Heisenberg's priciple of uncertainty

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Werner Heisenberg (5 December 1901– 1 February 1976) was a German theoretical physicist who made foundational contributions to quantum mechanics and is best known for asserting the uncertainty principle of quantum theory. In addition, he made important contributions to nuclear physics, quantum field theory, and particle physics.
Heisenberg, along with Max Born and Pascual Jordan, set forth the matrix formulation of quantum mechanics in 1925. Heisenberg was awarded the 1932 Nobel Prize in Physics for the creation of quantum mechanics, and its application especially to the discovery of the allotropic forms of hydrogen.

http://en.wikipedia.org/wiki/Werner_Heisenberg

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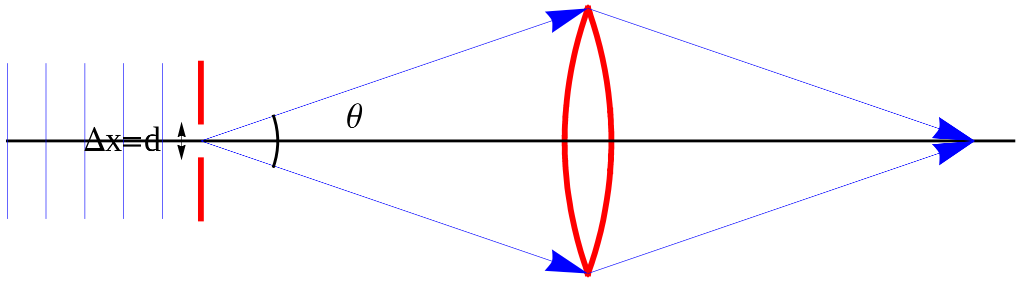
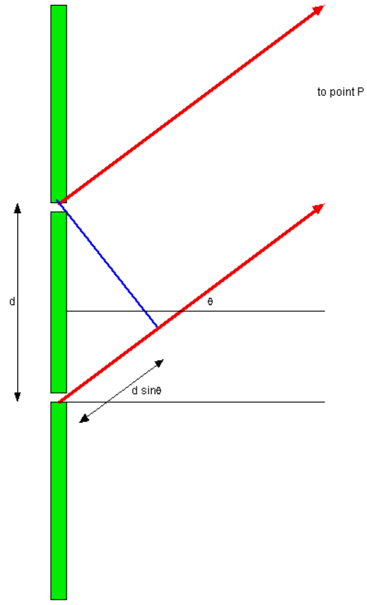
1. Heisenberg's uncertainty principle

An electron is moving in a horizontal direction (y). We wish to determine its coordinate in the perpendicular (x) direction. For this purpose, we set up a collimator perpendicular to the direction of motion with a slit of the width $d = \Delta x$.



Now we need to take into account of the wave nature of the electron. The emergent beam has a finite angle of divergence θ, which is, by the simplest laws of optics,

$$\Delta x = d = \frac{\lambda}{\sin\theta} \qquad \text{(constructive interference)}$$



We assume the momentum in the x axis to have been zero before passing through the slit. The momentum of the electron parallel to the screen (x direction) is uncertain, after passing through the slit, by an amount

$$\Delta p_x = \frac{h}{\lambda} \sin\theta$$

where h/λ is the momentum of the electron in the direction of the beam. Then we get the uncertainty relation,

$$\Delta p_x \Delta x \approx h$$

2. Application of microscope

The most famous thought experiment was introduced by Heisenberg and involves the measurement of an electron's position by means of a microscope, which forms an image of the electron on a screen.

