Chapter 11S
Poisson summation formula
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Poisson summation formula
Fourier series
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11S.1 Poisson summation formula

For appropriate functions $f(x)$, the Poisson summation formula may be stated as

$$\sum_{n=-\infty}^{\infty} f(x=n) = \sqrt{2\pi} \sum_{m=-\infty}^{\infty} F(k=m), \quad (1)$$

where $m$ and $n$ are integers, and $F(k)$ is the Fourier transform of $f(x)$ and is defined by

$$F(k) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx.$$

Note that the inverse Fourier transform is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} F(k) dk.$$

The factor of the right hand side of Eq.(1) arises from the definition of the Fourier transform.

((Proof)) The proof of Eq.(1) is given as follows.

$$\sum_{n=-\infty}^{\infty} f(x=n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) dk \sum_{n=-\infty}^{\infty} e^{ikn}.$$
We evaluate the factor

\[ I = \sum_{n=-\infty}^{\infty} e^{ikn}. \]

It is evident that \( I \) is not equal to zero only when \( k = 2\pi m \) \((m; \text{integer})\). Therefore \( I \) can be expressed by

\[ I = A \sum_{m=-\infty}^{\infty} \delta(k - 2\pi m), \]

where \( A \) is the normalization factor. Then

\[
\sum_{n=-\infty}^{\infty} f(x = n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) dk [A \sum_{m=-\infty}^{\infty} \delta(k - 2\pi m)]
\]

\[
= \frac{A}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} F(k) \delta(k - 2\pi m) dk
\]

\[
= \frac{A}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} F(k = 2\pi m)
\]

\[
= \frac{A}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} F(k = 2\pi n)
\]

The normalization factor, \( A \), is readily shown to be \( 2\pi \) by considering the symmetrical case

\[ f(x = n) = e^{-\pi x^2}, \]

\[ F(k = 2\pi m) = \frac{1}{\sqrt{2\pi}} e^{-\pi m^2} \]

((Mathematica))

2
\[ f(x) = \text{Exp}[-\pi x^2]; \]

\[
\text{FourierTransform}[f[x], x, k, \\
\text{FourierParameters} \to \{0, -1\}]
\]

\[
\frac{e^{-x^2/4}}{\sqrt{2\pi}}
\]

Since

\[
A = \frac{\sqrt{2\pi} f(x = n)}{F(k = 2\pi n)} = 2\pi.
\]

or

\[
I = \sum_{n=-\infty}^{\infty} e^{i\pi n} = 2\pi \sum_{m=-\infty}^{\infty} \delta(k - 2\pi n). \quad (2)
\]

Using this formula, we have

\[
\sum_{n=-\infty}^{\infty} f(x = n) = \sqrt{2\pi} \sum_{m=-\infty}^{\infty} F(k = 2\pi m). \quad \text{(Poisson sum formula)}
\]

11S.2 Summary

When we put \( k = 2\pi x \) in \( I \) of Eq.(2)

\[
\sum_{n=-\infty}^{\infty} e^{2\pi i n} = 2\pi \sum_{m=-\infty}^{\infty} \delta(2\pi x - 2\pi m)
\]

\[
= \frac{2\pi}{2\pi} \sum_{m=-\infty}^{\infty} \delta(x - m)
\]

\[
= \sum_{m=-\infty}^{\infty} \delta(x - m)
\]

or
\[
\sum_{n=-\infty}^{\infty} e^{2\pi i nx} = \sum_{m=-\infty}^{\infty} \delta(x-m). \quad (3)
\]

### 11.3. Convolution of Dirac comb: another method in the derivation of Poisson sum formula

The convolution of functions \( f(x) \) and \( g(x) \) is defined by

\[
f \star g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi) g(\xi) d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi
\]

The Fourier transform of the convolution is given by

\[
\mathcal{F}\{f \star g\} = \mathcal{F}\{f\} \mathcal{F}\{g\}.
\]

Here we assume that

\[
g(x) = \sum_{n=-\infty}^{\infty} \delta(x-na). \quad \text{(Dirac comb)}
\]

