

Chapter 30 Addition of angular momentum
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Clebsch-Gordan (CG) Coefficient
Orbital angular momentum
Spin angular momentum

30.1 Clebsch-Gordan (CG) Coefficient

Let $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$ be the two angular momenta, and let

$$[\hat{\mathbf{J}}_1, \hat{\mathbf{J}}_2] = 0,$$

or

$$[\hat{J}_{1i}, \hat{J}_{2j}] = 0. \quad (i, j = x, y, z)$$

Define the total angular momentum

$$\hat{\mathbf{J}} = \hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2,$$

where

$$\hat{\mathbf{J}}_1^2 |j_1, m_1\rangle = \hbar^2 j_1(j_1+1) |j_1, m_1\rangle$$

$$\hat{J}_{1z} |j_1, m_1\rangle = \hbar m_1 |j_1, m_1\rangle$$

$$\hat{J}_{1\pm} |j_1, m_1\rangle = \hbar \sqrt{j_1(j_1+1) - m_1(m_1 \pm 1)} |j_1, m_1 \pm 1\rangle$$

Similarly,

$$\hat{\mathbf{J}}_2^2 |j_2, m_2\rangle = \hbar^2 j_2(j_2+1) |j_2, m_2\rangle$$

$$\hat{J}_{2z} |j_2, m_2\rangle = \hbar m_2 |j_2, m_2\rangle$$

$$\hat{J}_{2\pm} |j_2, m_2\rangle = \hbar \sqrt{j_2(j_2+1) - m_2(m_2 \pm 1)} |j_2, m_2 \pm 1\rangle$$

We now consider a new ket defined by

$$|j_1, j_2; m_1, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle$$

The total angular momentum: commutation relation

$$\hat{\mathbf{J}} = \hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2$$

$$[\hat{J}_z, \hat{\mathbf{J}}_1^2] = [\hat{J}_z, \hat{\mathbf{J}}_2^2] = 0$$

$$[\hat{\mathbf{J}}^2, \hat{\mathbf{J}}_1^2] = [\hat{\mathbf{J}}^2, \hat{\mathbf{J}}_2^2] = 0$$

((Note))

$$[\hat{J}_z, \hat{\mathbf{J}}_1^2] = [\hat{J}_{1z} + \hat{J}_{2z}, \hat{\mathbf{J}}_1^2] = [\hat{J}_{1z}, \hat{\mathbf{J}}_1^2] = 0$$

$$[\hat{J}_z^2, \hat{\mathbf{J}}_1^2] = -[\hat{\mathbf{J}}_1^2, \hat{J}_z^2] = -([\hat{\mathbf{J}}_1^2, \hat{J}_z] \hat{J}_z + \hat{J}_z [\hat{\mathbf{J}}_1^2, \hat{J}_z]) = 0$$

$$[\hat{\mathbf{J}}^2, \hat{\mathbf{J}}_1^2] = [\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2, \hat{\mathbf{J}}_1^2] = 0$$

Furthermore, \hat{J}_{1z} and \hat{J}_{2z} commute with \hat{J}_z .

$$[\hat{J}_{1z}, \hat{J}_z] = [\hat{J}_{2z}, \hat{J}_z] = 0,$$

but not with $\hat{\mathbf{J}}^2$.

$$\hat{\mathbf{J}}^2 = \hat{\mathbf{J}}_1^2 + \hat{\mathbf{J}}_2^2 + 2\hat{\mathbf{J}}_1 \cdot \hat{\mathbf{J}}_2 = \hat{\mathbf{J}}_1^2 + \hat{\mathbf{J}}_2^2 + 2\hat{J}_{1z}\hat{J}_{2z} + \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}$$

$$\begin{aligned} [\hat{J}_{1z}, \hat{\mathbf{J}}^2] &= [\hat{J}_{1z}, \hat{\mathbf{J}}_1^2 + \hat{\mathbf{J}}_2^2 + 2\hat{J}_{1z}\hat{J}_{2z} + \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}] \\ &= [\hat{J}_{1z}, \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}] \\ &= [\hat{J}_{1z}, \hat{J}_{1+}]\hat{J}_{2-} + [\hat{J}_{1z}, \hat{J}_{1-}]\hat{J}_{2+} \\ &= \hbar\hat{J}_{1+}\hat{J}_{2-} - \hbar\hat{J}_{1-}\hat{J}_{2+} \end{aligned}$$

$$\begin{aligned} [\hat{J}_{2z}, \hat{\mathbf{J}}^2] &= [\hat{J}_{2z}, \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}] \\ &= \hat{J}_{1+}[\hat{J}_{2z}, \hat{J}_{2-}] + \hat{J}_{1-}[\hat{J}_{2z}, \hat{J}_{2+}] \\ &= -\hbar\hat{J}_{1+}\hat{J}_{2-} + \hbar\hat{J}_{1-}\hat{J}_{2+} \end{aligned}$$

Thus we have

$$[\hat{J}_z, \hat{\mathbf{J}}^2] = 0$$

Since

$$[\hat{\mathbf{J}}^2, \hat{J}_z] = [\hat{\mathbf{J}}^2, \hat{\mathbf{J}}_1^2] = [\hat{\mathbf{J}}^2, \hat{\mathbf{J}}_2^2] = [\hat{\mathbf{J}}_1^2, \hat{\mathbf{J}}_2^2] = [\hat{\mathbf{J}}_1^2, \hat{J}_z] = [\hat{\mathbf{J}}_2^2, \hat{J}_z] = 0,$$

We have simultaneous eigenkets of $\hat{\mathbf{J}}^2, \hat{\mathbf{J}}_1^2, \hat{\mathbf{J}}_2^2, \hat{J}_z$

We use

$$|j_1, j_2; j, m\rangle$$

to denote the basis.

$$\hat{\mathbf{J}}_1^2 |j_1, j_2; j, m\rangle = \hbar^2 j_1(j_1+1) |j_1, j_2; j, m\rangle$$

$$\hat{\mathbf{J}}_2^2 |j_1, j_2; j, m\rangle = \hbar^2 j_2(j_2+1) |j_1, j_2; j, m\rangle$$

$$\hat{\mathbf{J}}^2 |j_1, j_2; j, m\rangle = \hbar^2 j(j+1) |j_1, j_2; j, m\rangle$$

$$\hat{J}_z |j_1, j_2; j, m\rangle = \hbar m |j_1, j_2; j, m\rangle$$

Here we have the relation

$$|j_1, j_2; j, m\rangle = \sum_{m_1} \sum_{m_2} |j_1, j_2; m_1, m_2\rangle \langle j_1, j_2; m_1, m_2| j_1, j_2; j, m\rangle$$

where we use the closure relation.

$$\sum_{m_1} \sum_{m_2} |j_1, j_2; m_1, m_2\rangle \langle j_1, j_2; m_1, m_2| = 1 :$$

$\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle$: Clebsch Gordan co-efficient

30.2 Properties of CG co-efficient

(1) The co-efficient vanishes unless $m = m_1 + m_2$.

((Proof))

Note that

$$(\hat{J}_z - \hat{J}_{1z} - \hat{J}_{2z}) |j_1, j_2; j, m\rangle = 0$$

$$\langle j_1, j_2; m_1, m_2 | \hat{J}_z - \hat{J}_{1z} - \hat{J}_{2z} | j_1, j_2; j, m \rangle = 0$$

or

$$(m - m_1 - m_2) \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle = 0$$

For $m - m_1 - m_2 \neq 0$,

$$\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle = 0$$

(2) The CG co-efficient vanishes unless

$$|j_1 - j_2| \leq j \leq j_1 + j_2$$

triangle inequality, which tells us the possible range of values of j when we add two angular momenta j_1 and j_2 whose values are fixed.

