One dimensional barrier problems

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
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1. Overview on the boundary conditions for wave functions

(a) Behavior of a stationary wave function $\varphi(x)$

$$\frac{d^2}{dx^2}\varphi(x) + \frac{2m}{\hbar^2}(E - V)\varphi(x) = 0$$

where $V(x) = V$ in certain regions of space.

(i) $E > V$

$$E - V = \frac{\hbar^2}{2m}k^2 \qquad (k>0)$$

$$\frac{d^2}{dx^2}\varphi(x) + k^2\varphi(x) = 0$$

$$\varphi(x) = Ae^{ikx} + A'e^{-ikx}$$

where *A* and *A*' are complex constants.

(ii) $E<V$

$$-E + V = \frac{\hbar^2}{2m}\kappa^2 \qquad (\kappa > 0)$$

$$\frac{d^2}{dx^2}\varphi(x) - \kappa^2\varphi(x) = 0$$

$$\varphi(x) = Be^{\kappa x} + B'e^{-\kappa x}$$

where *B* and *B*' are complex constants.

(iii) $E = V$

$\varphi(x)$ is a linear function of *x*.

(b) Behavior of $\varphi(x)$ at a potential energy discontinuity

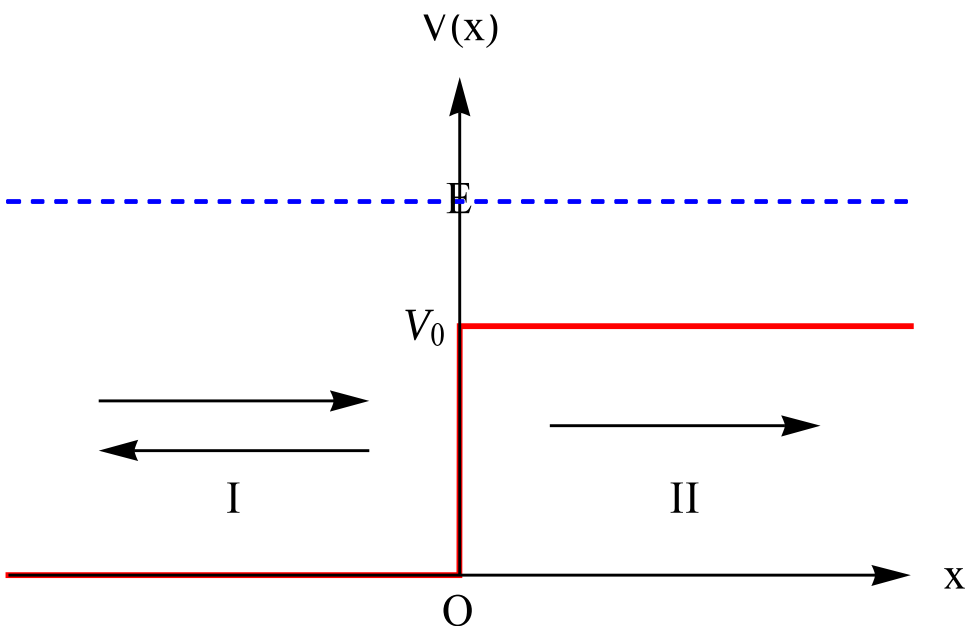
When the potential V(x) is discontinuous at $x = x_1$,

(i) $\varphi(x)$ and $\dfrac{d\varphi(x)}{dx}$ are continuous at $x = x_1$.

(ii) $\dfrac{d^2\varphi(x)}{dx^2}$ is discontinuous at $x = x_1$, if $V(x)$ remains bounded.

((**Note**)) $V(x) = a\delta(x)$; unbounded function whose integral remains finite. In the case $\varphi(x)$ remains continuous but $\dfrac{d\varphi(x)}{dx}$ does not.

2. 1D step barrier potential



(a) Case when $E > V_0$ (partial reflection)

The wave numbers:

$$k_1^2 = \frac{2mE}{\hbar^2} \qquad \text{for the region I}$$

$$k_2^2 = \frac{2m(E - V_0)}{\hbar^2} \qquad \text{for the region II}$$

The wave functions:

$\varphi_I(x) = A_1 e^{ik_1x} + A_1' e^{-ik_1x}$ for the region I

$\varphi_{II}(x) = A_2 e^{ik_2x} + A_2' e^{-ik_2x}$ for the region II

Suppose that $A_2' = 0$ (the wave propagates along the positive x axis in the region II). From the condition that $\varphi_I(x=0) = \varphi_{II}(x=0)$,

$$A_1 + A_1' = A_2 .$$

From the matching condition that $\frac{d\varphi_I}{dx}(x=0) = \frac{d\varphi_{II}}{dx}(x=0)$,

$$k_1(A_1 - A_1') = k_2 A_2$$

Then we have

$$\frac{A_1'}{A_1} = \frac{k_1 - k_2}{k_1 + k_2}, \qquad \frac{A_2}{A_1} = \frac{2k_1}{k_1 + k_2}$$

R: reflection coefficient
T: transmission coefficient

$$R = \frac{J_I'}{J_I} = \frac{\frac{\hbar k_1}{m}|A_1'|^2}{\frac{\hbar k_1}{m}|A_1|^2} = \frac{|A_1'|^2}{|A_1|^2} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 = 1 - \frac{4k_1k_2}{(k + k_{21})^2} .$$

$$T = \frac{J_{II}}{J_I} = \frac{\frac{\hbar k_2}{m}|A_2|^2}{\frac{\hbar k_1}{m}|A_1|^2} = \frac{k_2}{k_1}\frac{|A_2|^2}{|A_1|^2} = \frac{4k_1k_2}{(k + k_{21})^2} .$$

Thus we have the relation

$$R + T = 1$$

((Note))

Contrary to the prediction of classical mechanics, the incident particle has a non-zero probability of turning back.