The Fourier transform of \( g(x) \) is

\[
G(k) = \mathcal{F}\{g(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \sum_{n=-\infty}^{\infty} \delta(x-na) dx = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{-ikna}.
\]

The convolution \( f \star g \) is obtained as

\[
f \star g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi) \sum_{n=-\infty}^{\infty} \delta(\xi-na) d\xi = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} f(x-na). \quad (4)
\]

The Fourier transform of the convolution is

\[
\mathcal{F}\{f \star g\} = F(k)G(k) = \frac{1}{\sqrt{2\pi}} F(k) \sum_{n=-\infty}^{\infty} e^{-ikna}.
\]

We use the Poisson summation formula;
\[
\begin{align*}
\sum_{n=-\infty}^{\infty} e^{-i\pi n} & = \sum_{n=-\infty}^{\infty} e^{i\pi n} = \sum_{m=-\infty}^{\infty} \delta\left(\frac{ka}{2\pi} - m\right) \\
& = \sum_{m=-\infty}^{\infty} \delta\left[\frac{a}{2\pi}(k - \frac{2\pi m}{a})\right] \\
& = \frac{2\pi}{a} \sum_{m=-\infty}^{\infty} \delta(k - \frac{2\pi m}{a})
\end{align*}
\]

where
\[
\sum_{n=-\infty}^{\infty} e^{2\pi in} = \sum_{m=-\infty}^{\infty} \delta(x - m).
\]

with \( x = \frac{ka}{2\pi} \). Then we get
\[
\mathbf{F}[f * g] = \frac{1}{\sqrt{2\pi}} F(k) \sum_{n=-\infty}^{\infty} e^{-ikn} = \frac{1}{\sqrt{2\pi}} F(k) \sum_{n=-\infty}^{\infty} e^{ikn}
\]
\[
= \frac{1}{\sqrt{2\pi}} \frac{2\pi}{a} \sum_{m=-\infty}^{\infty} \delta(k - \frac{2\pi m}{a})F(k)
\]
\[
= \frac{1}{\sqrt{2\pi}} \frac{2\pi}{a} \sum_{m=-\infty}^{\infty} \delta(k - \frac{2\pi m}{a})F(k = \frac{2\pi m}{a})
\]

The inverse Fourier transform of \( \mathbf{F}[f * g] \) is obtained as
\[
f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{F}[f * g] e^{ikx} dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} dk \frac{2\pi}{a} \sum_{m=-\infty}^{\infty} \delta(k - \frac{2\pi m}{a})F(k = \frac{2\pi m}{a})
\]
\[
= \frac{1}{a} \sum_{m=-\infty}^{\infty} F(k = \frac{2\pi m}{a}) \int_{-\infty}^{\infty} e^{ikx} \delta(k - \frac{2\pi m}{a})
\]
\[
= \frac{1}{a} \sum_{m=-\infty}^{\infty} F(k = \frac{2\pi m}{a}) e^{\frac{i2\pi m x}{a}}
\]

where
Finally we get

\[ f \ast g = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} f(x-na) = \frac{1}{a} \sum_{m=-\infty}^{\infty} F(k = \frac{2\pi m}{a})e^{\frac{2\pi i}{a}nx}. \]  

(5)

or

\[ \sum_{n=-\infty}^{\infty} f(x-na) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} F(k = \frac{2\pi m}{a})e^{\frac{2\pi i}{a}nx}. \]

When \( a = 1 \), we get

\[ \sum_{n=-\infty}^{\infty} f(x-n) = \sqrt{2\pi} \sum_{m=-\infty}^{\infty} F(k = 2\pi m)e^{2\pi i nx}. \]  

(6)

When \( x = 0 \),

\[ \sum_{n=-\infty}^{\infty} f(x-n) = \sqrt{2\pi} \sum_{m=-\infty}^{\infty} F(k = 2\pi m). \]  

(7)

This is the Poisson sum formula.