(Proof)

For a proof of the triangle rule, let us consider the possible values of m . Since $m = m_1 + m_2$, its maximum value is $j_1 + j_2$. The value is realized for the single state $|j_1, j_2; j_1, j_2\rangle$. The total J of this state is $j = j_1 + j_2$. The next highest value of $m, j_1 + j_2 - 1$ is realized for the linear combinations of two states in the uncoupled representations; $|j_1, j_2; j_1 - 1, j_2\rangle$ and $|j_1, j_2; j_1, j_2 - 1\rangle$. One of the linear combinations belongs to $j = j_1 + j_2$ and the second one to $j = j_1 + j_2 - 1$.

For $m = j_1 + j_2 - 2$, there are three linear combinations of the three uncoupled states; $|j_1, j_2; j_1 - 2, j_2\rangle$, $|j_1, j_2; j_1 - 1, j_2 - 1\rangle$, and $|j_1, j_2; j_1, j_2 - 2\rangle$, corresponding to three of $j, j = j_1 + j_2, j = j_1 + j_2 - 1$, and $j = j_1 + j_2 - 2$.

If we continue this process, we can see that each time we lower m by 1, a new value of j appears. The argument continues until we reach a stage where we can no longer go one more step down in one of them to make new states. We assume that $j_1 > j_2$.

$$j = j_1 + j_2, m = j_1 + j_2, j_1 + j_2 - 1, \dots, -(j_1 + j_2)$$

$$j = j_1 + j_2 - 1, m = j_1 + j_2 - 1, j_1 + j_2 - 2, \dots, -(j_1 + j_2 - 1),$$

$$j = j_1 - j_2, m = j_1 - j_2, j_1 - j_2 - 1, \dots, -(j_1 - j_2).$$

Here we show that if the triangle rule is valid, then the dimension of the space spanned by $\{|j_1, j_2; j, m\rangle\}$ is the same as that of the space spanned by $\{|j_1, j_2; m_1, m_2\rangle\}$.

For the (m_1, m_2) way of counting, we obtain $N = (2j_1 + 1)(2j_2 + 1)$.

As for the $\{j, m\}$ way of counting, note that for each j , there are $(2j+1)$ states. According to the inequality ($|j_1 - j_2| \leq j \leq j_1 + j_2$), j runs (we assume $j_1 > j_2$) from $j_1 - j_2$ to $j_1 + j_2$.

$$\begin{aligned} N &= \sum_{j=j_1-j_2}^{j_1+j_2} (2j+1) = \sum_{j=0}^{j_1+j_2} (2j+1) - \sum_{j=0}^{j_1-j_2} (2j+1) \\ &= (j_1 + j_2 + 1)^2 - (j_1 - j_2)^2 \\ &= (2j_1 + 1)(2j_2 + 1) \end{aligned}$$

((Note))

$$D_{j_1} \times D_{j_2} = D_{j_1+j_2} + D_{j_1+j_2-1} + \dots + D_{|j_1-j_2|}$$

where

$$D_{j_1+j_2}; j = j_1 + j_2; m = j_1 + j_2, j_1 + j_2 - 1, \dots, -(j_1 + j_2),$$

$$D_{j_1+j_2-1}; j = j_1 + j_2 - 1; m = j_1 + j_2 - 1, j_1 + j_2 - 2, \dots, -(j_1 + j_2 - 1),$$

$$\dots$$

$$D_{|j_1-j_2|}; j = |j_1 - j_2|; m = |j_1 - j_2|, |j_1 - j_2| - 1, \dots, -|j_1 - j_2|,$$

The CG coefficients form an unitary matrix. Furthermore, the matrix elements are taken to be real by convention. A real unitary matrix is orthogonal. We have the orthogonal condition.

$$\langle j_1, j_2, j, m | j_1, j_2, m_1, m_2 \rangle = \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle^* = \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle$$

Closure relation

$$\begin{aligned} \langle j_1, j_2, m_1, m_1' | j_1, j_2, m_1, m_2' \rangle &= \sum_{j,m} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \langle j_1, j_2; j, m | j_1, j_2; m_1', m_2' \rangle \\ &= \sum_{j,m} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m \rangle \\ &= \delta_{m_1, m_1'} \delta_{m_2, m_2'} \end{aligned}$$

or

$$\sum_{j,m} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m \rangle = \delta_{m_1, m_1'} \delta_{m_2, m_2'}$$

Similarly,

$$\sum_{m_1, m_2} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \langle j_1, j_2; m_1, m_2 | j_1, j_2; j', m' \rangle = \delta_{j, j'} \delta_{m, m'}$$

As a special case of this, we may set $j = j'$, $m' = m = m_1 + m_2$.

$$\sum_{\substack{m_1, m_2 \\ m=m_1+m_2}} \left| \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \right|^2 = 1$$

which is just the normalization condition of $| j_1, j_2; j, m \rangle$.

30.3 Recursion relation-1 for the CG co-efficients

((First step))

$$\begin{aligned} \hat{J}_\pm | j_1, j_2; j, m \rangle &= (\hat{J}_{1\pm} + \hat{J}_{2\pm}) \sum_{\substack{m_1', m_2' \\ m_1' + m_2' = m}} | j_1, j_2; m_1', m_2' \rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m \rangle \\ &\quad \sqrt{(j \mp m)(j \pm m + 1)} | j_1, j_2; j, m \pm 1 \rangle \\ &= \sum_{\substack{m_1', m_2' \\ m_1' + m_2' = m}} [\sqrt{(j_1 \mp m_1')(j_1 \pm m_1' + 1)} | j_1, j_2; m_1' \pm 1, m_2' \rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m \rangle \\ &\quad + \sqrt{(j_2 \mp m_2')(j_2 \pm m_2' + 1)} | j_1, j_2; m_1', m_2' \pm 1 \rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m \rangle] \end{aligned}$$

((Second step))

Next step is to multiply by $\langle j_1, j_2; m_1, m_2 |$ on the left and to use orthonormality.

$$\begin{aligned}
& \sqrt{(j \mp m)(j \pm m+1)} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \pm 1 \rangle \\
&= \sum_{\substack{m_1', m_2' \\ m_1' + m_2' = m}} [\sqrt{(j_1 \mp m_1')(j_1 \pm m_1'+1)} \langle j_1, j_2; m_1, m_2 | j_1, j_2; m_1' \pm 1, m_2' \rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m \rangle \\
&\quad + \sqrt{(j_2 \mp m_2')(j_2 \pm m_2'+1)} \langle j_1, j_2; m_1, m_2 | j_1, j_2; m_1', m_2' \pm 1 \rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m \rangle]
\end{aligned}$$

or

$$\begin{aligned}
& \sqrt{(j \mp m)(j \pm m+1)} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \pm 1 \rangle \\
&= \sum_{\substack{m_1', m_2' \\ m_1' + m_2' = m}} [\sqrt{(j_1 \mp m_1')(j_1 \pm m_1'+1)} \delta_{m_1, m_1' \pm 1} \delta_{m_2, m_2'} \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m \rangle \\
&\quad + \sqrt{(j_2 \mp m_2')(j_2 \pm m_2'+1)} \delta_{m_1, m_1} \delta_{m_2, m_2' \pm 1} \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m \rangle]
\end{aligned}$$

