u1z[x, y, z] (v1x^{(\theta, 1, \theta)}[x, y, z] - v1y^{(1, \theta, \theta)}[x, y, z]) + \\ u1y[x, y, z] (-v1x^{(\theta, \theta, 1)}[x, y, z] + v1z^{(1, \theta, \theta)}[x, y, z])\end{aligned}$$

t2 = Diva[Cross[u11, v011]] // . rule2 // Simplify

$$\begin{aligned}v1x[x, y, z] (-u1y^{(\theta, \theta, 1)}[x, y, z] + u1z^{(\theta, 1, \theta)}[x, y, z]) + \\ v1z[x, y, z] (-u1x^{(\theta, 1, \theta)}[x, y, z] + u1y^{(1, \theta, \theta)}[x, y, z]) + \\ v1y[x, y, z] (u1x^{(\theta, \theta, 1)}[x, y, z] - u1z^{(1, \theta, \theta)}[x, y, z])\end{aligned}$$

t3 = Diva[Cross[u11, v11]] // Simplify

$$\begin{aligned}-u1y[x, y, z] v1x^{(\theta, \theta, 1)}[x, y, z] + \\ u1x[x, y, z] v1y^{(\theta, \theta, 1)}[x, y, z] - v1z[x, y, z] u1x^{(\theta, 1, \theta)}[x, y, z] + \\ v1x[x, y, z] (-u1y^{(\theta, \theta, 1)}[x, y, z] + u1z^{(\theta, 1, \theta)}[x, y, z]) + \\ u1z[x, y, z] v1x^{(\theta, 1, \theta)}[x, y, z] - u1x[x, y, z] v1z^{(\theta, 1, \theta)}[x, y, z] + \\ v1z[x, y, z] u1y^{(1, \theta, \theta)}[x, y, z] + v1y[x, y, z] (u1x^{(\theta, \theta, 1)}[x, y, z] - u1z^{(1, \theta, \theta)}[x, y, z]) - \\ u1z[x, y, z] v1y^{(1, \theta, \theta)}[x, y, z] + u1y[x, y, z] v1z^{(1, \theta, \theta)}[x, y, z]\end{aligned}$$

t4 = t3 - (t1 + t2) // Simplify

0

(2-2) Proof

$$\nabla_{\mathbf{u}} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{u} \cdot (\mathbf{v} \times \nabla_{\mathbf{u}}) = -\mathbf{u} \cdot (\nabla_{\mathbf{u}} \times \mathbf{v})$$

t1 = Diva[Cross[u011, v11]] // . rule1 // Simplify

$$u1x[x, y, z] (v1y^{(0,0,1)}[x, y, z] - v1z^{(0,1,0)}[x, y, z]) + \\ u1z[x, y, z] (v1x^{(0,1,0)}[x, y, z] - v1y^{(1,0,0)}[x, y, z]) + \\ u1y[x, y, z] (-v1x^{(0,0,1)}[x, y, z] + v1z^{(1,0,0)}[x, y, z])$$

t2 = -u11.Diva[v11] // Simplify

$$-\{u1x[x, y, z], u1y[x, y, z], u1z[x, y, z]\} \cdot \\ (v1z^{(0,0,1)}[x, y, z] + v1y^{(0,1,0)}[x, y, z] + v1x^{(1,0,0)}[x, y, z])$$

t12 = t1 - t2 // Simplify

0

(2-3) Proof

$$\nabla_v \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla_v \times \mathbf{u}) = \mathbf{v} \cdot (\nabla \times \mathbf{u})$$

t1 = Diva[Cross[u11, v011]] // . rule2 // Simplify

$$v1x[x, y, z] (-u1y^{(0,0,1)}[x, y, z] + u1z^{(0,1,0)}[x, y, z]) + \\ v1z[x, y, z] (-u1x^{(0,1,0)}[x, y, z] + u1y^{(1,0,0)}[x, y, z]) + \\ v1y[x, y, z] (u1x^{(0,0,1)}[x, y, z] - u1z^{(1,0,0)}[x, y, z])$$

t2 = v11.Curla[u11] // Simplify

$$v1x[x, y, z] (-u1y^{(0,0,1)}[x, y, z] + u1z^{(0,1,0)}[x, y, z]) + \\ v1z[x, y, z] (-u1x^{(0,1,0)}[x, y, z] + u1y^{(1,0,0)}[x, y, z]) + \\ v1y[x, y, z] (u1x^{(0,0,1)}[x, y, z] - u1z^{(1,0,0)}[x, y, z])$$

t12 = t1 - t2 // Simplify

0

(2-4) Proof

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = -\mathbf{u} \cdot (\nabla \times \mathbf{v}) + \mathbf{v} \cdot (\nabla \times \mathbf{u})$$

t1 = Diva[Cross[u11, v11]] // Simplify

$$\begin{aligned}
 & -u1y[x, y, z] v1x^{(\theta, \theta, 1)}[x, y, z] + \\
 & u1x[x, y, z] v1y^{(\theta, \theta, 1)}[x, y, z] - v1z[x, y, z] u1x^{(\theta, 1, \theta)}[x, y, z] + \\
 & v1x[x, y, z] (-u1y^{(\theta, \theta, 1)}[x, y, z] + u1z^{(\theta, 1, \theta)}[x, y, z]) + \\
 & u1z[x, y, z] v1x^{(\theta, 1, \theta)}[x, y, z] - \\
 & u1x[x, y, z] v1z^{(\theta, 1, \theta)}[x, y, z] + v1z[x, y, z] u1y^{(1, \theta, \theta)}[x, y, z] + \\
 & v1y[x, y, z] (u1x^{(\theta, \theta, 1)}[x, y, z] - u1z^{(1, \theta, \theta)}[x, y, z]) - \\
 & u1z[x, y, z] v1y^{(1, \theta, \theta)}[x, y, z] + u1y[x, y, z] v1z^{(1, \theta, \theta)}[x, y, z]
 \end{aligned}$$

t2 = -u11.Curla[v11] + v11.Curla[u11] // Simplify

$$\begin{aligned}
 & v1x[x, y, z] (-u1y^{(\theta, \theta, 1)}[x, y, z] + u1z^{(\theta, 1, \theta)}[x, y, z]) + \\
 & u1x[x, y, z] (v1y^{(\theta, \theta, 1)}[x, y, z] - v1z^{(\theta, 1, \theta)}[x, y, z]) + \\
 & v1z[x, y, z] (-u1x^{(\theta, 1, \theta)}[x, y, z] + u1y^{(1, \theta, \theta)}[x, y, z]) + \\
 & v1y[x, y, z] (u1x^{(\theta, \theta, 1)}[x, y, z] - u1z^{(1, \theta, \theta)}[x, y, z]) + \\
 & u1z[x, y, z] (v1x^{(\theta, 1, \theta)}[x, y, z] - v1y^{(1, \theta, \theta)}[x, y, z]) - \\
 & u1y[x, y, z] (v1x^{(\theta, \theta, 1)}[x, y, z] - v1z^{(1, \theta, \theta)}[x, y, z])
 \end{aligned}$$

t12 = t1 - t2 // Simplify

0

(iii)

$$\begin{aligned}
 \nabla(\mathbf{u} \cdot \mathbf{v}) &= \nabla_{\mathbf{u}}(\mathbf{u} \cdot \mathbf{v}) + \nabla_{\mathbf{v}}(\mathbf{u} \cdot \mathbf{v}) \\
 &= \mathbf{u} \times (\nabla_{\mathbf{u}} \times \mathbf{v}) + (\mathbf{u} \cdot \nabla_{\mathbf{u}})\mathbf{v} + \mathbf{v} \times (\nabla_{\mathbf{v}} \times \mathbf{u}) + (\mathbf{v} \cdot \nabla_{\mathbf{v}})\mathbf{u} \\
 &= [(\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{u} \times (\nabla \times \mathbf{v})] + [(\mathbf{v} \cdot \nabla)\mathbf{u} + \mathbf{v} \times (\nabla \times \mathbf{u})] \\
 &= (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u})
 \end{aligned}$$

(3-1) Proof

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = \nabla_{\mathbf{u}}(\mathbf{u} \cdot \mathbf{v}) + \nabla_{\mathbf{v}}(\mathbf{u} \cdot \mathbf{v})$$

t1 = Gra[u11.v11] // Simplify

$$\{v1x[x, y, z] u1x^{(1,0,0)}[x, y, z] + v1y[x, y, z] u1y^{(1,0,0)}[x, y, z] + v1z[x, y, z] u1z^{(1,0,0)}[x, y, z] + u1x[x, y, z] v1x^{(1,0,0)}[x, y, z] + u1y[x, y, z] v1y^{(1,0,0)}[x, y, z] + u1z[x, y, z] v1z^{(1,0,0)}[x, y, z], v1x[x, y, z] u1x^{(0,1,0)}[x, y, z] + v1y[x, y, z] u1y^{(0,1,0)}[x, y, z] + v1z[x, y, z] u1z^{(0,1,0)}[x, y, z] + u1x[x, y, z] v1x^{(0,1,0)}[x, y, z] + u1y[x, y, z] v1y^{(0,1,0)}[x, y, z] + u1z[x, y, z] v1z^{(0,1,0)}[x, y, z], v1x[x, y, z] u1x^{(0,0,1)}[x, y, z] + v1y[x, y, z] u1y^{(0,0,1)}[x, y, z] + v1z[x, y, z] u1z^{(0,0,1)}[x, y, z] + u1x[x, y, z] v1x^{(0,0,1)}[x, y, z] + u1y[x, y, z] v1y^{(0,0,1)}[x, y, z] + u1z[x, y, z] v1z^{(0,0,1)}[x, y, z]\}$$

t2 = Gra[u011.v11] //. rule1 // Simplify

$$\{u1x[x, y, z] v1x^{(1,0,0)}[x, y, z] + u1y[x, y, z] v1y^{(1,0,0)}[x, y, z] + u1z[x, y, z] v1z^{(1,0,0)}[x, y, z], u1x[x, y, z] v1x^{(0,1,0)}[x, y, z] + u1y[x, y, z] v1y^{(0,1,0)}[x, y, z] + u1z[x, y, z] v1z^{(0,1,0)}[x, y, z], u1x[x, y, z] v1x^{(0,0,1)}[x, y, z] + u1y[x, y, z] v1y^{(0,0,1)}[x, y, z] + u1z[x, y, z] v1z^{(0,0,1)}[x, y, z]\}$$

t3 = Gra[u11.v011] //. rule2 // Simplify

$$\{v1x[x, y, z] u1x^{(1,0,0)}[x, y, z] + v1y[x, y, z] u1y^{(1,0,0)}[x, y, z] + v1z[x, y, z] u1z^{(1,0,0)}[x, y, z], v1x[x, y, z] u1x^{(0,1,0)}[x, y, z] + v1y[x, y, z] u1y^{(0,1,0)}[x, y, z] + v1z[x, y, z] u1z^{(0,1,0)}[x, y, z], v1x[x, y, z] u1x^{(0,0,1)}[x, y, z] + v1y[x, y, z] u1y^{(0,0,1)}[x, y, z] + v1z[x, y, z] u1z^{(0,0,1)}[x, y, z]\}$$

t4 = t1 - (t2 + t3) // Simplify

$$\{0, 0, 0\}$$

(3-2) Proof

$$\nabla_u(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \times (\nabla_u \times \mathbf{v}) + (\mathbf{u} \cdot \nabla_u) \mathbf{v}$$

t1 = (Cross[u011, Curla[v11]] + Pc[u011, v11]) // . rule1 // Simplify

$$\{u1x[x, y, z] v1x^{(1,0,0)}[x, y, z] + u1y[x, y, z] v1y^{(1,0,0)}[x, y, z] + u1z[x, y, z] v1z^{(1,0,0)}[x, y, z], u1x[x, y, z] v1x^{(0,1,0)}[x, y, z] + u1y[x, y, z] v1y^{(0,1,0)}[x, y, z] + u1z[x, y, z] v1z^{(0,1,0)}[x, y, z], u1x[x, y, z] v1x^{(0,0,1)}[x, y, z] + u1y[x, y, z] v1y^{(0,0,1)}[x, y, z] + u1z[x, y, z] v1z^{(0,0,1)}[x, y, z]\}$$

t2 = Gra[u011.v11] // . rule1 // Simplify

$$\{u1x[x, y, z] v1x^{(1,0,0)}[x, y, z] + u1y[x, y, z] v1y^{(1,0,0)}[x, y, z] + u1z[x, y, z] v1z^{(1,0,0)}[x, y, z], u1x[x, y, z] v1x^{(0,1,0)}[x, y, z] + u1y[x, y, z] v1y^{(0,1,0)}[x, y, z] + u1z[x, y, z] v1z^{(0,1,0)}[x, y, z], u1x[x, y, z] v1x^{(0,0,1)}[x, y, z] + u1y[x, y, z] v1y^{(0,0,1)}[x, y, z] + u1z[x, y, z] v1z^{(0,0,1)}[x, y, z]\}$$

t3 = t1 - t2 // Simplify

$$\{0, 0, 0\}$$

(3-3) Proof

$$\nabla_v(\mathbf{u} \cdot \mathbf{v}) = \mathbf{v} \times (\nabla_v \times \mathbf{u}) + (\mathbf{v} \cdot \nabla_v) \mathbf{u}$$

t1 = Cross[v011, Curla[u11]] // . rule2 // Simplify

$$\{v1y[x, y, z] (-u1x^{(0,1,0)}[x, y, z] + u1y^{(1,0,0)}[x, y, z]) + v1z[x, y, z] (-u1x^{(0,0,1)}[x, y, z] + u1z^{(1,0,0)}[x, y, z]), v1z[x, y, z] (-u1y^{(0,0,1)}[x, y, z] + u1z^{(0,1,0)}[x, y, z]) + v1x[x, y, z] (u1x^{(0,1,0)}[x, y, z] - u1y^{(1,0,0)}[x, y, z]), v1y[x, y, z] (u1y^{(0,0,1)}[x, y, z] - u1z^{(0,1,0)}[x, y, z]) + v1x[x, y, z] (u1x^{(0,0,1)}[x, y, z] - u1z^{(1,0,0)}[x, y, z])\}$$

t2 = Pc[v011, u11] // . rule2 // Simplify

$$\{v1z[x, y, z] u1x^{(0,0,1)}[x, y, z] + v1y[x, y, z] u1x^{(0,1,0)}[x, y, z] + v1x[x, y, z] u1x^{(1,0,0)}[x, y, z], v1z[x, y, z] u1y^{(0,0,1)}[x, y, z] + v1y[x, y, z] u1y^{(0,1,0)}[x, y, z] + v1x[x, y, z] u1y^{(1,0,0)}[x, y, z], v1z[x, y, z] u1z^{(0,0,1)}[x, y, z] + v1y[x, y, z] u1z^{(0,1,0)}[x, y, z] + v1x[x, y, z] u1z^{(1,0,0)}[x, y, z]\}$$

t3 = Gra[u11.v011] // . rule2 // Simplify

$$\{v1x[x, y, z] u1x^{(1,0,0)}[x, y, z] + v1y[x, y, z] u1y^{(1,0,0)}[x, y, z] + v1z[x, y, z] u1z^{(1,0,0)}[x, y, z], v1x[x, y, z] u1x^{(0,1,0)}[x, y, z] + v1y[x, y, z] u1y^{(0,1,0)}[x, y, z] + v1z[x, y, z] u1z^{(0,1,0)}[x, y, z], v1x[x, y, z] u1x^{(0,0,1)}[x, y, z] + v1y[x, y, z] u1y^{(0,0,1)}[x, y, z] + v1z[x, y, z] u1z^{(0,0,1)}[x, y, z]\}$$