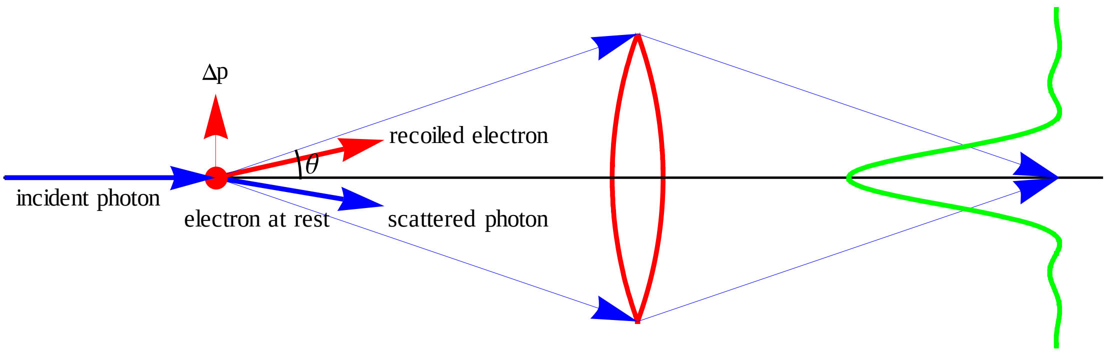


Fig. Seeing an electron" with a gamma-ray microscope. Electron (red point and red straight line) and one photon (blue straight line). The Compton scattering occurs at a point where an electron is at rest.

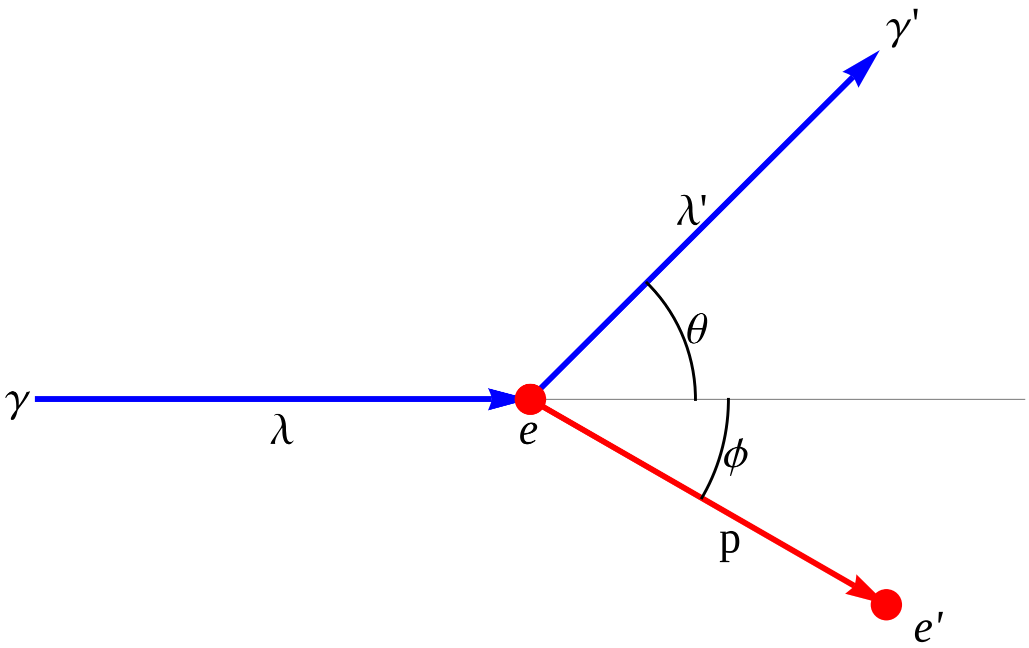


Fig. Compton scattering.

Because light can scatter from and perturb the electron, let us minimize this effect by considering the scattering of only a single photon from an electron initially at rest. To be collected by the lens, the photon must be scattered through an angle ranging from -θ to θ, which consequently imparts to the electron an x component value ranging from

$$-\frac{h}{\lambda}\sin\theta \le p_x \le \frac{h}{\lambda}\sin\theta$$

Thus the uncertainty in the electron's momentum is

$$\Delta p_x = \frac{2h}{\lambda}\sin\theta$$

After passing through the lens, the photon lands somewhere on the screen, but the image and consequently the position of the electron is fuzzy because the photon is diffracted on passing through the lens aperture. According to the Rayleigh's criterion, the resolution of a microscope or the uncertainty in the image of the electron, Δx is given by

$$\Delta x = \frac{\lambda}{\sin\theta}$$

Here 2θ is the angle subtended by the objective lens. Then we get the relation

$$\Delta p \Delta x = (\frac{2h}{\lambda}\sin\theta)(\frac{\lambda}{\sin\theta}) \approx 2h$$