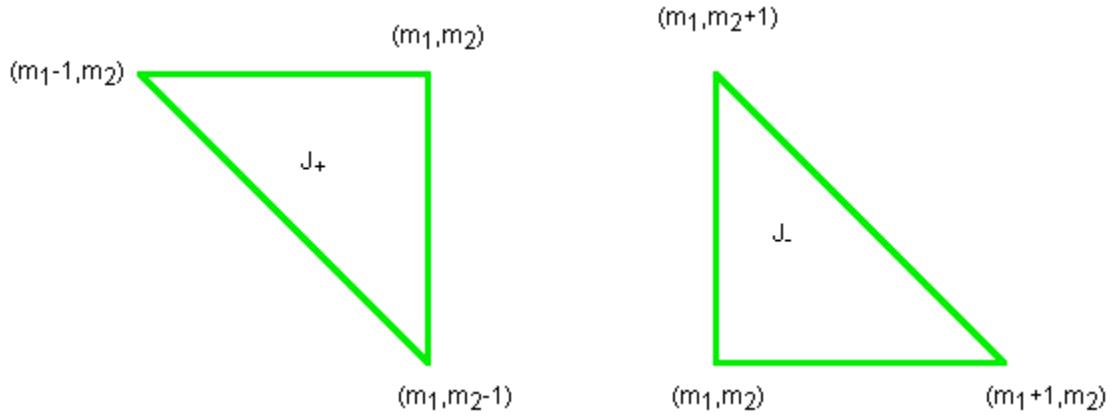
or

$$\begin{aligned}
& \sqrt{(j \mp m)(j \pm m+1)} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \pm 1 \rangle \\
&= \sqrt{[j_1 \mp (m_1 \mp 1))(j_1 \pm (m_1 \mp 1)+1)]} \langle j_1, j_2; m_1 \mp 1, m_2 | j_1, j_2; j, m \rangle \\
&\quad + \sqrt{[j_2 \mp (m_2 \mp 1))(j_2 \pm (m_2 \mp 1)+1)]} \langle j_1, j_2; m_1, m_2 \mp 1 | j_1, j_2; j, m \rangle
\end{aligned}$$

((Note)) We use a more convenient expression

$$\begin{aligned}
& \sqrt{(j \mp m+1)(j \pm m)} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \\
&= \sqrt{(j_1 \mp m_1+1)(j_1 \pm m_1)} \langle j_1, j_2; m_1 \mp 1, m_2 | j_1, j_2; j, m \mp 1 \rangle \\
&\quad + \sqrt{(j_2 \mp m_2+1)(j_2 \pm m_2)} \langle j_1, j_2; m_1, m_2 \mp 1 | j_1, j_2; j, m \mp 1 \rangle
\end{aligned}$$

This recursion relations are related to the CG coefficients at (m_1, m_2) , $(m_1 \mp 1, m_2)$, and $(m_1, m_2 \mp 1)$, where $m_1+m_2=m$.



30.4 Recursion relation-2 for the CG co-efficients

$$\hat{\mathbf{J}}^2 |j_1, j_2; j, m\rangle = \hat{\mathbf{J}}^2 \sum_{\substack{m'_1, m'_2 \\ m'_1 + m'_2 = m}} |j_1, j_2; m'_1, m'_2\rangle \langle j_1, j_2; m'_1, m'_2| j_1, j_2; j, m\rangle$$

$$\hat{\mathbf{J}}^2 = \hat{\mathbf{J}}_1^2 + \hat{\mathbf{J}}_2^2 + 2\hat{J}_{1z}\hat{J}_{2z} + \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}$$

$$\hat{\mathbf{J}}^2 |j_1, j_2; j, m\rangle = \hbar^2 j(j+1) |j_1, j_2; j, m\rangle$$

$$\begin{aligned} & \sum_{\substack{m'_1, m'_2 \\ m'_1 + m'_2 = m}} \hat{\mathbf{J}}^2 |j_1, j_2; m'_1, m'_2\rangle \langle j_1, j_2; m'_1, m'_2| j_1, j_2; j, m\rangle \\ &= \sum_{\substack{m'_1, m'_2 \\ m'_1 + m'_2 = m}} (\hat{\mathbf{J}}_1^2 + \hat{\mathbf{J}}_2^2 + 2\hat{J}_{1z}\hat{J}_{2z} + \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}) \\ & \quad \times |j_1, j_2; m'_1, m'_2\rangle \langle j_1, j_2; m'_1, m'_2| j_1, j_2; j, m\rangle \end{aligned}$$

Here

$$\begin{aligned} & \sum_{\substack{m'_1, m'_2 \\ m'_1 + m'_2 = m}} (\hat{\mathbf{J}}_1^2 + \hat{\mathbf{J}}_2^2 + 2\hat{J}_{1z}\hat{J}_{2z}) |j_1, j_2; m'_1, m'_2\rangle \langle j_1, j_2; m'_1, m'_2| j_1, j_2; j, m\rangle \\ &= \hbar^2 \sum_{\substack{m'_1, m'_2 \\ m'_1 + m'_2 = m}} \{j_1(j_1+1) + j_2(j_2+1) + 2m'_1 m'_2\} \\ & \quad \times |j_1, j_2; m'_1, m'_2\rangle \langle j_1, j_2; m'_1, m'_2| j_1, j_2; j, m\rangle \end{aligned}$$

$$\begin{aligned}
& \sum_{\substack{m_1', m_2' \\ m_1' + m_2' = m}} (\hat{J}_{1+} \hat{J}_{2-} + \hat{J}_{1-} \hat{J}_{2+}) |j_1, j_2; m_1', m_2'\rangle \langle j_1, j_2; m_1', m_2'| j_1, j_2; j, m\rangle \\
&= \hbar^2 \sum_{\substack{m_1', m_2' \\ m_1' + m_2' = m}} \{ \sqrt{(j_1 - m_1')(j_1 + m_1' + 1)} \sqrt{(j_2 + m_2')(j_2 - m_2' + 1)} \\
&\quad \times |j_1, j_2; m_1' + 1, m_2' - 1\rangle \langle j_1, j_2; m_1', m_2'| j_1, j_2; j, m\rangle \\
&+ \hbar^2 \sum_{\substack{m_1', m_2' \\ m_1' + m_2' = m}} \{ \sqrt{(j_1 + m_1')(j_1 - m_1' + 1)} \sqrt{(j_2 - m_2')(j_2 + m_2' + 1)} \\
&\quad \times |j_1, j_2; m_1' - 1, m_2' + 1\rangle \langle j_1, j_2; m_1', m_2'| j_1, j_2; j, m\rangle \}
\end{aligned}$$

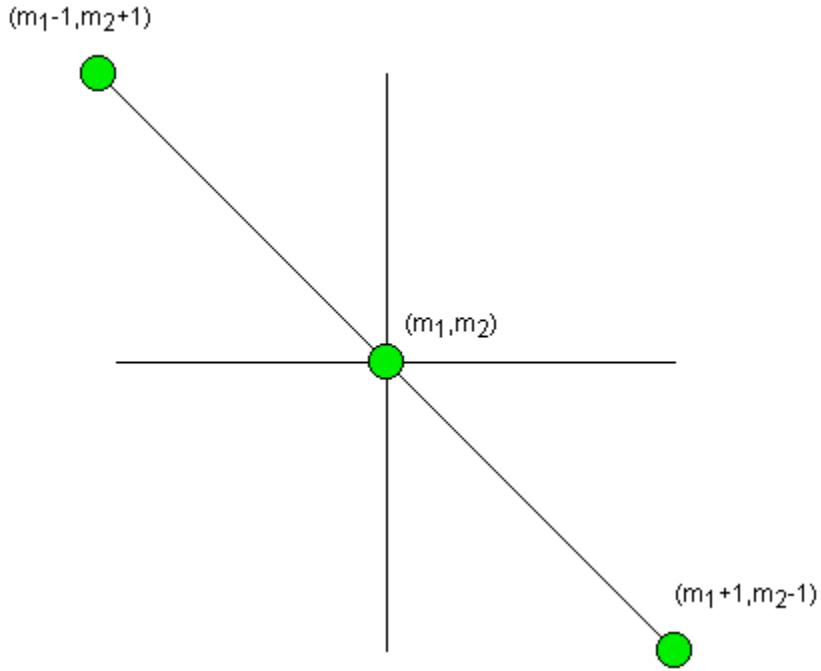
Note that

$$\langle j_1, j_2; m_1, m_2 | j_1, j_2; m_1', m_2' \rangle = \delta_{m_1, m_1'} \delta_{m_2, m_2'}$$