t4 = t3 - (t1 + t2) // Simplify

$$\{0, 0, 0\}$$

(3-4) Proof

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u})$$

t1 = Pc[u11, v11] + Pc[v11, u11] + Cross[u11, Curla[v11]] + Cross[v11, Curla[u11]] // Simplify

$$\{v1x[x, y, z] u1x^{(1,0,0)}[x, y, z] + v1y[x, y, z] u1y^{(1,0,0)}[x, y, z] + v1z[x, y, z] u1z^{(1,0,0)}[x, y, z] + u1x[x, y, z] v1x^{(1,0,0)}[x, y, z] + u1y[x, y, z] v1y^{(1,0,0)}[x, y, z] + u1z[x, y, z] v1z^{(1,0,0)}[x, y, z], v1x[x, y, z] u1x^{(0,1,0)}[x, y, z] + v1y[x, y, z] u1y^{(0,1,0)}[x, y, z] + v1z[x, y, z] u1z^{(0,1,0)}[x, y, z] + u1x[x, y, z] v1x^{(0,1,0)}[x, y, z] + u1y[x, y, z] v1y^{(0,1,0)}[x, y, z] + u1z[x, y, z] v1z^{(0,1,0)}[x, y, z], v1x[x, y, z] u1x^{(0,0,1)}[x, y, z] + v1y[x, y, z] u1y^{(0,0,1)}[x, y, z] + v1z[x, y, z] u1z^{(0,0,1)}[x, y, z] + u1x[x, y, z] v1x^{(0,0,1)}[x, y, z] + u1y[x, y, z] v1y^{(0,0,1)}[x, y, z] + u1z[x, y, z] v1z^{(0,0,1)}[x, y, z]\}$$

t2 = Gra[u11.v11] // Simplify

$$\{v1x[x, y, z] u1x^{(1,0,0)}[x, y, z] + v1y[x, y, z] u1y^{(1,0,0)}[x, y, z] + v1z[x, y, z] u1z^{(1,0,0)}[x, y, z] + u1x[x, y, z] v1x^{(1,0,0)}[x, y, z] + u1y[x, y, z] v1y^{(1,0,0)}[x, y, z] + u1z[x, y, z] v1z^{(1,0,0)}[x, y, z], v1x[x, y, z] u1x^{(0,1,0)}[x, y, z] + v1y[x, y, z] u1y^{(0,1,0)}[x, y, z] + v1z[x, y, z] u1z^{(0,1,0)}[x, y, z] + u1x[x, y, z] v1x^{(0,1,0)}[x, y, z] + u1y[x, y, z] v1y^{(0,1,0)}[x, y, z] + u1z[x, y, z] v1z^{(0,1,0)}[x, y, z], v1x[x, y, z] u1x^{(0,0,1)}[x, y, z] + v1y[x, y, z] u1y^{(0,0,1)}[x, y, z] + v1z[x, y, z] u1z^{(0,0,1)}[x, y, z] + u1x[x, y, z] v1x^{(0,0,1)}[x, y, z] + u1y[x, y, z] v1y^{(0,0,1)}[x, y, z] + u1z[x, y, z] v1z^{(0,0,1)}[x, y, z]\}$$

t3 = t1 - t2 // Simplify

$$\{0, 0, 0\}$$

(iv)

$$\begin{aligned}\nabla \times (\mathbf{u} \times \mathbf{v}) &= \nabla_{\mathbf{u}} \times (\mathbf{u} \times \mathbf{v}) + \nabla_{\mathbf{v}} \times (\mathbf{u} \times \mathbf{v}) \\ &= (\nabla_{\mathbf{u}} \cdot \mathbf{v})\mathbf{u} - (\nabla_{\mathbf{u}} \cdot \mathbf{u})\mathbf{v} + (\nabla_{\mathbf{v}} \cdot \mathbf{v})\mathbf{u} - (\nabla_{\mathbf{v}} \cdot \mathbf{u})\mathbf{v} \\ &= (\mathbf{v} \cdot \nabla_{\mathbf{v}})\mathbf{u} - (\mathbf{u} \cdot \nabla_{\mathbf{u}})\mathbf{v} + (\nabla_{\mathbf{u}} \cdot \mathbf{v})\mathbf{u} - (\nabla_{\mathbf{v}} \cdot \mathbf{u})\mathbf{v} \\ &= (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + (\nabla \cdot \mathbf{v})\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{v}\end{aligned}$$

(4-1) Proof

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = \nabla_{\mathbf{u}} \times (\mathbf{u} \times \mathbf{v}) + \nabla_{\mathbf{v}} \times (\mathbf{u} \times \mathbf{v})$$

`t1 = Curla[Cross[u11, v11]] // Simplify`

$$\begin{aligned}\{ &v1z[x, y, z] u1x^{(\theta, \theta, 1)}[x, y, z] - u1z[x, y, z] v1x^{(\theta, \theta, 1)}[x, y, z] + u1x[x, y, z] v1z^{(\theta, \theta, 1)}[x, y, z] + \\ &v1y[x, y, z] u1x^{(\theta, 1, \theta)}[x, y, z] - v1x[x, y, z] (u1z^{(\theta, \theta, 1)}[x, y, z] + u1y^{(\theta, 1, \theta)}[x, y, z]) - \\ &u1y[x, y, z] v1x^{(\theta, 1, \theta)}[x, y, z] + u1x[x, y, z] v1y^{(\theta, 1, \theta)}[x, y, z], \\ &v1z[x, y, z] u1y^{(\theta, \theta, 1)}[x, y, z] - u1z[x, y, z] v1y^{(\theta, \theta, 1)}[x, y, z] + u1y[x, y, z] v1z^{(\theta, \theta, 1)}[x, y, z] - \\ &v1y[x, y, z] (u1z^{(\theta, \theta, 1)}[x, y, z] + u1x^{(1, \theta, \theta)}[x, y, z]) + v1x[x, y, z] u1y^{(1, \theta, \theta)}[x, y, z] + \\ &u1y[x, y, z] v1x^{(1, \theta, \theta)}[x, y, z] - u1x[x, y, z] v1y^{(1, \theta, \theta)}[x, y, z], \\ &v1y[x, y, z] u1z^{(\theta, 1, \theta)}[x, y, z] + u1z[x, y, z] v1y^{(\theta, 1, \theta)}[x, y, z] - u1y[x, y, z] v1z^{(\theta, 1, \theta)}[x, y, z] - \\ &v1z[x, y, z] (u1y^{(\theta, 1, \theta)}[x, y, z] + u1x^{(1, \theta, \theta)}[x, y, z]) + v1x[x, y, z] u1z^{(1, \theta, \theta)}[x, y, z] + \\ &u1z[x, y, z] v1x^{(1, \theta, \theta)}[x, y, z] - u1x[x, y, z] v1z^{(1, \theta, \theta)}[x, y, z] \}\end{aligned}$$

`t2 = Curla[Cross[u011, v11]] // . rule1 // Simplify`

$$\begin{aligned}\{ &-u1z[x, y, z] v1x^{(\theta, \theta, 1)}[x, y, z] - \\ &u1y[x, y, z] v1x^{(\theta, 1, \theta)}[x, y, z] + u1x[x, y, z] (v1z^{(\theta, \theta, 1)}[x, y, z] + v1y^{(\theta, 1, \theta)}[x, y, z]), \\ &-u1z[x, y, z] v1y^{(\theta, \theta, 1)}[x, y, z] + u1y[x, y, z] (v1z^{(\theta, \theta, 1)}[x, y, z] + v1x^{(1, \theta, \theta)}[x, y, z]) - \\ &u1x[x, y, z] v1y^{(1, \theta, \theta)}[x, y, z], -u1y[x, y, z] v1z^{(\theta, 1, \theta)}[x, y, z] + \\ &u1z[x, y, z] (v1y^{(\theta, 1, \theta)}[x, y, z] + v1x^{(1, \theta, \theta)}[x, y, z]) - u1x[x, y, z] v1z^{(1, \theta, \theta)}[x, y, z] \}\end{aligned}$$

`t3 = Curla[Cross[u11, v011]] // . rule2 // Simplify`

$$\begin{aligned}\{ &v1z[x, y, z] u1x^{(\theta, \theta, 1)}[x, y, z] + \\ &v1y[x, y, z] u1x^{(\theta, 1, \theta)}[x, y, z] - v1x[x, y, z] (u1z^{(\theta, \theta, 1)}[x, y, z] + u1y^{(\theta, 1, \theta)}[x, y, z]), \\ &v1z[x, y, z] u1y^{(\theta, \theta, 1)}[x, y, z] - v1y[x, y, z] (u1z^{(\theta, \theta, 1)}[x, y, z] + u1x^{(1, \theta, \theta)}[x, y, z]) + \\ &v1x[x, y, z] u1y^{(1, \theta, \theta)}[x, y, z], v1y[x, y, z] u1z^{(\theta, 1, \theta)}[x, y, z] - \\ &v1z[x, y, z] (u1y^{(\theta, 1, \theta)}[x, y, z] + u1x^{(1, \theta, \theta)}[x, y, z]) + v1x[x, y, z] u1z^{(1, \theta, \theta)}[x, y, z] \}\end{aligned}$$

`t4 = t1 - (t2 + t3) // Simplify`

$$\{0, 0, 0\}$$

(4-2) Proof

$$\begin{aligned}\nabla_u \times (\mathbf{u} \times \mathbf{v}) &= (\nabla_u \cdot \mathbf{v})\mathbf{u} - (\nabla_u \cdot \mathbf{u})\mathbf{v} \\ &\rightarrow \\ &= (\nabla_u \cdot \mathbf{v})\mathbf{u} - (\mathbf{u} \cdot \nabla_u)\mathbf{v} \\ &= (\nabla \cdot \mathbf{v})\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v}\end{aligned}$$

`t1 = Curla[Cross[u011, v11]] // . rule1 // Simplify`

$$\begin{aligned}&\{-u1z[x, y, z] v1x^{(0,0,1)}[x, y, z] - \\ &\quad u1y[x, y, z] v1x^{(0,1,0)}[x, y, z] + u1x[x, y, z] (v1z^{(0,0,1)}[x, y, z] + v1y^{(0,1,0)}[x, y, z]), \\ &-u1z[x, y, z] v1y^{(0,0,1)}[x, y, z] + u1y[x, y, z] (v1z^{(0,0,1)}[x, y, z] + v1x^{(1,0,0)}[x, y, z]) - \\ &\quad u1x[x, y, z] v1y^{(1,0,0)}[x, y, z], -u1y[x, y, z] v1z^{(0,1,0)}[x, y, z] + \\ &\quad u1z[x, y, z] (v1y^{(0,1,0)}[x, y, z] + v1x^{(1,0,0)}[x, y, z]) - u1x[x, y, z] v1z^{(1,0,0)}[x, y, z]\}\end{aligned}$$

`t2 = (Diva[v11] u011 - Pc[u011, v11]) // . rule1 // Simplify`

$$\begin{aligned}&\{-u1z[x, y, z] v1x^{(0,0,1)}[x, y, z] - \\ &\quad u1y[x, y, z] v1x^{(0,1,0)}[x, y, z] + u1x[x, y, z] (v1z^{(0,0,1)}[x, y, z] + v1y^{(0,1,0)}[x, y, z]), \\ &-u1z[x, y, z] v1y^{(0,0,1)}[x, y, z] + u1y[x, y, z] (v1z^{(0,0,1)}[x, y, z] + v1x^{(1,0,0)}[x, y, z]) - \\ &\quad u1x[x, y, z] v1y^{(1,0,0)}[x, y, z], -u1y[x, y, z] v1z^{(0,1,0)}[x, y, z] + \\ &\quad u1z[x, y, z] (v1y^{(0,1,0)}[x, y, z] + v1x^{(1,0,0)}[x, y, z]) - u1x[x, y, z] v1z^{(1,0,0)}[x, y, z]\}\end{aligned}$$

`t3 = -Pc[u11, v11] + (Diva[v11]) u11 // Simplify`

$$\begin{aligned}&\{-u1z[x, y, z] v1x^{(0,0,1)}[x, y, z] - \\ &\quad u1y[x, y, z] v1x^{(0,1,0)}[x, y, z] + u1x[x, y, z] (v1z^{(0,0,1)}[x, y, z] + v1y^{(0,1,0)}[x, y, z]), \\ &-u1z[x, y, z] v1y^{(0,0,1)}[x, y, z] + u1y[x, y, z] (v1z^{(0,0,1)}[x, y, z] + v1x^{(1,0,0)}[x, y, z]) - \\ &\quad u1x[x, y, z] v1y^{(1,0,0)}[x, y, z], -u1y[x, y, z] v1z^{(0,1,0)}[x, y, z] + \\ &\quad u1z[x, y, z] (v1y^{(0,1,0)}[x, y, z] + v1x^{(1,0,0)}[x, y, z]) - u1x[x, y, z] v1z^{(1,0,0)}[x, y, z]\}\end{aligned}$$