((**Mathematica**))

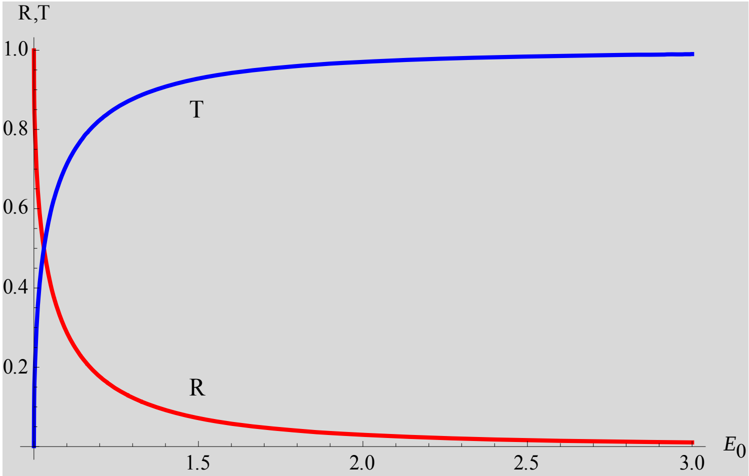
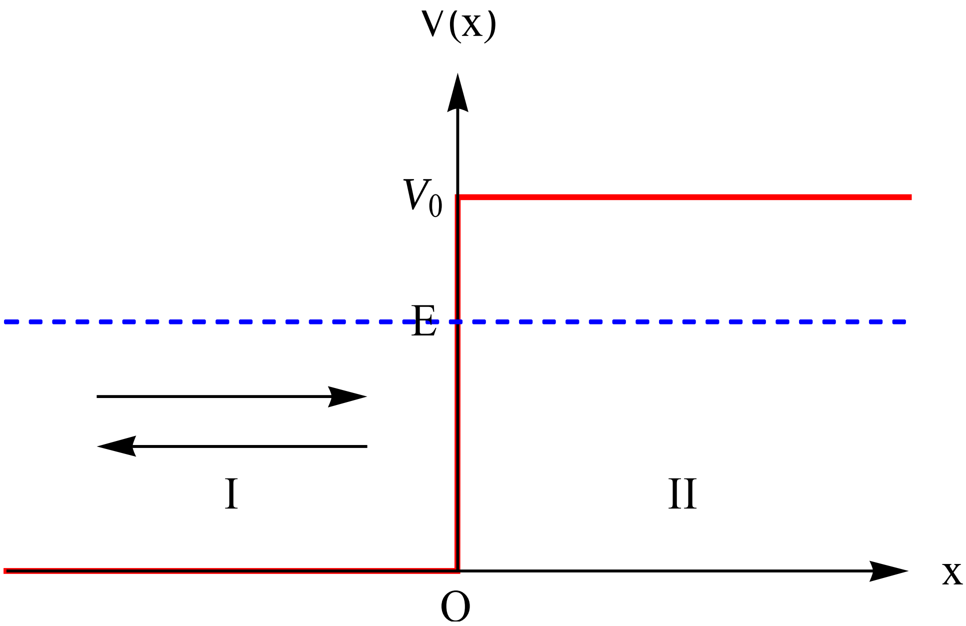


Fig. Plot of R and T as a function of E_0. $m = 1$. $\hbar = 1$. $V_0 = 1$. $E_0 > 1$.

(b) Case when $E < V_0$: $R = 1$. $T = 0$ (complete reflection)



The wave numbers:

$$k_1^2 = \frac{2mE}{\hbar^2}$$ for the region I

$$\rho_2^2 = \frac{2m(V_0 - E)}{\hbar^2}$$ for the region II

The wave functions:

$$\varphi_I(x) = A_1 e^{ik_1 x} + A_1' e^{-ik_1 x} \qquad \text{for the region I}$$

$$\varphi_{II}(x) = B_2 e^{\rho_2 x} + B_2' e^{-\rho_2 x} \qquad \text{for the region II}$$

Suppose that $B_2 = 0$ (the wave propagates along the positive *x* axis in the region II). From the condition that $\varphi_I(x=0) = \varphi_{II}(x=0)$,

$$A_1 + A_1' = B_2'$$

From the matching condition that $\frac{d\varphi_I}{dx}(x=0) = \frac{d\varphi_{II}}{dx}(x=0)$,

$$ik_1(A_1 - A_1') = -\rho_2 B_2'$$

Then we get

$$\frac{A_1'}{A_1} = \frac{k_1 - i\rho_2}{k_1 + i\rho_2}, \qquad \frac{B_2'}{A_1} = \frac{2k_1}{k_1 + i\rho_2}$$

The reflection coefficient *R* is obtained as

$$R = \frac{J_I'}{J_I} = \frac{\frac{\hbar k_1}{m}|A_1'|^2}{\frac{\hbar k_1}{m}|A_1|^2} = \frac{|A_1'|^2}{|A_1|^2} = \left|\frac{k_1 - i\rho_2}{k_1 + i\rho_2}\right|^2 = 1$$

Since $T + R = 1$, T=0. Since the wave function $\varphi_{II}(x)$ is a real wave function, we always have $J_{II} = 0$.

We consider the phase shift in the wave functions from the conditions,

$$A_1 + A_1' = B_2'$$

$$A_1 - A_1' = \frac{i\rho_2}{k_1} B_2'$$

When $\frac{\rho_2}{k_1} = \tan\phi$,

$$A_1 = \frac{1}{2}\left(1 + \frac{i\rho_2}{k_1}\right)B_2' = \frac{1}{2}(1 + i\tan\phi)B_2' = \frac{B_2'}{2\cos\phi}e^{i\phi} = \frac{I}{2}e^{i\phi}$$

and

$$A_1' = \frac{1}{2}(1 - \frac{i\rho_2}{k_1})B_2' = \frac{1}{2}(1 - i\tan\phi)B_2' = \frac{B_2'}{2\cos\phi}e^{-i\phi} = \frac{I}{2}e^{-i\phi}$$

where

$$\frac{B_2'}{2\cos\phi} = \frac{I}{2}, \qquad \text{or} \qquad B_2' = I\cos\phi$$