**11S.3 Fourier transform of periodic function**

We consider a periodic function \( N(x) \);

\[ N(x+a) = N(x), \]

where \( a \) is the periodicity. The function \( N(x) \) can be described by

\[ N(x) = \sum_{n=-\infty}^{\infty} f(x-na) = \frac{\sqrt{2\pi}}{a} \sum_{m=-\infty}^{\infty} F(k = \frac{2\pi m}{a})e^{\frac{2\pi i}{a}nx} = \sum_{n} N_{e}e^{iGx}. \]
Note that $f(x)$ is defined only in the limited region (for example, $-a/2 \leq x \leq a/2$). $G$ is the reciprocal lattice defined by

$$G = \frac{2\pi}{a} m.$$

The Fourier coefficient $N_G$ is given by

$$N_G = \frac{\sqrt{2\pi}}{a} F(k = G) = \frac{1}{a} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx = \frac{1}{a} \int_{-\pi/a}^{\pi/a} e^{-ikx} f(x) dx.$$

where $f(x)$ is just like a Gaussian distribution function around $x = 0$.

Fig. Plot of $N(x)$ as a function of $x$. $a$ is the lattice constant of the one-dimensional chain.

((Example))

Suppose that $f(x)$ is given by a Gaussian distribution,
\[ f(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{x^2}{2\sigma^2}\right). \]

Then we get

\[ N_G = \frac{1}{a} \int_{-a/2}^{a/2} e^{-iGx} f(x)dx = \frac{1}{2} \exp\left(-\frac{1}{2} G^2 \sigma^2\right) \left[ \text{erf}\left(\frac{1-2iG\sigma^2}{2\sqrt{2}\sigma}\right) + \text{erf}\left(\frac{1+2iG\sigma^2}{2\sqrt{2}\sigma}\right) \right], \]

where \( \text{erf}(x) \) is the error function and is defined by

\[ \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \]

Figure shows the intensity \(|N_G|^2\) vs \(n\), where \( G = \frac{2\pi}{a} n \).

Fig. \( a = 1, \sigma = 0.1, G = \frac{2\pi}{a} n \). The intensity \(|N_G|^2\) vs \(n\) (= integer).

11S.10 Fourier series
Suppose that the function \( N(x) \) is a periodic function of \( x \) with the periodicity \( a \). Then we have

\[
N(x) = \sum_{n=-\infty}^{\infty} f(x - na) = \frac{\sqrt{2\pi}}{a} \sum_{m=-\infty}^{\infty} F(k = \frac{2\pi m}{a}) e^{\frac{2\pi i m x}{a}}
\]

\[
= \sum_{G} N_G e^{iGx}
\]

where \( G = \frac{2\pi m}{a} \) (\( m \): integer).

((Example-1))

\[
f(x) = x \text{ for } |x| < a/2, \quad \text{with} \quad a = 2
\]

\[
F(k = G = \pi n) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} x \exp(-i\pi n x) dx = \frac{2i(-1)^n}{n\pi \sqrt{2\pi}} \quad \text{for } n \neq 0.
\]

\[
F(k = G = 0) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} x dx = 0 \quad \text{for } n = 0.
\]

\[
N(x) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{i(-1)^n}{n} e^{i\pi n x} = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{i(-1)^n}{n} e^{i\pi n x}
\]

We make a plot of \( N(x) \) as a function of \( x \) for the summation for \( n = -n_{\text{max}} \) and \( n_{\text{max}} \) with \( n_{\text{max}} = 50 \).
Fig. $n_{\text{max}} = 50$. The Gibbs phenomenon is clearly seen.