Then it follows that for the motion after the experiment,

$$\Delta p_x \Delta x \geq \frac{\hbar}{2}.$$

3. Energy-time uncertainty relation

We consider an electron moving along the x axis. The energy of the electron is given by

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

where

$$p = mv.$$

If p is uncertain by Δp, then the uncertainty in E is given by

$$\Delta E = \frac{p}{m}\Delta p = v\Delta p$$

Since

$$\frac{\Delta x}{\Delta t} = v$$

then we get

$$\Delta E = \frac{\Delta x}{\Delta t}\Delta p$$

or

$$\Delta E \Delta t = \Delta p \Delta x \geq \frac{\hbar}{2}$$

which is the uncertainty principle for energy vs time *t*.

4. Zero point energy of simple harmonics

The total energy *E* for the simple harmonics, is given by

$$E = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

where

$$\omega = \sqrt{\frac{k}{m}}$$

We use the Heisenberg's principle of uncertainty,

$$\Delta p \Delta x = \frac{\hbar}{2}$$

We calculate the minimum value of *E* in the following way.

$$E = \frac{(\Delta p)^2}{2m} + \frac{1}{2} m\omega^2 (\Delta x)^2 \geq 2\sqrt{\frac{(\Delta p)^2}{2m} \frac{1}{2} m\omega^2 (\Delta x)^2} =$$