Then we get

$$\varphi_I(x) = \frac{I}{2}(e^{i\phi}e^{ik_1x} + e^{-i\phi}e^{-ik_1x}) = I\cos(k_1x + \phi)$$

$$\varphi_{II}(x) = I\cos\phi e^{-\rho_2 x}$$

where

$$\frac{\rho_2}{k_1} = \tan\phi = \sqrt{\frac{V_0 - E}{E}}$$

((Note)) In the case of $V_0 \to \infty$

Then we have $\rho_2 \to \infty$,

or

$$\frac{\rho_2}{k_1} = \tan\phi \to \infty, \qquad \text{or } \phi = \frac{\pi}{2}.$$

$$\varphi_I(x) = I\cos(k_1x + \frac{\pi}{2}) = -I\sin(k_1x)$$

and

$$\varphi_{II}(x) = 0$$

At $x = 0$,

$$\varphi_I(x = 0) = 0$$

$$\varphi_{II}(x=0)=0\,.$$

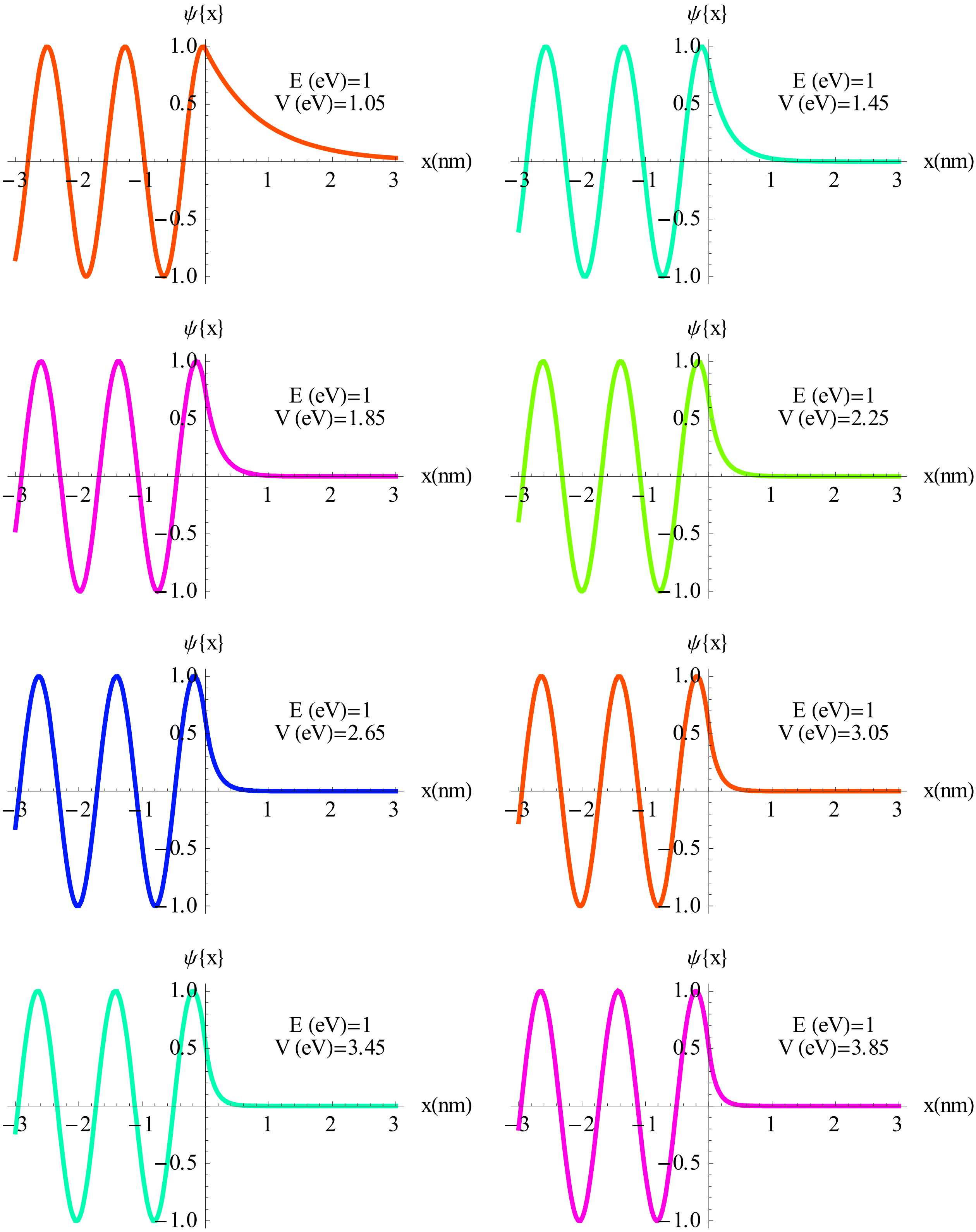
So it remains continuous. How about the matching condition?

$$\frac{d\varphi_I(x)}{dx}=-Ik_1\cos(k_1x)\text{, which is } -Ik_1 \text{ at } x=0.$$

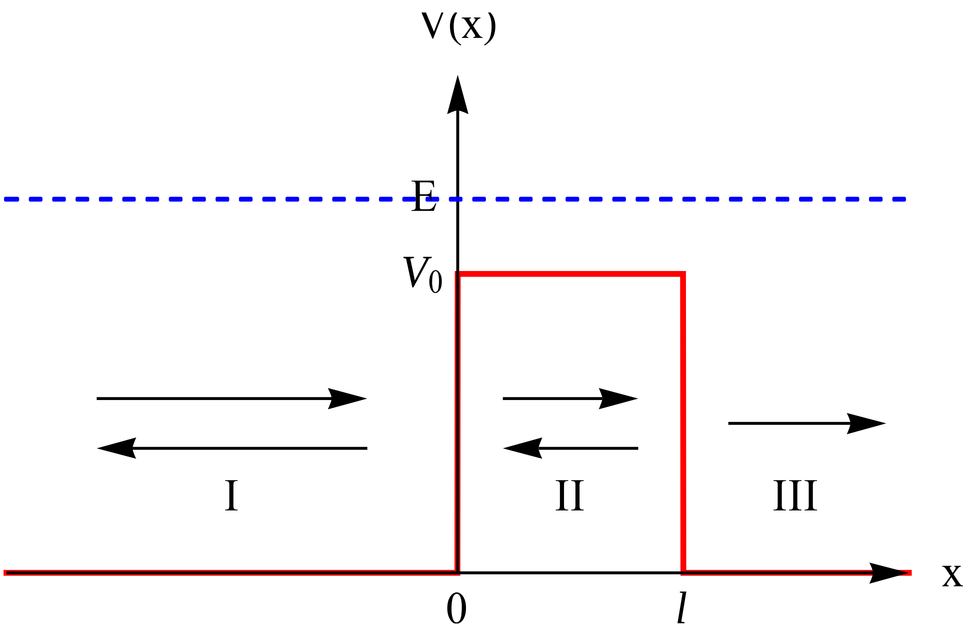
$$\frac{d\varphi_{II}(x)}{dx}=0$$

Therefore the derivative is no longer continuous. This is due to the fact that the potential jump is infinite at $x = 0$.

