Blackbody problem: Maxwell's equation
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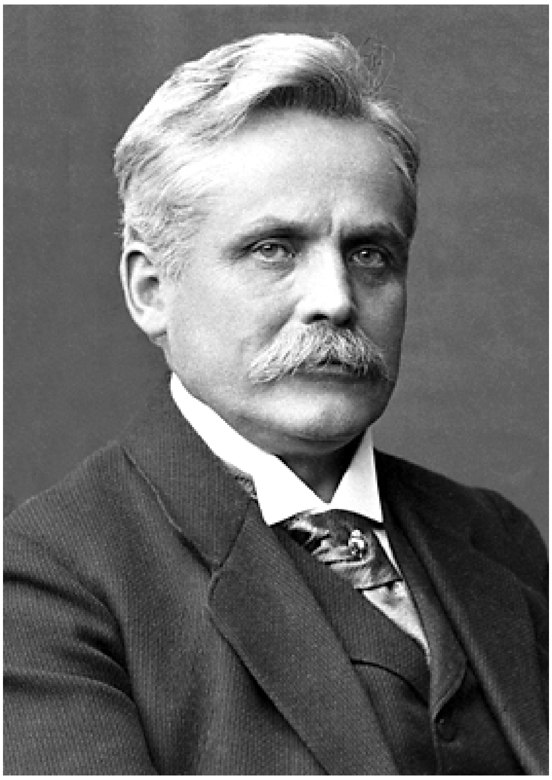
One of the most puzzling phenomena around 1900 was the spectral distribution of blackbody radiation. A blackbody is an ideal system that absorbs all the radiation incident on it. Max Planck proposed his theory that could explain the experimental data at all wavelengths. He assumed that the energy emitted and absorbed by the blackbody is not continuous but is instead emitted or absorbed in quanta. The size of an energy quantum is proportional to the frequency of the radiation.

Max Planck (April 23, 1858 – October 4, 1947) was a German physicist. He is considered to be the founder of the quantum theory, and thus one of the most important physicists of the twentieth century. Planck was awarded the Nobel Prize in Physics in 1918.

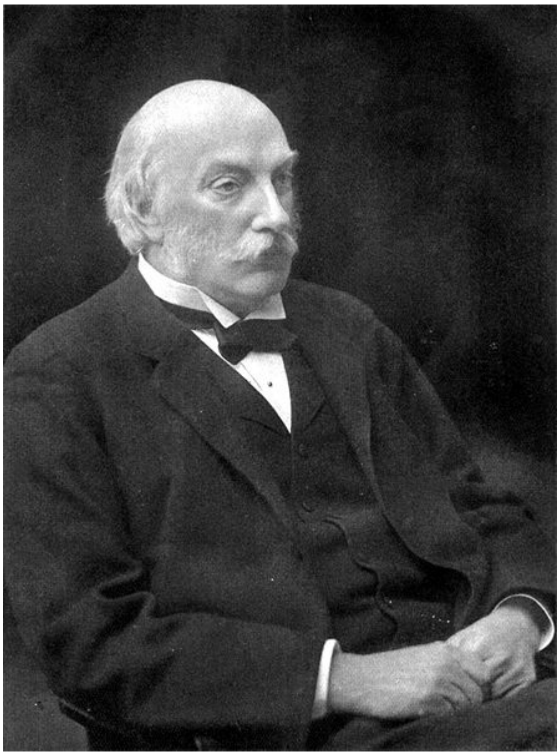
http://en.wikipedia.org/wiki/Max_Planck

Wilhelm Carl Werner Otto Fritz Franz Wien (13 January 1864 – 30 August 1928) was a German physicist who, in 1893, used theories about heat and electromagnetism to deduce Wien's displacement law, which calculates the emission of a blackbody at any temperature from the emission at any one reference temperature. He also formulated an expression for the black-body radiation which is correct in the photon-gas limit. His arguments were based on the notion of adiabatic invariance, and were instrumental for the formulation of quantum mechanics. Wien received the 1911 Nobel Prize for his work on heat radiation.



http://en.wikipedia.org/wiki/Wilhelm_Wien

John William Strutt, 3rd Baron Rayleigh, OM (12 November 1842 – 30 June 1919) was an English physicist who, with William Ramsay, discovered the element argon, an achievement for which he earned the Nobel Prize for Physics in 1904. He also discovered the phenomenon now called Rayleigh scattering, explaining why the sky is blue, and predicted the existence of the surface waves now known as Rayleigh waves. In 1910 Lord Rayleigh discovered that an electrical discharge in nitrogen gas produced "active nitrogen", an allotrope considered to be monatomic. The "whirling cloud of brilliant yellow light" produced by his apparatus reacted with quicksilver to produce explosive mercury nitride.



http://en.wikipedia.org/wiki/John_Strutt,_3rd_Baron_Rayleigh

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1 Blackbody problem

We start with the Maxwell's equation

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

We assume that

$$\mathbf{E} = \mathrm{Re}[\tilde{\mathbf{E}}_0 e^{-i\omega t}]$$
$$\mathbf{B} = \mathrm{Re}[\tilde{\mathbf{B}}_0 e^{-i\omega t)}]$$

$$\nabla \cdot \tilde{\mathbf{E}}_0 = 0$$
$$\nabla \cdot \tilde{\mathbf{B}}_0 = 0$$
$$\nabla \times \tilde{\mathbf{E}}_0 = i\omega \tilde{\mathbf{B}}_0$$

$$\nabla \times \tilde{\mathbf{B}}_0 = -i\frac{\omega}{c^2}\tilde{\mathbf{E}}_0$$

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -\nabla^2\mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}$$

or

$$\nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}$$

or

$$\nabla^2\tilde{\mathbf{E}}_0 + k^2\tilde{\mathbf{E}}_0 = 0$$

with $\omega = ck$. Similarly, we have

$$\nabla^2\mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla^2\tilde{\mathbf{B}}_0 + k^2\tilde{\mathbf{B}}_0 = 0$$

We now consider an electromagnetic wave in the closed cube with side *L*.

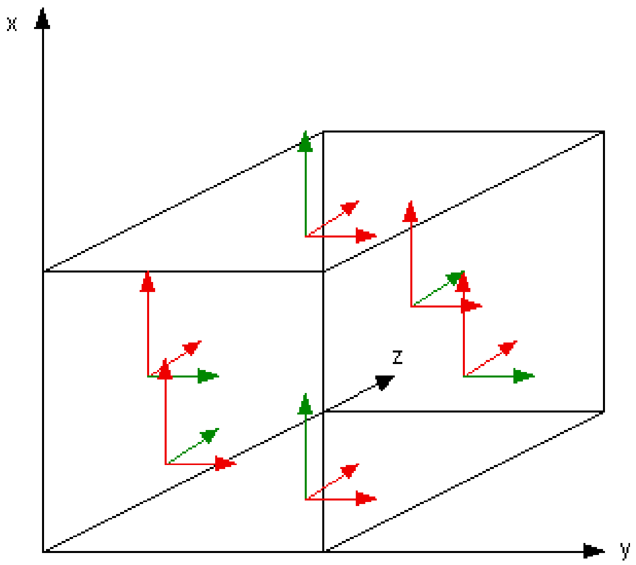


Fig. Boundary condition for the electric field (red) (tangential component continuous)) and the magnetic field (green) (normal component continuous).

