Theory of special relativity (review of Phys.132)

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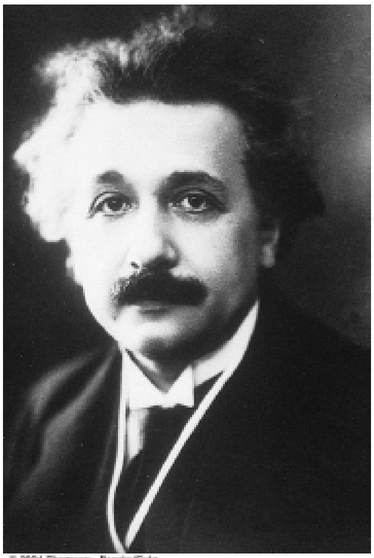
1. Basic postulate



Albert Einstein (1879 – 1955)

Neither the laws of mechanics nor the laws for the propagation of light reveal any intrinsic distinction between different inertial reference frames. This motivated Einstein to take a bold step and to propose a general hypothesis concerning all the laws of physics. This hypothesis is the principle of relativity.

The Principle of Relativity

1. Space is isotropic and uniform. The fundamental law of physics are identical for ant two observers in uniform relative motion.

2. The speed of light (in vacuum) is the same in all reference frames in uniform motion with respect to the source. It always has the value *c*.

2. Michelson-Morley experiments

2.1 Ether

Nineteenth-century physicists thought- reasonably at the time-that light must propagate in the ether analogously to the way that sound propagates in a material medium such as air. The speed of sound in air depends on properties of the air. Most important, the peed of sound an observer measures depends on the observer's motion relative to the air. If the ether carried light the way air carriers sound, then an observer moving relative to the ether would measure the speed of a light wave so that it would vary according to the observer's speed. The observer could "catch up with" or "fall back from" a propagating disturbance.

2.2 The questions of ether asked by Maxwell

In the 1870's Maxwell asked whether the velocity of Earth relative to the ether might affect the observed speed of light. Earth traces out an elliptical orbits as it moves around the Sun. This motion can be well approximated by uniform motion in a straight line with a speed over time intervals substantially shorter than a year. But how large is the velocity v? We can approximate Earth's orbit by a circle with a radius of 1 AU (astronomical unit = average distance between the Earth and the Sun, 1.49597870 x 10^8 km), around whose circumference our planet moves uniformly. With one year about equal to 365 x 24 x 60 x 60 = 3.1536 x 10^7 s, v is equal to 29.805 km/s, so that

$$v/c = 9.9421 \times 10^{-5}$$

2.3 Michelson-Morley experiment

Albert Abraham Michelson (December 19, 1852 – May 9, 1931) was a Prussian-born American physicist known for his work on the measurement of the speed of light and especially for the Michelson-Morley experiment. In 1907 he received the Nobel Prize in Physics. He became the first American to receive the Nobel Prize in sciences.

Attempts were made to determine the absolute velocity of the earth through the hypothetical "ether" that was supposed to pervade all space. The most famous of these experiments is one performed by Michelson ad Morley in 1887. It was 18 years later before the negative results of the experiment were finally explained, by Einstein. (Feynman Physics, volume 1, 15-3)

The Michelson-Morley experiment was performed with an apparatus like that shown schematically in the Fig. The system consists of a light source A, a partially silvered glass plate B, and two mirrors C and E, all mounted on a rigid base. The mirrors are placed at equal distance *L* from B. The plate B splits on a coming-beam of light, and the two resulting beam continue in mutually perpendicular directions to the mirrors, where they are reflected back to B. On arriving back at B, the two beams are recombined as two superposed beams, D and F. If the time taken for the light to go from B to E and back is the same as the time from B to C and back, the emerging beams D and F will be in phase and will reinforce each other, but if the two times differ slightly, the beams will be slightly out of phase and interference will result. If the apparatus is at rest in the ether, the times should be precisely equal, but if it is moving toward the right with a velocity u, there should be difference in the time.

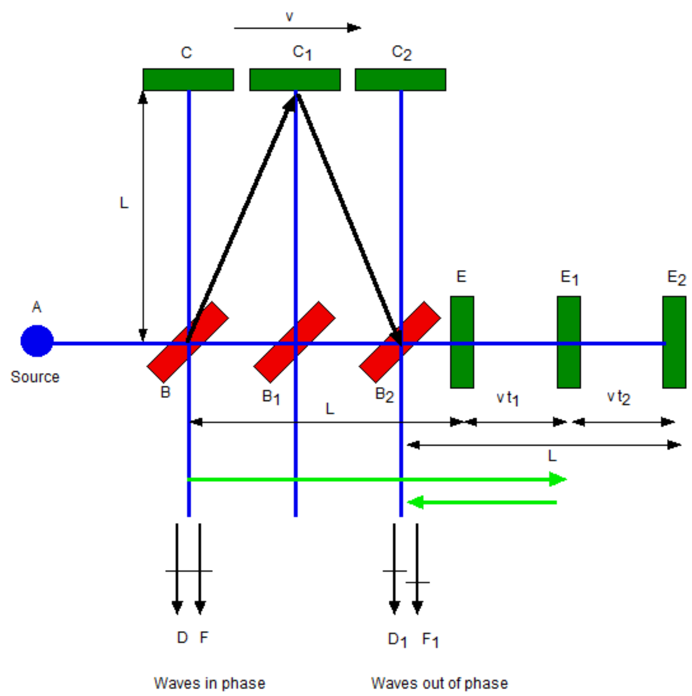


Fig. Schematic diagram of the Michael-Morley experiment.