((**Mathematica**))


ψ{x}
1.0
0.5
−0.5
−1.0
x(nm)
−3
−2
−1
1
2
3
E (eV)=1
V (eV)=1.05
E (eV)=1
V (eV)=1.45
E (eV)=1
V (eV)=1.85
E (eV)=1
V (eV)=2.25
E (eV)=1
V (eV)=2.65
E (eV)=1
V (eV)=3.05
E (eV)=1
V (eV)=3.45
E (eV)=1
V (eV)=3.85

3. One dimensional square barrier



(a) Fabry-Perot for $E > V_0$

The wave numbers:

$$k_1^2 = \frac{2mE}{\hbar^2} \qquad \text{for the region I and III}$$

$$k_2^2 = \frac{2m(-V_0 + E)}{\hbar^2} \qquad \text{for the region II}$$

The wave functions:

$$\varphi_I(x) = A_1 e^{ik_1 x} + A_1' e^{-ik_1 x} \qquad \text{for the region I}$$

$$\varphi_{II}(x) = A_2 e^{ik_2 x} + A_2' e^{-ik_2 x} \qquad \text{for the region II}$$

$$\varphi_{III}(x) = A_3 e^{ik_1 x} + A_3' e^{-ik_1 x} \qquad \text{for the region III}$$

Let us choose $A_3' = 0$ (neglecting particles coming from $x = +\infty$).

The matching condition at $x = 0$ and $x = l$.

(i)

$$\varphi_I(x = 0) = \varphi_{II}(x = 0)$$

leading to

$$A_1 + A_1' = A_2 + A_2' \qquad (1)$$

(ii)

$$\varphi_{II}(x=l) = \varphi_{III}(x=l)$$

leading to

$$A_2 e^{ik_2 l} + A_2' e^{-ik_2 l} = A_3 e^{ik_1 l} \qquad (2)$$

(iii)

$$\frac{d\varphi_I(x=0)}{dx} = \frac{d\varphi_{II}(x=0)}{dx}$$

leading to

$$A_1 ik_1 - A_1' ik_1 = A_2 ik_2 - A_2' ik_2 \qquad (3)$$

(iv)

$$\frac{d\varphi_{II}(x=l)}{dx} = \frac{d\varphi_{III}(x=l)}{dx}$$

leading to

$$A_2 ik_2 e^{ik_2 l} - A_2' ik_2 e^{-ik_2 l} = A_3 ik_1 e^{ik_1 l}\,. \qquad (4)$$

From Eqs.(1)-(4), we have

$$A_1 = A_3 e^{ik_1 l}[\cos(k_2 l) - i\left(\frac{k_1{}^2 + k_2{}^2}{2k_1 k_2}\right)\sin(k_2 l)]$$

$$A_1' = iA_3 e^{ik_1 l}\left(\frac{k_2{}^2 - k_1{}^2}{2k_1 k_2}\right)\sin(k_2 l)$$

$$A_2 = \frac{A_3}{2k_2} e^{i(k_1 - k_2)l}(k_1 + k_2)$$

$$A_2' = \frac{A_3}{2k_2} e^{i(k_1+k_2)l}(-k_1 + k_2)$$

Probability current density:

$$J_I = \frac{\hbar k_1}{m}\left(|A_1|^2 - |A_1'|^2\right)$$

$$J_{II} = \frac{\hbar k_2}{m}\left(|A_2|^2 - |A_2'|^2\right)$$

$$J_{III} = \frac{\hbar k_1}{m}|A_3|^2$$

Reflection co-efficient;

$$R = \left|\frac{A_1'}{A_1}\right|^2 = \frac{(k_1^2 - k_2^2)^2 \sin^2(k_2 l)}{4k_1^2 k_2^2 + (k_1^2 - k_2^2)^2 \sin^2(k_2 l)}$$

Transmission co-efficient:

$$T = \left|\frac{A_3}{A_1}\right|^2 = \frac{4k_1^2 k_2^2}{4k_1^2 k_2^2 + (k_1^2 - k_2^2)^2 \sin^2(k_2 l)}$$
$$= \frac{4E(E - V_0)}{4E(E - V_0) + V_0^2 \sin^2(k_2 l)}$$

for $E>V_0$. $k_2 l = l\sqrt{\frac{2m(-V_0 + E)}{\hbar^2}}$

When $k_2 l = n\pi$ (n: integer), T = 1.(**Fabrey-Perot**)