We multiply the above equation by bra $\langle j_1, j_2; m_1, m_2 |$

$$\begin{aligned}
& \{ j(j+1) - j_1(j_1+1) - j_2(j_2+1) - 2m_1m_2 \} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \\
&= \sqrt{(j_1 - m_1 + 1)(j_1 + m_1)} \sqrt{(j_2 + m_2 + 1)(j_2 - m_2)} \langle j_1, j_2; m_1 - 1, m_2 + 1 | j_1, j_2; j, m \rangle \\
&+ \sqrt{(j_1 + m_1 + 1)(j_1 - m_1)} \sqrt{(j_2 - m_2 + 1)(j_2 + m_2)} \langle j_1, j_2; m_1 + 1, m_2 - 1 | j_1, j_2; j, m \rangle
\end{aligned}$$

This recursion relations are related to the CG coefficients at (m_1, m_2) , (m_1-1, m_2+1) , and (m_1+1, m_2-1) , where $m_1 + m_2 = m$.



$$D_{j_1} \times D_{j_2} = D_{j_1+j_2} + D_{j_1+j_2-1} + \dots + D_{|j_1-j_2|}$$

For fixed j , $m = j, j-1, \dots, -j$.

In the (m_1, m_2) plane, we plot the boundary of the allowed region determined by

$$|m_1| \leq j_1, \quad |m_2| \leq j_2$$

$$m_1 + m_2 = m$$

The J_+ recursion relation tells us that the co-efficient at (m_1, m_2) is related to those at (m_1-1, m_2) and (m_1, m_2-1) . Likewise, the J_- recursion relation (lower sign) relates the three co-efficients whose m_1, m_2 values are given in Fig.

(Recursion relations + Normalization conditions) \rightarrow uniquely determine the CG co-efficients.

$$|l_1, l_2; j, m\rangle = |j, m\rangle = \sum_{\substack{m_1, m_2 \\ m_1 + m_2 = m}} C_{m_1, m_2} |l_1, m_1\rangle |l_2, m_2\rangle$$

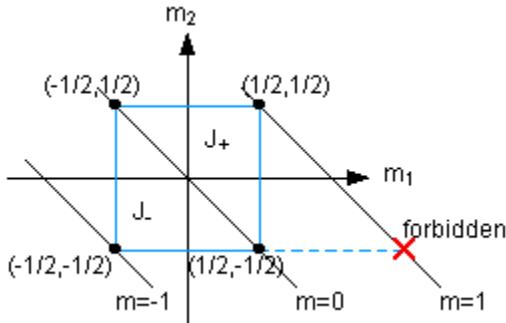
30.5 CG values

$$j_1 = 1/2, j_2 = 1/2 (\left|m_1\right| \leq 1/2, \left|m_2\right| \leq 1/2)$$

$$D_{1/2} \times D_{1/2} = D_1 + D_0$$

(i) $\mathbf{j} = \mathbf{1}$

$$m = m_1 + m_2$$



$$\left| \frac{1}{2}, \frac{1}{2} \right\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle \quad (m = 1)$$

$$\frac{\left| \frac{1}{2}, \frac{1}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle + \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle}{\sqrt{2}} \quad (m = 0)$$

$$\left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \quad (m = -1)$$

(ii) $\mathbf{j} = \mathbf{0} (m = 0)$

$$\frac{\left| \frac{1}{2}, \frac{1}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle - \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle}{\sqrt{2}} \quad (m = 0)$$

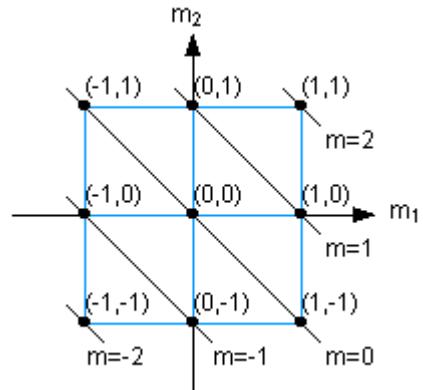
30.6 CG values

$$j_1 = 1, j_2 = 1 (\left|m_1\right| \leq 1, \left|m_2\right| \leq 1)$$

$$D_1 \times D_1 = D_2 + D_1 + D_0$$

(i) $j = 2 (\left|m\right| \leq 2)$

$$m = m_1 + m_2$$



$$|1,1\rangle |1,1\rangle \quad (m=2)$$

$$\frac{|1,1\rangle |1,0\rangle + |1,0\rangle |1,1\rangle}{\sqrt{2}} \quad (m=1)$$

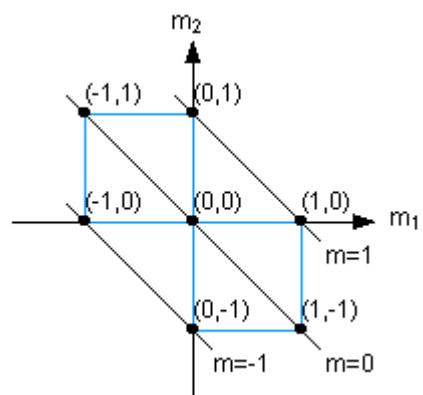
$$\frac{|1,1\rangle |1,-1\rangle + 2|1,0\rangle |1,0\rangle + |1,-1\rangle |1,1\rangle}{\sqrt{6}} \quad (m=0)$$

$$\frac{|1,0\rangle |1,-1\rangle + |1,-1\rangle |1,0\rangle}{\sqrt{2}} \quad (m=-1)$$

$$|1,-1\rangle |1,-1\rangle \quad (m=-2)$$

Symmetric for the particle exchange

(ii) $j = 1$ ($|m| \leq 1$)



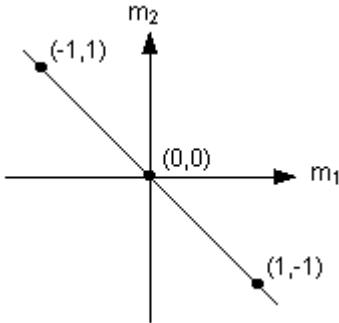
$$\frac{|1,1\rangle|1,0\rangle - |1,0\rangle|1,1\rangle}{\sqrt{2}} \quad (m = 1)$$

$$\frac{|1,1\rangle|1,-1\rangle - |1,-1\rangle|1,1\rangle}{\sqrt{2}} \quad (m = 0)$$

$$\frac{|1,0\rangle|1,-1\rangle - |1,-1\rangle|1,0\rangle}{\sqrt{2}} \quad (m = -1)$$

Anti-symmetric for the particle exchange

(iii) $j = 0 (m = 0)$



$$\frac{|1,1\rangle|1,-1\rangle - |1,0\rangle|1,0\rangle + |1,-1\rangle|1,1\rangle}{\sqrt{3}}$$