**((Example-2))**

\[ f(x) = 0 \text{ for } -1 < x < 0 \text{ and } 1 \text{ for } 0 < x < 1. \quad a = 2 \]

\[ F(k = G = \pi n) = \frac{1}{\sqrt{2\pi}} \int_0^1 \exp(-i\pi nx)dx = -\frac{i(-1)^n[-1 + (-1)^n]}{n\pi \sqrt{2\pi}} \quad \text{for } n \neq 0. \]

\[ F(k = G = 0) = \frac{1}{\sqrt{2\pi}} \int_0^1 dx = \frac{1}{\sqrt{2\pi}}, \quad \text{for } n = 0. \]

\[ N(x) = \frac{1}{2} - \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{i(-1)^n[-1 + (-1)^n]}{2n} e^{i n x} \]
Fig. \( n_{\text{max}} = 100 \). The Gibbs phenomenon is clearly seen.

11S.11 Fourier transform of function having values at integer \( x \)-value

We consider a function defined by

\[
f(x) = \sum_{m=-\infty}^{\infty} f(m) \delta(x - m).
\]  

(8)

Fig. Plot of \( f(x) \) which is described by a combination of the Dirac delta function with \( f(m) \) at \( x = m \).

The Fourier transform of \( f(x) \) is given by
\[ F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ikx} \, dx \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(m)\delta(x-m)e^{-ikx} \, dx \]

\[ = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} f(m)\int_{-\infty}^{\infty} \delta(x-m)e^{-ikx} \, dx \]

\[ = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-ikm} f(m) \]

We note that

\[ F(k + 2\pi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-i(k+2\pi)m} f(m) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-ikm} f(m) = F(k), \]

In other words, \( F(k) \) is a periodic function of \( k \) with a periodicity \( 2\pi \). We can also show that

\[ f(m) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(k)e^{imk} \, dk, \quad (9) \]

where

\[ \int_{-\pi}^{\pi} e^{ik(m-m')} \, dk = 2\pi\delta_{m,m'}, \quad (10) \]

since

\[ \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(k)e^{ikm} \, dk = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \sum_{m'=\infty}^{\infty} f(m')\int_{-\pi}^{\pi} e^{ik(m-m')} \, dk \]

\[ = \frac{1}{2\pi} \sum_{m'=\infty}^{\infty} f(m')2\pi\delta_{m,m'} \]

\[ = f(m) \]

Note that the inverse Fourier transform of \( F(k) \) is obtained as
\begin{align*}
f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)e^{ikx} \, dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikx} \frac{1}{\sqrt{2\pi}} \sum_{m=\infty}^{\infty} e^{-ikm} f(m) \\
&= \frac{1}{2\pi} \sum_{m=\infty}^{\infty} f(m) \int_{-\infty}^{\infty} dk e^{ik(x-m)} \\
&= \frac{1}{2\pi} \sum_{m=\infty}^{\infty} f(m) 2\pi \delta(x-m) \\
&= \sum_{m=\infty}^{\infty} f(m) \delta(x-m)
\end{align*}

(\textbf{Note}) The proof where the Poisson summation formula is used.

We have another method for the derivation of Eq.\((9)\):

\begin{align*}
f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)e^{ikx} \, dk = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \int_{(2n-1)\pi}^{(2n+1)\pi} F(k)e^{ikx} \, dk \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi nk} \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(k')e^{ik'x} \, dk' \\
&= \sum_{n=-\infty}^{\infty} \delta(x-m) \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(k)e^{ikx} \, dk \\
&= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \delta(x-m) \int_{-\pi}^{\pi} F(k)e^{ikm} \, dk \\
&= \sum_{n=-\infty}^{\infty} \delta(x-m) f(m)
\end{align*}

Here we use the Poisson summation formula
\[ \sum_{n=-\infty}^{\infty} e^{i2\pi n} = \sum_{m=-\infty}^{\infty} \delta(x - m). \]

### 11S.11 Random walk in the one-dimensional chain

We consider the case when a particle moves on the one-dimensional chain. The particle can be located only on a discrete position \([x = m \ (m; \text{integer})]\). the particle starts from the origin \(x = 0\). The particle can move from \(x = m\) to \(x = m+1\), or from \(x = m\) to \(x = m-1\) for each jump. We assume that the probability \(W(m, N)\) such that the particle reaches at the position \(x = m\) after jumps with \(N\) times. The probability \(W(m, N+1)\) can be described by

\[
W(m, N + 1) = \frac{1}{2} W(m - 1, N) + \frac{1}{2} W(m + 1, N),
\]

where one is a jump from \(x = m - 1\) to \(x = m\) for the \((N+1)\)-th jump, and the another is a jump from \(x = m + 1\) to \(x = m\) for the \((N+1)\)-th jump. The probability for these jumps is 1/2.