`t1 - t2 // Simplify`

$$\{0, 0, 0\}$$

`t1 - t3 // Simplify`

$$\{0, 0, 0\}$$

(4-3) Proof

$$\begin{aligned}
\nabla_v \times (\mathbf{u} \times \mathbf{v}) &= (\nabla_v \cdot \mathbf{v})\mathbf{u} - (\nabla_v \cdot \mathbf{u})\mathbf{v} \\
&\rightarrow \\
&= (\mathbf{v} \cdot \nabla_v)\mathbf{u} - (\nabla_v \cdot \mathbf{u})\mathbf{v} \\
&= (\mathbf{v} \cdot \nabla)\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{v}
\end{aligned}$$

t1 = Curla[Cross[u11, v011]] // . rule2

$$\begin{aligned}
&\{v1z[x, y, z] u1x^{(\theta, \theta, 1)}[x, y, z] - v1x[x, y, z] u1z^{(\theta, \theta, 1)}[x, y, z] + \\
&\quad v1y[x, y, z] u1x^{(\theta, 1, \theta)}[x, y, z] - v1x[x, y, z] u1y^{(\theta, 1, \theta)}[x, y, z], \\
&v1z[x, y, z] u1y^{(\theta, \theta, 1)}[x, y, z] - v1y[x, y, z] u1z^{(\theta, \theta, 1)}[x, y, z] - \\
&\quad v1y[x, y, z] u1x^{(1, \theta, \theta)}[x, y, z] + v1x[x, y, z] u1y^{(1, \theta, \theta)}[x, y, z], \\
&-v1z[x, y, z] u1y^{(\theta, 1, \theta)}[x, y, z] + v1y[x, y, z] u1z^{(\theta, 1, \theta)}[x, y, z] - \\
&\quad v1z[x, y, z] u1x^{(1, \theta, \theta)}[x, y, z] + v1x[x, y, z] u1z^{(1, \theta, \theta)}[x, y, z]\}
\end{aligned}$$

t2 = (Pc[v011, u11] - Diva[u11] v011) // . rule2 // Simplify

$$\begin{aligned}
&\{v1z[x, y, z] u1x^{(\theta, \theta, 1)}[x, y, z] + v1y[x, y, z] u1x^{(\theta, 1, \theta)}[x, y, z] - \\
&\quad v1x[x, y, z] (u1z^{(\theta, \theta, 1)}[x, y, z] + u1y^{(\theta, 1, \theta)}[x, y, z]), \\
&v1z[x, y, z] u1y^{(\theta, \theta, 1)}[x, y, z] - v1y[x, y, z] (u1z^{(\theta, \theta, 1)}[x, y, z] + u1x^{(1, \theta, \theta)}[x, y, z]) + \\
&\quad v1x[x, y, z] u1y^{(1, \theta, \theta)}[x, y, z], v1y[x, y, z] u1z^{(\theta, 1, \theta)}[x, y, z] - \\
&\quad v1z[x, y, z] (u1y^{(\theta, 1, \theta)}[x, y, z] + u1x^{(1, \theta, \theta)}[x, y, z]) + v1x[x, y, z] u1z^{(1, \theta, \theta)}[x, y, z]\}
\end{aligned}$$

t3 = Pc[v11, u11] - Diva[u11] v11 // Simplify

$$\begin{aligned}
&\{v1z[x, y, z] u1x^{(\theta, \theta, 1)}[x, y, z] + v1y[x, y, z] u1x^{(\theta, 1, \theta)}[x, y, z] - \\
&\quad v1x[x, y, z] (u1z^{(\theta, \theta, 1)}[x, y, z] + u1y^{(\theta, 1, \theta)}[x, y, z]), \\
&v1z[x, y, z] u1y^{(\theta, \theta, 1)}[x, y, z] - v1y[x, y, z] (u1z^{(\theta, \theta, 1)}[x, y, z] + u1x^{(1, \theta, \theta)}[x, y, z]) + \\
&\quad v1x[x, y, z] u1y^{(1, \theta, \theta)}[x, y, z], v1y[x, y, z] u1z^{(\theta, 1, \theta)}[x, y, z] - \\
&\quad v1z[x, y, z] (u1y^{(\theta, 1, \theta)}[x, y, z] + u1x^{(1, \theta, \theta)}[x, y, z]) + v1x[x, y, z] u1z^{(1, \theta, \theta)}[x, y, z]\}
\end{aligned}$$

t1 - t2 // Simplify

$$\{0, 0, 0\}$$

t1 - t3 // Simplify

$$\{0, 0, 0\}$$

(4-4) Proof

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + (\nabla \cdot \mathbf{v})\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{v}$$

`t1 = Curla[Cross[u11, v11]] // Simplify`

$$\begin{aligned} & \{v1z[x, y, z] u1x^{(\theta, \theta, 1)}[x, y, z] - u1z[x, y, z] v1x^{(\theta, \theta, 1)}[x, y, z] + \\ & \quad u1x[x, y, z] v1z^{(\theta, \theta, 1)}[x, y, z] + v1y[x, y, z] u1x^{(\theta, 1, \theta)}[x, y, z] - \\ & \quad v1x[x, y, z] (u1z^{(\theta, \theta, 1)}[x, y, z] + u1y^{(\theta, 1, \theta)}[x, y, z]) - \\ & \quad u1y[x, y, z] v1x^{(\theta, 1, \theta)}[x, y, z] + u1x[x, y, z] v1y^{(\theta, 1, \theta)}[x, y, z], \\ & \quad v1z[x, y, z] u1y^{(\theta, \theta, 1)}[x, y, z] - u1z[x, y, z] v1y^{(\theta, \theta, 1)}[x, y, z] + \\ & \quad u1y[x, y, z] v1z^{(\theta, \theta, 1)}[x, y, z] - \\ & \quad v1y[x, y, z] (u1z^{(\theta, \theta, 1)}[x, y, z] + u1x^{(1, \theta, \theta)}[x, y, z]) + \\ & \quad v1x[x, y, z] u1y^{(1, \theta, \theta)}[x, y, z] + u1y[x, y, z] v1x^{(1, \theta, \theta)}[x, y, z] - \\ & \quad u1x[x, y, z] v1y^{(1, \theta, \theta)}[x, y, z], v1y[x, y, z] u1z^{(\theta, 1, \theta)}[x, y, z] + \\ & \quad u1z[x, y, z] v1y^{(\theta, 1, \theta)}[x, y, z] - u1y[x, y, z] v1z^{(\theta, 1, \theta)}[x, y, z] - \\ & \quad v1z[x, y, z] (u1y^{(\theta, 1, \theta)}[x, y, z] + u1x^{(1, \theta, \theta)}[x, y, z]) + \\ & \quad v1x[x, y, z] u1z^{(1, \theta, \theta)}[x, y, z] + \\ & \quad u1z[x, y, z] v1x^{(1, \theta, \theta)}[x, y, z] - u1x[x, y, z] v1z^{(1, \theta, \theta)}[x, y, z] \} \end{aligned}$$

`t2 = Pc[v11, u11] - Pc[u11, v11] + (Diva[v11] u11) - (Diva[u11] v11) // Simplify`

$$\begin{aligned} & \{v1z[x, y, z] u1x^{(\theta, \theta, 1)}[x, y, z] - u1z[x, y, z] v1x^{(\theta, \theta, 1)}[x, y, z] + \\ & \quad u1x[x, y, z] v1z^{(\theta, \theta, 1)}[x, y, z] + v1y[x, y, z] u1x^{(\theta, 1, \theta)}[x, y, z] - \\ & \quad v1x[x, y, z] (u1z^{(\theta, \theta, 1)}[x, y, z] + u1y^{(\theta, 1, \theta)}[x, y, z]) - \\ & \quad u1y[x, y, z] v1x^{(\theta, 1, \theta)}[x, y, z] + u1x[x, y, z] v1y^{(\theta, 1, \theta)}[x, y, z], \\ & \quad v1z[x, y, z] u1y^{(\theta, \theta, 1)}[x, y, z] - u1z[x, y, z] v1y^{(\theta, \theta, 1)}[x, y, z] + \\ & \quad u1y[x, y, z] v1z^{(\theta, \theta, 1)}[x, y, z] - \\ & \quad v1y[x, y, z] (u1z^{(\theta, \theta, 1)}[x, y, z] + u1x^{(1, \theta, \theta)}[x, y, z]) + \\ & \quad v1x[x, y, z] u1y^{(1, \theta, \theta)}[x, y, z] + u1y[x, y, z] v1x^{(1, \theta, \theta)}[x, y, z] - \\ & \quad u1x[x, y, z] v1y^{(1, \theta, \theta)}[x, y, z], v1y[x, y, z] u1z^{(\theta, 1, \theta)}[x, y, z] + \\ & \quad u1z[x, y, z] v1y^{(\theta, 1, \theta)}[x, y, z] - u1y[x, y, z] v1z^{(\theta, 1, \theta)}[x, y, z] - \\ & \quad v1z[x, y, z] (u1y^{(\theta, 1, \theta)}[x, y, z] + u1x^{(1, \theta, \theta)}[x, y, z]) + \\ & \quad v1x[x, y, z] u1z^{(1, \theta, \theta)}[x, y, z] + \\ & \quad u1z[x, y, z] v1x^{(1, \theta, \theta)}[x, y, z] - u1x[x, y, z] v1z^{(1, \theta, \theta)}[x, y, z] \} \end{aligned}$$

`t1 - t2 // Simplify`

$$\{0, 0, 0\}$$

In summary, we can make proof of the formula derived by Gibbs and Feynman.

8. Application: Two examples: Force on electric dipole moment in homogeneous electric field and the magnetic moment in homogeneous magnetic field

In maser; a electric dipole moment (NH_3) passing through an inhomogeneous magnetic field, undergoes a force given by $F_E = \nabla(\mathbf{p} \cdot \mathbf{E})$. In Stern-Gerlach experiment, magnetic spin (magnetic moment) passing through an inhomogeneous magnetic field, undergoes a force given by

$$F_B = \nabla(\mathbf{m} \cdot \mathbf{B}).$$

In quantum physics, the **Stern–Gerlach experiment** demonstrated that the spatial orientation of angular momentum is quantized. Thus an atomic-scale system was shown to have intrinsically quantum properties. In the original experiment, silver atoms were sent through a spatially-varying magnetic field, which deflected them before they struck a detector screen, such as a glass slide. Particles with non-zero magnetic moment were deflected, owing to the magnetic field gradient, from a straight path. The screen revealed discrete points of accumulation, rather than a continuous distribution, owing to their quantized spin. Historically, this experiment was decisive in convincing physicists of the reality of angular-momentum quantization in all atomic-scale systems.

The experiment is based on the interaction of particles with magnetic fields. When a charged particle with a magnetic moment passes through a non-uniform magnetic field, it experiences a force due to the **gradient of the field**. This force, proportional to the magnetic moment and the field gradient, causes the particle's trajectory to deviate.

9. Application: Purcell's discussion on the force on magnetic moment

We consider a magnetic moment \mathbf{m} in the presence of a magnetic field \mathbf{B} . The potential energy of this system is given by

$$U_{mag} = -\mathbf{m} \cdot \mathbf{B}.$$

The force exerted on the magnetic moment is

$$\mathbf{F}_{mag} = -\nabla U_{mag} = \nabla(\mathbf{m} \cdot \mathbf{B}).$$

We note that the force \mathbf{F}_{mag} can be rewritten as

$$\begin{aligned} \mathbf{F}_{mag} &= \nabla(\mathbf{m} \cdot \mathbf{B}) \\ &= \nabla_{\mathbf{m}}(\mathbf{m} \cdot \mathbf{B}) + \nabla_{\mathbf{B}}(\mathbf{m} \cdot \mathbf{B}) \end{aligned} \quad (1)$$

When \mathbf{m} is constant and \mathbf{B} depends on the position vector \mathbf{r} , we get

$$\begin{aligned} \mathbf{F}_{mag} &= \nabla_{\mathbf{m}}(\mathbf{m} \cdot \mathbf{B}) \\ &= (\mathbf{m} \cdot \nabla)\mathbf{B} + \mathbf{m} \times (\nabla \times \mathbf{B}) \end{aligned} \quad (2)$$

Here we note that $\nabla \times \mathbf{B}$ is given by the Maxwell's equation as

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right). \quad (\text{Ampere-Maxwell law})$$

If we deal only with static setups where $\frac{\partial \mathbf{E}}{\partial t} = 0$ and $\mathbf{J} = 0$, the force \mathbf{F}_{mag} can be obtained as

$$\begin{aligned} \mathbf{F}_{\text{mag}} &= (\mathbf{m} \cdot \nabla) \mathbf{B} \\ &= P_c[\mathbf{m}, \mathbf{B}] \end{aligned}$$

where $(\mathbf{m} \cdot \nabla) \mathbf{B}$ is a scalar differential operator.

10. Purcell's discussion on the force on electric dipole moment in electric field \mathbf{E}

How about the force on the electric dipole moment in the presence of electric field \mathbf{E} ? We consider an electric dipole moment \mathbf{p} in the presence of a magnetic field \mathbf{E} . The potential energy of this system is given by

$$U_{\text{elec}} = -\mathbf{p} \cdot \mathbf{E}.$$

The force exerted on the electric dipole moment is

$$\mathbf{F}_{\text{elec}} = -\nabla U_{\text{elec}} = \nabla(\mathbf{p} \cdot \mathbf{E}).$$

We note that the force \mathbf{F}_{elec} can be rewritten as

$$\begin{aligned} \mathbf{F}_{\text{elec}} &= \nabla(\mathbf{p} \cdot \mathbf{E}) \\ &= \nabla_{\mathbf{p}}(\mathbf{p} \cdot \mathbf{E}) + \nabla_{\mathbf{E}}(\mathbf{p} \cdot \mathbf{E}) \end{aligned}$$

When \mathbf{p} is constant and \mathbf{E} depends on the position vector \mathbf{r} , we have

$$\begin{aligned} \mathbf{F}_{\text{elec}} &= \nabla_{\mathbf{p}}(\mathbf{p} \cdot \mathbf{E}) \\ &= (\mathbf{p} \cdot \nabla) \mathbf{E} + \mathbf{p} \times (\nabla \times \mathbf{E}) \end{aligned} \quad (2)$$

Here we note that $\nabla \times \mathbf{E}$ is given by the Maxwell's equation as

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (\text{Faraday law})$$

If we deal only with static setups where $\frac{\partial \mathbf{B}}{\partial t} = 0$, the force can be obtained as

$$\begin{aligned} \mathbf{F}_{elec} &= (\mathbf{p} \cdot \nabla) \mathbf{E} \\ &= P_c[\mathbf{p}, \mathbf{E}] \end{aligned}$$

According to this equation, an electric dipole moment aligned with the field is pulled in the direction of increasing field strength, whereas an electric dipole moment aligned against the field is pushed in the opposite direction. (H.C. Ohanian).

11. ((Feynman)): energy conservation law with Poynting vector

Here we follow the discussion of Chapter 27 vol II (Feynman's Lectures on Physics) on the energy conservation law. We start with

$$\begin{aligned} \mathbf{J} \cdot \mathbf{E} &= \mathbf{E} \cdot \left(\frac{1}{\mu_0} \nabla \times \mathbf{B} - \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \\ &= \frac{1}{\mu_0} \mathbf{E} \cdot (\nabla \times \mathbf{B}) - \varepsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

where \mathbf{E} is the electric field, \mathbf{B} is the magnetic field, \mathbf{J} is the current density. We note that

$$\begin{aligned} \nabla \cdot (\mathbf{E} \times \mathbf{B}) &= \nabla_{\mathbf{E}} \cdot (\mathbf{E} \times \mathbf{B}) + \nabla_{\mathbf{B}} \cdot (\mathbf{E} \times \mathbf{B}) \\ &= \mathbf{B} \cdot (\nabla_{\mathbf{E}} \times \mathbf{E}) + \mathbf{E} \cdot (\mathbf{B} \times \nabla_{\mathbf{B}}) \\ &= \mathbf{B} \cdot (\nabla_{\mathbf{E}} \times \mathbf{E}) - \mathbf{E} \cdot (\nabla_{\mathbf{B}} \times \mathbf{B}) \\ &= \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B}) \end{aligned}$$

So that,

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{B}).$$

Thus, we have

$$\begin{aligned} \mu_0 (\mathbf{J} \cdot \mathbf{E}) &= \mathbf{E} \cdot (\nabla \times \mathbf{B}) - \varepsilon_0 \mu_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \\ &= \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{B}) - \varepsilon_0 \mu_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \\ &= -\nabla \cdot (\mathbf{E} \times \mathbf{B}) - \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \varepsilon_0 \mu_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

The Poynting vector and energy density are defined by

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}, \quad (\text{Poynting vector})$$

$$u = \frac{1}{2} \epsilon_0 \mathbf{E}^2 + \frac{1}{2\mu_0} \mathbf{B}^2, \quad (\text{energy density})$$

The energy conservation law:

$$\begin{aligned} \mathbf{J} \cdot \mathbf{E} &= -\frac{1}{\mu_0} \nabla \cdot (\mathbf{E} \times \mathbf{B}) - \frac{1}{\mu_0} \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \\ &= -\nabla \cdot \mathbf{S} - \frac{\partial u}{\partial t} \end{aligned}$$

or

$$\mathbf{J} \cdot \mathbf{E} + \nabla \cdot \mathbf{S} + \frac{\partial u}{\partial t} = 0.$$