From the boundary conditions we have

$$E_x = E_1 \begin{pmatrix} \sin(k_1 x) \\ \cos(k_1 x) \end{pmatrix} \begin{pmatrix} \sin(k_2 y) \\ \end{pmatrix} \begin{pmatrix} \sin(k_3 z) \\ \end{pmatrix}$$

$$E_y = E_2 \begin{pmatrix} \sin(k_1 x) \\ \end{pmatrix} \begin{pmatrix} \sin(k_2 y) \\ \cos(k_2 y) \end{pmatrix} \begin{pmatrix} \sin(k_3 z) \\ \end{pmatrix}$$

$$E_z = E_3 \begin{pmatrix} \sin(k_1 x) \\ \end{pmatrix} \begin{pmatrix} \sin(k_2 y) \\ \end{pmatrix} \begin{pmatrix} \sin(k_3 z) \\ \cos(k_3 y) \end{pmatrix}$$

where

$$k_1 = \frac{\pi}{L} n_x , \qquad k_2 = \frac{\pi}{L} n_y , \qquad k_3 = \frac{\pi}{L} n_z$$

$(n_{\mathrm{x}}, n_{\mathrm{y}}, n_{\mathrm{z}} = 1, 2, 3, \ldots)$

Note that

$E_{\mathrm{x}} = 0$ for $y = 0$ and $y = L$ planes and $z = 0$ and $z = L$ planes.
$E_{\mathrm{y}} = 0$ for $z = 0$ and $z = L$ planes and $x = 0$ and $x = L$ planes.
$E_{\mathrm{z}} = 0$ for $x = 0$ and $x = L$ planes and $y = 0$ and $y = L$ planes.

From the condition

$$\nabla \cdot \tilde{\mathbf{E}} = 0$$

we have

$$E_x = E_1 \cos(k_1 x) \sin(k_2 y) \sin(k_3 z) ,$$

$$E_y = E_2 \sin(k_1 x) \cos(k_2 y) \sin(k_3 z) ,$$

$$E_z = E_3 \sin(k_1 x) \sin(k_2 y) \cos(k_3 y)$$

From the condition

$$\nabla \times \tilde{\mathbf{E}}_0 = i\omega \tilde{\mathbf{B}}_0 ,$$

we have

$$B_x = B_1 \sin(k_1 x) \cos(k_2 y) \cos(k_3 z) ,$$

$$B_y = B_2 \cos(k_1 x)\sin(k_2 y)\cos(k_3 z)\,,$$

$$B_z = B_3 \cos(k_1 x)\cos(k_2 y)\sin(k_3 z)$$

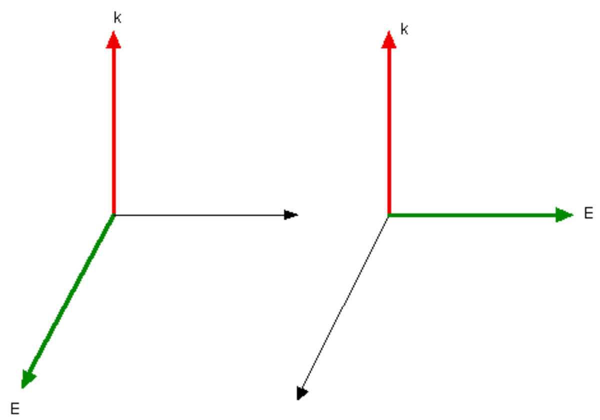
where

$B_x = 0$	for $x = 0$ and $x = L$ planes
$B_y = 0$	for $y = 0$ and $y = L$ planes.
$B_z = 0$	for $z = 0$ and $z = L$ planes.

We note that

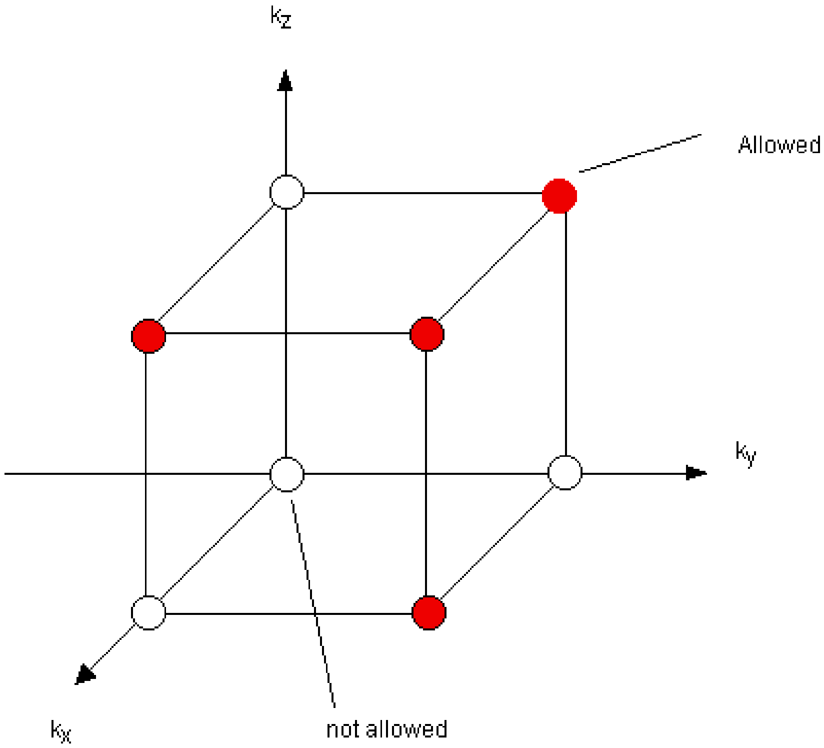
$$\nabla.\mathbf{E} = (E_1 k_1 + E_2 k_2 + E_3 k_3)\sin(k_1 x)\sin(k_2 y)\sin(k_3 z) = 0$$

This means that the vector (E_1, E_2, E_3) is perpendicular to the wave vector $\boldsymbol{k} = (k_1, k_2, k_3)$. For each $\boldsymbol{k}$, there are two independent directions for (E_1, E_2, E_3); polarization.