The time required for the light to go from plate B to mirror E is t_1, and the time for the return is t_2. While the light is on its way from B to the mirror, the system moves a distance vt_1, so the light must transverse a distance $L + vt_1$ at the speed c.

$$L + vt_1 = ct_1 \qquad \text{or} \qquad t_1 = \frac{L}{c - v}$$

During the time t_2, the plate B advances a distance vt_2, so the return distance of the light is $L - vt_2$. Then we have

$$L - vt_2 = ct_2 \qquad \text{or} \qquad t_2 = \frac{L}{c+v}$$

The total time is

$$t_1 + t_2 = \frac{2Lc}{c^2 - v^2} = \frac{2L/c}{1 - \frac{v^2}{c^2}} \qquad (1)$$

The time required for the light to go from B to the mirror C is t_3. During the time t_3, the mirror C moves to the right a distance vt_3 to the position C_1; in the same time, the light travels a distance ct_3 along the hypotenuse of a triangle, which is BC_1. Then we have

$$(ct_3)^2 = L^2 + (vt_3)^2 \qquad \text{or} \qquad t_3 = \frac{L}{\sqrt{c^2 - v^2}} = \frac{L/c}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The total time for the round trip is $2t_3$.

$$2t_3 = \frac{2L/c}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (2)$$

The numerators of Eq.(1) and Eq.(2) are identical, and represent the time that would be taken if the system were at rest. The difference time Δt is given by

$$\Delta t = t_1 + t_2 - 2t_3 = \frac{2L}{c}\left(\frac{1}{1 - \frac{v^2}{c^2}} - \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = \frac{2L}{c}\left(\frac{1}{2}\frac{v^2}{c^2} + \frac{5}{8}\frac{v^4}{c^4} + \frac{11}{16}\frac{v^6}{c^6} + ...\right)$$

For $L = 1$ m and $v = 2.9805 \times 10^4$ m/s, we have

$$\Delta t = 3.297 \times 10^{-17} s.$$

For the Na D-line ($\lambda = 590$ nm), the period T is obtained as

$$T = \frac{\lambda}{c} = \frac{590 \times 10^{-9} m}{2.998 \times 10^8} = 1.97 \times 10^{-15} \text{ s}$$

or

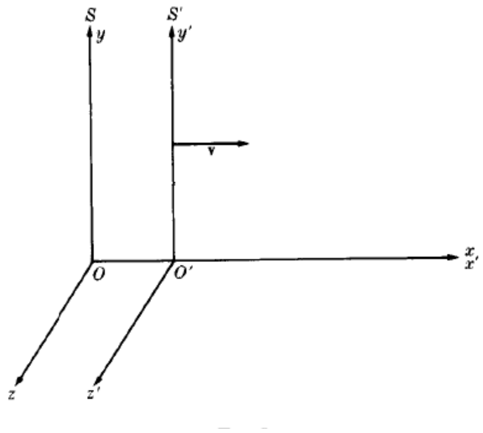
$$\frac{\Delta t}{T} = \frac{3.297 \times 10^{-17}}{1.97 \times 10^{-15}} = 0.01675$$

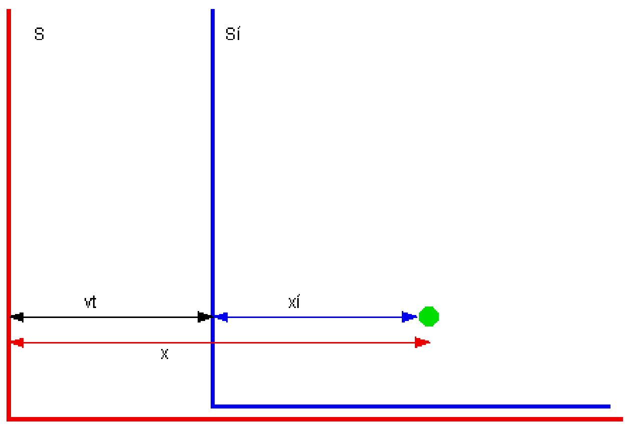
This is 1.675 % of a fringe shift and the Michelson instrument could detect that. This was something that was absolutely measurable; indeed it could not be missed. But when they did the experiment, they found nothing! There was no fringe shift, indicating no phase shift between the waves and hence no time difference Δt. They repeated the experiment later in the year, when the Earth was in a different position in its orbit, just in case the planet has been accidentally at rest in the ether the first time. Still they found nothing; there was no shift at all.

In conclusion, the speed of light is the same, no matter how the frame in which its speed is measured moves, at least for the kind of approximately uniform motion an earthbound instrument undergoes.

3. Lorentz transformation

3.1 Derivation of Lorentz transformation





We consider a Galilean transformation given by

$$x' = x - vt$$
$$x = x' + vt'$$
$$t' = t$$

$$\frac{dx'}{dt'} = \frac{dx}{dt'} - v\frac{dt}{dt'} = \frac{dx}{dt} - v$$
$$u' = u - v$$

We know that the velocity of light remains unchanged under a transformation (so-called the Lorentz transformation) satisfying the principle of relativity. This implies that the Lorentz transformation is not the same as the Galilean transformation.

Here we assume that

$$x' = \gamma(x - vt)$$
$$x = \gamma(x' + vt')$$

from the symmetry of transformation

What is the value of γ?

(i) The light is emitted at

$$t = t' = 0$$
$$x = x' = 0$$

(initially). The speed of light (in vacuum) is the same in all internal reference frames; it always has the value c.

$$\frac{x'}{t'} = \frac{x}{t} = c$$

The determination of the Lorentz factor γ.