12. Maxwell stress tensor (Griffiths)

The **Maxwell stress tensor** (named after James Clerk Maxwell) is a symmetric second-order tensor used in classical electromagnetism to represent the interaction between electromagnetic forces and mechanical momentum. In simple situations, such as a point charge moving freely in a homogeneous magnetic field, it is easy to calculate the forces on the charge from the Lorentz force law. When the situation becomes more complicated, this ordinary procedure can become impractically difficult, with equations spanning multiple lines. It is therefore convenient to collect many of these terms in the Maxwell stress tensor, and to use tensor arithmetic to find the answer to the problem at hand. In the relativistic formulation of electromagnetism, the Maxwell's tensor appears as a part of the electromagnetic stress–energy tensor which is the electromagnetic component of the total stress–energy tensor. The latter describes the density and flux of energy and momentum in spacetime. (**Wikipedia**)

https://en.wikipedia.org/wiki/Maxwell_stress_tensor

((Griffiths))

Force density:

$$\mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}.$$

Using the Maxwell's equation

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right),$$

or

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E}, \quad \mathbf{J} = \frac{1}{\mu_0} \nabla \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

we have

$$\mathbf{f} = \epsilon_0 (\nabla \cdot \mathbf{E}) \mathbf{E} + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}.$$

Here we introduce the Poynting vector \mathbf{S} as

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}.$$

The time derivative of $\mathbf{E} \times \mathbf{B}$ is

$$\begin{aligned} \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) &= \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} + \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} \\ &= \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} - \mathbf{E} \times (\nabla \times \mathbf{E}) \end{aligned}$$

where we use the Faraday's law, $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$. So, we can rewrite \mathbf{f} as

$$\begin{aligned} \mathbf{f} &= \epsilon_0 [(\nabla \cdot \mathbf{E}) \mathbf{E} - \mathbf{E} \times (\nabla \times \mathbf{E})] - \epsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) \\ &\quad + \frac{1}{\mu_0} [(\nabla \cdot \mathbf{B}) \mathbf{B} - \mathbf{B} \times (\nabla \times \mathbf{B})] - \frac{1}{\mu_0} (\nabla \cdot \mathbf{B}) \mathbf{B} \\ &= \epsilon_0 [(\nabla \cdot \mathbf{E}) \mathbf{E} - \mathbf{E} \times (\nabla \times \mathbf{E})] + \frac{1}{\mu_0} [(\nabla \cdot \mathbf{B}) \mathbf{B} - \mathbf{B} \times (\nabla \times \mathbf{B})] - \epsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) \end{aligned}$$

where we use

$$\nabla \cdot \mathbf{B} = 0, \quad \frac{1}{\mu_0} (\nabla \cdot \mathbf{B}) \mathbf{B} = 0$$

and

$$\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} = \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times (\nabla \times \mathbf{E})$$

Now we use the formula

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{u} \cdot \nabla) \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u})$$

When $\mathbf{u} = \mathbf{v} = \mathbf{E}$,

$$\frac{1}{2} \nabla \mathbf{E}^2 = (\mathbf{E} \cdot \nabla) \mathbf{E} + \mathbf{E} \times (\nabla \times \mathbf{E})$$

or

$$\mathbf{E} \times (\nabla \times \mathbf{E}) = \frac{1}{2} \nabla \mathbf{E}^2 - (\mathbf{E} \cdot \nabla) \mathbf{E}$$

Similarly, when $\mathbf{u} = \mathbf{v} = \mathbf{B}$,

$$\frac{1}{2} \nabla \mathbf{B}^2 = (\mathbf{B} \cdot \nabla) \mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{B})$$

or

$$\mathbf{B} \times (\nabla \times \mathbf{B}) = \frac{1}{2} \nabla \mathbf{B}^2 - (\mathbf{B} \cdot \nabla) \mathbf{B}$$

Thus, we have

$$\begin{aligned} \mathbf{f} &= -\varepsilon_0 [\mathbf{E} \times (\nabla \times \mathbf{E}) - (\nabla \cdot \mathbf{E}) \mathbf{E}] - \frac{1}{\mu_0} [\mathbf{B} \times (\nabla \times \mathbf{B}) - (\nabla \cdot \mathbf{B}) \mathbf{B}] - \varepsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) \\ &= -\varepsilon_0 \left[\frac{1}{2} \nabla \mathbf{E}^2 - (\mathbf{E} \cdot \nabla) \mathbf{E} - (\nabla \cdot \mathbf{E}) \mathbf{E} \right] - \frac{1}{\mu_0} \left[\frac{1}{2} \nabla \mathbf{B}^2 - (\mathbf{B} \cdot \nabla) \mathbf{B} - (\nabla \cdot \mathbf{B}) \mathbf{B} \right] - \varepsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) \end{aligned}$$

or

$$\begin{aligned}\mathbf{f} &= -\nabla\left(\frac{1}{2}\varepsilon_0\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2\right) + \varepsilon_0[(\mathbf{E}\cdot\nabla)\mathbf{E} + (\nabla\cdot\mathbf{E})\mathbf{E}] + \frac{1}{\mu_0}[(\mathbf{B}\cdot\nabla)\mathbf{B} + (\nabla\cdot\mathbf{B})\mathbf{B}] - \varepsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) \\ &= -\nabla u - \varepsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t} + \varepsilon_0[(\mathbf{E}\cdot\nabla)\mathbf{E} + (\nabla\cdot\mathbf{E})\mathbf{E}] + \frac{1}{\mu_0}[(\mathbf{B}\cdot\nabla)\mathbf{B} + (\nabla\cdot\mathbf{B})\mathbf{B}]\end{aligned}$$

or

$$\mathbf{f} + \varepsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t} = -\nabla u + \varepsilon_0[(\mathbf{E}\cdot\nabla)\mathbf{E} + (\nabla\cdot\mathbf{E})\mathbf{E}] + \frac{1}{\mu_0}[(\mathbf{B}\cdot\nabla)\mathbf{B} + (\nabla\cdot\mathbf{B})\mathbf{B}]$$

Clearly we find two terms related to the Feynman (Gibbs) subscript notation in Maxwell stress tensor. Thus, we have

$$\begin{aligned}&\varepsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t} + \mathbf{f} \\ &= \varepsilon_0[(\mathbf{E}\cdot\nabla)\mathbf{E} + \mathbf{E}(\nabla\cdot\mathbf{E})] + \frac{1}{\mu_0}[(\mathbf{B}\cdot\nabla)\mathbf{B} + \mathbf{B}(\nabla\cdot\mathbf{B})] \\ &\quad - \nabla\left(\frac{\varepsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2\right) \\ &= \mathbf{I}\end{aligned}$$

Note that the i -component of the vector \mathbf{I} is defined by

$$\begin{aligned}
I_i &= \sum_j [\varepsilon_0 (E_j \nabla_j E_i + E_i \nabla_j E_j)] + \frac{1}{\mu_0} (B_j \nabla_j B_i + B_i \nabla_j B_j) \\
&\quad - \frac{1}{2} \nabla_i (\varepsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0} \mathbf{B}^2) \\
&= \sum_j [\varepsilon_0 (E_j \nabla_j E_i + E_i \nabla_j E_j)] + \frac{1}{\mu_0} (B_j \nabla_j B_i + B_i \nabla_j B_j) + \\
&\quad - \frac{1}{2} \nabla_i (\varepsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0} \mathbf{B}^2) \\
&= \sum_j \nabla_j [\varepsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \delta_{ij} (\frac{1}{2} \varepsilon_0 \mathbf{E}^2 + \frac{1}{2\mu_0} \mathbf{B}^2)] \\
&= \sum_j \nabla_j [\varepsilon_0 (E_i E_j - \frac{1}{2} \delta_{ij} \mathbf{E}^2) + \frac{1}{\mu_0} (B_i B_j - \frac{1}{2} \delta_{ij} \mathbf{B}^2)] \\
&= \sum_j \nabla_j T_{ij}
\end{aligned}$$

where $i = 1$ (x), 2 (y), and 3 (z).

$$\vec{T} = \sum_{i,j} T_{ij} \mathbf{e}_i \mathbf{e}_j \quad (\text{Maxwell stress tensor})$$

$$\begin{aligned}
\nabla \cdot \vec{T} &= \sum_{i,j,k} \nabla_k T_{ij} (\mathbf{e}_k \cdot \mathbf{e}_i) \mathbf{e}_j \\
&= \sum_{i,j,k} \nabla_k T_{ij} \delta_{k,i} \mathbf{e}_j \\
&= \sum_{i,j} \nabla_i T_{ij} \mathbf{e}_j
\end{aligned}$$

or

$$(\nabla \cdot \vec{T})_j = \sum_i \nabla_i T_{ij}$$

and

$$\begin{aligned}
 T_{ij} &= \varepsilon_0 (E_i E_j - \frac{1}{2} \delta_{ij} \mathbf{E}^2) + \frac{1}{\mu_0} (B_i B_j - \frac{1}{2} \delta_{ij} \mathbf{B}^2) \\
 &= \varepsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \delta_{ij} (\frac{1}{2} \varepsilon_0 \mathbf{E}^2 + \frac{1}{2\mu_0} \mathbf{B}^2)
 \end{aligned}$$

Using the tensor, we get the final results

$$\varepsilon_0 \mu_0 \frac{\partial \mathbf{S}}{\partial t} + \mathbf{f} = \nabla \cdot \vec{T},$$

or

$$\frac{1}{c^2} \frac{\partial S_i}{\partial t} + f_i = \sum_j \nabla_j T_{ij}.$$

((**Mathematica**))

We prove the above result using Mathematica.

Maxwell Stress Tensor using Mathematica (Cartesian coordinates)

```

Clear["Global`"];
SetCoordinates[Cartesian[x, y, z]];
ux = {1, 0, 0};
uy = {0, 1, 0};
uz = {0, 0, 1};
r = {x, y, z};
R =  $\sqrt{\mathbf{r} \cdot \mathbf{r}}$ ;
Lap := Laplacian[#, {x, y, z}, "Cartesian"] &;
Gra := Grad[#, {x, y, z}, "Cartesian"] &;
Curla := Curl[#, {x, y, z}, "Cartesian"] &;
Diva := Div[#, {x, y, z}, "Cartesian"] &;
Pc[u_, v_] := (u[[1]]  $\times$  D[v, x] + u[[2]]  $\times$  D[v, y] + u[[3]] D[v, z]);
Es = {Ex[x, y, z], Ey[x, y, z], Ez[x, y, z]}; Bs = {Bx[x, y, z], By[x, y, z], Bz[x, y, z]};

```

$$T_{ij} = \varepsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \delta_{ij} \left(\frac{1}{2} \varepsilon_0 \mathbf{E}^2 + \frac{1}{2\mu_0} \mathbf{B}^2 \right)$$

$$\mathbf{F}_s = \varepsilon_0 [(\mathbf{E} \cdot \nabla) \mathbf{E} + \mathbf{E}(\nabla \cdot \mathbf{E})] + \frac{1}{\mu_0} [(\mathbf{B} \cdot \nabla) \mathbf{B} + \mathbf{B}(\nabla \cdot \mathbf{B})] - \nabla \left(\frac{\varepsilon_0}{2} \mathbf{E}^2 + \frac{1}{2\mu_0} \mathbf{B}^2 \right)$$

$$\bar{\mathbf{T}} = \sum_{i,j} T_{ij} \mathbf{e}_i \mathbf{e}_j \quad (\text{Maxwell stress tensor})$$

$$\begin{aligned}
\nabla \cdot \vec{T} &= \sum_{i,j,k} \nabla_k T_{ij} (\mathbf{e}_k \cdot \mathbf{e}_i) \mathbf{e}_j \\
&= \sum_{i,j,k} \nabla_k T_{ij} \delta_{k,i} \mathbf{e}_j \\
&= \sum_{i,j} \nabla_i T_{ij} \mathbf{e}_j \\
&= \sum_j \left[\sum_i \nabla_i T_{ij} \right] \mathbf{e}_j \\
&= \sum_j A_j \mathbf{e}_j
\end{aligned}$$

$$A_j = (\nabla \cdot \vec{T})_j = \sum_i \nabla_i T_{ij}$$

$$\mathbf{Us}[E1_, B1_] := \frac{1}{2} \left(\epsilon_0 E1.E1 + \frac{1}{\mu_0} B1.B1 \right);$$

$$\mathbf{Fs} = -\text{Gra}[\mathbf{Us}[\mathbf{Es}, \mathbf{Bs}]] + \epsilon_0 (\text{Diva}[\mathbf{Es}] \mathbf{Es} + \text{Pc}[\mathbf{Es}, \mathbf{Es}]) + \frac{1}{\mu_0} (\text{Diva}[\mathbf{Bs}] \mathbf{Bs} + \text{Pc}[\mathbf{Bs}, \mathbf{Bs}]) //$$

Simplify;

$$\mathbf{T}[i_, j_, E1_, B1_] := \epsilon_0 E1[[i]] E1[[j]] + \frac{1}{\mu_0} B1[[i]] B1[[j]] - \text{KroneckerDelta}[i, j] (\mathbf{Us}[E1, B1])$$

$$\mathbf{A1} = \text{D}[\mathbf{T}[1, 1, \mathbf{Es}, \mathbf{Bs}], \mathbf{x}] + \text{D}[\mathbf{T}[2, 1, \mathbf{Es}, \mathbf{Bs}], \mathbf{y}] + \text{D}[\mathbf{T}[3, 1, \mathbf{Es}, \mathbf{Bs}], \mathbf{z}] // \text{FullSimplify};$$

$$\mathbf{A2} = \text{D}[\mathbf{T}[1, 2, \mathbf{Es}, \mathbf{Bs}], \mathbf{x}] + \text{D}[\mathbf{T}[2, 2, \mathbf{Es}, \mathbf{Bs}], \mathbf{y}] + \text{D}[\mathbf{T}[3, 2, \mathbf{Es}, \mathbf{Bs}], \mathbf{z}] // \text{FullSimplify};$$

$$\mathbf{A3} = \text{D}[\mathbf{T}[1, 3, \mathbf{Es}, \mathbf{Bs}], \mathbf{x}] + \text{D}[\mathbf{T}[2, 3, \mathbf{Es}, \mathbf{Bs}], \mathbf{y}] + \text{D}[\mathbf{T}[3, 3, \mathbf{Es}, \mathbf{Bs}], \mathbf{z}] // \text{FullSimplify};$$

$\mathbf{Fs}[[1]] // \text{Simplify}$

$$\begin{aligned}
& \frac{1}{\mu\theta} \left(\epsilon\theta \mu\theta \text{Ez}[x, y, z] \text{Ex}^{(\theta, \theta, 1)}[x, y, z] + \right. \\
& \epsilon\theta \mu\theta \text{Ex}[x, y, z] \text{Ez}^{(\theta, \theta, 1)}[x, y, z] + \text{By}[x, y, z] \text{Bx}^{(\theta, 1, \theta)}[x, y, z] + \\
& \epsilon\theta \mu\theta \text{Ey}[x, y, z] \text{Ex}^{(\theta, 1, \theta)}[x, y, z] + \epsilon\theta \mu\theta \text{Ex}[x, y, z] \text{Ey}^{(\theta, 1, \theta)}[x, y, z] + \\
& \text{Bx}[x, y, z] \left(\text{Bz}^{(\theta, \theta, 1)}[x, y, z] + \text{By}^{(\theta, 1, \theta)}[x, y, z] + \text{Bx}^{(1, \theta, \theta)}[x, y, z] \right) - \\
& \text{By}[x, y, z] \text{By}^{(1, \theta, \theta)}[x, y, z] + \text{Bz}[x, y, z] \left(\text{Bx}^{(\theta, \theta, 1)}[x, y, z] - \text{Bz}^{(1, \theta, \theta)}[x, y, z] \right) + \\
& \epsilon\theta \mu\theta \text{Ex}[x, y, z] \text{Ex}^{(1, \theta, \theta)}[x, y, z] - \\
& \left. \epsilon\theta \mu\theta \text{Ey}[x, y, z] \text{Ey}^{(1, \theta, \theta)}[x, y, z] - \epsilon\theta \mu\theta \text{Ez}[x, y, z] \text{Ez}^{(1, \theta, \theta)}[x, y, z] \right)
\end{aligned}$$