2. Density of states for the modes

Since $E_1 k_1 + E_2 k_2 + E_3 k_3 = 0$, only one of k_1,k_2, k_3 can be zero at a time. Since if two or three are zero, $E_1 = E_2 = E_3 = 0$. There is no electromagnetic field in the cavity. Each set of integers (n_x, n_y, n_z) defines a mode of the radiation field and corresponds to two degrees of freedom of the field when two polarization directions are taken into account.



There are 2 states per $\left(\frac{\pi}{L}\right)^3$.

$$\omega = ck = c\sqrt{k_x^{\;2} + k_y^{\;2} + k_z^{\;2}}$$

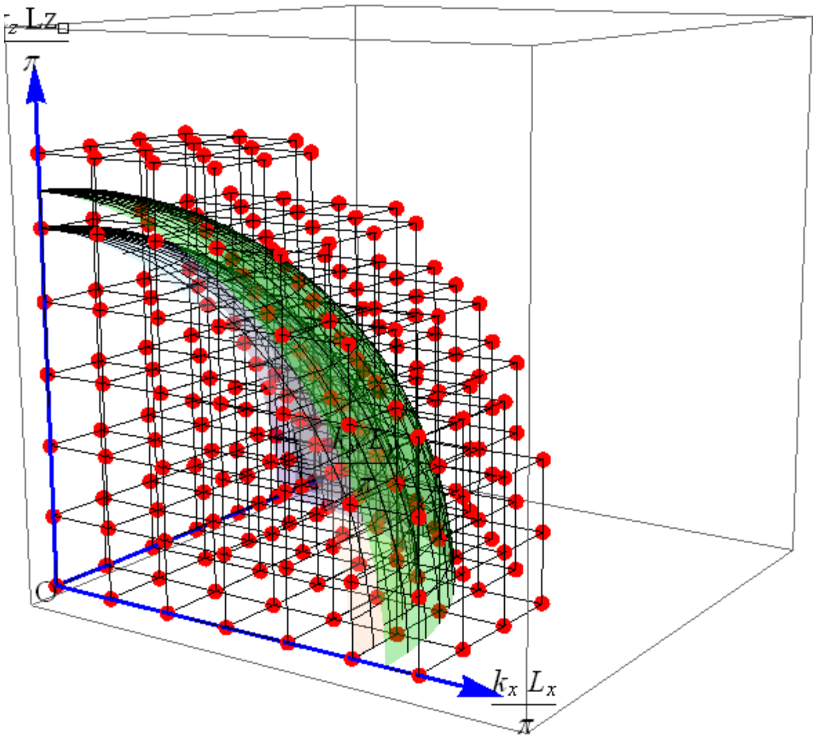
or

$$\frac{\omega^2}{c^2} = k_x^{\;2} + k_y^{\;2} + k_z^{\;2}$$

The density of states (k to k +dk)

$$\rho_k dk = \frac{1}{8}\frac{4\pi k^2 dk}{\left(\frac{\pi}{L}\right)^3}2 = \frac{Vk^2 dk}{\pi^2}$$

where $V = L^3$.



Since $\omega = ck$,

$$\rho_\omega d\omega = \frac{V\left(\frac{\omega}{c}\right)^2 d\frac{\omega}{c}}{\pi^2} = \frac{V\omega^2 d\omega}{\pi^2 c^3}$$

of modes having their frequencies between ω and ω+dω.

$$\rho_\omega = \frac{V\omega^2}{\pi^2 c^3} = VD(\omega) \quad \text{(density of modes)}$$

where c is the velocity of light and

$$D(\omega) = \frac{\omega^2}{\pi^2 c^3}$$

We have the following formula;

$$\sum_k \to \int \rho_k dk$$

or

$$\sum_k \to \int \rho_\omega d\omega = V\int D(\omega)d\omega$$

For single mode $|\mathbf{k}\rangle$, the energy is given by

$$E_{n.\mathbf{k}} = (n_k + \frac{1}{2})\hbar\omega_{\mathbf{k}} .$$

We use the Planck distribution. The total energy is given by

$$E_{tot} = \sum_{\mathbf{k}} n_{\mathbf{k}}\hbar\omega_{\mathbf{k}} = \int \frac{V\omega^2}{\pi^2 c^3} d\omega n_\omega \hbar\omega = V\int u(\omega)d\omega$$

or the energy density by

$$\frac{E_{tot}}{V} = \int_0^\infty u(\omega)d\omega = \int_0^\infty u(\lambda)d\lambda$$

where

$$u(\omega) = \overline{W_T} = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{\exp(\frac{\hbar\omega}{k_B T})-1} = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{\exp(x)-1} = \frac{k_B{}^3 T^3}{\pi^2\hbar^2 c^3}\frac{x^3}{\exp(x)-1}$$

(Planck's law for the radiation energy density). It is clear that

$$\frac{u(\omega)}{\frac{k_B{}^3 T^3}{\pi^2\hbar^2 c^3}} = f(x) = \frac{x^3}{\exp(x)-1}$$

is dependent on a variable x given by

$$x = \frac{\hbar\omega}{k_B T} .$$

(scaling relation). The experimentally observed spectral distribution of the black body radiation is very well fitted by the formula discovered by Planck.

(1) Region of Wien ($x = \frac{\hbar\omega}{k_B T} >> 1$),

$$u_W(\omega) = \frac{k_B{}^3 T^3}{\pi^2 \hbar^2 c^3} x^3 e^{-x}$$

(2) Region of Rayleigh-Jeans ($x = \frac{\hbar\omega}{k_B T} >> 1$),

$$u_{RJ}(\omega) = \frac{k_B{}^3 T^3}{\pi^2 \hbar^2 c^3} \frac{x^3}{\exp(x) - 1} \approx \frac{k_B{}^3 T^3}{\pi^2 \hbar^2 c^3} x^2$$

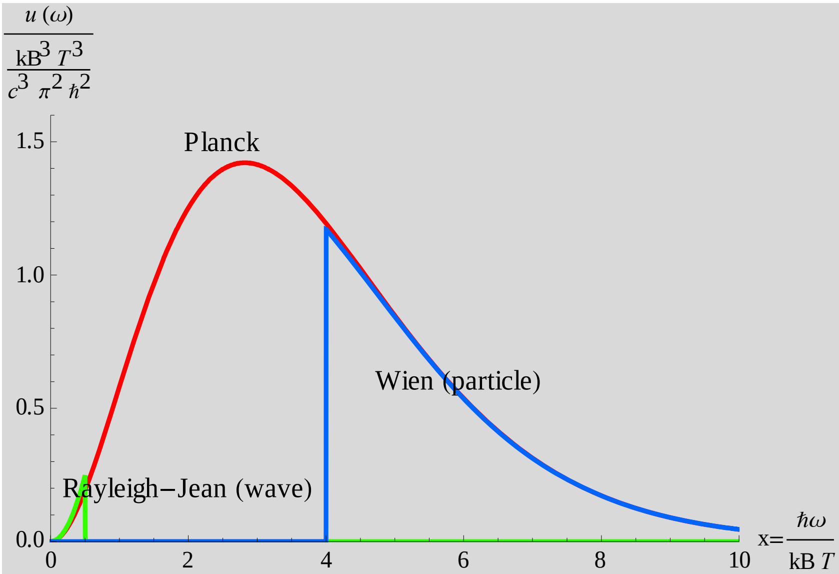


Fig. Scaling plot of $f(x)$ vs x for the Planck's law for the energy density of electromagnetic radiation at angular frequency ω and temperature T. Planck (red). Wien (blue, particle-like). Rayleigh-Jean (green, wave-like).