Symmetric for the particle exchange

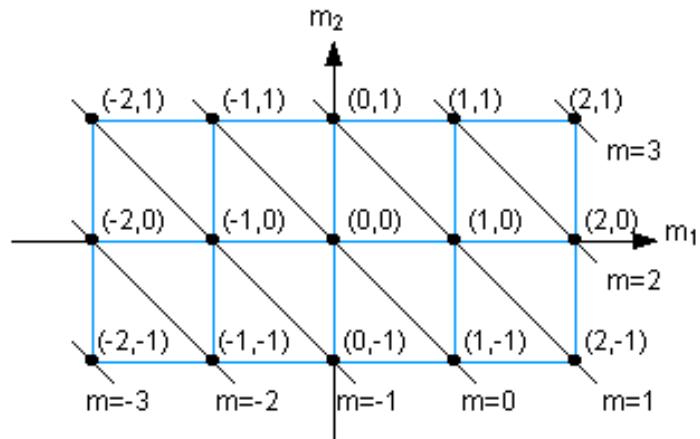
30.7 CG values

$$j_1 = 2, j_2 = 1 (\left|m_1\right| \leq 2, \left|m_2\right| \leq 1)$$

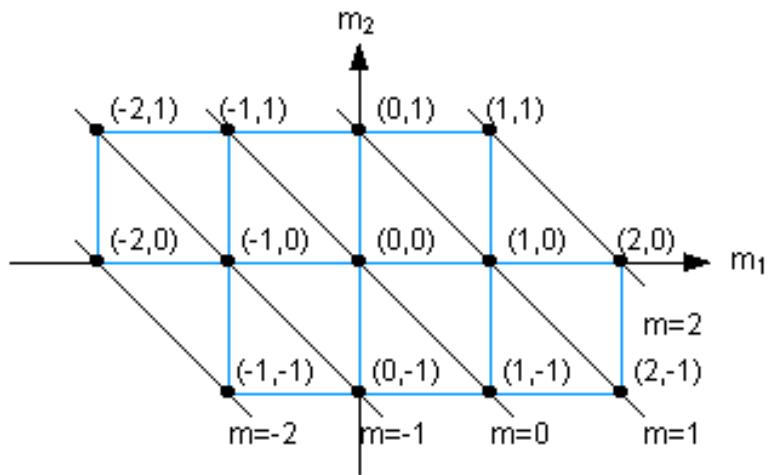
$$D_2 \times D_1 = D_3 + D_2 + D_1$$

(i) $j = 3 (\left|m\right| \leq 3)$

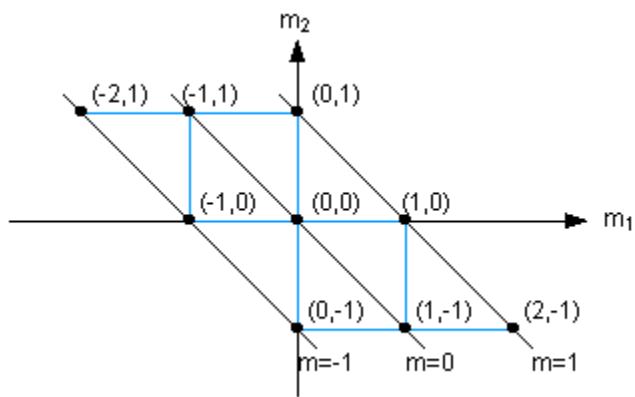
$$m = m_1 + m_2$$



(ii) $j = 2$ ($|m| \leq 2$)



(iii) $j = 1$ ($|m| \leq 1$)



30.8 Addition of orbital and spin angular momentum

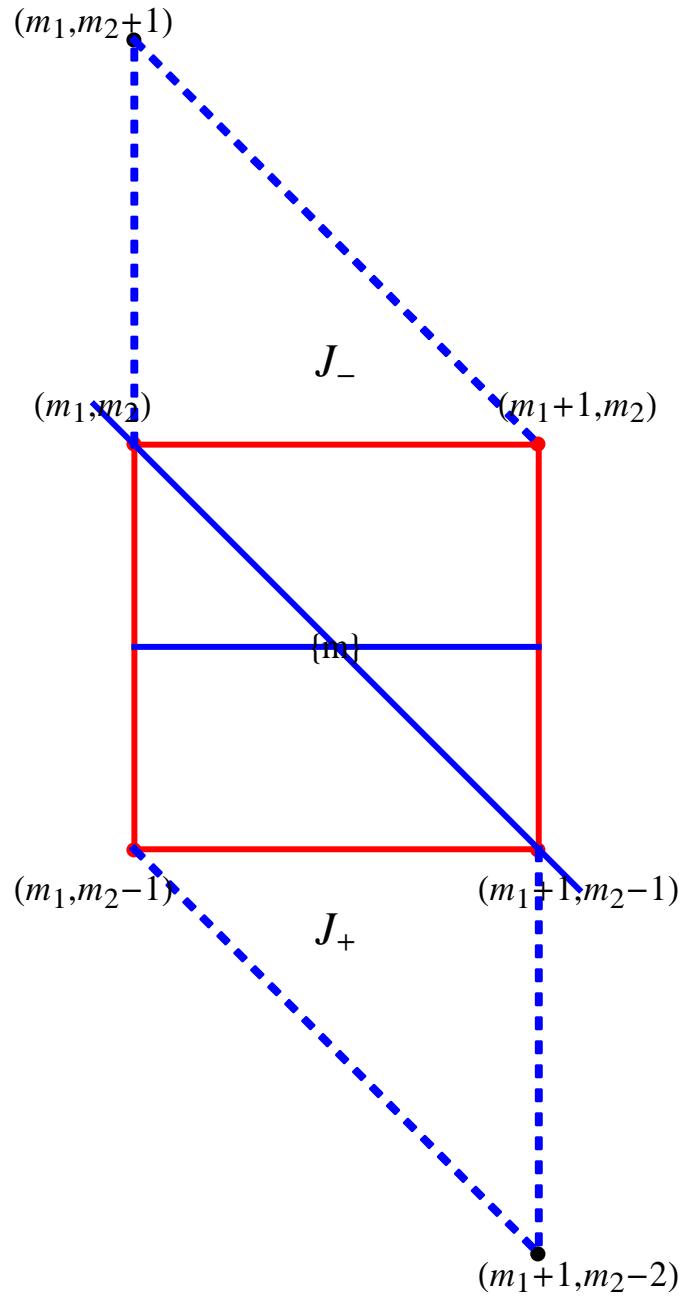
We consider one electron with the orbital angular momentum \mathbf{L} (typically $l = 1$, p electron) and the spin angular momentum \mathbf{S} . The total angular momentum J is an addition of \mathbf{L} and \mathbf{S} .

$$\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$$

$$D_l \times D_{1/2} = D_{l+1/2} + D_{l-1/2}$$

30.9 J . recursion

We use the J . recursion for the diagram below.



$$\begin{aligned}
 & \sqrt{(j+m+1)(j-m)} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \\
 &= \sqrt{(j_1 + m_1 + 1)(j_1 - m_1)} \langle j_1, j_2; m_1 + 1, m_2 | j_1, j_2; j, m + 1 \rangle \\
 &+ \sqrt{(j_2 + m_2 + 1)(j_2 - m_2)} \langle j_1, j_2; m_1, m_2 + 1 | j_1, j_2; j, m + 1 \rangle
 \end{aligned}$$

We note that

$$\langle j_1, j_2; m_1, m_2 + 1 | j_1, j_2; j, m + 1 \rangle = 0$$

Then we have

$$\begin{aligned} & \sqrt{(j+m+1)(j-m)} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \\ &= \sqrt{(j_1+m_1+1)(j_1-m_1)} \langle j_1, j_2; m_1+1, m_2 | j_1, j_2; j, m+1 \rangle \end{aligned}$$