A1 // Simplify

$$\begin{aligned}
& \frac{1}{\mu\theta} \left(\text{Bx}[x, y, z] \left(\text{Bz}^{(\theta, \theta, 1)}[x, y, z] + \text{By}^{(\theta, 1, \theta)}[x, y, z] + \text{Bx}^{(1, \theta, \theta)}[x, y, z] \right) + \right. \\
& \text{By}[x, y, z] \left(\text{Bx}^{(\theta, 1, \theta)}[x, y, z] - \text{By}^{(1, \theta, \theta)}[x, y, z] \right) + \\
& \text{Bz}[x, y, z] \left(\text{Bx}^{(\theta, \theta, 1)}[x, y, z] - \text{Bz}^{(1, \theta, \theta)}[x, y, z] \right) + \\
& \epsilon\theta \mu\theta \left(\text{Ex}[x, y, z] \left(\text{Ez}^{(\theta, \theta, 1)}[x, y, z] + \text{Ey}^{(\theta, 1, \theta)}[x, y, z] + \text{Ex}^{(1, \theta, \theta)}[x, y, z] \right) + \right. \\
& \text{Ey}[x, y, z] \left(\text{Ex}^{(\theta, 1, \theta)}[x, y, z] - \text{Ey}^{(1, \theta, \theta)}[x, y, z] \right) + \\
& \left. \left. \text{Ez}[x, y, z] \left(\text{Ex}^{(\theta, \theta, 1)}[x, y, z] - \text{Ez}^{(1, \theta, \theta)}[x, y, z] \right) \right) \right)
\end{aligned}$$

A2 // Simplify

$$\begin{aligned}
& \frac{1}{\mu\theta} \left(\text{Bz}[x, y, z] \left(\text{By}^{(\theta, \theta, 1)}[x, y, z] - \text{Bz}^{(\theta, 1, \theta)}[x, y, z] \right) + \right. \\
& \text{By}[x, y, z] \left(\text{Bz}^{(\theta, \theta, 1)}[x, y, z] + \text{By}^{(\theta, 1, \theta)}[x, y, z] + \text{Bx}^{(1, \theta, \theta)}[x, y, z] \right) + \\
& \text{Bx}[x, y, z] \left(-\text{Bx}^{(\theta, 1, \theta)}[x, y, z] + \text{By}^{(1, \theta, \theta)}[x, y, z] \right) + \\
& \epsilon\theta \mu\theta \left(\text{Ez}[x, y, z] \left(\text{Ey}^{(\theta, \theta, 1)}[x, y, z] - \text{Ez}^{(\theta, 1, \theta)}[x, y, z] \right) + \right. \\
& \text{Ey}[x, y, z] \left(\text{Ez}^{(\theta, \theta, 1)}[x, y, z] + \text{Ey}^{(\theta, 1, \theta)}[x, y, z] + \text{Ex}^{(1, \theta, \theta)}[x, y, z] \right) + \\
& \left. \left. \text{Ex}[x, y, z] \left(-\text{Ex}^{(\theta, 1, \theta)}[x, y, z] + \text{Ey}^{(1, \theta, \theta)}[x, y, z] \right) \right) \right)
\end{aligned}$$

A3 // Simplify

$$\begin{aligned}
& \frac{1}{\mu\theta} \left(\text{By}[x, y, z] \left(-\text{By}^{(\theta, \theta, 1)}[x, y, z] + \text{Bz}^{(\theta, 1, \theta)}[x, y, z] \right) + \right. \\
& \text{Bz}[x, y, z] \left(\text{Bz}^{(\theta, \theta, 1)}[x, y, z] + \text{By}^{(\theta, 1, \theta)}[x, y, z] + \text{Bx}^{(1, \theta, \theta)}[x, y, z] \right) + \\
& \text{Bx}[x, y, z] \left(-\text{Bx}^{(\theta, \theta, 1)}[x, y, z] + \text{Bz}^{(1, \theta, \theta)}[x, y, z] \right) + \\
& \epsilon\theta \mu\theta \left(\text{Ey}[x, y, z] \left(-\text{Ey}^{(\theta, \theta, 1)}[x, y, z] + \text{Ez}^{(\theta, 1, \theta)}[x, y, z] \right) + \right. \\
& \text{Ez}[x, y, z] \left(\text{Ez}^{(\theta, \theta, 1)}[x, y, z] + \text{Ey}^{(\theta, 1, \theta)}[x, y, z] + \text{Ex}^{(1, \theta, \theta)}[x, y, z] \right) + \\
& \left. \left. \text{Ex}[x, y, z] \left(-\text{Ex}^{(\theta, \theta, 1)}[x, y, z] + \text{Ez}^{(1, \theta, \theta)}[x, y, z] \right) \right) \right)
\end{aligned}$$

Fs[[1]] - A1 // Simplify

0

Fs[[2]] - A2 // Simplify

0

Fs[[3]] - A3 // Simplify

0

13. Application: Vector potential due to magnetic moment

Here we discuss the magnetic field due to the magnetic moment. The vector potential due to the magnetic moment \mathbf{m} is

$$\mathbf{A}_{dip} = \frac{\mu_0}{4\pi r^3} \mathbf{m} \times \mathbf{r} = \frac{\mu_0}{4\pi r^2} \mathbf{m} \times \hat{\mathbf{r}}.$$

So that, the magnetic field due to the magnetic moment is

$$\begin{aligned} \mathbf{B}_{dip} &= \nabla \times \mathbf{A}_{dip} \\ &= \frac{\mu_0}{4\pi} \nabla \times \left(\mathbf{m} \times \frac{\hat{\mathbf{r}}}{r^2} \right) \\ &= \frac{\mu_0}{4\pi r^3} [3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] \end{aligned}$$

((Proof))

We use the Feynman subscript notation

$$\begin{aligned} \nabla \times (\mathbf{u} \times \mathbf{v}) &= \nabla_{\mathbf{u}} \times (\mathbf{u} \times \mathbf{v}) + \nabla_{\mathbf{v}} \times (\mathbf{u} \times \mathbf{v}) \\ &= (\nabla_{\mathbf{u}} \cdot \mathbf{v})\mathbf{u} - (\nabla_{\mathbf{u}} \cdot \mathbf{u})\mathbf{v} + (\nabla_{\mathbf{v}} \cdot \mathbf{v})\mathbf{u} - (\nabla_{\mathbf{v}} \cdot \mathbf{u})\mathbf{v} \\ &= (\mathbf{v} \cdot \nabla_{\mathbf{v}})\mathbf{u} - (\mathbf{u} \cdot \nabla_{\mathbf{u}})\mathbf{v} + (\nabla_{\mathbf{u}} \cdot \mathbf{v})\mathbf{u} - (\nabla_{\mathbf{v}} \cdot \mathbf{u})\mathbf{v} \\ &= (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + (\nabla \cdot \mathbf{v})\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{v} \end{aligned}$$

which is exactly the same as the formula of vector analysis,

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{u}(\nabla \cdot \mathbf{v}) - \mathbf{v}(\nabla \cdot \mathbf{u})$$

We now calculate the magnetic field \mathbf{B} with

$$\begin{aligned}
\mathbf{A}_{dip} &= \frac{\mu_0}{4\pi r^3} \mathbf{m} \times \mathbf{r} \\
&= \mathbf{m} \times \frac{\mu_0}{4\pi r^3} \mathbf{r} \\
&= \mathbf{m} \times \mathbf{A}_1
\end{aligned}$$

and

$$\mathbf{A}_1 = \frac{\mu_0}{4\pi r^3} \mathbf{r},$$

$$\begin{aligned}
\nabla \times (\mathbf{m} \times \mathbf{A}_1) &= (\mathbf{A}_1 \cdot \nabla) \mathbf{m} - (\mathbf{m} \cdot \nabla) \mathbf{A}_1 + (\nabla \cdot \mathbf{A}_1) \mathbf{m} - (\nabla \cdot \mathbf{m}) \mathbf{A}_1 \\
&= -(\mathbf{m} \cdot \nabla) \mathbf{A}_1 + (\nabla \cdot \mathbf{A}_1) \mathbf{m} \\
&= -P_c[\mathbf{m}, \mathbf{A}_1] + (\nabla \cdot \mathbf{A}_1) \mathbf{m}
\end{aligned}$$

So that, we have the magnetic field due to the magnetic dipole moment

$$\begin{aligned}
\mathbf{B}_{dipole} &= \nabla \times (\mathbf{m} \times \mathbf{A}_1) \\
&= -P_c[\mathbf{m}, \mathbf{A}_1] + (\nabla \cdot \mathbf{A}_1) \mathbf{m} \\
&= -P_c[\mathbf{m}, \mathbf{A}_1] \\
&= \frac{\mu_0}{4\pi r^5} [3(\mathbf{m} \cdot \mathbf{r}) \mathbf{r} - \mathbf{m} r^2]
\end{aligned}$$

where

$$\nabla \cdot \mathbf{A}_1 = 0,$$

and

$$-P_c[\mathbf{m}, \mathbf{A}_1] = \frac{\mu_0}{4\pi r^5} [3(\mathbf{m} \cdot \mathbf{r}) \mathbf{r} - \mathbf{m} r^2].$$

In summary, the magnetic field due to the magnetic moment is given by

$$\mathbf{B}_{dipole} = -\frac{\mu_0}{4\pi} P_c\left[\mathbf{m}, \frac{\mathbf{r}}{r^3}\right] = -\frac{\mu_0}{4\pi} (\mathbf{m} \cdot \nabla) \frac{\mathbf{r}}{r^3}$$

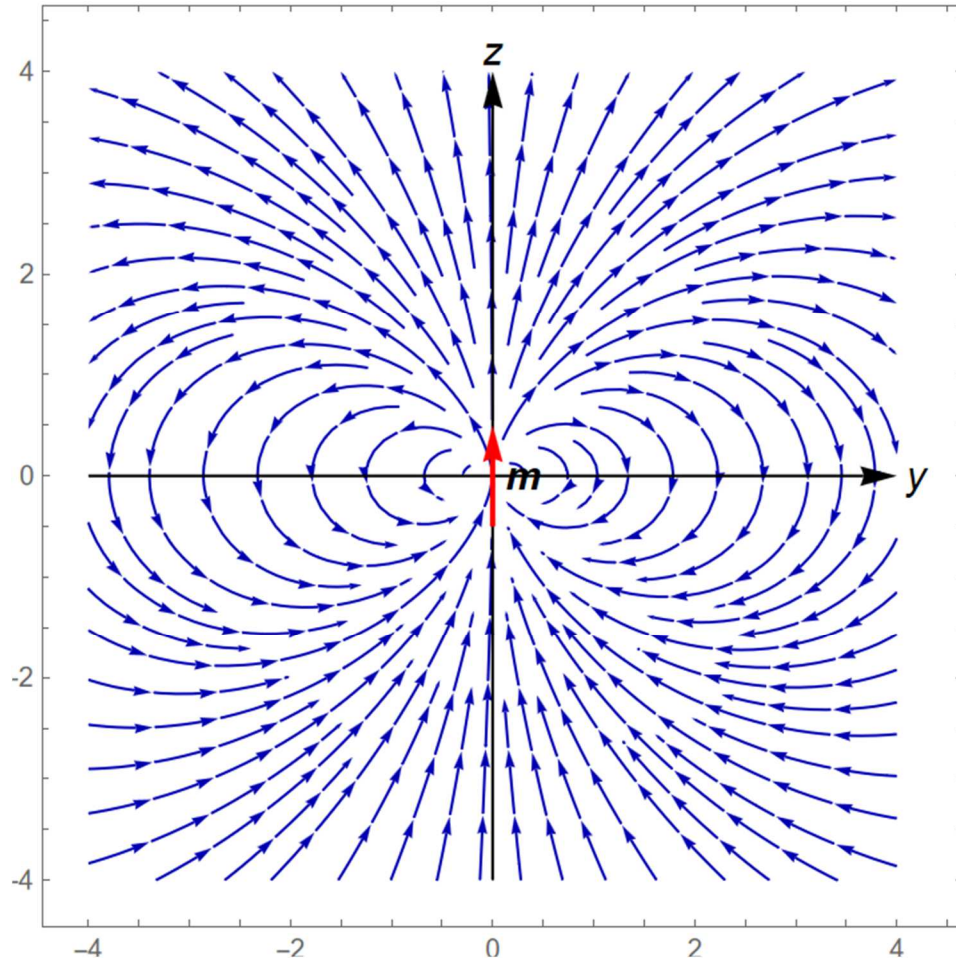


Fig.3 StreamPlot. Magnetic field distribution due to the magnetic moment $\mathbf{m} (= m_0 \hat{\mathbf{z}})$ along the z axis. $\frac{\mu_0 m_0}{(4\pi)} = a$. $a=1$.