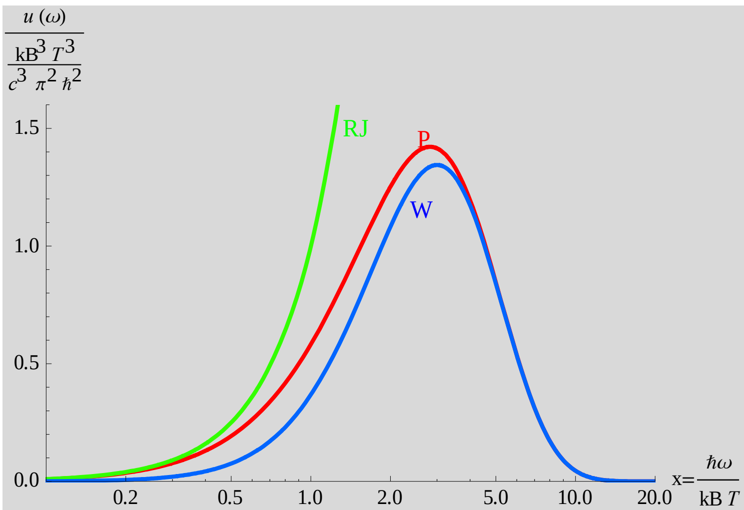


Fig. Scaling plot of Planck's law. Wien's law, and Rayleigh-Jean's law.

3. Deivation of $u(\lambda, T)$

$$\int_0^\infty u(\omega)d\omega = \int_0^\infty \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{\exp(\frac{\hbar\omega}{k_B T})-1}d\omega$$

Since $\omega = \frac{2\pi c}{\lambda}$, $\qquad d\omega = -2\pi c\frac{d\lambda}{\lambda^2}$

$$\int_0^\infty u(\omega)d\omega = \int_0^\infty \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{\exp(\frac{\hbar\omega}{k_B T})-1}d\omega$$

$$= \int_0^\infty \frac{\hbar\left(\frac{2\pi c}{\lambda}\right)^3}{\pi^2 c^3}\frac{1}{\exp(\frac{2\pi\hbar c}{\lambda k_B T})-1}2\pi c\frac{d\lambda}{\lambda^2}$$

or

$$\int_0^\infty u(\omega)d\omega = \int_0^\infty u(\lambda)d\lambda = \int_0^\infty 16\pi^2\hbar c\frac{1}{\lambda^5}\frac{1}{\exp(\frac{2\pi\hbar c}{\lambda k_B T})-1}d\lambda$$

Then we have

$$u(\lambda) = \frac{16\pi^2\hbar c}{\lambda^5}\frac{1}{\exp(\frac{2\pi\hbar c}{\lambda k_B T})-1}$$

where

$\hbar$ = 1.054571596 x 10^{-27} erg s, $\qquad k_B$ = 1.380650324 x 10^{-16} erg/K
c = 2.99792458 x 10^{10} cm/s.
J = 10^7 erg

4. Wien's displacement law

$u(\lambda)$ has a maximum at

$$\frac{2\pi\hbar c}{\lambda k_B T} = 4.96511, \qquad \text{(dimensionless)}$$

or

$$\lambda = \frac{0.28977}{T(K)} \qquad (\lambda \text{ in the units of cm})$$

or

$$\lambda = \frac{2.897768551}{T(K)} \times 10^6 . \qquad (\lambda \text{ in the units of nm})$$

T is the temperature in the units of K. λ is the wave-length in the unit of nm

T(K)	λ (nm)
1000	2897.77
1500	1931.85
2000	1448.89
2500	1159.11
3000	965.924
3500	827.935
4000	724.443
4500	643.949
5000	579.554
5500	526.867
6000	482.962
6500	445.811
7000	413.967
7500	386.369
8000	362.221
8500	340.914
9000	321.975
9500	305.029
10 000	289.777

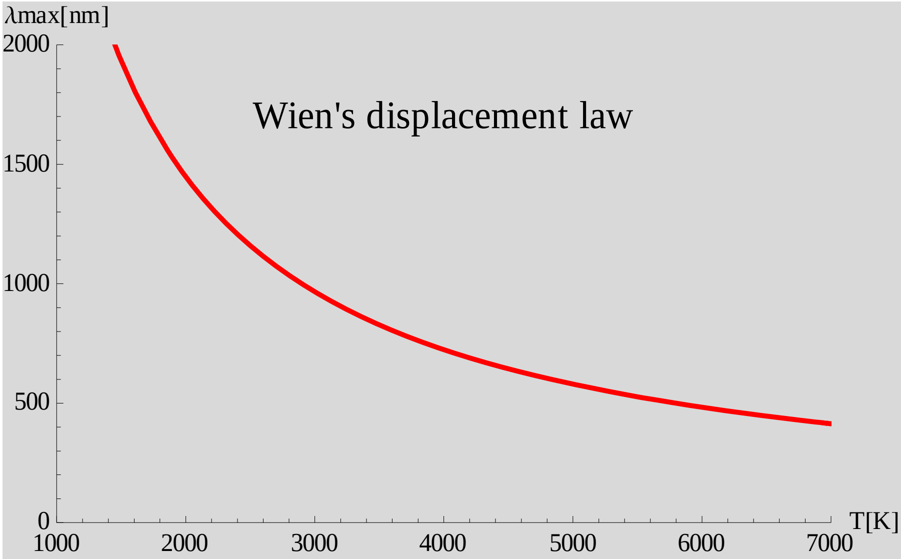


Fig. Wien's displacement law. The peak wavelength vs temperature T(K).

