de Broglie waves

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Louis-Victor-Pierre-Raymond, 7th duc de Broglie, FRS (15 August 1892 – 19 March 1987) was a French physicist and a Nobel laureate. He was the sixteenth member elected to occupy seat 1 of the Académie française in 1944, and served as Perpetual Secretary of the Académie des sciences, France.

http://en.wikipedia.org/wiki/Louis_de_Broglie

1. de Broglie hypothesis: duality of wave and particle

1.1 Over view

In 1923, Louis de Broglie put forth the following hypothesis. Material particles, just like photons, can have a wave-like aspect. He then derived the Bohr-Sommerfeld quantization rules as a consequence of his hypothesis. the various permitted energy levels appearing as analogues of the normal modes of a vibrating string. Electron diffraction experiments (Davisson and Germer, 1927) strikingly confirmed the existence of a wave-like aspect of matter by showing that interference patterns could be obtained with material particles such as electrons.

1.2 Duality of particle and wave

From experiments on the interference and diffraction of particles, we infer the very simple law that the infinite harmonic plane waves associated with the motion of a free particle of momentum p propagate in the direction of motion.

Particle:

E (energy), $\boldsymbol{p}$ (momentum)

Wave:

$$\omega = 2\pi\nu, \qquad k = \frac{2\pi}{\lambda} \text{(wave number)}$$

where λ is the wavelength, ω is the angular frequency, ν is the frequency. From the relations,

$$E = \hbar\omega = h\nu, \qquad p = \hbar k$$

using the Planck's constant (h) or the Dirac constant ($\hbar$), the de Broglie wavelength is derived as

$$\lambda = \frac{2\pi}{k} = \frac{h}{p}$$

since

$$p = \frac{h}{2\pi}\frac{2\pi}{\lambda} = \frac{h}{\lambda}$$

1.3 Finiteness of *h* in the quantum mechanics

This relation establishes contact between the wave and the particle pictures. The finiteness of Planck's constant h is the basic point here. If the constant h were zero, then no matter what momentum a particle would follow the laws of classical mechanics. A free particle would not be diffracted but would go on a straight rectilinear path.

If x is a characteristic length involved in describing the motion of momentum p, the wave aspect of matter will be hidden from our sight, if

$$\frac{\lambda}{x} = \frac{h}{px} << 1$$

i.e., if h is negligible compared with px;

$$px >> h$$

This condition satisfies well in the classical mechanics. In other words, the classical mechanics is contained in quantum mechanics as a limiting form ($h \rightarrow 0$)

1.4 Experimental evidence

The particle-nature:

- Einstein's photoelectric effect (particle)
- Compton effect (particle)

Wave-like nature (the diffraction and interference):

Young's double slit interference (wave)
Davisson-Germer experiment (wave)
x-ray diffraction (wave)
Neutron scattering (wave)

2. de Broglie's explanation of quantization of the angular momentum.

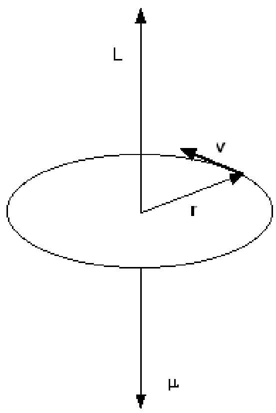

Fig.1 Orbital (circular) motion of electron with mass m and a charge *–e*. The direction of orbital angular momentum *L* is perpendicular to the plane of the motion (*x*-*y* plane).

The orbital angular momentum of an electron (charge *–e* and mass *m*) *L* is defined by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times (m\mathbf{v}),$$