((Mathematica)) Fabry-Perot

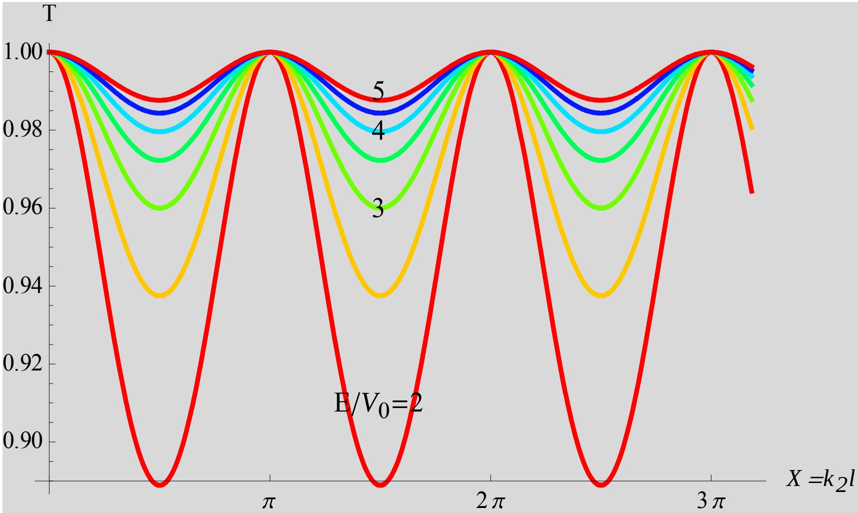
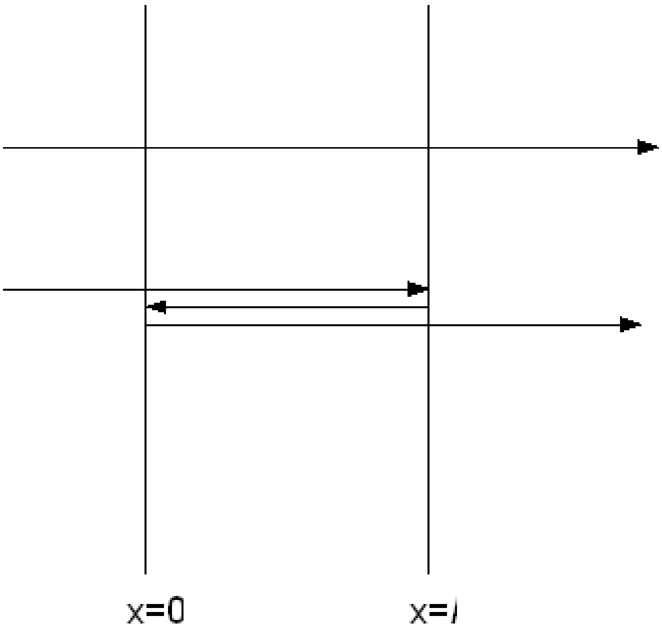


Fig. Plot of T as a function of $x = k_2 l$. The parameter E/V_0 is changed as 2, 2.5. 3, 3.5, 4, 4.5 and 5.

((**Note**))



We consider the case of $k_2 2l = 2n\pi$, or $k_2 l = n\pi$ (resonance condition).

$T = 1$, $R = 0$

$A_1' = 0$

$$A_1 = \cos(n\pi) e^{ik_1 l} A_3 = (-1)^n e^{ik_1 l} A_3$$

$$A_2 = \frac{A_3}{2k_2}(k_1 + k_2) e^{ik_1 l}(-1)^n = \frac{1}{2}(1 + \frac{k_1}{k_2}) A_1$$

$$A_2' = \frac{A_3}{2k_2}(-k_1 + k_2) e^{ik_1 l}(-1)^n = \frac{1}{2}(1 - \frac{k_1}{k_2}) A_1$$

Then we have

$$\varphi_I(x) = A_1 e^{ik_1 x}$$

$$\varphi_{II}(x) = A_2 e^{ik_2 x} + A_2' e^{-ik_2 x} = \frac{1}{2}(1 + \frac{k_1}{k_2}) A_1 e^{ik_2 x} + \frac{1}{2}(1 - \frac{k_1}{k_2}) A_1 e^{-ik_2 x}$$

$$= A_1[\cos(k_2 x) + \frac{k_1}{k_2} 2i \sin(k_2 x)]$$

(resonance scattering)

$$\varphi_{III}(x) = A_3 e^{ik_1 x} = (-1)^n e^{-ik_1 l} A_1 e^{ik_1 x}$$

(b) Tunneling effect for $E < V_0$

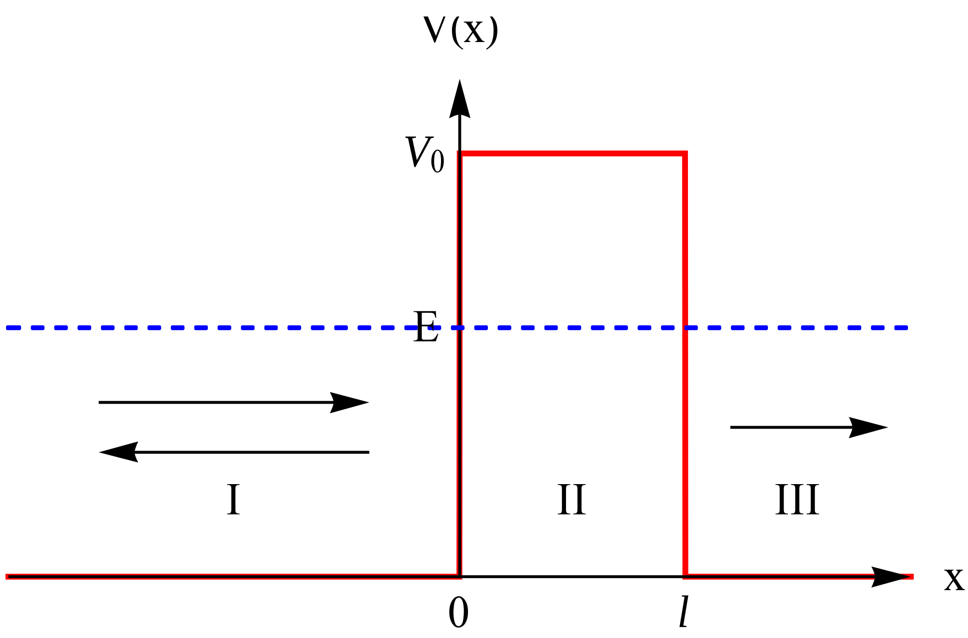


Fig. A beam of particles represented by a plane wave is incident on a potential barrier. Most particles are reflected, but some are transmitted by quantum mechanical tunneling.