5. Rate of the energy flux density

It is assumed that the thermal equilibrium of the electromagnetic waves is not disturbed even when a small hole is bored through the wall of the box. The area of the hole is dS. The energy which passes in unit time through a solid angle dΩ, making an angle θ with the normal to dS is

$$J(\lambda, T, \theta) d\lambda d\Omega dS = cu(\lambda, T) d\lambda \cos\theta \frac{d\Omega}{4\pi} dS,$$

where c is the velocity of light. The right hand side is divided by 4π, because the energy density u comprises all waves propagating along different directions. The emitted energy unit time, per unit area is

$$\begin{aligned}\iint J(\lambda, T, \theta) d\lambda d\Omega &= \int cu(\lambda, T) d\lambda \int \cos\theta \frac{d\Omega}{4\pi} = \int \frac{cu(\lambda, T)}{4} d\lambda \\ &\equiv \frac{c}{4} \int u(\lambda, T) d\lambda \\ &= \frac{c}{4} \varepsilon\end{aligned}$$

where

$$\varepsilon = \int u(\lambda, T) d\lambda,$$

$$\int \cos\theta \frac{d\Omega}{4\pi} = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta d\theta$$

$$= \frac{1}{4\pi} 2\pi \int_0^{\pi/2} \frac{1}{2}\sin(2\theta) d\theta$$

$$= \frac{1}{4}\frac{1}{2}[-\cos(2\theta)]|_0^{\pi/2} = \frac{1}{4}$$

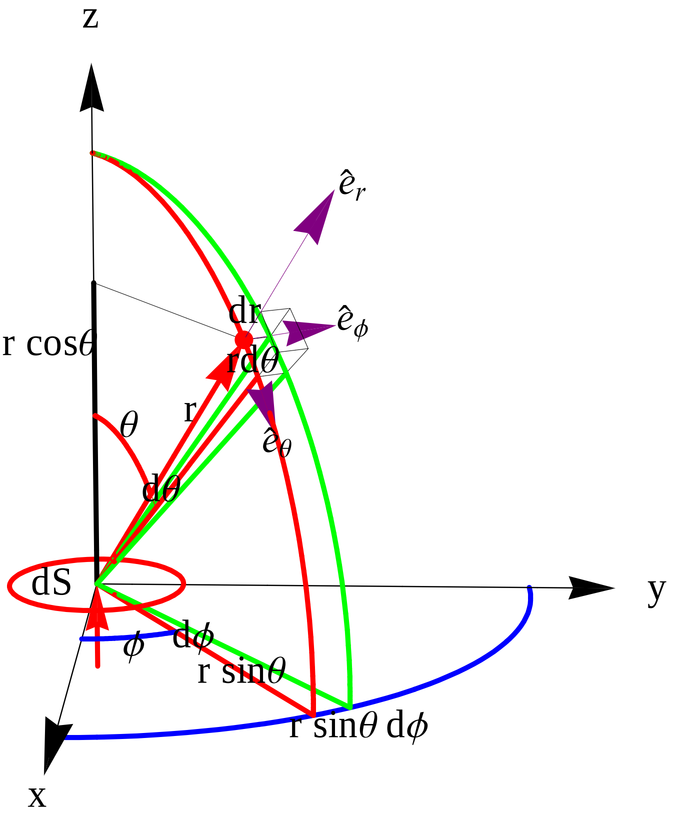


Fig. Radiation intensity is used to describe the variation of radiation energy with direction.

In other words, the geometrical factor is equal to 1/4. Then we have a measure for the intensity of radiation (the rate of energy flux density);

$$S(\lambda, T) = \frac{cu(\lambda, T)}{4} = \frac{4\pi^2 \hbar c^2}{\lambda^5} \frac{1}{\exp(\frac{2\pi\hbar c}{\lambda k_B T}) - 1}$$

where

$S(\lambda, T)d\lambda$ = power radiated per unit area in $(\lambda, \lambda + d\lambda)$

Unit

$$[\hbar c^2 \frac{1}{\lambda^5}] = \frac{erg.s}{cm^5}\frac{cm^2}{s^2} = \frac{erg}{cm^3}\frac{1}{s} = \frac{10^{-7} J}{(10^{-2} m)^3}\frac{1}{s} = 10^{-1}\frac{W}{m^3} = [\frac{W}{m^3}]$$

The energy flux density $S(\lambda, T)$ is defined as the rate of energy emission per unit area.

((Note)) The unit of the poynting vector <S> is [W/m^2]. $\langle S\rangle$ is the energy flux (energy per unit area per unit time).

(1) Rayleigh-Jeans law (in the long-wavelength limit)

$$S_{RJ}(\lambda) = \frac{1}{4}cu_{RJ}(\lambda) = 4\pi^2\hbar c\frac{1}{\lambda^5}\frac{1}{\frac{2\pi\hbar c}{\lambda k_B T}} = \frac{2\pi c k_B T}{\lambda^4}$$

for

$$\frac{\lambda k_B T}{2\pi\hbar c} >> 1$$

(2) Wien's law (in short-wavelength limit)

$$S_W(\lambda) = \frac{1}{4}cu_W(\lambda) = \frac{\pi^2\hbar c}{\lambda^5}\exp(-\frac{2\pi\hbar c}{\lambda k_B T})$$

$$\frac{\lambda k_B T}{2\pi\hbar c} << 1$$

We make a plot of $S(\lambda, T)$ as a function of the wavelength, where $S(\lambda, T)$ is in the units of W/m^3 and the wavelength is in the units of nm.

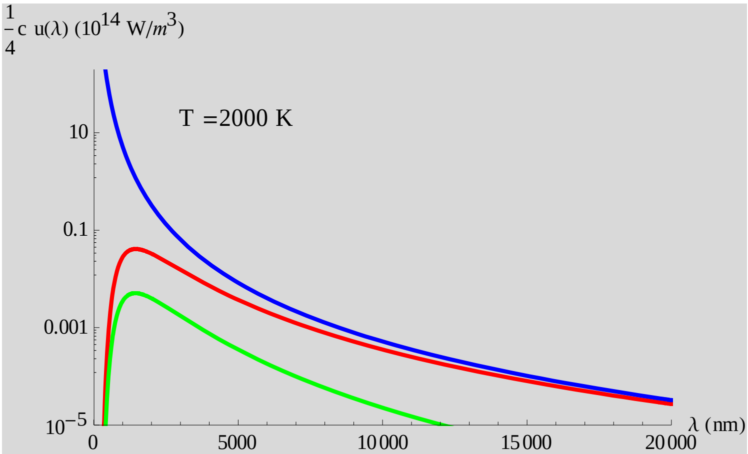
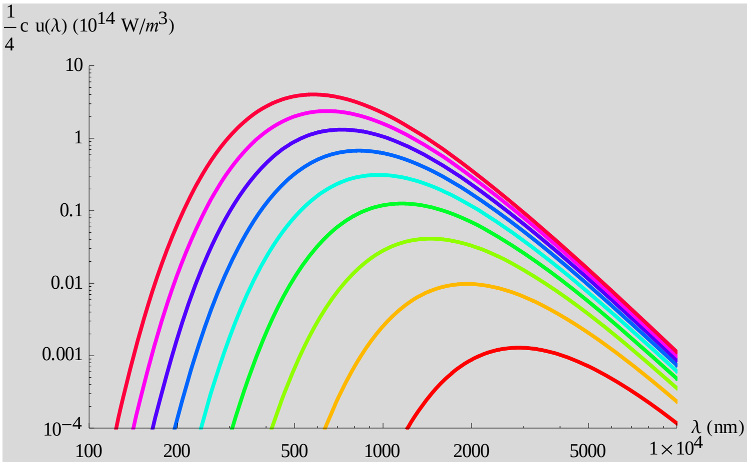