In order to find the expression of \(W(m, N)\) we define the Fourier transform

\[
\Phi(k, N) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-ikm} W(m, N).
\]

Note that \(\Phi(k, N)\) is a periodic function of \(k\) with a periodicity \(2\pi\), and

\[
\frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-ikm} W(m - 1, N) = \frac{1}{\sqrt{2\pi}} e^{-ik} \sum_{m=-\infty}^{\infty} e^{-ik(m-1)} W(m - 1, N)
\]

\[= e^{-ik} \Phi(k, N).\]
\[
\frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-ikm} W(m+1, N) = \frac{1}{\sqrt{2\pi}} e^{ik} \sum_{m=-\infty}^{\infty} e^{-ik(m+1)} W(m+1, N).
\]

Then we have

\[
\Phi(k, N+1) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-ikm} W(m, N+1)
= \frac{1}{2} (e^{-ik} + e^{-ik}) \Phi(k, N)
= (\cos k) \Phi(k, N)
\]

or

\[
\Phi(k, N) = (\cos k)^N \Phi(k, N = 0).
\]

We use the initial condition that the particle is located at \(x = 0\) at the probability of 1;

\[
W(m, N = 0) = \delta_{m,0}.
\]

or

\[
\Phi(k, N = 0) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-ikm} W(m, N = 0) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-ikm} \delta_{m,0} = \frac{1}{\sqrt{2\pi}}.
\]

So we get

\[
\Phi(k, N) = \frac{1}{\sqrt{2\pi}} (\cos k)^N.
\]
Fig. Plot of $\Phi(k, N)$ vs $k$, where $N = 200, 400, 600, 800,$ and $1000$. In the large limit of $N$, $\Phi(k, N)$ approaches a Gaussian distribution.

The inverse Fourier transform of $\Phi(k, N)$ is

$$W(m, N) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \Phi(k, N) e^{ikm} dk = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} dke^{ikm} \frac{1}{\sqrt{2\pi}} (\cos k)^N$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} dke^{ikm}(\cos k)^N$$

Here we use the binomial theorem to get

$$(\cos k)^N = \left(\frac{1}{2}\right)^N (e^{ik} + e^{-ik})^N$$

$$= \left(\frac{1}{2}\right)^N \sum_{l=0}^{N} \frac{N!}{(N-l)!l!} e^{ik(2l-N)}$$

Then we have
\[ W(m, N) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dke^{ikm} \left( \frac{1}{2} \right)^N \sum_{l=0}^{N} \frac{N!}{(N-l)!l!} e^{ik(2l-N)} \]

\[ = \left( \frac{1}{2} \right)^N \sum_{l=0}^{N} \frac{N!}{(N-l)!l!} \frac{1}{2\pi} \int_{-\pi}^{\pi} dke^{ik(m+2l-N)} \]

\[ = \left( \frac{1}{2} \right)^N \sum_{l=0}^{N} \frac{N!}{(N-l)!l!} \frac{1}{2\pi} (2\pi)\delta_{m+2l-N,0} \]

\[ = \left( \frac{1}{2} \right)^N \sum_{l=0}^{N} \frac{N!}{(N-l)!l!} \delta_{m+2l-N,0} \]

\[ = \left( \frac{1}{2} \right)^N \sum_{l=0}^{N} \frac{N!}{(N-l)!l!} \frac{1}{2\pi} \int_{-\pi}^{\pi} dke^{ikm} (\cos k)^N \]

\[ = \frac{1}{2}\left( \frac{N+m}{2} \right) \left( \frac{N-m}{2} \right)! \]

where

\[ m = N - 2l, \quad \text{or} \quad l = \frac{N - m}{2}. \]