or

$$L_z = mvr\ . \qquad (1)$$

According to the de Broglie relation, we have

$$2\pi r = n\lambda$$

and

$$p = \frac{h}{\lambda}$$

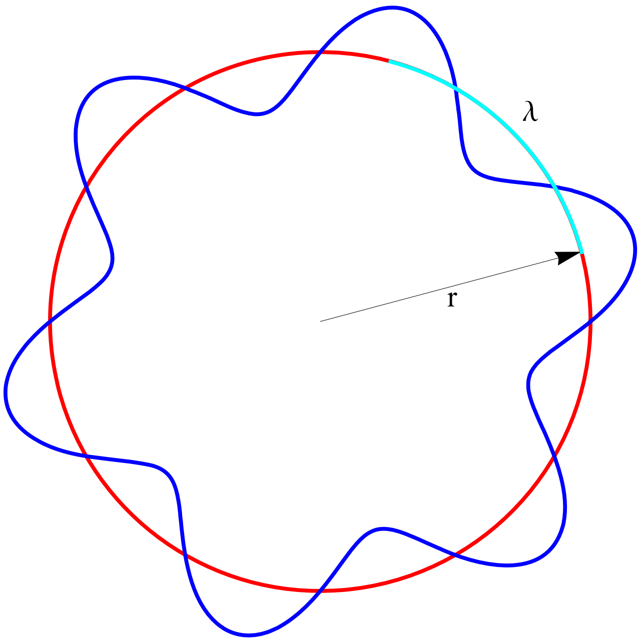

Fig. de Broglie waves for the circular orbits of an electron about the nucleus of an atom. The condition for $2\pi r = n\lambda$ with n = 6. Note that n is a positive integer.

From these we have the relation,

$$p(2\pi r) = \frac{h}{\lambda} 2\pi r = nh \, , \tag{2}$$

where p (= mv) is the momentum ($p = \frac{h}{\lambda}$), n is integer, h is the Planck constant, and λ is the wavelength.