Fig. $cu(\lambda)/4$ (W/m^3) vs λ (nm). T = 2 x 10^3 K. Red [Planck]. Green [Wien]. Blue [Rayleigh-Jean]. Wien's displacement law: The peak appears at λ = 1448.89 nm for T = 2 x 10^3 K. This figure shows the misfit of Wien's law at long wavelength and the failure of the Rayleigh-Jean's law at short wavelength.



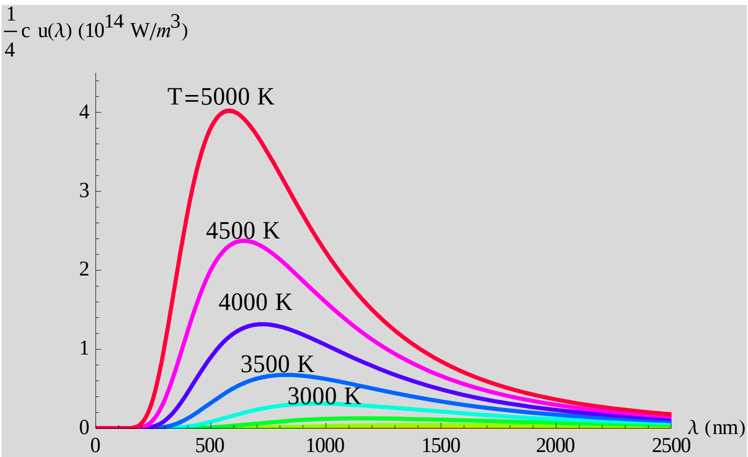


Fig. (a) and (b) $cu(\lambda)/4$ (W/m^3) vs λ (nm) for the Plank's law. T = 1000 K (red), 1500 K, 2000 K, 2500 K, 3000 K (blue), 3500 K, 4000 K (purple), 4500 K, and 5000 K. The peak shifts to the higher wavelength side as T decreases according to the Wien's displacement law.

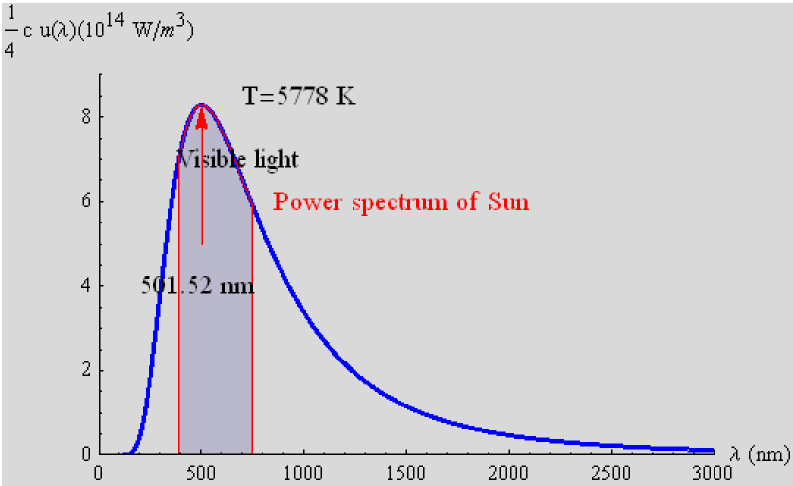


Fig. Power spectrum of sun. $cu(\lambda)/4$ (W/m^3) vs λ (nm). T = 5778 K. The peak wavelength is 501.52 nm according to the Wien's displacement law.

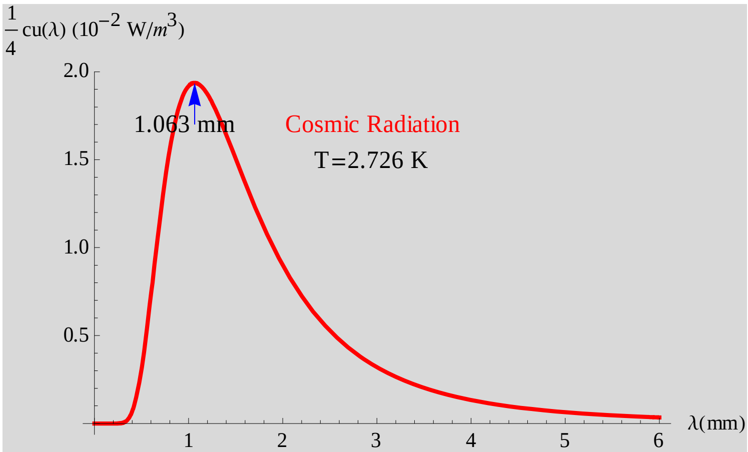


Fig. Power spectrum of cosmic blackbody radiation at T = 2.726 K. The peak wavelength is 1.063 mm (Wien's displacement law.

6. Stefan-Boltzmann radiation law for a black body (1879).

Joseph Stefan (24 March 1835 – 7 January 1893) was a physicist, mathematician and poet of Slovene mother tongue and Austrian citizenship.

http://en.wikipedia.org/wiki/Joseph_Stefan

Ludwig Eduard Boltzmann (February 20, 1844 – September 5, 1906) was an Austrian physicist famous for his founding contributions in the fields of statistical mechanics and statistical thermodynamics. He was one of the most important advocates for atomic theory at a time when that scientific model was still highly controversial.

http://en.wikipedia.org/wiki/Ludwig_Boltzmann

The total energy per unit volume is given by

$$\varepsilon = \frac{E_{tot}}{V} = \int u(\omega)d\omega = \int u(\lambda)d\lambda = \frac{(k_B T)^4}{\pi^2 \hbar^3 c^3}\int_0^\infty \frac{x^3}{\exp(x)-1} = \frac{\pi^2 (k_B T)^4}{15\hbar^3 c^3}$$