How about the interaction between magnetic moment \mathbf{m}_1 and \mathbf{m}_2 , where \mathbf{m}_1 is at the origin and \mathbf{m}_2 are located at \mathbf{r} ? The interaction energy is given by

$$\begin{aligned}
 U &= -\mathbf{m}_2 \cdot \mathbf{B}_2 \\
 &= -\mathbf{m}_2 \cdot \frac{\mu_0}{4\pi r^5} [3(\mathbf{m}_1 \cdot \mathbf{r})\mathbf{r} - \mathbf{m}_1 r^2] \\
 &= \frac{\mu_0}{4\pi r^5} [-3(\mathbf{m}_1 \cdot \mathbf{r})(\mathbf{m}_2 \cdot \mathbf{r}) + (\mathbf{m}_1 \cdot \mathbf{m}_2)r^2] \\
 &= \frac{\mu_0}{4\pi r^3} [\mathbf{m}_1 \cdot \mathbf{m}_2 - 3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}})]
 \end{aligned}$$

where

$$\mathbf{B}_2 = \frac{\mu_0}{4\pi r^5} [3(\mathbf{m}_1 \cdot \mathbf{r})\mathbf{r} - m_1 r^2]$$

(i) Cartesian coordinates

Magnetic field due to magnetic moment (Cartesian coordinates)

$$\mathbf{m1} = \{\mathbf{mx}, \mathbf{my}, \mathbf{mz}\};$$

$$\mathbf{A1} = \left\{ \frac{\mathbf{x}}{\mathbf{R}^3}, \frac{\mathbf{y}}{\mathbf{R}^3}, \frac{\mathbf{z}}{\mathbf{R}^3} \right\};$$

`Diva[A1] // FullSimplify`

\emptyset

`-Pc[m1, A1] // FullSimplify`

$$\left\{ \begin{aligned} & (3x(my y + mz z) + mx(2x^2 - y^2 - z^2)) / (x^2 + y^2 + z^2)^{5/2}, \\ & (3y(mx x + mz z) - my(x^2 - 2y^2 + z^2)) / (x^2 + y^2 + z^2)^{5/2}, \\ & (3(mx x + my y)z - mz(x^2 + y^2 - 2z^2)) / \\ & (x^2 + y^2 + z^2)^{5/2} \end{aligned} \right\}$$

(2). Spherical coordinates

Magnetic field due to magnetic moment (spherical coordinates)

$$\mathbf{m1} = \{m\theta \cos[\theta], -m\theta \sin[\theta], \theta\}; \quad \mathbf{A1} = \frac{\mu\theta}{4\pi r^2} \{1, \theta, \theta\};$$

`-Pc[m1, A1] // Simplify`

$$\left\{ \frac{m\theta \mu\theta \cos[\theta]}{2\pi r^3}, \frac{m\theta \mu\theta \sin[\theta]}{4\pi r^3}, \theta \right\}$$

((Note)) You need to use the Mathematica based on the spherical coordinates (main program shown later)

14. Application: Electric field due to electric dipole moment

The electric potential due to the electric dipole moment p is

$$V = \frac{\mathbf{p} \cdot \mathbf{r}}{4\pi\epsilon_0 r^3}.$$

So that, the electric field is obtained as

$$\mathbf{E}_{dipole} = -\nabla V = -\frac{1}{4\pi\epsilon_0} \nabla \left(\mathbf{p} \cdot \frac{\mathbf{r}}{r^3} \right).$$

Using the formula

$$\begin{aligned} \nabla \left(\mathbf{p} \cdot \frac{\mathbf{r}}{r^3} \right) &= \nabla_{\mathbf{p}} \left(\mathbf{p} \cdot \frac{\mathbf{r}}{r^3} \right) \\ &= P_c \left[\mathbf{p}, \frac{\mathbf{r}}{r^3} \right] + \mathbf{p} \times (\nabla \times \frac{\mathbf{r}}{r^3}) \\ &= P_c \left[\mathbf{p}, \frac{\mathbf{r}}{r^3} \right] \end{aligned}$$

since $\nabla \times \frac{\mathbf{r}}{r^3} = 0$. The electric field \mathbf{E}_{dipole} due to the electric dipole moment \mathbf{p} is

$$\mathbf{E}_{dipole} = -\frac{1}{4\pi\epsilon_0} P_c \left[\mathbf{p}, \frac{\mathbf{r}}{r^3} \right].$$

So that the electric field \mathbf{E}_{dipole} and \mathbf{B}_{dipole} have mathematically the same structure in mathematics.

15. A scalar differential operator $P_c[\mathbf{A}\mathbf{B}]=(\mathbf{A}\cdot\nabla)\mathbf{B}$:

In this section, we make a Mathematica program to evaluate the scalar differential operator

$$(\mathbf{A}\cdot\nabla)\mathbf{B}$$

In terms of three coordinate systems (Cartesian, Cylindrical, and Spherical coordinates)

(a) $(\mathbf{A}\cdot\nabla)\mathbf{B}$: Cartesian coordinates

$$\begin{aligned}
(\mathbf{A} \cdot \nabla) \mathbf{B} &= (A_x \nabla_x + A_y \nabla_y + A_z \nabla_z)(B_x \hat{x} + B_y \hat{y} + B_z \hat{z}) \\
&= (A_x \nabla_x + A_y \nabla_y + A_z \nabla_z)(B_x \hat{x}) + (A_x \nabla_x + A_y \nabla_y + A_z \nabla_z)(B_y \hat{y}) \\
&\quad + (A_x \nabla_x + A_y \nabla_y + A_z \nabla_z)(B_z \hat{z}) \\
&= (A_x \nabla_x)(B_x \hat{x}) + (A_y \nabla_y)(B_x \hat{x}) + (A_z \nabla_z)(B_x \hat{x}) \\
&\quad + (A_x \nabla_x)(B_y \hat{y}) + (A_y \nabla_y)(B_y \hat{y}) + (A_z \nabla_z)(B_y \hat{y}) \\
&\quad + (A_x \nabla_x)(B_z \hat{z}) + (A_y \nabla_y)(B_z \hat{z}) + (A_z \nabla_z)(B_z \hat{z}) \\
&= (A_x \nabla_x B_x) \hat{x} + (A_y \nabla_y B_x) \hat{x} + (A_z \nabla_z B_x) \hat{x} \\
&\quad + (A_x \nabla_x B_y) \hat{y} + (A_y \nabla_y B_y) \hat{y} + (A_z \nabla_z B_y) \hat{y} \\
&\quad + (A_x \nabla_x B_z) \hat{z} + (A_y \nabla_y B_z) \hat{z} + (A_z \nabla_z B_z) \hat{z} \\
&= (A_x \nabla_x B_x + A_y \nabla_y B_x + A_z \nabla_z B_x) \hat{x} \\
&\quad + (A_x \nabla_x B_y + A_y \nabla_y B_y + A_z \nabla_z B_y) \hat{y} \\
&\quad + (A_x \nabla_x B_z + A_y \nabla_y B_z + A_z \nabla_z B_z) \hat{z}
\end{aligned}$$

Here we use the definition of differential operator for the Cartesian coordinates

$$\nabla_x = \frac{\partial}{\partial x}, \quad \nabla_y = \frac{\partial}{\partial y}, \quad \nabla_z = \frac{\partial}{\partial z},$$

We note that

$$\begin{aligned}
\nabla_x \hat{x} &= \nabla_y \hat{x} = \nabla_z \hat{x} = 0, \\
\nabla_x \hat{y} &= \nabla_y \hat{y} = \nabla_z \hat{y} = 0, \\
\nabla_x \hat{z} &= \nabla_y \hat{z} = \nabla_z \hat{z} = 0
\end{aligned}$$

$$\begin{aligned}
(\mathbf{A} \cdot \nabla) \mathbf{B} &= (A_x \frac{\partial}{\partial x} B_x + A_y \frac{\partial}{\partial y} B_x + A_z \frac{\partial}{\partial z} B_x) \hat{x} \\
&\quad + (A_x \frac{\partial}{\partial x} B_y + A_y \frac{\partial}{\partial y} B_y + A_z \frac{\partial}{\partial z} B_y) \hat{y} \\
&\quad + (A_x \frac{\partial}{\partial x} B_z + A_y \frac{\partial}{\partial y} B_z + A_z \frac{\partial}{\partial z} B_z) \hat{z}
\end{aligned}$$

((Mathematica))

For simplicity we define the

$$\mathbf{Pc}[u_ , v_] := (u[[1]] \times \mathbf{D}[v, x] + u[[2]] \times \mathbf{D}[v, y] + u[[3]] \times \mathbf{D}[v, z]);$$

(b) **(A·∇)B: Cylindrical coordinates**

$$\begin{aligned}
(\mathbf{A} \cdot \nabla) \mathbf{B} &= (A_\rho \nabla_\rho + A_\phi \nabla_\phi + A_z \nabla_z)(B_\rho \hat{\rho} + B_\phi \hat{\phi} + B_z \hat{z}) \\
&= (A_\rho \nabla_\rho + A_\phi \nabla_\phi + A_z \nabla_z)(B_\rho \hat{\rho}) + (A_\rho \nabla_\rho + A_\phi \nabla_\phi + A_z \nabla_z)(B_\phi \hat{\phi}) \\
&\quad + (A_\rho \nabla_\rho + A_\phi \nabla_\phi + A_z \nabla_z)(B_z \hat{z}) \\
&= (A_\rho \nabla_\rho)(B_\rho \hat{\rho}) + (A_\phi \nabla_\phi)(B_\rho \hat{\rho}) + (A_z \nabla_z)(B_\rho \hat{\rho}) \\
&\quad + (A_\rho \nabla_\rho)(B_\phi \hat{\phi}) + (A_\phi \nabla_\phi)(B_\phi \hat{\phi}) + (A_z \nabla_z)(B_\phi \hat{\phi}) \\
&\quad + (A_\rho \nabla_\rho)(B_z \hat{z}) + (A_\phi \nabla_\phi)(B_z \hat{z}) + (A_z \nabla_z)(B_z \hat{z}) \\
&= (A_\rho \nabla_\rho B_\rho) \hat{\rho} + (A_\phi \nabla_\phi B_\rho) \hat{\rho} + (A_z \nabla_z B_\rho) \hat{\rho} \\
&\quad + (A_\rho \nabla_\rho B_\phi) \hat{\phi} + (A_\phi \nabla_\phi B_\phi) \hat{\phi} + (A_z \nabla_z B_\phi) \hat{\phi} \\
&\quad + (A_\rho \nabla_\rho B_z) \hat{z} + (A_\phi \nabla_\phi B_z) \hat{z} + (A_z \nabla_z B_z) \hat{z} \\
&\quad + (A_\rho B_\rho) \nabla_\rho \hat{\rho} + (A_\phi B_\rho) \nabla_\phi \hat{\rho} + (A_z B_\rho) \nabla_z \hat{\rho} \\
&\quad + (A_\rho B_\phi) \nabla_\rho \hat{\phi} + (A_\phi B_\phi) \nabla_\phi \hat{\phi} + (A_z B_\phi) \nabla_z \hat{\phi} \\
&\quad + (A_\rho B_z) \nabla_\rho \hat{z} + (A_\phi B_z) \nabla_\phi \hat{z} + (A_z B_z) \nabla_z \hat{z}
\end{aligned}$$

Here we use the definition of differential operator for the Cylindrical coordinates

$$\nabla_\rho = \frac{\partial}{\partial \rho}, \quad \nabla_\phi = \frac{1}{\rho} \frac{\partial}{\partial \phi}, \quad \nabla_z = \frac{\partial}{\partial z}.$$

We note that

$$\mathbf{r} = \rho \hat{\rho} + z \hat{z} = \rho(\cos \phi \hat{x} + \sin \phi \hat{y}) + z \hat{z}, \quad \mathbf{A} = A_\rho \hat{\rho} + A_\phi \hat{\phi} + A_z \hat{z}.$$

The relation of $\hat{\rho}$, $\hat{\phi}$, \hat{z} with \hat{x} , \hat{y} , \hat{z} is as follows,

$$\hat{\rho} = \frac{\partial \mathbf{r}}{\partial \rho} = \cos \phi \hat{x} + \sin \phi \hat{y}, \quad \hat{\phi} = \frac{1}{\rho} \frac{\partial \mathbf{r}}{\partial \phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}, \quad \hat{z} = \hat{z},$$

and

$$\nabla_\rho \hat{\rho} = \frac{\partial}{\partial \rho} \hat{\rho} = 0, \quad \nabla_\rho \hat{\phi} = \frac{\partial}{\partial \rho} \hat{\phi} = 0, \quad \nabla_\rho \hat{z} = \frac{\partial}{\partial \rho} \hat{z} = 0.$$

$$\nabla_{\phi}\hat{\rho} = \frac{1}{\rho} \frac{\partial}{\partial\phi} \hat{\rho} = \frac{1}{\rho} \hat{\phi}, \quad \nabla_{\phi}\hat{\phi} = \frac{1}{\rho} \frac{\partial}{\partial\phi} \hat{\phi} = -\frac{1}{\rho} \hat{\rho}, \quad \nabla_{\phi}\hat{z} = \frac{1}{\rho} \frac{\partial}{\partial\phi} \hat{z} = 0,$$

$$\nabla_z\hat{\rho} = \frac{\partial}{\partial z} \hat{\rho} = 0, \quad \nabla_z\hat{\phi} = \frac{\partial}{\partial z} \hat{\phi} = 0, \quad \nabla_z\hat{z} = \frac{\partial}{\partial z} \hat{z} = 0,$$

Thus, we have

$$\begin{aligned} (\mathbf{A}\cdot\nabla)\mathbf{B} &= (A_{\rho}\nabla_{\rho}B_{\rho})\hat{\rho} + (A_{\phi}\nabla_{\phi}B_{\rho})\hat{\rho} + (A_z\nabla_zB_{\rho})\hat{\rho} \\ &\quad + (A_{\rho}\nabla_{\rho}B_{\phi})\hat{\phi} + (A_{\phi}\nabla_{\phi}B_{\phi})\hat{\phi} + (A_z\nabla_zB_{\phi})\hat{\phi} \\ &\quad + (A_{\rho}\nabla_{\rho}B_z)\hat{z} + (A_{\phi}\nabla_{\phi}B_z)\hat{z} + (A_z\nabla_zB_z)\hat{z} \\ &\quad + (A_{\phi}B_{\rho})\nabla_{\phi}\hat{\rho} + (A_{\phi}B_{\phi})\nabla_{\phi}\hat{\phi} \\ &= (A_{\rho}\nabla_{\rho}B_{\rho})\hat{\rho} + (A_{\phi}\nabla_{\phi}B_{\rho})\hat{\rho} + (A_z\nabla_zB_{\rho})\hat{\rho} \\ &\quad + (A_{\rho}\nabla_{\rho}B_{\phi})\hat{\phi} + (A_{\phi}\nabla_{\phi}B_{\phi})\hat{\phi} + (A_z\nabla_zB_{\phi})\hat{\phi} \\ &\quad + (A_{\rho}\nabla_{\rho}B_z)\hat{z} + (A_{\phi}\nabla_{\phi}B_z)\hat{z} + (A_z\nabla_zB_z)\hat{z} \\ &\quad + (A_{\phi}B_{\rho})\frac{1}{\rho}\hat{\phi} - (A_{\phi}B_{\phi})\frac{1}{\rho}\hat{\rho} \end{aligned}$$

or

$$\begin{aligned} (\mathbf{A}\cdot\nabla)\mathbf{B} &= [(A_{\rho}\nabla_{\rho}B_{\rho}) + (A_{\phi}\nabla_{\phi}B_{\rho}) + (A_z\nabla_zB_{\rho}) - (A_{\phi}B_{\phi})\frac{1}{\rho}]\hat{\rho} \\ &\quad + [(A_{\rho}\nabla_{\rho}B_{\phi}) + (A_{\phi}\nabla_{\phi}B_{\phi}) + (A_z\nabla_zB_{\phi}) + (A_{\phi}B_{\rho})\frac{1}{\rho}]\hat{\phi} \\ &\quad + [(A_{\rho}\nabla_{\rho}B_z) + (A_{\phi}\nabla_{\phi}B_z) + (A_z\nabla_zB_z)]\hat{z} \end{aligned}$$

with

$$\nabla_{\rho} = \frac{\partial}{\partial\rho}, \quad \nabla_{\phi} = \frac{1}{\rho} \frac{\partial}{\partial\phi}, \quad \nabla_z = \frac{\partial}{\partial z}$$