The substitution of $x = ct$ and $x' = ct'$ yields

$$ct' = \gamma(ct - vt) = \gamma(c - v)t$$
$$ct = \gamma(ct' + vt') = \gamma(c + v)t'$$

or

$$c^2 = \gamma^2(c - v)(c + v) = \gamma^2(c^2 - v^2)$$

or

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \text{and} \quad \beta = \frac{v}{c}$$

((**Mathematica**))

Derivation of Lorentz transformation

```
eq1 = x == γ (x' + v t'); eq2 = x' == γ (x - v t)
```

$x' == (-t\,v + x)\,\gamma$

```
eq3 = Solve[{eq1, eq2}, {x', t'}] // Simplify // Flatten
```

$\left\{t' \to \frac{x\left(\frac{1}{\gamma} - \gamma\right)}{v} + t\,\gamma,\ x' \to (-t\,v + x)\,\gamma\right\}$

```
eq4 = x'/t' == x/t /. eq3 // Simplify
```

$\frac{v\,(-t\,v + x)\,\gamma^2}{x + t\,v\,\gamma^2 - x\,\gamma^2} == \frac{x}{t}$

```
eq5 = eq4 /. {x → c t} // Simplify
```

$-\frac{v\,(-c + v)\,\gamma^2}{c - c\,\gamma^2 + v\,\gamma^2} == c$

```
eq6 = Solve[eq5, γ]
```

$\left\{\left\{\gamma \to -\frac{i\,c}{\sqrt{-c^2 + v^2}}\right\}, \left\{\gamma \to \frac{i\,c}{\sqrt{-c^2 + v^2}}\right\}\right\}$

```
t1' = t' /. eq3 /. eq6[[2]] // Simplify
```

$\frac{i\,(c^2\,t - v\,x)}{c\,\sqrt{-c^2 + v^2}}$

```
x' /. eq3
```

$(-t\,v + x)\,\gamma$

```
x1' = x' /. eq3 /. eq6[[2]] // Simplify
```

$\frac{i\,c\,(-t\,v + x)}{\sqrt{-c^2 + v^2}}$

Then we have

$$x' = \gamma(x - vt)$$
$$x = \gamma(x' + vt')$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \text{and} \qquad \beta = \frac{v}{c}$$

Here we have

$$\frac{x}{\gamma} = x' + vt' = \gamma(x - vt) + vt'$$
$$vt' = -\gamma(1 - \frac{1}{\gamma^2})x + \gamma vt$$
$$t' = \gamma t - \frac{\gamma}{v}(1 - \frac{1}{\gamma^2})x$$
$$= \gamma(t - \frac{\beta}{c}x)$$

or

$$t' = \gamma(t - \frac{\beta}{c}x)$$

When v is changed into $-v$, the Lorentz transformation from the S' to S frames can be described as

$$x = \gamma(x' + vt') = \gamma(x' + \beta ct')$$
$$t = \gamma(t' + \frac{\beta}{c}x')$$

noting that $x \rightarrow x'$, $t \rightarrow t'$, v $-v$.

In summary, we have the Lorentz transformation

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t - \frac{\beta}{c}x) \end{aligned} \quad , \qquad \begin{aligned} x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \gamma(t' + \frac{\beta}{c}x') \end{aligned}$$

((**Mathematica**))

```
γ = 1/Sqrt[1 - β^2]
```

$$\frac{1}{\sqrt{1-\beta^2}}$$

```
Series[γ, {β, 0, 5}]
```

$$1+\frac{\beta^2}{2}+\frac{3\beta^4}{8}+O[\beta]^6$$

3.2 Lorentz contraction (length contraction)

Imagine a stick moving to the right at the velocity *v*. Its rest length (that is, its length measured in S') is $\Delta x'$.
We measure the distance of the stick under the condition that $\Delta t = 0$. Since

$$\Delta x' = \gamma(\Delta x - \beta c \Delta t) = \gamma \Delta x$$

we have

$$\Delta x = \frac{1}{\gamma}\Delta x' = \sqrt{1-\beta^2}\,\Delta x' = \sqrt{1-\beta^2}\,\Delta x_0$$

The length of the stick measure in S (Δx) is shorter than that observed in S' ($\Delta x' = \Delta x_0$ proper length at the rest-frame)

3.3 Time dilation

3.3.1 Derivation of the time dilation from the Lorentz transformation

We are watching one moving clock moving to the right at the velocity *v*.

$$\Delta t = \gamma(\Delta t' + \frac{\beta}{c}\Delta x')$$

with $\Delta x' = 0$. Then we have

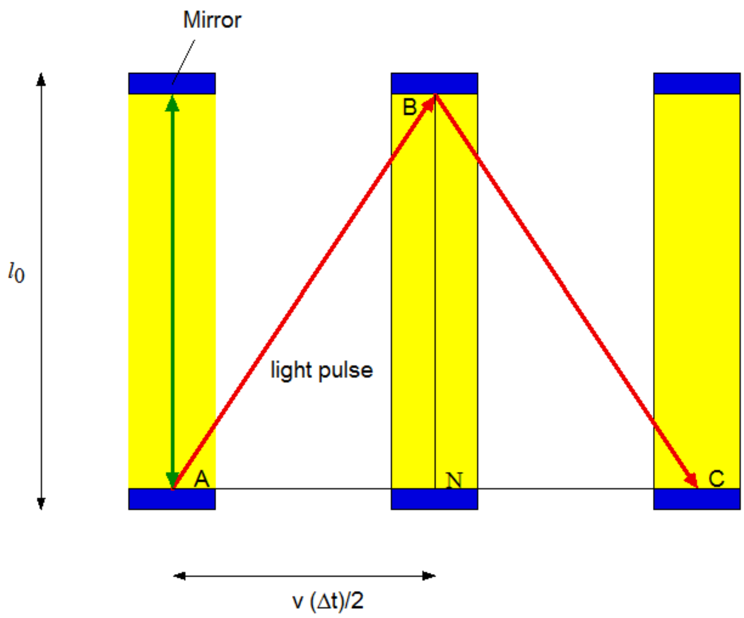
$$\Delta t = \gamma \Delta t' > \Delta t'.$$

or

$$\Delta t = \frac{1}{\sqrt{1-\beta^2}}\Delta t' = \frac{1}{\sqrt{1-\beta^2}}\Delta t_0 \geq \Delta t_0$$

The time in S (Δt) is longer than that observed in S' (Δt_0, proper time). The moving clocks run slow. The time measured with a clock that stays at one place in its reference frame (like the clock in S') measures proper time.