((**Mathematica**))

$$\int_0^\infty \frac{x^3}{e^x - 1}\, dx$$

$$\frac{\pi^4}{15}$$

A spherical enclosure is in equilibrium at the temperature T with a radiation field that it contains. The power emitted through a hole of unit area in the wall of enclosure is

$$P = \frac{1}{4} c\varepsilon = \frac{\pi^2 k_B{}^4}{60\hbar^3 c^2} T^4 = \sigma T^4$$

where σ is the Stefan-Boltzmann constant

$$\sigma = \frac{\pi^2 k_B{}^4}{60\hbar^3 c^2} = 0.5670400 \times 10^{-4} \text{ erg/s-cm}^2\text{-K}^4 = 5.670400 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$$

and the geometrical factor is equal to 1/4. The application of the Stefan-Boltzmann law is discussed in lecture notes of Phys.131 (Chapter 18) (see URL at

http://bingweb.binghamton.edu/~suzuki/GeneralPhysLN.html

7. Duality of wave and particle

Region of Rayleigh-Jeans: wave-like nature
Region of Wien: particle-like nature

The mean energy contained in a volume ΔV in the frequency range between ω and $\omega+\Delta\omega$, is given by

$$\overline{E}(\omega) = \langle E(\omega) \rangle = \Delta V \overline{W_T}(\omega)\Delta\omega = \Delta V D(\omega)\hbar\omega\Delta\omega\overline{n} = \Delta V D(\omega)\hbar\omega\Delta\omega \frac{1}{e^{\beta\hbar\omega} - 1}$$

where

$$\overline{n} = \frac{1}{e^{\beta\hbar\omega} - 1},$$

and

$$D(\omega) = \frac{\omega^2}{\pi^2 c^3}$$

The mean-square of the fluctuation in energy is obtained as

$$\left\langle [\Delta E(\omega)]^2 \right\rangle = \left\langle [E(\omega)]^2 \right\rangle - \left\langle E(\omega) \right\rangle^2 = k_B T^2 \frac{\partial}{\partial T} \left\langle E(\omega) \right\rangle$$

from the general theory of thermodynamics,

or

$$\left\langle [\Delta E(\omega)]^2 \right\rangle = \Delta V \Delta \omega D(\omega) \hbar \omega k_B T^2 \frac{\partial}{\partial T} \frac{1}{e^{\beta\hbar\omega} - 1} = \Delta V \Delta \omega D(\omega) \hbar^2 \omega^2 [\frac{1}{e^{\beta\hbar\omega} - 1} + \frac{1}{\left(e^{\beta\hbar\omega} - 1\right)^2}]$$

or

$$\left\langle [\Delta E(\omega)]^2 \right\rangle = \Delta V \Delta \omega D(\omega) \hbar^2 \omega^2 [\overline{n} + \overline{n}^2] = \Delta V \Delta \omega D(\omega) \hbar^2 \omega^2 (\Delta n)^2$$

where

$$\overline{n} + \overline{n}^2 = (\Delta n)^2$$

(See the Appendix for the detail). Note that

$$(\Delta n)^2 = < (n - \overline{n})^2 > = < n^2 > - \overline{n}^2 \qquad \text{(from the definition).}$$

(i) Rayleigh-Jean (wave-like)

For $\frac{\hbar\omega}{k_B T} = \beta\hbar\omega << 1$, $\qquad \overline{n}^2 >> \overline{n}$

$$(\Delta n)^2 \approx \overline{n}^2, \quad \text{or} \quad (\Delta n) \approx \overline{n} \qquad \text{(wave-like, Rayleigh-Jeans)}$$

$$\left\langle [\Delta E(\omega)]^2 \right\rangle \approx \Delta V \Delta \omega D(\omega) \hbar^2 \omega^2 \overline{n}^2$$

Then we have

$$\frac{\left\langle [\Delta E(\omega)]^2 \right\rangle}{\left\langle E(\omega) \right\rangle^2} = \frac{\Delta V \Delta \omega D(\omega) \hbar^2 \omega^2 \overline{n}^2}{(\Delta V \Delta \omega D(\omega) \hbar \omega \overline{n})^2} = \frac{1}{\Delta V \Delta \omega D(\omega)} = \frac{1}{\Delta V \Delta \omega} \frac{c^3}{\omega^2}$$

(ii) Wien (particle-like)

For $\dfrac{\hbar\omega}{k_B T} = \beta\hbar\omega >> 1$, $\qquad \overline{n}^2 < \overline{n}$

$$(\Delta n)^2 \approx \overline{n} \qquad \text{(particle-like, corpuscle, Wien)}$$

$$\langle [\Delta E(\omega)]^2 \rangle \approx \Delta V \Delta\omega D(\omega)\hbar^2\omega^2\overline{n} = \hbar\omega(\Delta V \Delta\omega D(\omega)\hbar\omega\overline{n}) = \hbar\omega\langle E(\omega)\rangle$$

or

$$\frac{\langle [\Delta E(\omega)]^2 \rangle}{\langle E(\omega)\rangle} = \hbar\omega$$

(iii) Planck

$$\langle [\Delta E(\omega)]^2 \rangle = \hbar\omega\langle E(\omega)\rangle + \frac{1}{\Delta V \Delta\omega}\frac{c^3}{\omega^2}\langle E(\omega)\rangle^2$$

8. Einstein A and B coefficient

$$\overline{W}_T(\omega) = \varepsilon = \frac{\omega^2}{\pi^2 c^3}\frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}$$

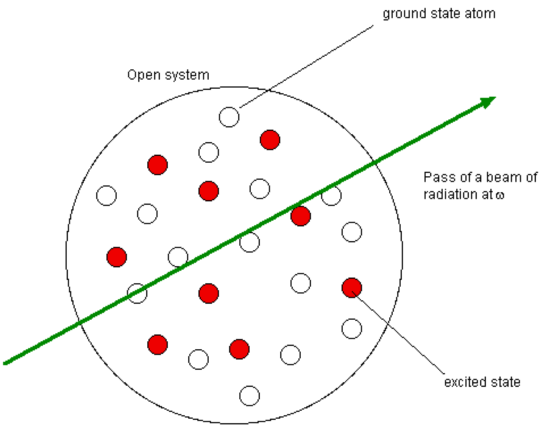
Planck’s law for the radiative energy density (Black body)

Suppose that a gas of N identical atoms is placed in the interior of the cavity:

$$\hbar\omega = E_2 - E_1 .$$

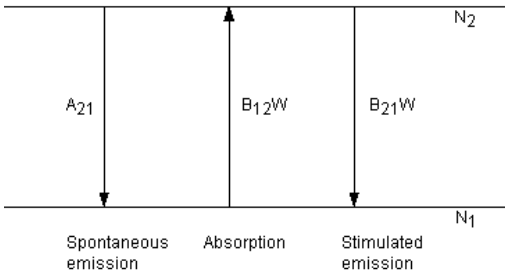
Two atomic levels are not degenerate.