We assume that

$$j_1 = l, \quad j_2 = 1/2, \quad j = l + 1/2$$

$$m_1 = m - 1/2, \quad m_2 = 1/2$$

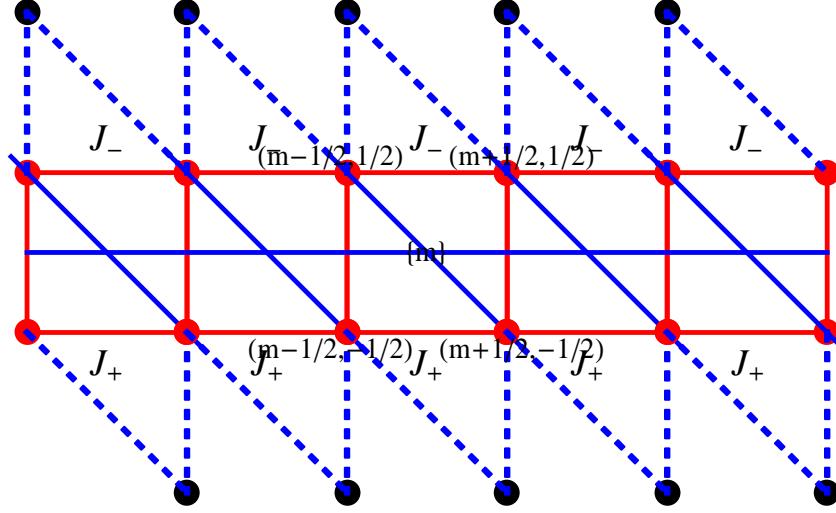
Then we have

$$\begin{aligned} & \sqrt{(l+\frac{1}{2}+m+1)(l+\frac{1}{2}-m)} \left\langle j_1 = l, j_2 = \frac{1}{2}; m - \frac{1}{2}, \frac{1}{2} \middle| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m \right\rangle \\ &= \sqrt{(l+m+\frac{1}{2})(l-m+\frac{1}{2})} \left\langle j_1 = l, j_2 = \frac{1}{2}; m + \frac{1}{2}, \frac{1}{2} \middle| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m + 1 \right\rangle \end{aligned}$$

or

$$\begin{aligned} & \left\langle j_1 = l, j_2 = \frac{1}{2}; m - \frac{1}{2}, \frac{1}{2} \middle| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m \right\rangle \\ &= \sqrt{\frac{l+m+\frac{1}{2}}{l+m+\frac{3}{2}}} \left\langle j_1 = l, j_2 = \frac{1}{2}; m + \frac{1}{2}, \frac{1}{2} \middle| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m + 1 \right\rangle \end{aligned}$$

This procedure can be continued until $m = l + 1/2$.



Then we have

$$\begin{aligned}
 & \left\langle j_1 = l, j_2 = \frac{1}{2}; m - \frac{1}{2}, \frac{1}{2} \middle| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m \right\rangle \\
 &= \sqrt{\frac{l+m+\frac{1}{2}}{l+m+\frac{3}{2}}} \sqrt{\frac{l+m+\frac{3}{2}}{l+m+\frac{5}{2}}} \left\langle j_1 = l, j_2 = \frac{1}{2}; m + \frac{3}{2}, \frac{1}{2} \middle| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m+2 \right\rangle \\
 &= \sqrt{\frac{l+m+\frac{1}{2}}{l+m+\frac{3}{2}}} \sqrt{\frac{l+m+\frac{3}{2}}{l+m+\frac{5}{2}}} \sqrt{\frac{l+m+\frac{5}{2}}{l+m+\frac{7}{2}}} \left\langle j_1 = l, j_2 = \frac{1}{2}; m + \frac{5}{2}, \frac{1}{2} \middle| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m+3 \right\rangle \\
 &= \dots \\
 &= \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}} \left\langle j_1 = l, j_2 = \frac{1}{2}; l, \frac{1}{2} \middle| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, l + \frac{1}{2} \right\rangle
 \end{aligned}$$

We choose

$$\left\langle j_1 = l, j_2 = \frac{1}{2}; l, \frac{1}{2} \middle| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, l + \frac{1}{2} \right\rangle = 1$$

Then we have

$$\left\langle j_1 = l, j_2 = \frac{1}{2}; m - \frac{1}{2}, \frac{1}{2} \middle| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m \right\rangle = \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}}$$

30.10 J_+ recursion

We use the J_+ recursion for the diagram.

$$\begin{aligned} & \sqrt{(j-m+1)(j+m)} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \\ &= \sqrt{(j_1 - m_1 + 1)(j_1 + m_1)} \langle j_1, j_2; m_1 - 1, m_2 | j_1, j_2; j, m - 1 \rangle \\ &+ \sqrt{(j_2 - m_2 + 1)(j_2 + m_2)} \langle j_1, j_2; m_1, m_2 - 1 | j_1, j_2; j, m - 1 \rangle \end{aligned}$$

We assume that

$$j_1 = l, \quad j_2 = 1/2, \quad j = l + 1/2$$

$$m_1 = m + 1/2, \quad m_2 = -1/2$$

Then we have

$$\begin{aligned} & \sqrt{(l-m+\frac{2}{3})(l+m+\frac{1}{2})} \left\langle j_1 = l, j_2 = \frac{1}{2}; m + \frac{1}{2}, -\frac{1}{2} \middle| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m \right\rangle \\ &= \sqrt{(l-m+\frac{1}{2})(l+m+\frac{1}{2})} \left\langle j_1 = l, j_2 = \frac{1}{2}; m - \frac{1}{2}, -\frac{1}{2} \middle| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m - 1 \right\rangle \end{aligned}$$

or

$$\begin{aligned} & \left\langle j_1 = l, j_2 = \frac{1}{2}; m + \frac{1}{2}, -\frac{1}{2} \middle| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m \right\rangle \\ &= \sqrt{\frac{l-m+\frac{1}{2}}{l-m+\frac{3}{2}}} \left\langle j_1 = l, j_2 = \frac{1}{2}; m - \frac{1}{2}, -\frac{1}{2} \middle| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m - 1 \right\rangle \end{aligned}$$

This procedure can be continued until $m = -l - 1/2$.

$$\left\langle j_1 = l, j_2 = \frac{1}{2}; m + \frac{1}{2}, -\frac{1}{2} \middle| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m \right\rangle$$

$$\begin{aligned}
&= \sqrt{\frac{l-m+\frac{1}{2}}{l-m+\frac{3}{2}}} \left\langle j_1 = l, j_2 = \frac{1}{2}; m - \frac{1}{2}, -\frac{1}{2} \middle| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m - 1 \right\rangle \\
&= \sqrt{\frac{l-m+\frac{1}{2}}{l-m+\frac{3}{2}}} \sqrt{\frac{l-m+\frac{3}{2}}{l-m+\frac{5}{2}}} \left\langle j_1 = l, j_2 = \frac{1}{2}; m - \frac{3}{2}, -\frac{1}{2} \middle| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m - 2 \right\rangle \\
&= \dots \\
&= \sqrt{\frac{l-m+\frac{1}{2}}{2l+1}} \left\langle j_1 = l, j_2 = \frac{1}{2}; -l, -\frac{1}{2} \middle| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, -l - \frac{1}{2} \right\rangle
\end{aligned}$$

We choose

$$\left\langle j_1 = l, j_2 = \frac{1}{2}; -l, -\frac{1}{2} \middle| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, -l - \frac{1}{2} \right\rangle = 1$$

Then we have

$$\left\langle j_1 = l, j_2 = \frac{1}{2}; m + \frac{1}{2}, -\frac{1}{2} \middle| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m \right\rangle = \sqrt{\frac{l-m+\frac{1}{2}}{2l+1}}$$