3.3.2 Experiment for the time dilation



Our system consists of a box containing two mirrors between which a light pulse bounces back and forth. If the distance between mirrors is l_0, the interval between successive counts is

$$\Delta t' = \frac{2l_0}{c} \qquad (1)$$

in the frame S'. We assume that this frame S' is moving at constant velocity (v) with respect to another frame S. The path of the light pulse with respect to the frame S is ABC, and takes a time Δt.

AN = NC = $v\,(\Delta t/2)$.
BN = l_0.

Therefore,

$$AB + BC = 2\sqrt{l_0^{\,2} + (\frac{v\Delta t}{2})^2} = c\Delta t$$

or

$$\Delta t = \frac{2}{c}\sqrt{l_0{}^2 + (\frac{v\Delta t}{2})^2} \qquad (2)$$

From Eqs.(1) and (2), we have

$$(\frac{c\Delta t}{2})^2 = (\frac{c\Delta t'}{2})^2 + (\frac{v\Delta t}{2})^2$$

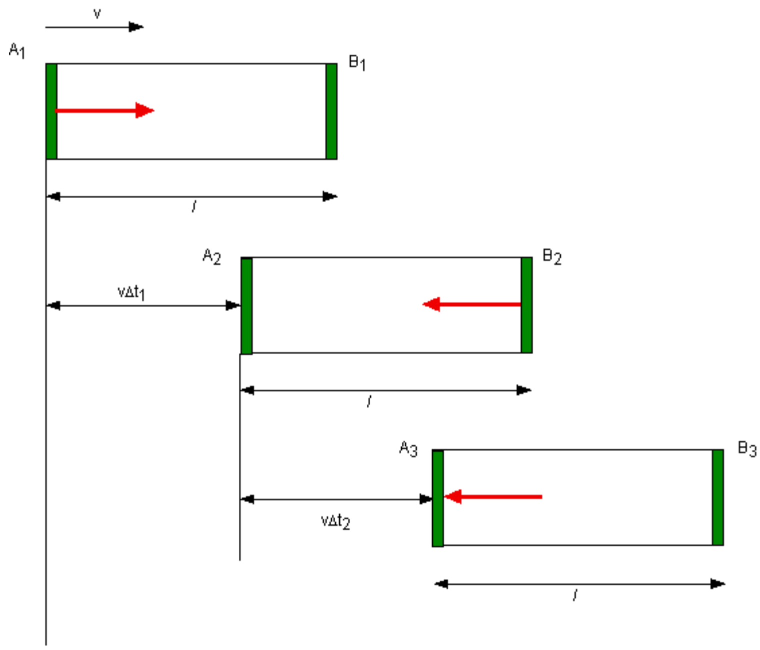
using the relation $\frac{c\Delta t'}{2} = l_0$

or

$$\Delta t = \frac{\Delta t'}{\sqrt{1-\beta^2}} = \gamma\Delta t'$$

with $\beta = v/c$.

3.3.3 Length contraction derived from the time dilation



The proper time for one round trip of the light is

$$\Delta t_0 = \frac{2l_0}{c}.$$

Note that the definition of the proper time is given later.

We assume that the length of the system is l as measured in S. Let the light take a time Δt_1 to travel from A_1 to B_2 (at which point it is reflected) and a time Δt_2 to travel from B_2 to A_3. Then we have

$$l + v\Delta t_1 = c\Delta t_1 \qquad \text{or} \qquad \Delta t_1 = \frac{l}{c - v}$$

$$l - v\Delta t_2 = c\Delta t_2 \qquad \text{or} \qquad \Delta t_2 = \frac{l}{c + v}$$

The total time Δt is obtained as

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{l}{c - v} + \frac{l}{c + v} = \frac{2cl}{c^2 - v^2} = \frac{2l/c}{1 - \frac{v^2}{c^2}}$$

Using the relation for the time dilation

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2l_0/c}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2l/c}{1 - \frac{v^2}{c^2}}$$

we get

$$l = \sqrt{1 - \frac{v^2}{c^2}}\, l_0$$

4. Four-dimensional representation of the Lorentz transformation

4.1 Lorentz transformation

For convenience, we introduce

$$x_4 = ict$$

or

$$-i\frac{x_4}{c} = t$$

and

$x_1 = x$, $x_2 = y$, and $x_3 = z$,

where i is the pure imaginary. $i = \sqrt{-1}$.

Then we have

$$
\begin{aligned}
x_1' &= \gamma(x_1 + i\beta x_4) \\
x_2' &= x_2 \\
x_3' &= x_3 \\
x_4' &= \gamma(-i\beta x_1 + x_4)
\end{aligned}
$$

or

$$
\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}
$$

or

$$
x' = ax \qquad \text{or} \qquad x_\mu' = a_{\mu\nu} x_\nu
$$

with

$$
a = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix}
$$

4.2. Inverse Lorentz transformation

Inversely, we have the inverse Lorentz transformation

$$
\begin{aligned}
x_1 &= \gamma(x_1' - i\beta x_4') \\
x_4 &= \gamma(i\beta x_1' + x_4')
\end{aligned}
$$

or

$$
\begin{aligned}
x_1 &= \gamma(x_1' + c\beta t') = \gamma(x_1' + vt') \\
t &= \gamma(t' + \frac{\beta}{c} x_1') = \gamma(t' + \frac{v}{c^2} x_1')
\end{aligned}
$$

or in the matrix form,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix}$$

or

$$x = a^{-1}x'$$

with

$$a^{-1} = a^T = \begin{pmatrix} \gamma & 0 & 0 & -i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

where a^T is the transpose matrix of a.