N_1, N_2: level population



$$\overline{W}(\omega) = \overline{W_T}(\omega) + \overline{W_E}(\omega)$$

$\overline{W}(\omega)$: cycle-average energy density of radiation at ω
$\overline{W_T}(\omega)$: thermal part
$\overline{W_E}(\omega)$: contribution from some external source of electromagnetic radiation



$$\begin{cases} \dfrac{dN_1}{dt} = A_{21}N_2 - N_1B_{12}\overline{W}(\omega) + N_2B_{21}\overline{W}(\omega) \\ \dfrac{dN_2}{dt} = -A_{21}N_2 + N_1B_{12}\overline{W}(\omega) - N_2B_{21}\overline{W}(\omega) \end{cases}$$

Case of thermal equilibrium

$$\frac{dN_1}{dt} = \frac{dN_2}{dt} = 0$$

or

$$N_2A_{21} - N_1B_{12}\overline{W}(\omega) + N_2B_{21}\overline{W}(\omega) = 0$$

For thermal equilibrium with no external radiation introduced into the cavity

$$\overline{W}(\omega) = \overline{W}_T(\omega)$$

$$\overline{W}_T(\omega) = \frac{A_{21}}{\left(\dfrac{N_1}{N_2}B_{12} - B_{21}\right)}$$

The level populations N_1 and N_2 are related in thermal equilibrium by Boltzman's law

$$\frac{N_1}{N_2} = \frac{e^{-\beta E_1}}{e^{-\beta E_2}} = \exp(\beta\hbar\omega), \ (\beta = 1/k_B T)$$

Then

$$\overline{W}_T(\omega) = \frac{A_{21}}{B_{12}e^{\beta\hbar\omega} - B_{21}}$$

which is compared with the Planck's law

$$\overline{W}_T(\omega) = \frac{\left(\dfrac{\hbar\omega^3}{\pi^2 c^3}\right)}{e^{\beta\hbar\omega} - 1}$$

$$\Rightarrow \qquad \begin{cases} B_{12} = B_{21} \\ \dfrac{A_{21}}{B_{12}} = \dfrac{\hbar\omega^3}{\pi^2 c^3} \end{cases}$$

$$\overline{W}_T(\omega) = \frac{A_{21}}{B_{12}}\overline{n}, \text{ where } \overline{n} = \frac{1}{e^{\beta\hbar\omega} - 1}$$

or

$$\frac{A_{21}}{B_{21}\overline{W}_T(\omega)} = e^{\beta\hbar\omega} - 1$$

((Example)) $\hbar\omega = k_B T$

For T = 300 K, $\nu_T = 6 \times 10^{12}$ Hz = 6 THz
For $\hbar\omega \ll k_B T$, $A_{21} \ll B_{21}\overline{W}_T(\omega)$ $\qquad (\nu \ll \nu_T)$
For $\hbar\omega \gg k_B T$, $A_{21} \gg B_{21}\overline{W}_T(\omega)$ $\qquad (\nu \gg \nu_T)$

For optical experiments that use electromagnetic radiation in the near-infrared, we have visible, ultraviolet region of the spectrum ($\nu \gg 5$ THz).

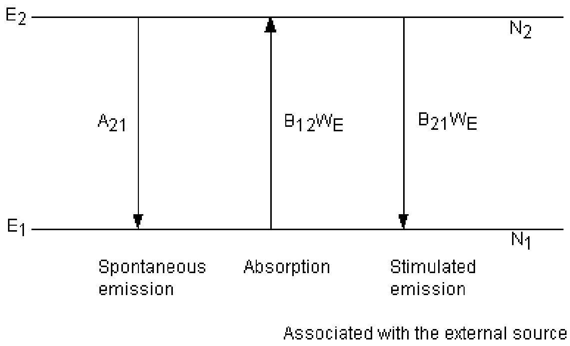
We have

(i) $A_{21} \gg B_{21} \overline{W}_T(\omega)$

A_{21}: spontaneous emission rate
B_{21}: rate of thermally stimulated emission

(ii) $\overline{W}(\omega) = \overline{W_T}(\omega) + \overline{W_E}(\omega) \cong \overline{W_E}(\omega)$

Therefore the radioactive process of interest involve the absorption and stimulated emission associated with the external source.



((Note))

Calculation of $\dfrac{A_{21}}{B_{12}\overline{W}_T(\omega)} = \dfrac{1}{\overline{n}} = e^{\hbar\omega/k_B T} - 1$ at T = 300 K as a typical example. This factor is larger than 1 when ν = 4.333 THz.

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APPENDIX

Planck's law

In thermal equilibrium at temperature T, the probability P_n that the mode oscillator is thermally excited to the n-th excited state is given by the usual Boltzmann factor

$$P_n = \frac{\exp(-\frac{E_n}{k_B T})}{\sum_n \exp(-\frac{E_n}{k_B T})} .$$

The zero-point energy cancels when the quantized energy expression is substituted and, with the shorthand notation

$$U = \exp(-\frac{\hbar\omega}{k_B T})$$

the thermal probability becomes

$$P_n = \frac{U}{\sum_{n=0}^{\infty} U^n} = \frac{1}{1-U}$$

where 0<U<1. We define that

$$< n^m > = \sum_{n=0}^{\infty} n^m P_n .$$

Then we have

$$\overline{n} = < n > = \sum_{n=0}^{\infty} n^m P_n = \frac{U}{1-U} = \frac{1}{\exp(\frac{\hbar\omega}{k_B T}) - 1}$$

$$< n^2 > = \frac{U}{(1-U)^2}$$

$$< n^3 > = \frac{U(1+4U+U^2)}{(1-U)^3}$$

The fluctuation in the number is characterized by the root-mean square deviation Δn of the distribution.

$$(\Delta n)^2 =< (n-\overline{n})^2 >=< n^2 > -\overline{n}^2 = \frac{U}{(1-U)^2}$$

Since

$$\overline{n}^2 + \overline{n} = \frac{U}{(1-U)^2}$$

we get the relation

$$(\Delta n)^2 = \overline{n}^2 + \overline{n} \, .$$

((**Mathematica**))

Fluctuation in photon number (Planck distribution)

```
P[n_] = (1 - U) U^n;
```

$$K[m\_] := \sum_{n=0}^{\infty} \left(n^m P[n]\right) \text{ // Simplify}[\#, 0 < U < 1] \&;$$

```
K[1] // Simplify
```

$$-\frac{U}{-1+U}$$

```
K[2] // Simplify
```

$$\frac{U\,(1+U)}{(-1+U)^2}$$

```
K[3] // Simplify
```

$$-\frac{U\left(1+4\,U+U^2\right)}{(-1+U)^3}$$

```
K[2] - K[1]^2 // Simplify
```

$$\frac{U}{(-1+U)^2}$$

```
K[1] + K[1]^2 // Simplify
```

$$\frac{U}{(-1+U)^2}$$