((**Mathematica**))

$$\begin{aligned}
\text{Pc}[u_ , v_] &:= \text{Module} \left[\{ \text{J}\rho, \text{J}\phi, \text{Jz}, \text{h1} \}, \right. \\
\text{J}\rho &= u[[1]] \times \text{D}[v[[1]], \rho] + \frac{u[[2]]}{\rho} \text{D}[v[[1]], \phi] + u[[3]] \times \text{D}[v[[1]], z] - \\
&\quad \frac{1}{\rho} u[[2]] \times v[[2]]; \\
\text{J}\phi &= u[[1]] \times \text{D}[v[[2]], \rho] + \frac{u[[2]]}{\rho} \text{D}[v[[2]], \phi] + u[[3]] \times \text{D}[v[[2]], z] + \\
&\quad \frac{1}{\rho} u[[2]] \times v[[1]]; \\
\text{Jz} &= u[[1]] \times \text{D}[v[[3]], \rho] + \frac{u[[2]]}{\rho} \text{D}[v[[3]], \phi] + \\
&\quad u[[3]] \times \text{D}[v[[3]], z]; \text{h1} = \{ \text{J}\rho, \text{J}\phi, \text{Jz} \} \left. \right];
\end{aligned}$$

(c) (A·∇)B: **Spherical coordinates**

$$\begin{aligned}
(\mathbf{A} \cdot \nabla) \mathbf{B} &= (A_r \nabla_r + A_\theta \nabla_\theta + A_\phi \nabla_\phi)(B_r \hat{r} + B_\theta \hat{\theta} + B_\phi \hat{\phi}) \\
&= (A_r \nabla_r + A_\theta \nabla_\theta + A_\phi \nabla_\phi)(B_r \hat{r}) + (A_r \nabla_r + A_\theta \nabla_\theta + A_\phi \nabla_\phi)(B_\theta \hat{\theta}) \\
&\quad + (A_r \nabla_r + A_\theta \nabla_\theta + A_\phi \nabla_\phi)(B_\phi \hat{\phi}) \\
&= (A_r \nabla_r)(B_r \hat{r}) + (A_\theta \nabla_\theta)(B_r \hat{r}) + (A_\phi \nabla_\phi)(B_r \hat{r}) \\
&\quad + (A_r \nabla_r)(B_\theta \hat{\theta}) + (A_\theta \nabla_\theta)(B_\theta \hat{\theta}) + (A_\phi \nabla_\phi)(B_\theta \hat{\theta}) \\
&\quad + (A_r \nabla_r)(B_\phi \hat{\phi}) + (A_\theta \nabla_\theta)(B_\phi \hat{\phi}) + (A_\phi \nabla_\phi)(B_\phi \hat{\phi}) \\
&= (A_r \nabla_r B_r) \hat{r} + (A_\theta \nabla_\theta B_r) \hat{r} + (A_\phi \nabla_\phi B_r) \hat{r} \\
&\quad + (A_r \nabla_r B_\theta) \hat{\theta} + (A_\theta \nabla_\theta B_\theta) \hat{\theta} + (A_\phi \nabla_\phi B_\theta) \hat{\theta} \\
&\quad + (A_r \nabla_r B_\phi) \hat{\phi} + (A_\theta \nabla_\theta B_\phi) \hat{\phi} + (A_\phi \nabla_\phi B_\phi) \hat{\phi} \\
&\quad + (A_r B_r) \nabla_r \hat{r} + (A_\theta B_r) \nabla_\theta \hat{r} + (A_\phi B_r) \nabla_\phi \hat{r} \\
&\quad + (A_r B_\theta) \nabla_r \hat{\theta} + (A_\theta B_\theta) \nabla_\theta \hat{\theta} + (A_\phi B_\theta) \nabla_\phi \hat{\theta} \\
&\quad + (A_r B_\phi) \nabla_r \hat{\phi} + (A_\theta B_\phi) \nabla_\theta \hat{\phi} + (A_\phi B_\phi) \nabla_\phi \hat{\phi}
\end{aligned}$$

Here we use the definition of differential operator for the Cylindrical coordinates

$$\nabla_r = \frac{\partial}{\partial r}, \quad \nabla_\theta = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad \nabla_\phi = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}.$$

We note that

$$\mathbf{r} = r\hat{r} = r(\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}), \quad \mathbf{A} = A_\rho\rho + A_\phi\phi + A_z\hat{z}.$$

The relation of $\hat{\rho}$, $\hat{\phi}$, \hat{z} with \hat{x} , \hat{y} , \hat{z} is as follows,

$$\hat{r} = \frac{\partial\mathbf{r}}{\partial r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z},$$

$$\hat{\theta} = \frac{1}{r}\frac{\partial\mathbf{r}}{\partial\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z},$$

$$\hat{\phi} = \frac{1}{r\sin\theta}\frac{\partial\mathbf{r}}{\partial\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y},$$

$$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta,$$

and

$$\nabla_r\hat{r} = \frac{\partial}{\partial r}\hat{r} = 0, \quad \nabla_r\hat{\theta} = \frac{\partial}{\partial r}\hat{\theta} = 0, \quad \nabla_r\hat{\phi} = \frac{\partial}{\partial r}\hat{\phi} = 0,$$

$$\begin{aligned} \nabla_\theta\hat{r} &= \frac{1}{r}\frac{\partial}{\partial\theta}\hat{r} \\ &= \frac{1}{r}(\cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}) \\ &= \frac{1}{r}\hat{\theta} \end{aligned}$$

$$\begin{aligned} \nabla_\theta\hat{\theta} &= \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} \\ &= \frac{1}{r}(-\sin\theta\cos\phi\hat{x} - \sin\theta\sin\phi\hat{y} - \cos\theta\hat{z}) \\ &= -\frac{1}{r}\hat{r} \end{aligned}$$

$$\nabla_\theta\hat{\phi} = \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\phi} = 0$$

$$\begin{aligned}
\nabla_\phi \hat{r} &= \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \hat{r} \\
&= \frac{1}{r} (-\sin \phi \hat{x} + \cos \phi \hat{y}) \\
&= \frac{1}{r} \hat{\phi}
\end{aligned}$$

$$\begin{aligned}
\nabla_\theta \hat{\theta} &= \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \hat{\theta} \\
&= \frac{\cot \theta}{r} (-\sin \phi \hat{x} + \cos \phi \hat{y}) \\
&= \frac{\cot \theta}{r} \hat{\phi}
\end{aligned}$$

$$\begin{aligned}
\nabla_\phi \hat{\phi} &= \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \hat{\phi} \\
&= -\frac{1}{r \sin \theta} (\cos \phi \hat{x} + \sin \phi \hat{y}) \\
&= -\frac{1}{r \sin \theta} (\sin \theta \hat{r} + \cos \theta \hat{\theta}) \\
&= -\frac{1}{r} (\hat{r} + \cot \theta \hat{\theta})
\end{aligned}$$

So that,

$$\begin{aligned}
(\mathbf{A} \cdot \nabla) \mathbf{B} &= (A_r \nabla_r B_r) \hat{r} + (A_\theta \nabla_\theta B_r) \hat{r} + (A_\phi \nabla_\phi B_r) \hat{r} \\
&\quad + (A_r \nabla_r B_\theta) \hat{\theta} + (A_\theta \nabla_\theta B_\theta) \hat{\theta} + (A_\phi \nabla_\phi B_\theta) \hat{\theta} \\
&\quad + (A_r \nabla_r B_\phi) \hat{\phi} + (A_\theta \nabla_\theta B_\phi) \hat{\phi} + (A_\phi \nabla_\phi B_\phi) \hat{\phi} \\
&\quad + (A_\theta B_r) \frac{1}{r} \hat{\theta} + (A_\phi B_r) \frac{1}{r} \hat{\phi} \\
&\quad - (A_\theta B_\theta) \frac{1}{r} \hat{r} + (A_\phi B_\theta) \frac{\cot \theta}{r} \hat{\phi} \\
&\quad - (A_\phi B_\phi) \frac{1}{r} (\hat{r} + \cot \theta \hat{\theta})
\end{aligned}$$

or

$$\begin{aligned}
(\mathbf{A} \cdot \nabla) \mathbf{B} &= [(A_r \nabla_r B_r) + (A_\theta \nabla_\theta B_r) + (A_\phi \nabla_\phi B_r) - (A_\theta B_\theta) \frac{1}{r} - (A_\phi B_\phi) \frac{1}{r}] \hat{r} \\
&+ [(A_r \nabla_r B_\theta) + (A_\theta \nabla_\theta B_\theta) + (A_\phi \nabla_\phi B_\theta) + (A_\theta B_r) \frac{1}{r} - (A_\phi B_\phi) \frac{1}{r} \cot \theta] \hat{\theta} \\
&+ [(A_r \nabla_r B_\phi) + (A_\theta \nabla_\theta B_\phi) + (A_\phi \nabla_\phi B_\phi) + (A_\theta B_r) \frac{1}{r} + (A_\phi B_\theta) \frac{\cot \theta}{r}] \hat{\phi}
\end{aligned}$$

with

$$\nabla_r = \frac{\partial}{\partial r}, \quad \nabla_\theta = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad \nabla_\phi = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}.$$

((Mathematica))

```

Pc[u_, v_] := Module[{Jr, Jθ, Jφ, h1},
  Jr = u[[1]] × D[v[[1]], r] +  $\frac{u[[2]]}{r}$  D[v[[1]], θ] -  $\frac{u[[2]]}{r}$  v[[2]] +  $\frac{u[[3]]}{r \text{Sin}[\theta]}$  D[v[[1]], φ] -
   $\frac{1}{r}$  u[[3]] × v[[3]];
  Jθ = u[[1]] × D[v[[2]], r] +  $\frac{u[[2]]}{r}$  D[v[[2]], θ] +  $\frac{u[[2]] \times v[[1]]}{r}$  +  $\frac{u[[3]]}{r \text{Sin}[\theta]}$  D[v[[2]], φ] -
   $\frac{1}{r}$  u[[3]] × v[[3]]  $\frac{\text{Cos}[\theta]}{\text{Sin}[\theta]}$ ;
  Jφ = u[[1]] × D[v[[3]], r] +  $\frac{u[[2]]}{r}$  D[v[[3]], θ] +  $\frac{u[[3]]}{r \text{Sin}[\theta]}$  D[v[[3]], φ] +
   $\frac{u[[3]] \times v[[1]]}{r}$  + u[[3]]  $\frac{v[[2]]}{r \text{Sin}[\theta]}$  Cos[θ]; h1 = {Jr, Jθ, Jφ}];

```

16. Example from exercises

We solve typical problems of Arfken using Mathematica discussed above.

(a) Cylindrical coordinates

((Arfken 3-10-17))

A calculation of the magnetohydrodynamic punch effect involves the evaluation of $(\mathbf{B} \cdot \nabla) \mathbf{B}$. If the magnetic induction \mathbf{B} is taken to be

$$\mathbf{B} = \hat{\phi} \frac{1}{\rho} B_{\phi}^2$$

Show that

$$(\mathbf{B} \cdot \nabla) \mathbf{B} = -\hat{\rho} \frac{1}{\rho} B_{\phi}^2$$

((Solution))

$$\mathbf{B1} = \{\mathbf{0}, B_{\phi}[\rho], \mathbf{0}\}; \text{Pc}[\mathbf{B1}, \mathbf{B1}] // \text{Simplify}$$

$$\left\{ -\frac{B_{\phi}[\rho]^2}{\rho}, \mathbf{0}, \mathbf{0} \right\}$$

which is equal to $(\mathbf{B} \cdot \nabla) \mathbf{B} = -\hat{\rho} \frac{1}{\rho} B_{\phi}^2$.

(b) Spherical coordinates

((Arfken 3-10-26))

With \mathbf{A} any vector

$$(\mathbf{A} \cdot \nabla) \mathbf{r} = \mathbf{A}$$

Verify this result using spherical polare coordinates.

((Solution))

$$\mathbf{A1} = \{\mathbf{Ar}[\mathbf{r}, \theta, \phi], \mathbf{A}\theta[\mathbf{r}, \theta, \phi], \mathbf{A}\phi[\mathbf{r}, \theta, \phi]\}; \mathbf{r11} = \{\mathbf{r}, \theta, \theta\}; \text{Pc}[\mathbf{A1}, \mathbf{r11}]$$

$$\{\mathbf{Ar}[\mathbf{r}, \theta, \phi], \mathbf{A}\theta[\mathbf{r}, \theta, \phi], \mathbf{A}\phi[\mathbf{r}, \theta, \phi]\}$$

17. Scalar differential operator: Formula

Here the formula which we obtain are listed here.

$$\mathbf{A} = r^n \hat{\mathbf{r}}, \quad \mathbf{B} = r^m \hat{\mathbf{r}},$$

$$(\mathbf{A} \cdot \nabla) \mathbf{B} = m r^{m+n-1} \hat{\mathbf{r}}.$$

So that, we have

$$(\hat{r} \cdot \nabla)\hat{r} = 0,$$

$$(\hat{r} \cdot \nabla)\mathbf{r} = \hat{r},$$

$$(\hat{r} \cdot \nabla)\hat{r} = 0,$$

$$(\mathbf{r} \cdot \nabla)\mathbf{r} = r\hat{r} = \mathbf{r},$$

$$(\mathbf{r} \cdot \nabla)r^2\hat{r} = 2r^2\hat{r}.$$

((Note-1)) n : integers

$$\nabla \cdot \frac{\mathbf{r}}{r^3} = 0.$$

$$\nabla \cdot \frac{\mathbf{r}}{r^n} = (3-n)r^{-n},$$

$$\nabla \frac{1}{r^n} = -\frac{n}{r^{n+1}}\hat{\mathbf{r}} = -\frac{n}{r^{n+2}}\mathbf{r},$$

$$\nabla \times \hat{\mathbf{r}} = 0,$$

$$\nabla \times \left(\frac{\hat{\mathbf{r}}}{r^n}\right) = \nabla \times \left(\frac{\mathbf{r}}{r^{n+1}}\right) = 0.$$