In summary, we have

$$x_\mu = (a^{-1})_{\mu\nu} x'_\nu = (a^T)_{\mu\nu} x'_\nu = a_{\nu\mu} x'_\nu$$

Note that

$$a_{\mu\nu} a_{\mu\lambda} = \delta_{\nu\lambda}$$

since

$$\left(a^{-1}a\right)_{\lambda\nu} = (a^{-1})_{\lambda\mu} a_{\mu\nu} = (a^T)_{\lambda\mu} a_{\mu\nu} = a_{\mu\lambda} a_{\mu\nu} = \delta_{\lambda\nu}$$

(i) Invariant under the Lorentz transformation

$$x'_\mu x'_\mu = a_{\mu\nu} x_\nu a_{\mu\lambda} x_\lambda = a_{\mu\nu} a_{\mu\lambda} x_\nu x_\lambda = \delta_{\mu\lambda} x_\nu x_\lambda = x_\mu x_\mu$$

(ii) Inverse Lorentz transformation

$$x_\mu' = a_{\mu\lambda} x_\lambda$$

$$a_{\mu\nu} x_\mu' = a_{\mu\nu} a_{\mu\lambda} x_\lambda = \delta_{\nu\lambda} x_\lambda = x_\nu$$

or

$$x_\nu = a_{\mu\nu} x_\mu ' \qquad \text{or} \qquad x_\mu = a_{\nu\mu} x_\nu '$$

((Mathematica))

Property of Lorentz transformation

Matrix of Lorentz reansformation a[μ,ν]

```
Clear[v, c, β, x, y, z, t];
β = v / c;
L = {{1 / Sqrt[1 - β^2], 0, 0, I β / Sqrt[1 - β^2]},
   {0, 1, 0, 0}, {0, 0, 1, 0}, {-I β / Sqrt[1 - β^2], 0, 0, 1 / Sqrt[1 - β^2]}};
```

```
L // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} & 0 & 0 & \frac{i\,v}{c\sqrt{1-\frac{v^2}{c^2}}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{i\,v}{c\sqrt{1-\frac{v^2}{c^2}}} & 0 & 0 & \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \end{pmatrix}$$

Determinant of the matrix of Lorentz transformation

```
Det[L] // Simplify
```

1

Proof of a[μ, λ] a[ν, λ] = δ[μ, ν]

$$\texttt{K[}\mu\texttt{\_, }\nu\texttt{\_] := }\sum_{\lambda=1}^{4}\texttt{L[[}\mu\texttt{, }\lambda\texttt{]] L[[}\nu\texttt{, }\lambda\texttt{]]}$$

```
I1 = Table[K[μ, ν], {μ, 1, 4}, {ν, 1, 4}] // Simplify
```

{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}

```
MatrixForm[I1]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Inverse of Lorentz transformation matrix

```
Linv = Inverse[L] // Simplify
```

$$\left\{\left\{\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, 0, 0, -\frac{i\,v}{c\sqrt{1-\frac{v^2}{c^2}}}\right\}, \{0, 1, 0, 0\}, \{0, 0, 1, 0\}, \left\{\frac{i\,v}{c\sqrt{1-\frac{v^2}{c^2}}}, 0, 0, \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right\}\right\}$$

```
Linv // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} & 0 & 0 & -\frac{i\,v}{c\sqrt{1-\frac{v^2}{c^2}}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{i\,v}{c\sqrt{1-\frac{v^2}{c^2}}} & 0 & 0 & \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \end{pmatrix}$$

Transpose of the Lorentz transformation matrix

```
Ltrans = Transpose[L]
```

$$\left\{\left\{\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, 0, 0, -\frac{i\,v}{c\sqrt{1-\frac{v^2}{c^2}}}\right\}, \{0, 1, 0, 0\}, \{0, 0, 1, 0\}, \left\{\frac{i\,v}{c\sqrt{1-\frac{v^2}{c^2}}}, 0, 0, \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right\}\right\}$$

```
Ltrans // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} & 0 & 0 & -\frac{i\,v}{c\sqrt{1-\frac{v^2}{c^2}}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{i\,v}{c\sqrt{1-\frac{v^2}{c^2}}} & 0 & 0 & \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \end{pmatrix}$$

```
Ltrans - Linv // Simplify
```

{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

Proof : Ltrans = Linv

```
L.Ltrans // Simplify
```

{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}

```
x = {x1, x2, x3, x4}
```

{x1, x2, x3, x4}

```
x' = {x1', x2', x3', x4'}
```

{x1′, x2′, x3′, x4′}

```
L.x // Simplify
```

$$\left\{\frac{c\,x1 + i\,v\,x4}{c\sqrt{1-\frac{v^2}{c^2}}}, x2, x3, \frac{-i\,v\,x1 + c\,x4}{c\sqrt{1-\frac{v^2}{c^2}}}\right\}$$

```
L.x /. x4 → i c t // Simplify
```

$$\left\{\frac{-t\,v + x1}{\sqrt{1-\frac{v^2}{c^2}}}, x2, x3, \frac{i\,(c^2\,t - v\,x1)}{c\sqrt{1-\frac{v^2}{c^2}}}\right\}$$

4.3 Lorentz transformation of four vector

We introduce the four vector notation

$$A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ iA_0 \end{pmatrix}$$

A_μ (μ = 1, 2, 3, and 4)

where

A_1, A_2, A_3:	real
$A_4 = iA_0$	purely imaginary

The new four-vector in the S' frame is related to the old four-vector in the S-frame through the Lorentz transformation by

$$A_\mu' = a_{\mu\nu} A_\nu$$

or

$$A_\mu = a_{\nu\mu} A_\nu'$$

since

$$a_{\mu\nu} A_\mu' = a_{\mu\nu} a_{\mu\lambda} A_\lambda = \delta_{\nu\lambda} A_\lambda = A_\nu .$$