This implies that for \( N = \text{even}, m \) should be even. In other words, \( W(N = \text{even}, m = \text{odd}) = 0. \)

For \( N = \text{odd}, m \) should be odd. In other words, \( W(N = \text{odd}, m = \text{even}) = 0. \)

**11S.13 Numerical calculation of \( W(m, N) \)**

Using the Mathematica, we calculate numerically the value of \( W(m, N) \) given by

\[ W(m, N) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dke^{ikm} (\cos k)^N = \frac{1}{\pi} \int_{0}^{\pi} dk \cos(km)(\cos k)^N, \]

for the cases with \( N = 12 \) (even) and \( N = 13 \) (odd).

(i) \( N = 12. \)
This figure clearly shows that $W(m, N = 12) = 0$ for $m = -11, -9, -7, -5, -3, -1, 1, 3, 5, 7, 9,$ and 11 ($m = \text{odd}$).

(ii) $N = 13$.

This figure clearly shows that $W(m, N = 13) = 0$ for $m = -12, -10, -8, -6, -4, -2, 0, 2, 4, 6, 8,$ 10, and 12 ($m = \text{even}$).

((Mathematica))
Using the Stirling's formula,
\[\ln(x!) = \frac{1}{2} \ln(2\pi) + (x + \frac{1}{2})\ln(x) - x \quad \text{for large } x,\]

and the Mathematica, we have the series expansion to the order of \((m/N)^8\),

\[
\ln W(m, N) = -N \ln(2) + \ln(N!) - \ln\left(\frac{N + m}{2}\right)! - \ln\left(\frac{N - m}{2}\right)!
\approx -\frac{1}{2} \ln\left(\frac{\pi N}{2}\right) + \frac{1}{2} \left(1 - N\right)\left(\frac{m}{N}\right)^2 + \frac{1}{12} \left(3 - N\right)\left(\frac{m}{N}\right)^4
\]
\[+ \frac{1}{50} \left(5 - N\right)\left(\frac{m}{N}\right)^6 + \frac{1}{56} \left(7 - N\right)\left(\frac{m}{N}\right)^8 + ...\]
\[
\approx -\frac{1}{2} \ln\left(\frac{\pi N}{2}\right) - \frac{1}{2} \frac{m^2}{N}
\]

or

\[
W(m, N) = \exp\left[-\frac{1}{2} \ln\left(\frac{\pi N}{2}\right) - \frac{1}{2} \frac{m^2}{N}\right]
\]
\[
= \exp\left[\ln\left(\frac{\pi N}{2}\right)^{-\frac{1}{2}} - \frac{1}{2} \frac{m^2}{N}\right]
\]
\[
= \left(\frac{\pi N}{2}\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{m^2}{N}\right)
\]
\[
= \left(\frac{2}{\pi N}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{m^2}{N}\right)
\]

((Mathematica))
We assume that the distance between the nearest neighbor lattices is \( \Delta x \) and the time taken for each jump is \( \Delta t \). Then we have

\[
\Delta x = \frac{N}{2}, \\
\Delta t = \frac{1}{2} \log[2\pi] + \left(x + \frac{1}{2}\right) \log[x] - x;
\]

\[
g_l = -N \log[2] + f_1[N] - f_1 \left[ \frac{N + m}{2} \right] - f_1 \left[ \frac{N - m}{2} \right];
\]

\[
\text{rule1} = \{ m \to N \}
\]

\[
g_{11} = g_1 / . \text{rule1}; \text{Series}[g_{11}, \{x, 0, 8\}] // \text{Simplify} // \text{Normal}
\]

\[
\frac{1}{2} \left( 1 - N \right) x^2 + \frac{1}{12} \left( 3 - N \right) x^4 + \frac{1}{30} \left( 5 - N \right) x^6 + \frac{1}{56} \left( 7 - N \right) x^8 - \frac{1}{2} \log \left[ \frac{N \pi}{2} \right]
\]