We set

$$k_2 = -i\rho_2 \qquad \text{or} \qquad \rho_2 = ik_2$$

where

$$\rho_2{}^2 = \frac{2m}{\hbar^2}(V_0 - E) \qquad \text{for the region II}$$

The wave functions:

$$\varphi_I(x) = A_1 e^{ik_1 x} + A_1' e^{-ik_1 x} \qquad \text{for the region I}$$

$$\varphi_{II}(x) = A_2 e^{\rho_2 x} + A_2' e^{-\rho_2 x} \qquad \text{for the region II}$$

$$\varphi_{III}(x) = A_3 e^{ik_1 x} \qquad \text{for the region III}$$

where

$$k_1{}^2 = \frac{2m}{\hbar^2}E \qquad \text{for the regions I and III}$$

Boundary conditions: continuity of the wave function and its derivative with respect to x at $x = 0$ and $x = l$.

$$A_1 + A_1' = A_2 + A_2'$$

$$A_2 e^{\rho_2 l} + A_2' e^{-\rho_2 l} = A_3 e^{ik_1 l}$$

$$ik_1(A_1 - A_1') = \rho_2(A_2 - A_2')$$

$$\rho_2(A_2 e^{\rho_2 l} - A_2' e^{-\rho_2 l}) = A_3 ik_1 e^{ik_1 l}$$

When A_3 is a fixed parameter, we get A_1, A_1', A_2, and A_2' as

$$A_1 = \frac{A_3}{4k_1\rho_2} e^{(ik_1 - \rho_2)l}[-i(e^{2\rho_2 l} - 1)k_1{}^2 + 2(e^{2\rho_2 l} + 1)k_1\rho_2 + i(e^{2\rho_2 l} - 1)\rho_2{}^2]$$

$$A_1' = -\frac{A_3}{4k_1\rho_2} e^{(ik_1-\rho_2)l}(e^{2\rho_2 l}-1)(k_1+i\rho_2)(ik_1+\rho_2)$$

$$A_2 = \frac{A_3}{2\rho_2} e^{(ik_1-\rho_2)l}(ik_1+\rho_2)$$

$$A_2' = \frac{A_3}{2\rho_2} e^{(ik_1+\rho_2)l}(-ik_1+\rho_2)$$

Noting that

$$\cos(\pm i\theta) = \cosh\theta \text{ and } \sin(\pm i\theta) = \mp i \sinh\theta,$$

$$\cosh(2x) = 1+2\sinh^2(x), \qquad \cosh^2(x) - \sinh^2(x) = 1.$$

we have

$$A_1 = -A_3 e^{ik_1 l}[\cosh(\rho_2 l) + \frac{i}{2}\left(\frac{k_1^{\,2}-\rho_2^{\,2}}{k_1\rho_2}\right)\sinh(\rho_2 l)]$$

$$A_1' = -iA_3 e^{ik_1 l}\left(\frac{k_1^{\,2}+\rho_2^{\,2}}{2k_1\rho_2}\right)\sinh(\rho_2 l)$$

The wave functions in the regions I, II, and II are obtained as

$$\psi_I = \frac{A_3}{k_1\rho_2} e^{ik_1 l}[k_1\rho_2 e^{ik_1 x}\cosh(\rho_2 l) - (ik_1^{\,2}\cos(k_1 x) + \rho_2^{\,2}\sin(k_1 x))\sinh(\rho_2 l)]$$

$$\psi_{II} = \frac{A_3}{\rho_2} e^{ik_1 l}[\rho_2\cosh(\rho_2(l-x)) - ik_1\sinh(\rho_2(l-x))]$$

$$\psi_{II} = A_3 e^{ik_1 x}$$

where A_3 is fixed parameter. A typical example is shown below.

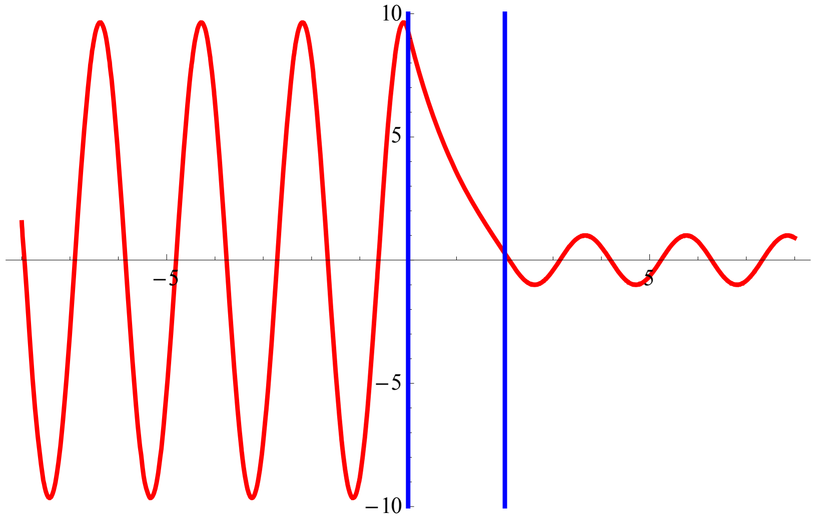


Fig. Real part of the wave functions in the tunneling through a barrier (denoted by blue lines).

((Probability current density))

$$J_I = \frac{\hbar k_1}{m}\left(|A_1|^2 - |A_1'|^2\right)$$

$$J_{II} = \frac{-i\hbar}{m}\rho_2\left(A_2 A_2'^* - A_2^* A_2'\right)$$

$$J_{III} = \frac{\hbar k_1}{m}|A_3|^2$$

where

$$J_I = J_{II} = J_{III}$$

The transmission coefficient is given by

$$T = \left|\frac{A_3}{A_1}\right|^2 = \frac{4k_1^2\rho_2^2}{4k_1^2\rho_2^2\cosh^2(\rho_2 l) + (k_1^2 - \rho_2^2)^2\sinh^2(\rho_2 l)}$$
$$= \frac{4k_1^2\rho_2^2}{4k_1^2\rho_2^2[1+\sinh^2(\rho_2 l)] + (k_1^2 - \rho_2^2)^2\sinh^2(\rho_2 l)}$$
$$= \frac{4k_1^2\rho_2^2}{4k_1^2\rho_2^2 + (k_1^2 + \rho_2^2)^2\sinh^2(\rho_2 l)}$$
$$= \frac{4E(V_0 - E)}{4E(V_0 - E) + V_0^2\sinh^2(\rho_2 l)}$$

where $E<V_0$ and $\rho_2 l = l\sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$. The reflection coefficient is given by

$$R = \left|\frac{A_1'}{A_1}\right|^2 = \left|\frac{A_1'}{A_3}\right|^2 \left|\frac{A_3}{A_1}\right|^2$$
$$= \frac{(k_1^2 + \rho_2^2)^2\sinh^2(\rho_2 l)}{4k_1^2\rho_2^2\cosh^2(\rho_2 l) + (k_1^2 - \rho_2^2)^2\sinh^2(\rho_2 l)}$$
$$= \frac{(k_1^2 + \rho_2^2)^2\sinh^2(\rho_2 l)}{4k_1^2\rho_2^2 + (k_1^2 + \rho_2^2)^2\sinh^2(\rho_2 l)}$$