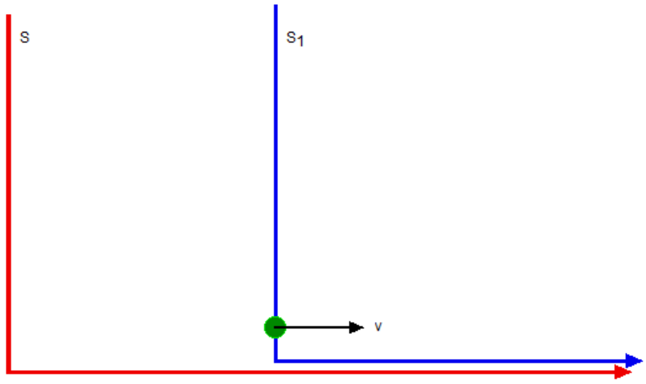
Here we define the scalar product by

$$A \cdot B = A_\mu B_\mu$$

It is seen that this scalar product is invariant (Lorentz scalar) under the Lorentz transformation,

$$A'_\mu B'_\mu = a_{\mu\nu} A_\nu a_{\mu\lambda} B_\lambda = a_{\mu\nu} a_{\mu\lambda} A_\nu B_\lambda = \delta_{\nu\lambda} A_\nu B_\lambda = A \cdot B$$

4.4 Proper time



In simple terms, the proper time is the time on a clock that moves with a particle. The system moves with the velocity $\boldsymbol{v}$ in the S-frame. The system is at rest in the S'-frame. Here we show that $(dx_\mu)^2$ is relativistic *invariant* under the Lorentz transformation.

$$(dx_\mu')^2 = a_{\mu\lambda} a_{\mu\sigma} dx_\lambda dx_\sigma = \delta_{\lambda\sigma} dx_\lambda dx_\sigma = (dx_\mu)^2 = -(ds)^2$$

The proper time is defined as follows,

$$(ds)^2 = c^2(dt)^2 - (dx_1)^2 - (dx_2)^2 - (dx_3)^2 = c^2(dt')^2 - (dx_1')^2 - (dx_2')^2 - (dx_3')^2$$

$$(ds)^2 = c^2(dt)^2\{1 - \frac{1}{c^2}[(\frac{dx_1}{dt})^2 + (\frac{dx_2}{dt})^2 + (\frac{dx_3}{dt})^2]\} = c^2(dt)^2[1-\beta^2] =$$
$$= c^2(dt')^2\{1 - \frac{1}{c^2}[(\frac{dx_1'}{dt'})^2 + (\frac{dx_2'}{dt'})^2 + (\frac{dx_3}{dt'})^2]\}$$

or

$$(ds)^2 = c^2(dt)^2\{1 - \frac{1}{c^2}[(\frac{dx_1}{dt})^2 + (\frac{dx_2}{dt})^2 + (\frac{dx_3}{dt})^2]\} = c^2(dt)^2[1-\beta^2] =$$
$$= c^2(dt')^2$$

or

$$dt' = d\tau = \frac{ds}{c} = dt\sqrt{1-\beta^2}$$

where τ is a proper time. *The proper time is a time recorded by the standard clock moving with a uniform velocity v relative to an inertial system S.* Since ds is invariant under the Lorentz transformation, the proper time is also invariant.

4.5 Four dimensional Laplacian operator

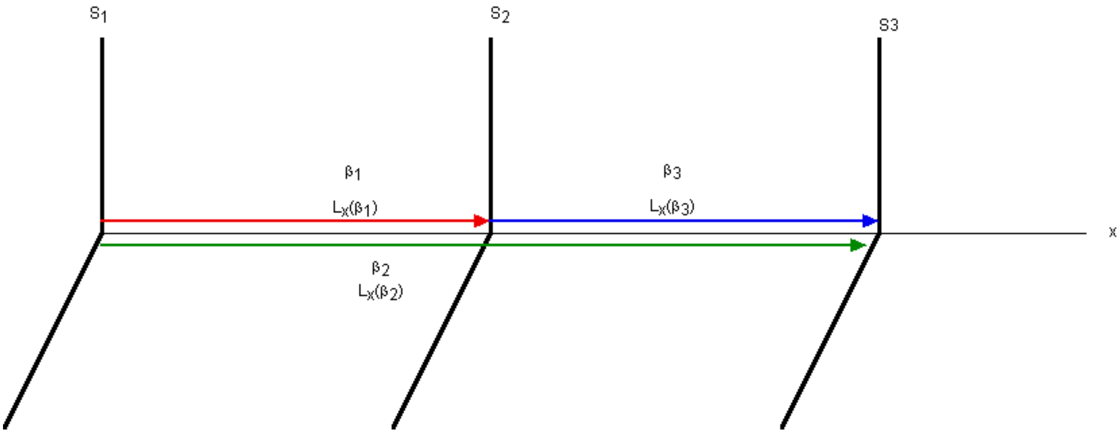
$$\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\mu} = \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \frac{\partial^2}{\partial x_3{}^2} + \frac{\partial^2}{\partial x_4{}^2} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

is invariant under the Lorentz transformation: Lorentz scalar

$$\frac{\partial}{\partial x_\mu'}\frac{\partial}{\partial x_\mu'} = a_{\mu\nu}\frac{\partial}{\partial x_\nu}a_{\mu\lambda}\frac{\partial}{\partial x_\lambda} = a_{\mu\nu}a_{\mu\lambda}\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\lambda} = \delta_{\nu\lambda}\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\lambda} = \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\nu} = \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\mu}$$

4.6 Successive Lorentz boots

4.6.1 Lorentz boot I



We define two Lorentz transformations as

$$x' = a_x(\beta_1)x$$

with

$$a_x(\beta_1) = \begin{pmatrix} \gamma_1 & 0 & 0 & i\beta_1\gamma_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta_1\gamma_1 & 0 & 0 & \gamma_1 \end{pmatrix},$$

from the S frame to the S' frame (along the *x* direction).