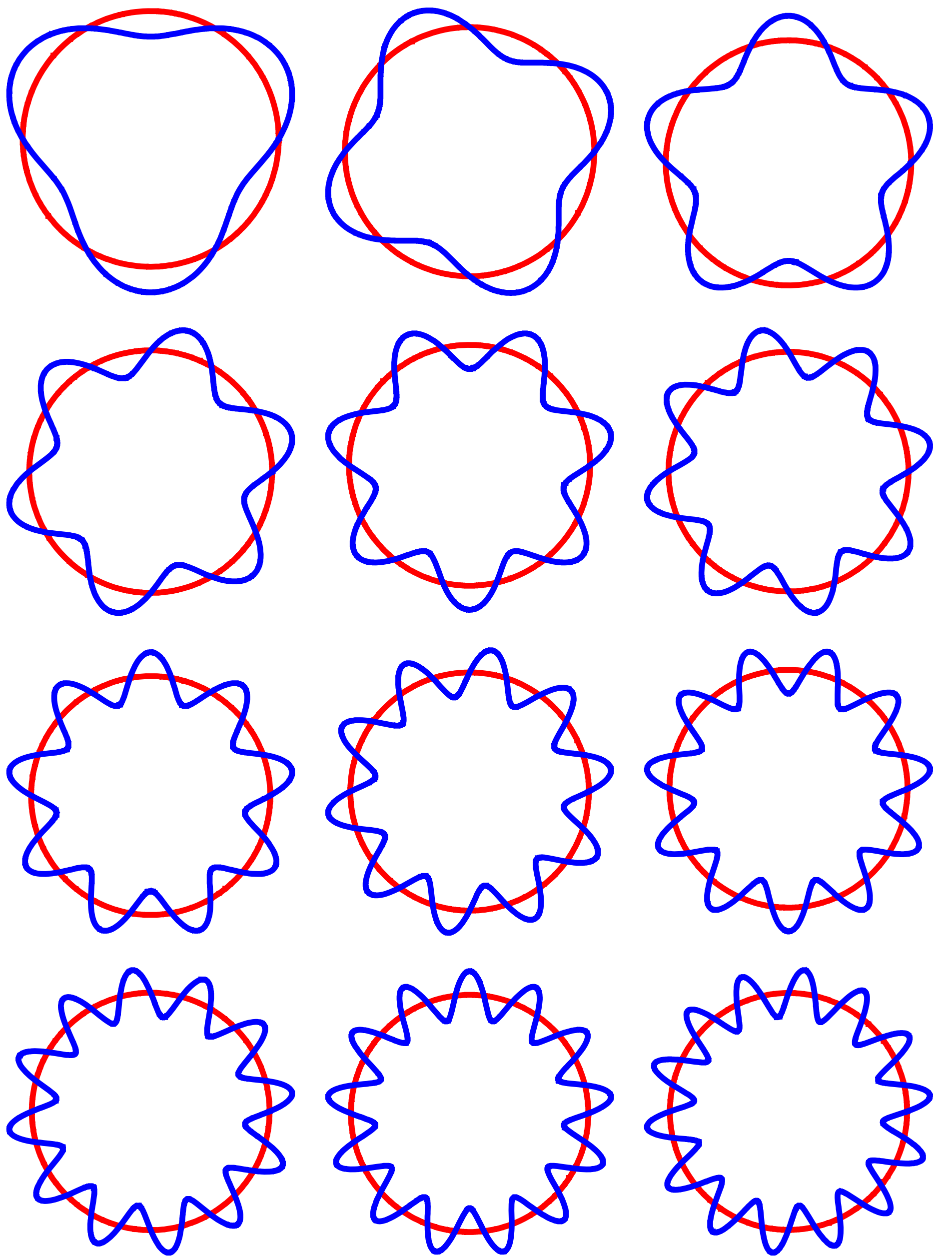

Fig. Acceptable wave on the ring (circular orbit). The circumference should be equal to the integer n (=1, 2, 3,…) times the de Broglie wavelength λ. The picture of fitting the de Broglie waves onto a circle makes clear the reason why the orbital angular momentum is quantized. Only integral numbers of wavelengths can be fitted. Otherwise, there would be destructive interference between waves on successive cycles of the ring.

Then the angular momentum L_z is described by

$$L_z = pr = mvr = \frac{nh}{2\pi} = n\hbar\,. \qquad (3)$$

The magnetic moment of the electron is given by

$$\mu_z = \frac{1}{c} I_\theta A\,, \qquad (4)$$

where c is the velocity of light, $A = \pi r^2$ is the are of the electron orbit, and I_θ is the current due to the circular motion of the electron. Note that the direction of the current is opposite to that of the velocity because of the negative charge of the electron. The current I_θ is given by

$$I_\theta = -\frac{e}{T} = -\frac{e}{(2\pi r / v)} = -\frac{ev}{2\pi r}\,, \qquad (5)$$

where T is the period of the circular motion. Then the magnetic moment is derived as

$$\mu_z = \frac{1}{c} I_\theta A = -\frac{evr}{2c} = -\frac{e}{2mc} L_z = -\frac{e\hbar}{2mc}\frac{L_z}{\hbar} = -\frac{\mu_B}{\hbar} L_z \;\; (e > 0), \qquad (6)$$

where μ_B $(= \frac{e\hbar}{2mc})$ is the Bohr magneton.

$\mu_B = 9.27400915 \times 10^{-21}$ emu. emu=erg/Oe. (cgs units)

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3. de Broglie wave length

We consider the de Broglie wavelength of a particle m and the kinetic energy for a relativistic particle.

$$E = \sqrt{E_0^{\;2} + c^2 p^2} = E_0 + K$$

where E_0 is the rest energy;

$$E_0 = mc^2$$

The kinetic energy K is

$$K = E - E_0 = \sqrt{E_0^{\;2} + c^2 p^2} - E_0$$

Then the momentum p is obtained as

$$p=\frac{1}{c}\sqrt{(K+E_0)^2-E_0{}^2}=\frac{1}{c}\sqrt{K^2+2KE_0}=\frac{1}{c}\sqrt{K(K+2E_0)}$$

Using the de Broglie relation, we have the de Broglie wavelength

$$\lambda=\frac{h}{p}=\frac{hc}{\sqrt{K(K+2E_0)}}$$

We find that the wavelength is a scaling function of K/E_0 as

$$\lambda=\frac{\frac{hc}{E_0}}{\sqrt{\frac{K}{E_0}\left(\frac{K}{E_0}+2\right)}}$$

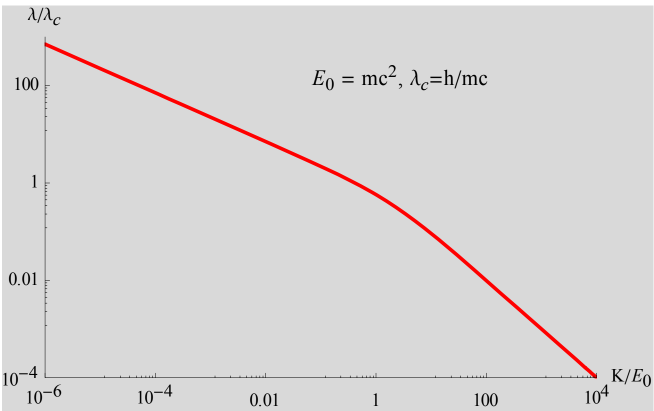
We consider the case of electron. In this case, the above formula is

$$\frac{\lambda}{\lambda_c}=\frac{1}{\sqrt{\frac{K}{E_0}\left(\frac{K}{E_0}+2\right)}}$$

Note that λ_c is the Compton wavelength for the particle and is given by

$$\lambda_c=\frac{h}{mc}=2.4263102389\times10^{-12}\text{ m.}$$

for the electron.



The nonrelativistic case.