Then we get

$$T + R = 1$$

We make a plot of T as a function of $x = \rho_2 l$, where $E/V_0 = 0.9$

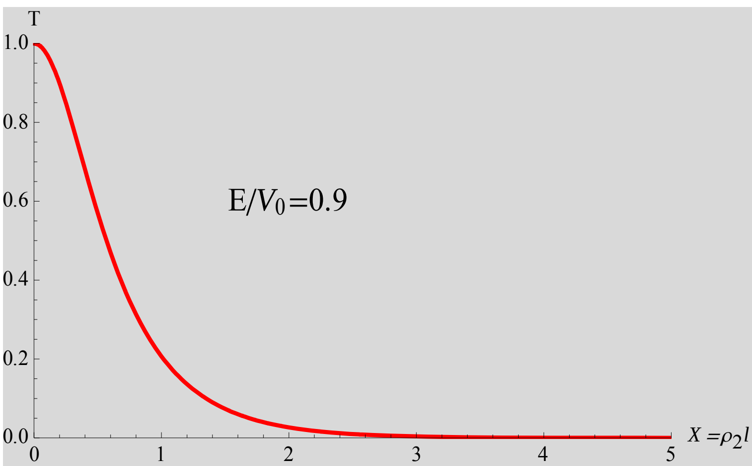


Fig. Plot of T as a function of $X = \rho_2 l$. $E/V_0 = 0.9$.

((Note))

When $\rho_2 l >> 1$,

$$\sinh(\rho_2 l) = \frac{e^{\rho_2 l} - e^{-\rho_2 l}}{2} = \frac{e^{\rho_2 l}}{2}$$

Then we have

$$T \approx \frac{16E(V_0 - E)}{V_0^{\ 2}} e^{-2\rho_2 l}$$

The particle has a non-zero probability of crossing the potential barrier.

((Parameter ρ_2 for electron))

$$\rho_2 = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$$

or

$$\frac{1}{\rho_2} = \frac{1.95192}{\sqrt{(V_0 - E)[eV]}} \text{Å}$$

When V_0 = 2 eV, l = 1 Å, and E = 1 eV, we have $1/\rho_2$ = 1.96 Å.

or

$$T = 0.78$$

The electron must then have a considerable probability of crossing the barrier.

((Parameter ρ_2 for proton))

$$\frac{1}{\rho_2} = \sqrt{\frac{2M}{\hbar^2}(V_0 - E)}$$

or

$$\frac{1}{\rho_2} = \frac{4.5552 \times 10^{-2}}{\sqrt{(V_0 - E)[eV]}} \text{ Å.}$$

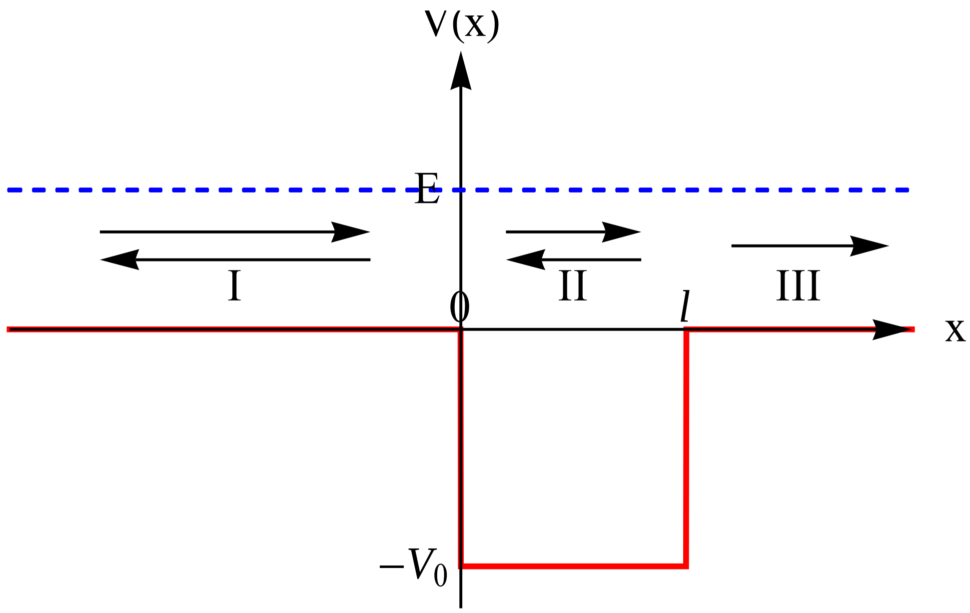
When $V_0 = 2$ eV, $l = 1$ Å, and $E = 1$ eV, we have $1/\rho_2 = 4.5552\text{x}10^{-2}$ Å.

or

$$T = 3.94 \text{ x } 10^{-19},$$

which is negligibly small.

4. One dimensional square-well potential: Ramsuer effect



The wave numbers:

$$k_2^2 = \frac{2m}{\hbar^2}(E + V_0)$$

$$k_1^2 = \frac{2m}{\hbar^2}E$$

$$A_1 = A_3 e^{ik_1 l}[\cos(k_2 l) - i\left(\frac{k_1^2 + k_2^2}{2k_1 k_2}\right)\sin(k_2 l)]$$