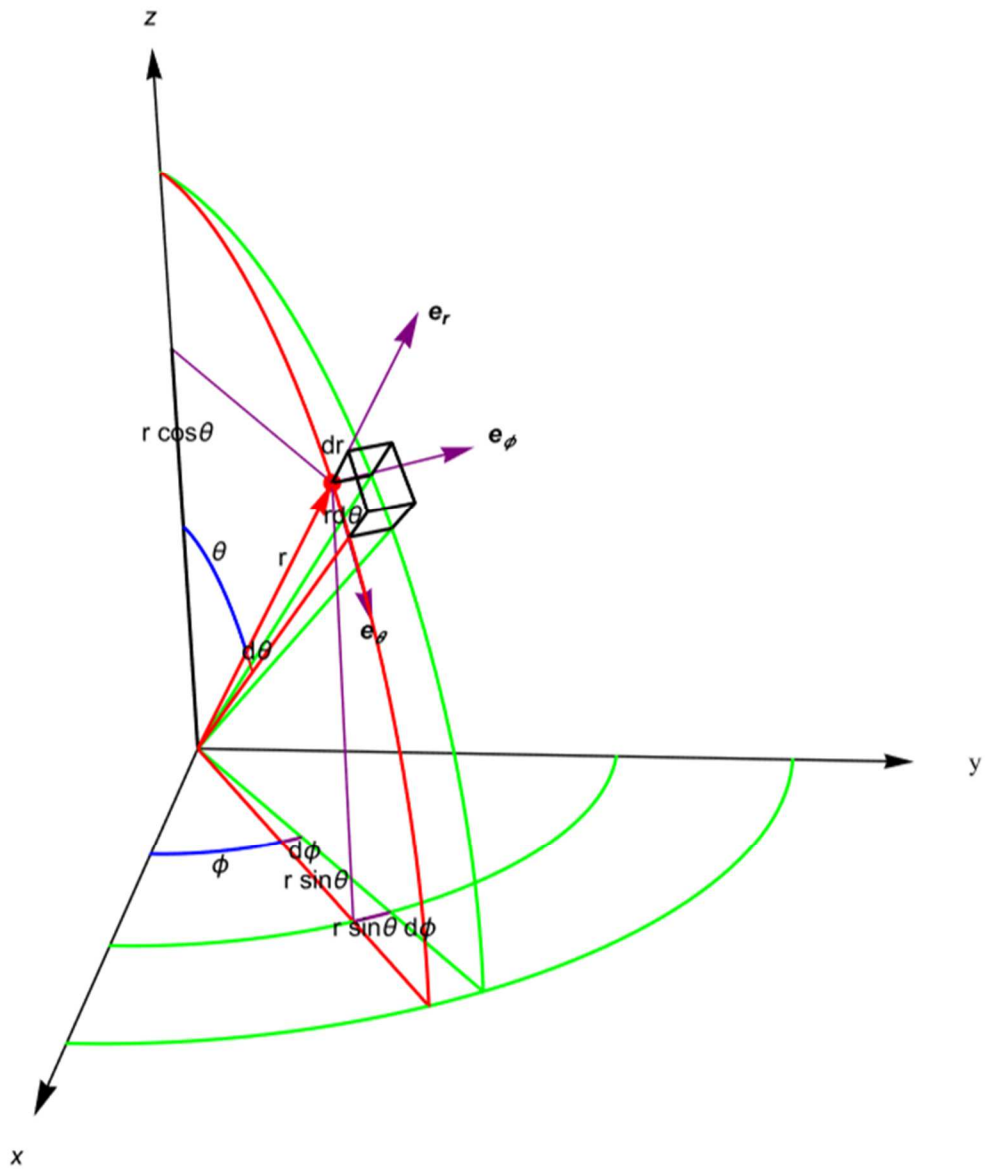
((Note-2)) n : integers

$$\begin{aligned}\nabla \times \left(\frac{\hat{\mathbf{r}}}{r^n}\right) &= \frac{1}{r^n} \nabla \times \hat{\mathbf{r}} - \hat{\mathbf{r}} \times \nabla \frac{1}{r^n} \\ &= -\hat{\mathbf{r}} \times \nabla \frac{1}{r^n} \\ &= nr^{-(n+1)}\hat{\mathbf{r}} \times \hat{\mathbf{r}} \\ &= 0\end{aligned}$$

((Mathematica Program)) Spherical coordinates

Mathematica Program

Feynman's subscript identities in vector product (spherical coordinates)



```

Clear["Global`"];
SetCoordinates[Spherical[r,  $\theta$ ,  $\phi$ ]];
ux = {Sin[ $\theta$ ] Cos[ $\phi$ ], Cos[ $\theta$ ] Cos[ $\phi$ ], -Sin[ $\phi$ ]};
uy = {Sin[ $\theta$ ] Sin[ $\phi$ ], Cos[ $\theta$ ] Sin[ $\phi$ ], Cos[ $\phi$ ]};
uz = {Cos[ $\theta$ ], -Sin[ $\theta$ ], 0};
ur = {1,  $\theta$ , 0};
x = r ur.ux;
y = r ur.uy;
z = r ur.uz;
Lap := Laplacian[#, {r,  $\theta$ ,  $\phi$ }, "Spherical"] &;
Gra := Grad[#, {r,  $\theta$ ,  $\phi$ }, "Spherical"] &;
Diva := Div[#, {r,  $\theta$ ,  $\phi$ }, "Spherical"] &;
Curla := Curl[#, {r,  $\theta$ ,  $\phi$ }, "Spherical"] &;
Pc[u_, v_] := Module[{Jr, J $\theta$ , J $\phi$ , h1},
  Jr = u[[1]]  $\times$  D[v[[1]], r] +  $\frac{u[[2]]}{r}$  D[v[[1]],  $\theta$ ] -  $\frac{u[[2]]}{r}$  v[[2]] +  $\frac{u[[3]]}{r \text{Sin}[\theta]}$  D[v[[1]],  $\phi$ ] -  $\frac{1}{r}$  u[[3]]  $\times$  v[[3]];
  J $\theta$  = u[[1]]  $\times$  D[v[[2]], r] +  $\frac{u[[2]]}{r}$  D[v[[2]],  $\theta$ ] +  $\frac{u[[2]] \times v[[1]]}{r}$  +  $\frac{u[[3]]}{r \text{Sin}[\theta]}$  D[v[[2]],  $\phi$ ] -
     $\frac{1}{r}$  u[[3]]  $\times$  v[[3]]  $\frac{\text{Cos}[\theta]}{\text{Sin}[\theta]}$ ;
  J $\phi$  = u[[1]]  $\times$  D[v[[3]], r] +  $\frac{u[[2]]}{r}$  D[v[[3]],  $\theta$ ] +  $\frac{u[[3]]}{r \text{Sin}[\theta]}$  D[v[[3]],  $\phi$ ] +  $\frac{u[[3]] \times v[[1]]}{r}$  +
    u[[3]]  $\frac{v[[2]]}{r \text{Sin}[\theta]}$  Cos[ $\theta$ ]; h1 = {Jr, J $\theta$ , J $\phi$ };
FIGS[a_, b_] := (Pc[a, b] + Cross[a, Curla[b]]) + (Pc[b, a] + Cross[b, Curla[a]]);
FIDC[a_, b_] := -a.Curla[b] + b.Curla[a];

```

Feynman's subscript identities in vector product

$$\text{Pc}[A,B] = (A \cdot \nabla)B$$

$$\text{Grad}[A \cdot B] = \text{FIGS}[A,B]$$

$$\text{FIGS}[A,B] = (A \cdot \nabla)B + A \times (\nabla \times B) + (B \cdot \nabla)A + B \times (\nabla \times A)$$

Grad[A.B] is equal to FIGS[A,B]

$$A1 = \{Ar[r, \theta, \phi], A\theta[r, \theta, \phi], A\phi[r, \theta, \phi]\}; B1 = \{Br[r, \theta, \phi], B\theta[r, \theta, \phi], B\phi[r, \theta, \phi]\};$$

$$j11 = \text{FIGS}[A1, B1];$$

$$j11 - \text{Gra}[A1, B1] // \text{FullSimplify}$$

$$\{0, 0, 0\}$$

$$\text{Pc}[A1, B1] // \text{FullSimplify}$$

$$\left\{ \frac{1}{r} \left(-A\phi[r, \theta, \phi]^2 + A\phi[r, \theta, \phi] \text{Csc}[\theta] Br^{(0,0,1)}[r, \theta, \phi] + A\theta[r, \theta, \phi] \left(-A\theta[r, \theta, \phi] + Br^{(0,1,0)}[r, \theta, \phi] \right) + r Ar[r, \theta, \phi] Br^{(1,0,0)}[r, \theta, \phi] \right), \right. \\ \frac{1}{r} \left(-A\phi[r, \theta, \phi]^2 \text{Cot}[\theta] + A\phi[r, \theta, \phi] \text{Csc}[\theta] A\theta^{(0,0,1)}[r, \theta, \phi] + A\theta[r, \theta, \phi] \left(Br[r, \theta, \phi] + A\theta^{(0,1,0)}[r, \theta, \phi] \right) + r Ar[r, \theta, \phi] A\theta^{(1,0,0)}[r, \theta, \phi] \right), \\ \left. \frac{1}{r} \left(A\phi[r, \theta, \phi] \left(Br[r, \theta, \phi] + A\theta[r, \theta, \phi] \text{Cot}[\theta] + \text{Csc}[\theta] A\phi^{(0,0,1)}[r, \theta, \phi] \right) + A\theta[r, \theta, \phi] A\phi^{(0,1,0)}[r, \theta, \phi] + r Ar[r, \theta, \phi] A\phi^{(1,0,0)}[r, \theta, \phi] \right) \right\}$$

Example

$$S1 = \{r, \theta, \theta\};$$

$$\text{Pc}[S1, S1]$$

$$\{r, \theta, \theta\}$$

$$S2 = \{1, \theta, \theta\}; \text{Pc}[S2, S2]$$

$$\{0, \theta, \theta\}$$

((Arfken)) Application (solving problems in Arfken)

(a) ((Arfken 3-10-23))

3.10.23 A rigid body is rotating about a fixed axis with a constant angular velocity ω . Take ω to be along the z-axis. Using spherical polar coordinates,

- (a) calculate $\mathbf{v} = \omega \times \mathbf{r}$.
- (b) calculate $\nabla \times \mathbf{v}$.

ANS. (a) $\mathbf{v} = \hat{\mathbf{e}}_\phi \omega r \sin \theta$.
 (b) $\nabla \times \mathbf{v} = 2\omega$.

$$\mathbf{W11} = \{\omega \cos[\theta], -\omega \sin[\theta], 0\}; \mathbf{R11} = \{r, \theta, \theta\}; \mathbf{v11} = \text{Cross}[\mathbf{W11}, \mathbf{R11}]$$

$$\{0, 0, r \omega \sin[\theta]\}$$

$$\text{Curl1a}[\mathbf{v11}]$$

$$\{2 \omega \cos[\theta], -2 \omega \sin[\theta], 0\}$$

(b) ((Arfken 3-10-26))

3.10.26 With \mathbf{A} any vector,

$$\mathbf{A} \cdot \nabla \mathbf{r} = \mathbf{A}.$$

- (a) Verify this result in Cartesian coordinates.
- (b) Verify this result using spherical polar coordinates. Equation (3.156) provides ∇ .

$$\mathbf{A1} = \{A_r[r, \theta, \phi], A_\theta[r, \theta, \phi], A_\phi[r, \theta, \phi]\}; \mathbf{r11} = \{r, \theta, \theta\}; \text{Pc}[\mathbf{A1}, \mathbf{r11}]$$

$$\{A_r[r, \theta, \phi], A_\theta[r, \theta, \phi], A_\phi[r, \theta, \phi]\}$$

18. Examples: Spherical coordinates

(a) ((Arfken 3-10-26))

With \mathbf{A} any vector

$$(\mathbf{A} \cdot \nabla) \mathbf{r} = \mathbf{A}$$

Verify this result using spherical polare coordinates.

((Solution))

$$\mathbf{A1} = \{A_r[r, \theta, \phi], A_\theta[r, \theta, \phi], A_\phi[r, \theta, \phi]\}; \mathbf{r11} = \{r, \theta, \theta\}; \text{Pc}[\mathbf{A1}, \mathbf{r11}]$$

$$\{A_r[r, \theta, \phi], A_\theta[r, \theta, \phi], A_\phi[r, \theta, \phi]\}$$

(b) ((Arfken 3-10-37))

An electric dipole of moment \mathbf{p} is located at the origin. The dipole creates an electric potential at \mathbf{r} given by

$$\psi(\mathbf{r}) = \frac{\mathbf{p} \cdot \mathbf{r}}{4\pi\epsilon_0 r^3}$$

Find the electric field, $\mathbf{E} = -\nabla\psi(\mathbf{r})$ at \mathbf{r} .

((Solution))

$$\psi_1 = \frac{p_1 \cos[\theta]}{4\pi\epsilon_0 r^2}; \mathbf{E}_{11} = -\text{Gra}[\psi_1]$$

$$\left\{ \frac{p_1 \cos[\theta]}{2\pi r^3 \epsilon_0}, \frac{p_1 \sin[\theta]}{4\pi r^3 \epsilon_0}, \theta \right\}$$

(c) ((Arfken 3-10-35))

A certain force field is given in spherical coordinates by

$$\mathbf{F} = \hat{\mathbf{r}} \frac{2P \cos \theta}{r^3} + \hat{\boldsymbol{\theta}} \frac{P}{r^3} \sin \theta \quad r \geq P/2$$

- Examine $\nabla \times \mathbf{F}$ to see if a potential exists.
- Calculate $\oint \mathbf{F} \cdot d\mathbf{r}$ for a unit circle in the plane $\theta = \pi/2$. What does this indicate about the force being conservative or nonconservative?
- If you believe that \mathbf{F} may be described by $\mathbf{F} = -\nabla\psi$, find ψ . Otherwise simply state that no acceptable potential exists.

$$\mathbf{F}_{11} = \left\{ \frac{2 P_1}{r^3} \cos[\theta], \frac{P_1}{r^3} \sin[\theta], \theta \right\}; \text{Curla}[\mathbf{F}_{11}] // \text{Simplify}$$

$$\{0, 0, 0\}$$

$$\mathbf{A}_{11} = \{\mathbf{F}_1[r], \theta, \theta\}; \mathbf{B}_{11} = \{\mathbf{G}[r], \theta, \theta\}; \text{Pc}[\mathbf{A}_{11}, \mathbf{B}_{11}]$$

$$\{\mathbf{F}_1[r] \mathbf{G}'[r], \theta, \theta\}$$

19. CONCLUSION

Using Mathematica, we show that the Feynman (Gibbs) subscript notations are mathematically correct. Here we only show the case of Cartesian coordinates. We also make programs of these expressions in terms of three coordinates such as cylindrical; coordinates and spherical coordinates

as well as the Cartesian coordinates. During the above discussions, we realize that Feynman (Gibbs) subscript notations in vector product plays significant roles to understanding the essential concepts in electricity and magnetism;

1. The energy conservation with Poynting vector,
2. The magnetic force due to the magnetic moment.
3. The electric force due to the electric dipole moment.
4. The magnetic field from the vector potential due to magnetic moment.
5. The electric field from the electric dipole moment.
6. Magnetic field due to the magnetic moment and electric field due to electric dipole moment. The mathematical structures are the same and expressed by scalar differential operator.
7. Maxwell's stress tensor (actually, the derivation of this formula is our first motivation).

Using the Mathematica, one can calculate a scalar differential operator

$$P_c[\mathbf{u}, \mathbf{v}] = (\mathbf{u} \cdot \nabla) \mathbf{v},$$

in terms of Cartesian, Cylindrical, and Spherical coordinates.

REFERENCES

Mathematica (version 14.0).

R.P. Feynman, R.B. Leighton, and M. Sands, *The Feynman Lectures on Physics volume II* (Basic Books, 2010)

E.M. Purcell and D.J. Morin, *Electricity and Magnetism*, 3rd edition (Cambridge University Press, 2013).

D.J. Griffiths, *Introduction to Electrodynamics*, 5th edition (Cambridge University Press, 2024).

J.D. Jackson, *Classical Electrodynamics*, 3rd edition (Wiley, 1998).

G.B. Arfken, H.J. Weber, F.E. Harris, *Mathematical Methods for Physicists; A Comprehensive Guide*, 7th edition (Elsevier, 2013).

Edwin Bidwell Wilson, Yale Bicentennial Publications, *Vector Analysis: A Textbook for the use of Students of Mathematics and Physics, Founded upon the Lectures of J. Willard Gibbs* Ph. D. (New York: Charles Scribners' sons, London, 1901).

H.C. Ohanian, *Classical Electrodynamics*, 2nd edition (Infinity Science Press LLC, 2007).

Tai L. Chow, *Introduction to Electromagnetic Theory; A Modern Perspective* (Jones and Bartlett Publishers, 2006).

J.R. Reitz, F.J. Milford, and R.W. Christy, *Foundations of Electromagnetic Theory*, 3rd edition (Addison Wesley 1980).

TensorTime@2024 by Joe Heafner

Did Feynman Invent Feynman subscript notation?

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