30.11 Derivation of Clebsch-Gordan coefficient)

We assume that

$$|j = l + 1/2, m\rangle = \alpha |m_l = m - 1/2, m_s = 1/2\rangle + \beta |m_l = m + 1/2, m_s = -1/2\rangle$$

$$|j = l - 1/2, m\rangle = \gamma |m_l = m - 1/2, m_s = 1/2\rangle + \delta |m_l = m + 1/2, m_s = -1/2\rangle$$

where α, β, γ , and δ are real. The normalization condition:

$$\alpha^2 + \beta^2 = 1$$

$$\gamma^2 + \delta^2 = 1$$

The condition of the orthogonality:

$$\alpha\gamma + \beta\delta = 0$$

We already know that

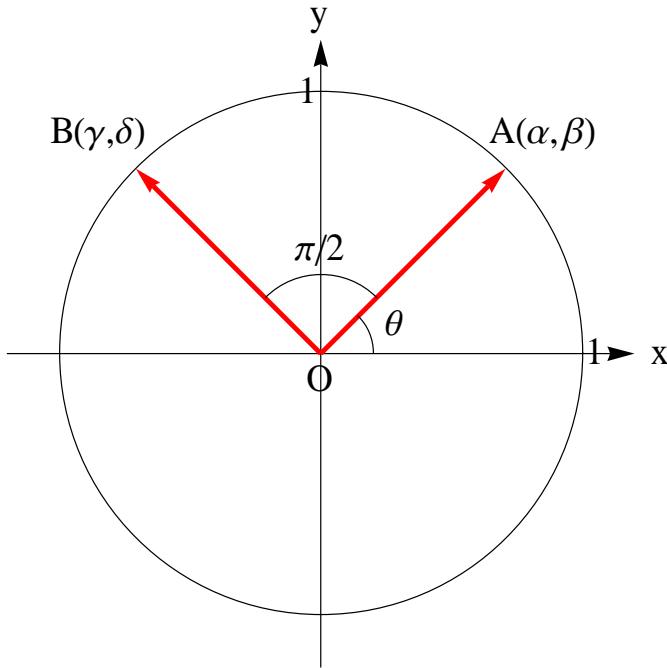
$$\alpha = \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}}$$

We want to determine the values of β , γ , and δ . To this end, we assume that

$$\alpha = \cos \theta,$$

$$\beta = \sin \theta$$

$$\gamma = \cos(\theta + \pi/2) = -\sin \theta, \delta = \sin(\theta + \pi/2) = \cos \theta$$



Since

$$\alpha^2 + \beta^2 = 1$$

we have

$$\beta = \sqrt{1 - \frac{l+m+\frac{1}{2}}{2l+1}} = \sqrt{\frac{l-m+\frac{1}{2}}{2l+1}}$$

Then we get

$$\alpha = \delta = \sqrt{\frac{l+m+1/2}{2l+1}}, \quad \beta = -\gamma = \sqrt{\frac{l-m+1/2}{2l+1}}$$

The final result is

$$\begin{aligned} |j=l+1/2, m\rangle &= \sqrt{\frac{l+m+1/2}{2l+1}} |m_l=m-1/2, m_s=1/2\rangle \\ &\quad + \sqrt{\frac{l-m+1/2}{2l+1}} |m_l=m+1/2, m_s=-1/2\rangle \\ |j=l-1/2, m\rangle &= -\sqrt{\frac{l-m+1/2}{2l+1}} |m_l=m-1/2, m_s=1/2\rangle \\ &\quad + \sqrt{\frac{l+m+1/2}{2l+1}} |m_l=m+1/2, m_s=-1/2\rangle \end{aligned}$$

or

$$\begin{aligned} |j=l+1/2, m\rangle &= \begin{pmatrix} \sqrt{\frac{l+m+1/2}{2l+1}} \\ \sqrt{\frac{l-m+1/2}{2l+1}} \end{pmatrix}, \\ |j=l-1/2, m\rangle &= \begin{pmatrix} -\sqrt{\frac{l-m+1/2}{2l+1}} \\ \sqrt{\frac{l+m+1/2}{2l+1}} \end{pmatrix} \end{aligned}$$

30.12 Mathematica: The use of Standard Conventions of Edmonds

It is much easier to calculate the Clebsch-Gordan co-efficient by using Mathematica.

Fig. $j = l + 1/2$ and $j = l - 1/2$.

`ClebschGordan[{l,m-1/2},{1/2,1/2},{l+1/2,m}]`

$$\alpha = -\frac{(-1)^{2(m+l)} \sqrt{1+2m+2l}}{\sqrt{2+4l}}$$

Since l is an integer and m is an half-integer, we have $(-1)^{2(m+l)} = -1$. Then α is simplified as

$$\alpha = \frac{\sqrt{1+2m+2l}}{\sqrt{2+4l}}$$

ClebschGordan[$\{l,m+1/2\}, \{1/2,-1/2\}, \{l+1/2,m\}$]

$$\beta = -\frac{(-1)^{2(m+l)} \sqrt{1-2m+2l}}{\sqrt{2+4l}}$$

or

$$\beta = \frac{\sqrt{1-2m+2l}}{\sqrt{2+4l}}$$

ClebschGordan[$\{l,m-1/2\}, \{1/2, 1/2\}, \{l-1/2,m\}$]

$$\gamma = -\frac{\sqrt{1-2m+2l}}{\sqrt{2+4l}}.$$

ClebschGordan[$\{l,m+1/2\}, \{1/2, -1/2\}, \{l-1/2,m\}$]

$$\delta = \frac{\sqrt{1+2m+2l}}{\sqrt{2+4l}}.$$

((Note)) Clebsch-Gordan coefficient with l and $s = 1$

$$D_l \times D_1 = D_{l+1} + D_l + D_{l-1}$$

ClebschGordan[$\{l,m-1\}, \{1, 1\}, \{l+1,m\}$]

$$\frac{(-1)^{2(m+l)} \sqrt{m+l} \sqrt{1+m+l}}{\sqrt{2} \sqrt{1+l} \sqrt{1+2l}}.$$

ClebschGordan[$\{l,m\}, \{1, 0\}, \{l+1,m\}$]

$$\frac{(-1)^{2(m+l)} \sqrt{1-m+l} \sqrt{1+m+l}}{\sqrt{1+l} \sqrt{1+2l}}.$$

ClebschGordan[$\{l,m+1\}, \{1, -1\}, \{l+1, m\}$]

$$\frac{(-1)^{2(m+l)} \sqrt{-m+l} \sqrt{1-m+l}}{\sqrt{2} \sqrt{l+1} \sqrt{1+2l}}.$$

ClebschGordan[$\{l,m-1\}, \{1, 1\}, \{l, m\}$]

$$-\frac{\sqrt{m+l} \sqrt{1-m+l}}{\sqrt{2} \sqrt{l} \sqrt{1+2l}}.$$

ClebschGordan[$\{l,m\}, \{1, 0\}, \{l, m\}$]

$$\frac{m}{\sqrt{l} \sqrt{1+2l}}$$

ClebschGordan[$\{l,m+1\}, \{1, -1\}, \{l, m\}$]

$$\frac{(-1)^{2(m+l)} \sqrt{-m+l} \sqrt{1+m+l}}{\sqrt{2} \sqrt{l} \sqrt{1+2l}}$$

ClebschGordan[$\{l,m-1\}, \{1, 1\}, \{l-1, m\}$]

$$\frac{\sqrt{-m+l} \sqrt{1-m+l}}{\sqrt{2} \sqrt{l} \sqrt{1+2l}}$$

ClebschGordan[$\{l,m\}, \{1, 0\}, \{l-1, m\}$]

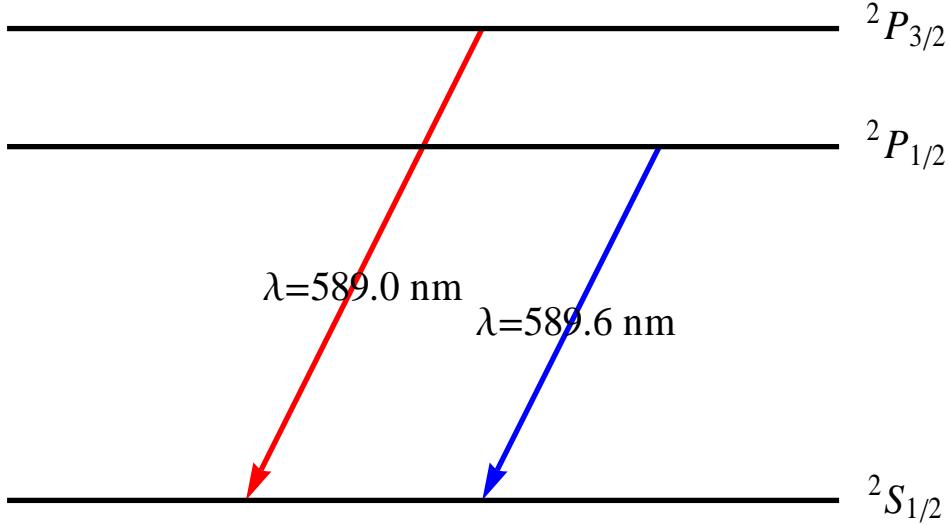
$$-\frac{\sqrt{-m+l} \sqrt{m+l}}{\sqrt{l} \sqrt{1+2l}}$$