When $E_0 >> K$, λ can be approximated by

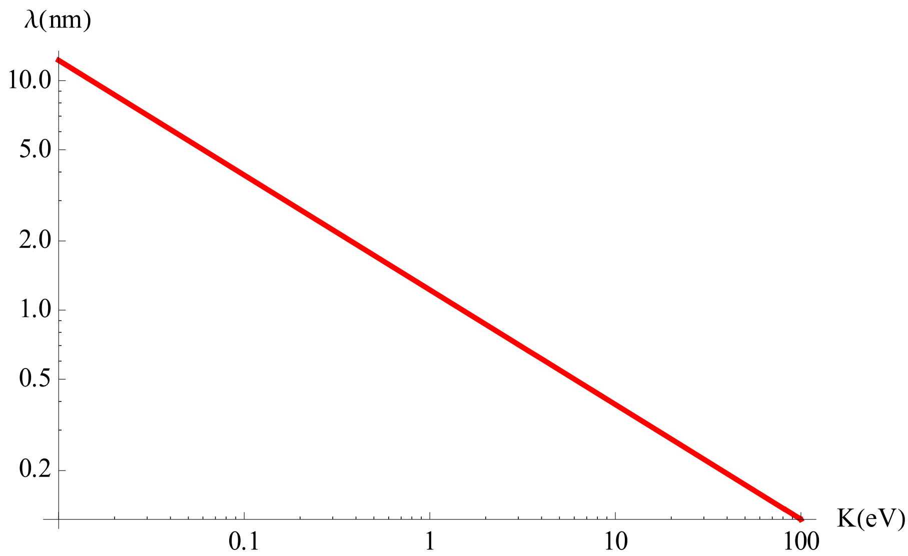
$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{2KE_0}} = \frac{hc}{\sqrt{2Kmc^2}} = \frac{h}{\sqrt{2mK}}$$

or

$$\lambda = \frac{h}{\sqrt{2mK}}$$

4. Electron: Classical limit

The de Broglie wavelength (relativistic) vs the kinetic energy for electron



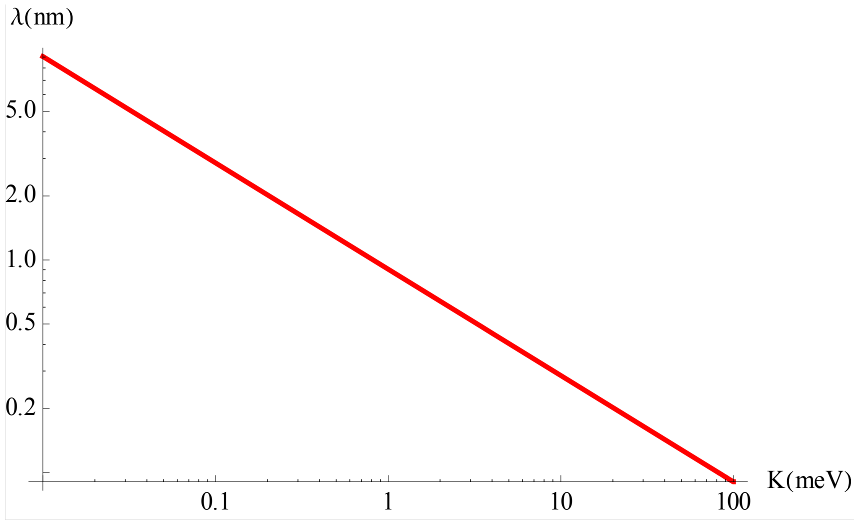
In the nonrelativistic case,

$$\lambda = \frac{12.2643}{\sqrt{K(eV)}} \text{Å}$$

When K = 100 eV, the wavelength is λ = 1.22643 Å.

5. Neutron: classical limit

The de Broglie wavelength (relativistic) vs the kinetic energy for neutron.



In the nonrelativistic case,

$$\lambda = \frac{0.286014}{\sqrt{K(eV)}} \text{Å}$$

or

$$\lambda = \frac{9.04457}{\sqrt{K(meV)}} \text{Å}$$

When K = 80 meV, the wavelength is λ = 1.0112 Å.