$$A_1' = iA_3 e^{ik_1 l}\left(\frac{k_2^2 - k_1^2}{2k_1 k_2}\right)\sin(k_2 l)$$

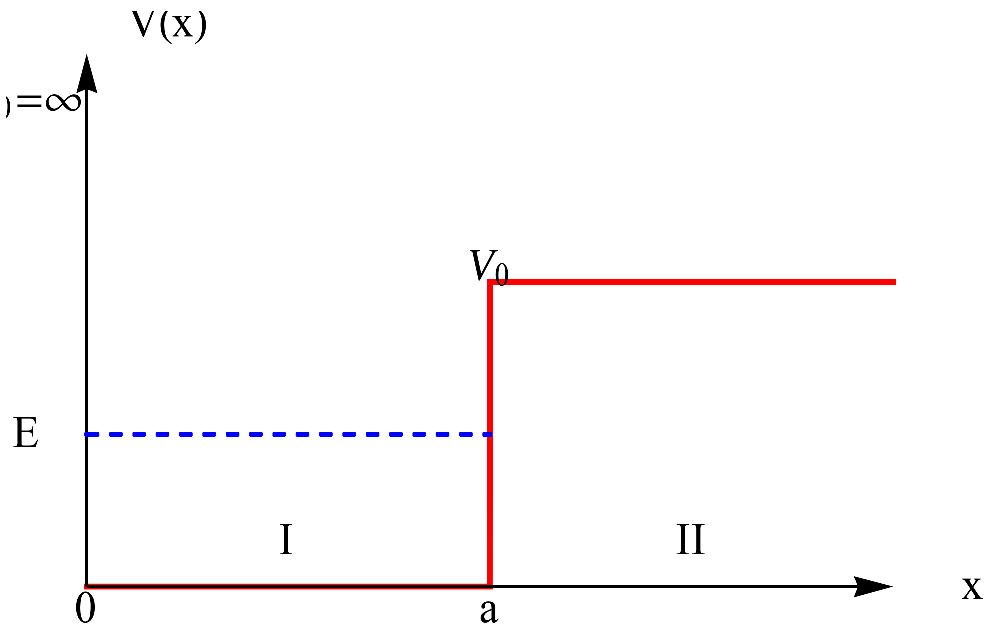
Reflection co-efficient

$$R = \left|\frac{A_1'}{A_1}\right|^2 = \frac{(k_1^2 - k_2^2)^2 \sin^2(k_2 l)}{4k_1^2 k_2^2 + (k_1^2 - k_2^2)^2 \sin^2(k_2 l)}$$

Transmission co-efficient

$$T = \left|\frac{A_3}{A_1}\right|^2 = \frac{4k_1^2 k_2^2}{4k_1^2 k_2^2 + (k_1^2 - k_2^2)^2 \sin^2(k_2 l)}$$

5. Semi infinite well potential



For 0<*x*<*a* (region I)

$$H\varphi_I(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\varphi_I(x) = E\varphi_I(x) = \frac{\hbar^2 k^2}{2m}\varphi_I(x)$$

The solution of this equation is

$$\varphi_I(x) = A\sin(kx) + A_1\cos(kx)$$

where

$$k = \sqrt{\frac{2m}{\hbar^2}E}$$

Using the boundary condition:

$$\varphi_I(x=0)=0$$

we have

$A_1 = 0$ and $A \neq 0$.

Then we get

$$\varphi_I(x)=A\sin(kx)$$

For $x>a$ (region-II)

$$(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V_0)\varphi_{II}(x)=E\varphi_{II}(x)$$

or

$$\frac{d^2}{dx^2}\varphi_{II}(x)-\kappa^2\varphi_{II}(x)=0$$

where

$$\kappa=\sqrt{\frac{2m}{\hbar^2}(V_0-E)}\,.$$

The solution of $\varphi_{II}(x)$ is given by

$$\varphi_{II}(x)=Be^{-\kappa(x-a)}$$

The condition for the continuity of $\varphi(x)$ and $\frac{d\varphi(x)}{dx}$ at $x = a$

$$A\sin(ka)=B$$

$$kA\cos(ka)=-B\kappa$$

From these two equations we have

$$\frac{1}{ka}\tan(ka) = -\frac{1}{\kappa a} \qquad \text{or} \qquad ka = -\kappa a \tan(ka)$$

with

$$(\kappa a)^2 + (ka)^2 = \frac{2m}{\hbar^2} a^2 V_0$$

For simplicity, we assume that

$$x = ka, \qquad y = \kappa a, \qquad R = \sqrt{\frac{2m}{\hbar^2} a^2 V_0}$$

Then we need to solve the equations by using graphs,

$$x = -y\tan(x), \qquad x^2 + y^2 = R^2$$

where $x>0$ and $y>0$.

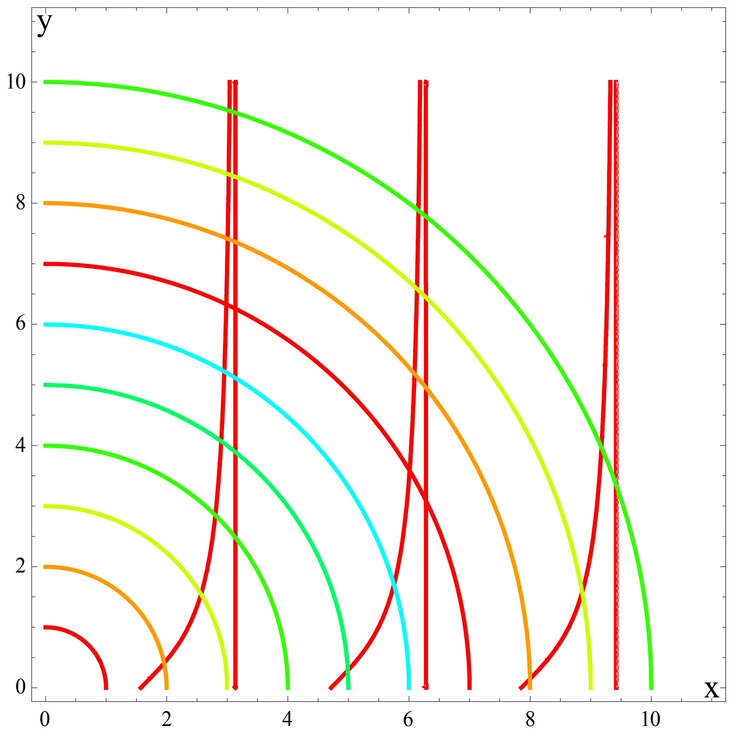


Fig. The intersections of the curve $y = -x/\tan x$ and the circle ($x^2+y^2 = R^2$). The radius R is changed as a parameter. Note that $y = -x/\tan x$ changes the sign from negative to positive at $x = \pi/2$, $5\pi/2$, $7\pi/2$,... When $\pi/2<R<3\pi/2$, there are two intersections, leading to the two energy levels. When $5\pi/2<R<7\pi/2$, there are three intersections, leading to the three energy levels.