$$a_x(-\beta_2) = \begin{pmatrix} \gamma_2 & 0 & 0 & -i\beta_2\gamma_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta_2\gamma_2 & 0 & 0 & \gamma_2 \end{pmatrix},$$

from S” frame to the S frame (along the *x* direction), where

$$a_x(\beta_2)^{-1} = a_x(\beta_2)^T = \begin{pmatrix} \gamma_2 & 0 & 0 & -i\beta_2\gamma_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta_2\gamma_2 & 0 & 0 & \gamma_2 \end{pmatrix} = a_x(-\beta_2)$$

from the S" frame to the S' frame (along the *x* direction). Then Lorentz transformation (S' →S") [= $a(\beta_3)$] can be expressed by

$$a_x(\beta_3) = a_x(\beta_2)a_x(-\beta_1) = \begin{pmatrix} (1-\beta_1\beta_2)\gamma_1\gamma_2 & 0 & 0 & i(\beta_1-\beta_2)\gamma_1\gamma_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i(\beta_1-\beta_2)\gamma_1\gamma_2 & 0 & 0 & (1-\beta_1\beta_2)\gamma_1\gamma_2 \end{pmatrix}$$

from the S' frame to the S'' frame (along the *x* direction), where

$$\gamma_3 = (1-\beta_1\beta_2)\gamma_1\gamma_2, \quad \text{and} \quad \beta_3\gamma_3 = (\beta_1-\beta_2)\gamma_1\gamma_2$$

leading to

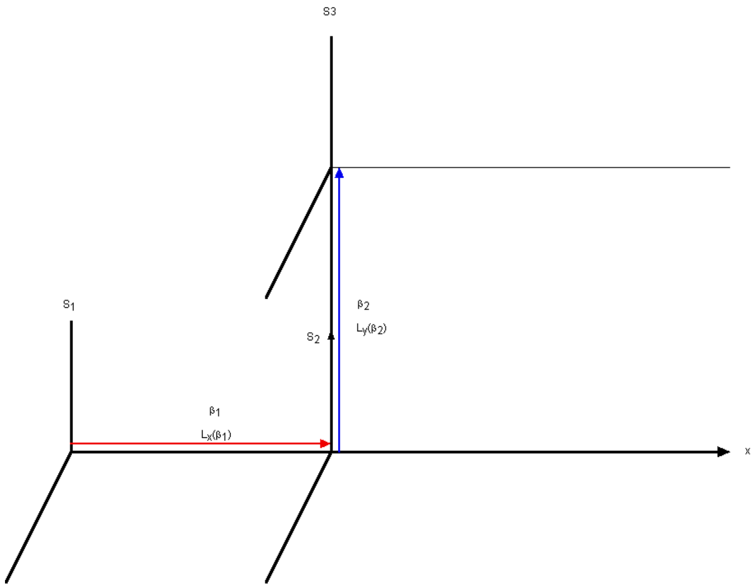
$$\beta_3 = \frac{(\beta_1-\beta_2)}{1-\beta_1\beta_2}.$$

It is easily proved that

$$a_x(-\beta_2)a_x(\beta_1) = a_x(\beta_1)a_x(-\beta_2)$$

In other words, $a_x(-\beta_2)$ and $a_x(\beta_1)$ commute.

4.6.2 Lorentz boots II



We define the Lorentz transformations from the frame S to S' (along the *y* direction), and from S to S' (along the *z* direction) as

$$a_y(\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & i\beta\gamma \\ 0 & 0 & 1 & 0 \\ 0 & -i\beta\gamma & 0 & \gamma \end{pmatrix}, \qquad a_z(\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{pmatrix}$$

respectively.

(1) $a_x(\beta_1)$ and $a_y(\beta_2)$ do not commute.

(2)

$$a_x(\beta_1)a_y(\beta_2) = \begin{pmatrix} \gamma_1 & 0 & 0 & i\beta_1\gamma_1 \\ \beta_1\beta_2\gamma_1\gamma_2 & \gamma_2 & 0 & i\beta_2\gamma_1\gamma_2 \\ 0 & 0 & 1 & 0 \\ -i\beta_1\gamma_1\gamma_2 & -i\beta_2\gamma_2 & 0 & \gamma_1\gamma_2 \end{pmatrix}$$

$$a_y(\beta_2)a_x(\beta_1) = \begin{pmatrix} \gamma_1 & \beta_1\beta_2\gamma_1\gamma_2 & 0 & i\beta_1\gamma_1\gamma_2 \\ 0 & \gamma_2 & 0 & i\beta_2\gamma_2 \\ 0 & 0 & 1 & 0 \\ -i\beta_1\gamma_1 & -i\beta_2\gamma_1\gamma_2 & 0 & \gamma_1\gamma_2 \end{pmatrix}$$

Neither the matrix $a_x(\beta_1)a_y(\beta_2)$ nor the matrix $a_y(\beta_2)a_x(\beta_1)$ can be replaced by the transforming matrix required by a single boost.

4.6.3. General Lorentz transformation without rotation

If v_x, v_y, and v_z denote the components of the velocity of the system S' relative to S, we have a general Lorentz transformation without rotation,

$$a(\boldsymbol{\beta}) = \begin{pmatrix} 1+(\gamma-1)\dfrac{\beta_x\beta_x}{\beta^2} & (\gamma-1)\dfrac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\dfrac{\beta_x\beta_z}{\beta^2} & i\gamma\beta_x \\ (\gamma-1)\dfrac{\beta_y\beta_x}{\beta^2} & 1+(\gamma-1)\dfrac{\beta_y\beta_y}{\beta^2} & (\gamma-1)\dfrac{\beta_y\beta_z}{\beta^2} & i\gamma\beta_y \\ (\gamma-1)\dfrac{\beta_z\beta_x}{\beta^2} & (\gamma-1)\dfrac{\beta_z\beta_y}{\beta^2} & 1+(\gamma-1)\dfrac{\beta_z\beta_z}{\beta^2} & i\gamma\beta_z \\ -i\gamma\beta_x & -i\gamma\beta_y & -i\gamma\beta_z & \gamma \end{pmatrix}$$

where $\beta^2 = \beta_x{}^2 + \beta_y{}^2 + \beta_z{}^2$ and $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$