6. Duality of wave and particle seen in the blackbody problem

In the Planck's theory, the energy density is given by

$$\frac{E_{tot}}{V} = \int_0^\infty u(\omega)d\omega = \int_0^\infty u(\lambda)d\lambda$$

where

$$u(\omega) = \overline{W_T} = \frac{\hbar\omega^3}{\pi^2c^3}\frac{1}{\exp(\frac{\hbar\omega}{k_BT})-1} = \frac{\hbar\omega^3}{\pi^2c^3}\frac{1}{\exp(x)-1} = \frac{k_B{}^3T^3}{\pi^2\hbar^2c^3}\frac{x^3}{\exp(x)-1}$$

(Planck’s law for the radiation energy density). It is clear that

$$\frac{u(\omega)}{\frac{k_B{}^3 T^3}{\pi^2 \hbar^2 c^3}} = f(x) = \frac{x^3}{\exp(x) - 1}$$

is dependent on a variable x given by

$$x = \frac{\hbar\omega}{k_B T}.$$

(1) Region of Wien ($x = \frac{\hbar\omega}{k_B T} >> 1$),

$$u_W(\omega) = \frac{k_B{}^3 T^3}{\pi^2 \hbar^2 c^3} x^3 e^{-x}$$

(2) Region of Rayleigh-Jeans ($x = \frac{\hbar\omega}{k_B T} >> 1$),

$$u_{RJ}(\omega) = \frac{k_B{}^3 T^3}{\pi^2 \hbar^2 c^3} \frac{x^3}{\exp(x) - 1} \approx \frac{k_B{}^3 T^3}{\pi^2 \hbar^2 c^3} x^2$$

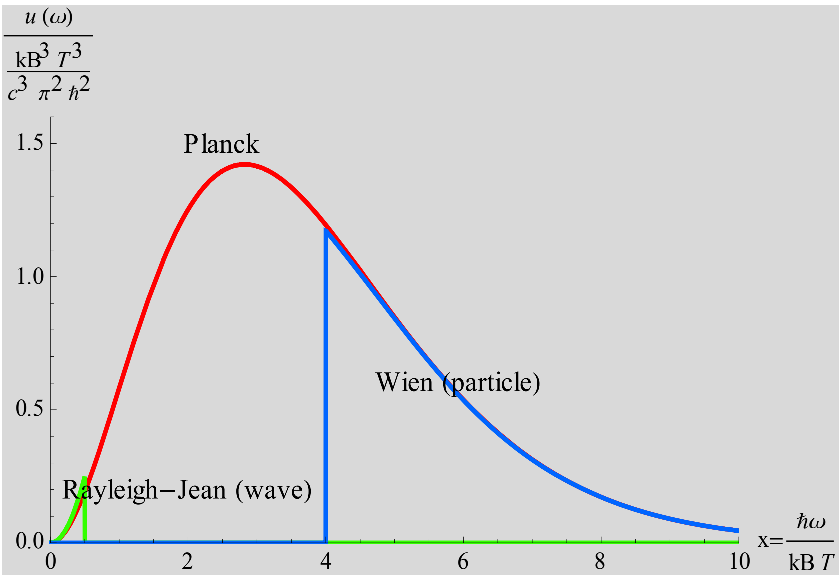


Fig. Scaling plot of $f(x)$ vs x for the Planck's law for the energy density of electromagnetic radiation at angular frequency ω and temperature T. Planck (red). Wien (blue, particle-like). Rayleigh-Jean (green, wave-like).

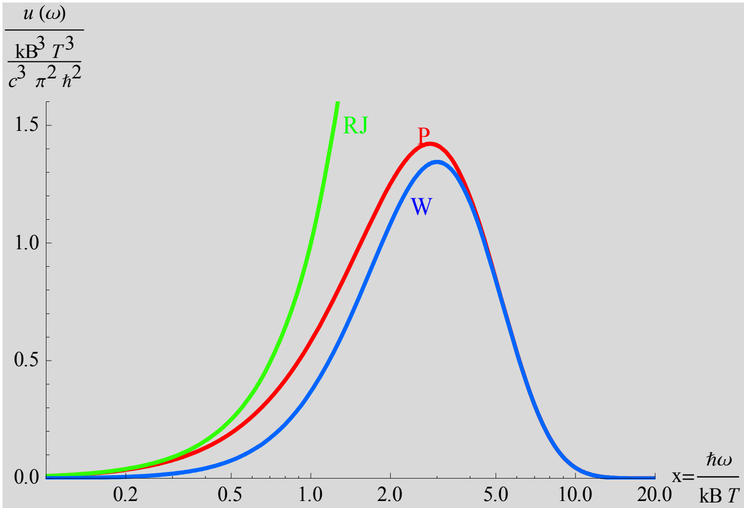


Fig. Scaling plot of Planck's law. Wien's law, and Rayleigh-Jean's law.