ClebschGordan[$\{l,m+1\}, \{1, -1\}, \{l-1, m\}$]

$$\frac{\sqrt{m+l} \sqrt{1+m+l}}{\sqrt{2} \sqrt{l} \sqrt{1+2l}}$$

30.13 Sodium D lines

The sodium D lines correspond to the $3p \rightarrow 3s$ transition. In the absence of a magnetic field B , the spin orbit interaction splits the upper $3p$ state into $^2P_{3/2}$ and $^2P_{1/2}$ terms separated by 17 cm^{-1} . The lower $^2S_{1/2}$ has no spin-orbit interaction.



The electron configuration of Na is $(1s)^2(2s)^2(2p)^6(3s)^1$. For the electron with 3s state ($l = 0, s = 1/2$)

$$D_0 \times D_{1/2} = D_{1/2}$$

Thus we have $j = 1/2$. The state is described by $^2S_{1/2}$.

$$|j = 1/2, m = 1/2\rangle = |m_l = 0, m_s = 1/2\rangle$$

$$|j = 1/2, m = -1/2\rangle = |m_l = 0, m_s = -1/2\rangle$$

For the electron with 3p state ($l = 1, s = 1/2$)

$$D_1 \times D_{1/2} = D_{3/2} + D_{1/2}$$

Thus we have $j = 3/2$ and $j = 1/2$. The state is described by $^2P_{3/2}$ and $^2P_{1/2}$

$$|j = 3/2, m = -3/2\rangle = |m_l = -1, m_s = -1/2\rangle$$

$$|j = 3/2, m = -1/2\rangle = \sqrt{\frac{2}{3}} |m_l = 0, m_s = -1/2\rangle + \frac{1}{\sqrt{3}} |m_l = -1, m_s = 1/2\rangle$$

$$|j = 3/2, m = 1/2\rangle = \frac{1}{\sqrt{3}} |m_l = 1, m_s = -1/2\rangle + \sqrt{\frac{2}{3}} |m_l = 0, m_s = 1/2\rangle$$

$$|j = 3/2, m = 3/2\rangle = |m_l = 1, m_s = 1/2\rangle$$

$$|j=1/2, m=-1/2\rangle = \frac{1}{\sqrt{3}} |m_l=0, m_s=-1/2\rangle - \sqrt{\frac{2}{3}} |m_l=-1, m_s=1/2\rangle$$

$$|j=1/2, m=1/2\rangle = \sqrt{\frac{2}{3}} |m_l=1, m_s=-1/2\rangle - \frac{1}{\sqrt{3}} |m_l=0, m_s=1/2\rangle$$

The Lande g -factor

$$g_J = \frac{3}{2} + \frac{s(s+1) - l(l+1)}{2j(j+1)}$$

Table

Term	j	l	S	g_J
$^2P_{3/2}$	$3/2$	1	$1/2$	$4/3$
$^2P_{1/2}$	$1/2$	1	$1/2$	$2/3$
$^2S_{1/2}$	$1/2$	0	$1/2$	2

D₁ line (589.6 nm)

$$^2P_{1/2} \rightarrow ^2S_{1/2}$$

D₂ line (589.0 nm)

$$^2P_{3/2} \rightarrow ^2S_{1/2}$$

The Zeeman energy for the magnetic moment;

$$\hat{H}_B = \frac{g_J \mu_B B}{\hbar} \hat{J}_z$$

$$\hat{H}_B |j, m\rangle = \frac{g_J \mu_B B}{\hbar} \hat{J}_z |j, m\rangle = g_J \mu_B B m |j, m\rangle$$

The spectra can be understood by applying the following selection rules on j and m :

$$\begin{aligned} \Delta j &= 0, \pm 1 \\ \Delta m &= 0, \pm 1 \end{aligned}$$

There rules have to be applied in addition to the usual ones for l and s namely

$$\Delta l = \pm 1$$

$$\Delta s = 0$$

There are no selection rule for m_l and m_s here because l_z and s_z are not constants of the motion when l and s are coupled by the spin-orbit interaction.

$\Delta j = 0$ transitions are forbidden when $j = 0$ for both states, and $\Delta m = 0$ transitions are forbidden in a $\Delta j = 0$ transition.

Table: Anomalous Zeeman effect for the sodium D lines. The energy shifts are quoted in units of $\mu_B B$.

$$\Delta E(D_2) = g_{3/2}m(j=3/2)\mu_B B - g_{1/2}m(j=1/2)\mu_B B$$

$$= [(4/3)m_{j=3/2} - 2m_{j=1/2}]\mu_B B$$

$$\Delta E(D_1) = g_{1/2}m_{j=1/2}(\text{upper})\mu_B B - g_{1/2}m_{j=1/2}(\text{lower})\mu_B B$$

$$= [(2/3)m_{j=1/2} - 2m_{j=1/2}]\mu_B B$$

m_j^{upper}	m_j^{lower}	Δm_j	$\Delta E(D_1)$	$\Delta E(D_2)$
3/2	1/2	1		1
1/2	1/2	0	-2/3	-1/3
1/2	-1/2	-1	4/3	5/3
-1/2	1/2	1	-4/3	-5/3
-1/2	-1/2	0	2/3	1/3
-3/2	-1/2	1		-1

In the normal Zeeman effect, we observe three lines with an energy spacing equal to $\mu_B B$. In the anomalous effect, there are more than three lines, and the spacing is different to the classical values: in fact, the lines are not evenly spaced. Furthermore, none of the lines occur at the same frequency as the unperturbed line at $B = 0$.

APPENDIX

Mathematica: CG efficients

j1=1 and j2=1

```

Clear["Global`*"]

CG[j_, m_, j1_, j2_] :=
Sum[ClebschGordan[{j1, m1}, {j2, m - m1}, {j, m}] a[j1, m1] b[j2, m - m1],
{m1, -j1, j1}]

CG[2, 2, 1, 1]
ClebschGordan::phy: ThreeJSymbol[{1, -1}, {1, 3}, {2, -2}] is not physical. >>
ClebschGordan::phy: ThreeJSymbol[{1, 0}, {1, 2}, {2, -2}] is not physical. >>
a[1, 1] b[1, 1]

CG[2, 1, 1, 1]
ClebschGordan::phy: ThreeJSymbol[{1, -1}, {1, 2}, {2, -1}] is not physical. >>
a[1, 1] b[1, 0] + a[1, 0] b[1, 1]
────────────────────────────────────────────────────────────────────────────────────────
      √2          √2

CG[2, 0, 1, 1]
a[1, 1] b[1, -1] + √(2/3) a[1, 0] b[1, 0] + a[1, -1] b[1, 1]
────────────────────────────────────────────────────────────────────────────────
      √6

CG[2, -1, 1, 1]
ClebschGordan::phy: ThreeJSymbol[{1, 1}, {1, -2}, {2, 1}] is not physical. >>
a[1, 0] b[1, -1] + a[1, -1] b[1, 0]
────────────────────────────────────────
      √2

```

REFERENCES

J.J. Sakurai *Modern Quantum Mechanics*, Revised Edition (Addison-Wesley, Reading Massachusetts, 1994).