$$= \omega \Delta p \Delta x = \frac{1}{2}\hbar\omega$$

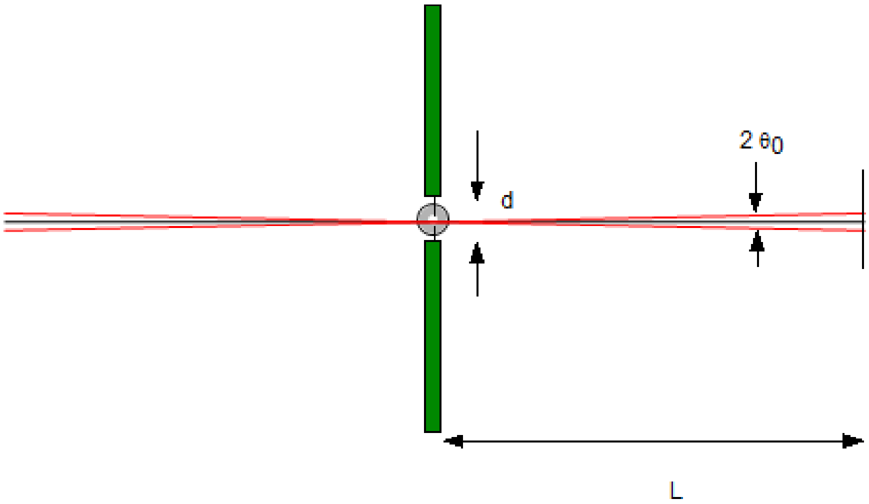
Here we use the inequality,

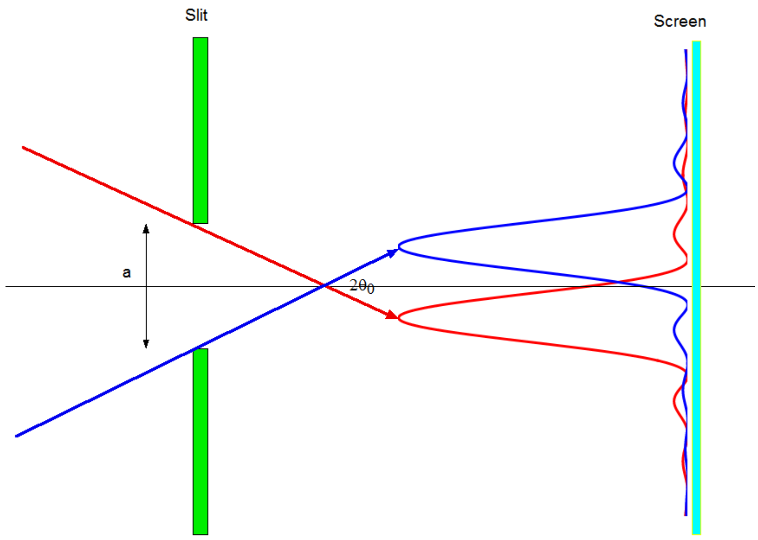
If $a>0$ and $b>0$,

$$\frac{a+b}{2} \geq \sqrt{ab} .$$

APPENDIX

Rayleigh criterion: diffraction by a circular aperture





We consider the angular separation of the two point sources (centered at $\theta = \theta_0$ and $\theta = -\theta_0$) for the single slit with the width a, where x_0 is a parameter given by

$$x_0 = \frac{a}{\lambda}\sin\theta_0 \approx \frac{a}{\lambda}\theta_0$$

For $x_0 = 0.25$ two sources cannot be distinguished.
For $x_0 = 0.5$, they can be marginally distinguished.
For $x_0 = 0.60$ they are clearly distinguished.

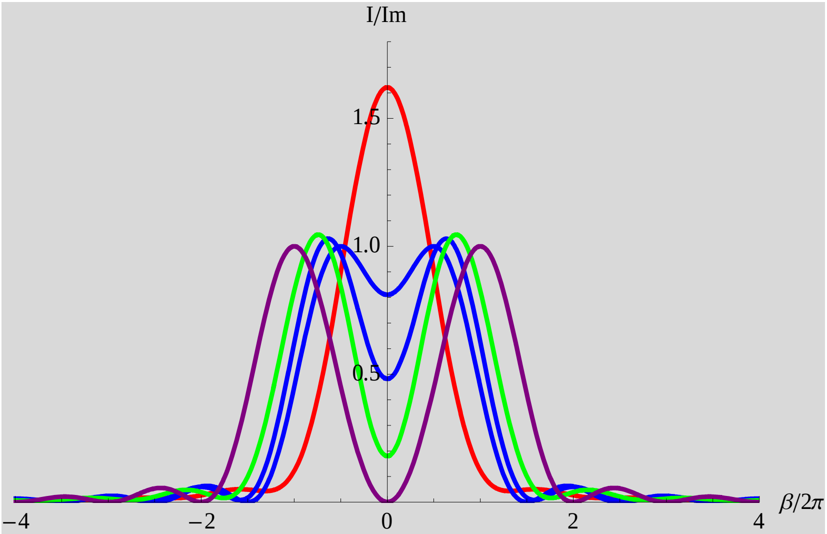


Fig. Superposition of the intensity I/I_{max} centered with $\theta = \theta_0$ and the intensity I/I_{max} centered with $\theta = -\theta_0$ as a function of $\beta/(2\pi)$, where $\frac{\beta}{2\pi} = \frac{a}{\lambda}(\theta \pm \theta_0)$. $x_0 = 0.25$ (red), 0.5 (blue), 0.61 (blue), 0.75 (green), and 1.00 (purple).

The Rayleigh's criterion is satisfied for $x_0 = 0.5$.

$$2x_0 = 1 = 2\frac{a}{\lambda}\theta_0 = \frac{a}{\lambda}(2\theta_0) = \frac{a}{\lambda}\theta_R \approx \frac{a}{\lambda}\sin\theta_R$$

or

$$\boxed{\sin\theta_R = \frac{\lambda}{a}}$$

where

$$\theta_R = 2\theta_0$$

For the circular aperture, we have

$$\sin\theta_R = 1.220\frac{\lambda}{d}$$

where *d* is the diameter of circular aperture.

REFERENCES

W. Heisenberg, The Physical Principles of the Quantum Theory, translated by C.Eckart and F.C. Hoyt, Dover Publications, Inc.

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