### 11S.15 Diffusion

We assume that the distance between the nearest neighbor lattices is \( \Delta x \) and the time taken for each jump is \( \Delta t \). Then we have

\[
t = N \Delta t, \quad x = m \Delta x.
\]

The probability of finding particle between \( x \) and \( x + dx \) is

\[
P(x, t)dx = \exp\left( \frac{x^2}{4D} \right) \cdot \frac{1}{\sqrt{4D \pi t}} \cdot \exp\left( -\frac{1}{4D} \frac{x^2}{\Delta t} \right) dx
\]

where \( D \) is the diffusion constant and is defined by

\[
D = \frac{(\Delta x)^2}{2 \Delta t}.
\]
Here we note that the factor $2\Delta x$ of the $W\left(\frac{x}{\Delta x}, \frac{t}{\Delta t}\right) \frac{dx}{2\Delta x}$ arises from the fact that (i) for every jump ($N = \text{even}$), the probability of finding the particle at the sites with even $m$ is zero, and that (ii) for every jump ($N = \text{odd}$), the probability of finding the particle at the sites with odd $m$ is zero. The distance for the jump is $2\Delta x$, but not $\Delta x$.

The final form of $P(x, t)$ is obtained as

$$P(x, t) = \frac{1}{\sqrt{4D\pi t}} \exp\left(-\frac{x^2}{4Dt}\right).$$

We note that $P(x, t)$ satisfies the diffusion equation given by

$$\frac{\partial P(x, t)}{\partial t} = D \frac{\partial^2 P(x, t)}{\partial x^2},$$

with the initial condition

$$P(x, t = 0) = \delta(x).$$

11.S16  Gaussian distribution in magnetization; analogy of random walk

We consider a system consisting of $N$ independent spins. Each spin has a magnetic moment $\mu$. In the absence of an external magnetic field, each spin has the magnetic moment ($\pm \mu$) along the $z$ axis. We assume that the number of spins having the $z$ component magnetic moment ($+\mu$) is $N_\uparrow$ and the number of spins having the $z$-component magnetic moment ($-\mu$):

$$N_\uparrow = \frac{1}{2}(N + n), \quad N_\downarrow = \frac{1}{2}(N - n)$$

where

$$N = N_\uparrow + N_\downarrow.$$

Here we discuss the probability distribution of total magnetic moment, $M$, which is given by

$$\text{...}$$
$M = \mu (N_\uparrow - N_\downarrow) = n\mu$.

The probability that the total magnetization has $M = n\mu$ is obtained as

$$W(M) = \frac{1}{2^N} \frac{N!}{N^!N^!} = \frac{1}{2^N} \frac{N!}{[\frac{1}{2}(N+n)]![\frac{1}{2}(N-n)]!}$$

Using the Stirling's formula, we have

$$\ln W(M) = -\frac{1}{2} \ln(\frac{\pi N}{2}) - \frac{n^2}{2N}$$

for $n<<N$. Then we get the probability as

$$W(M) = \left(\frac{2}{\pi N}\right)^{1/2} \exp\left(-\frac{n^2}{2N}\right)$$

where $M = \mu n$. The average of magnetization is equal to zero.

$$<M> = \mu <n> = \mu \int \int nW(M)dn = 0.$$ 

Since

$$<M^2> = \mu^2 <n> = \mu^2 \int \int n^2W(M)dn = 2\mu^2 N,$$

the standard deviation is obtained as

$$\Delta M = \sqrt{<M^2> - <M>^2} = \sqrt{2N}\mu.$$

Since

$$\frac{\Delta M}{2N\mu} = \frac{\sqrt{2N}}{2N} = \frac{1}{\sqrt{2N}}.$$
the relative width of the Gaussian distribution becomes sharp as $N$ increases. We make a plot of $W(M = n\mu)$ as a function of $N$, where $N = 100$.

Fig. Plot of $W(M)\text{ vs } n. M = 2\mu n. N = 100.$

REFERENCES