5. Velocity and acceleration

5.1 Lorentz velocity transformation

$$x_1' = \gamma(x_1 - vt) \qquad x_1 = \gamma(x_1' + vt')$$
$$x_2' = x_2 \qquad x_2 = x_2'$$
$$x_3' = x_3 \qquad x_3 = x_3'$$
$$t' = \gamma(t - \frac{\beta}{c}x_1) \qquad t = \gamma(t' + \frac{\beta}{c}x_1')$$

Suppose that an object has velocity components as measured in *S*' and *S*.

$$u_1' = \frac{dx_1'}{dt'} = \frac{u_1 - v}{1 - \dfrac{\beta}{c}u_1} \qquad u_1 = \frac{dx_1}{dt} = \frac{u_1' + v}{1 + \dfrac{\beta}{c}u_1'}$$

$$u_2' = \frac{dx_2'}{dt'} = \frac{1}{\gamma}\frac{u_2}{1 - \dfrac{\beta}{c}u_1} \qquad u_2 = \frac{dx_2}{dt} = \frac{1}{\gamma}\frac{u_2'}{1 + \dfrac{\beta}{c}u_1'}$$

$$u_3' = \frac{dx_3'}{dt'} = \frac{1}{\gamma}\frac{u_3}{1 - \dfrac{\beta}{c}u_1} \qquad u_3 = \frac{dx_3}{dt} = \frac{1}{\gamma}\frac{u_3'}{1 + \dfrac{\beta}{c}u_1'}$$

((**Note**))

(i) If $u_1 = c$, $u_1' = c$.

(ii) If $u_2 = c$ and $u_1 = 0$,

$$u_1' = -v$$
$$u_2' = \frac{c}{\gamma} = c\sqrt{1-\beta^2}$$

(iii) *The Lorentz transformation of a velocity less than c never leads to a velocity greater than c.*

((**Example**))

$$u_1' = \frac{9}{10}c \qquad u_1 = \frac{u_1' + v}{1 + \frac{\beta}{c}u_1'} = \frac{\frac{9}{10}c + \frac{9}{10}c}{1 + \frac{81}{100}} = \frac{180}{181}c < c$$
$$v = \frac{9}{10}c$$

whereas the Galilean transformation would have given

$$u_1 = u_1' + v = \frac{9}{10}c + \frac{9}{10}c = \frac{9}{5}c > c$$

5.2 Lorentz acceleration transformation

Similarly we have the acceleration components as measured in *S*' and *S*.

$$a_1 = \frac{du_1}{dt} = \frac{dt'}{dt}\frac{d}{dt'}\left(\frac{u_1'+v}{1+\frac{\beta}{c}u_1'}\right) = \frac{1}{\gamma}\frac{1}{(1+\frac{\beta}{c}u_1')}\frac{d}{dt'}\left(\frac{u_1'+v}{1+\frac{\beta}{c}u_1'}\right) = \frac{1}{\gamma}\frac{(1-\beta^2)a_1'}{(1+\frac{\beta}{c}u_1')^3} = \frac{1}{\gamma^3}\frac{a_1'}{(1+\frac{\beta}{c}u_1')^3}$$

$$a_2 = \frac{du_2}{dt} = \frac{1}{\gamma}\frac{dt'}{dt}\frac{d}{dt'}\left(\frac{u_2'}{1+\frac{\beta}{c}u_1'}\right) = \frac{1}{\gamma^2}\frac{1}{(1+\frac{\beta}{c}u_1')}\frac{d}{dt'}\left(\frac{u_2'}{1+\frac{\beta}{c}u_1'}\right) = \frac{1}{\gamma^2}\frac{a_2'}{(1+\frac{\beta}{c}u_1')^2} - \frac{1}{\gamma^2}\frac{a_1'a_2\frac{\beta}{c}}{(1+\frac{\beta}{c}u_1')^3}$$

$$a_3 = \frac{du_3}{dt} = \frac{1}{\gamma}\frac{dt'}{dt}\frac{d}{dt'}\left(\frac{u_3'}{1+\frac{\beta}{c}u_1'}\right) = \frac{1}{\gamma^2}\frac{1}{(1+\frac{\beta}{c}u_1')}\frac{d}{dt'}\left(\frac{u_3'}{1+\frac{\beta}{c}u_1'}\right) = \frac{1}{\gamma^2}\frac{a_3'}{(1+\frac{\beta}{c}u_1')^3} - \frac{1}{\gamma^2}\frac{a_1'a_3\frac{\beta}{c}}{(1+\frac{\beta}{c}u_1')^3}$$

where

$$\frac{dt'}{dt} = \frac{1}{\gamma}\frac{1}{(1+\frac{\beta}{c}u_1')}$$

The acceleration is a quantity of limited and questionable value in special relativity. Not only is it not an invariant, but the expressions for it are in general cumbersome, and moreover its different components transform in different ways.

6. Relativistic dynamics

6.1 Universal function f(*u*) for mass

First we show that the relativistically correct definition of momentum of a particle of mass *m* and velocity *u* is given by

$$\mathbf{p} = \gamma(u)m_0\mathbf{u} = \frac{m_0\mathbf{u}}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}}$$

where m_0 is the mass when the particle is at rest. For a particle moving with the velocity ***u*** relative to a system of inertia S, we shall assign a momentum vector ***p*** proportional to ***u***,

$$\mathbf{p} = m(u)\mathbf{u}$$

The proportionality factor $m(u)$ is called the mass of the particle. Here we assume that $m(u)$ is a universal function f(u) of the magnitude $u = |\mathbf{u}|$ of the velocity vector. Thus

$$m(u) = f(u)$$

If the velocity of the particle relative to another system of inertia S' is ***u***', the momentum and mass of the particle relative to S' must be given by

$$\mathbf{p}' = f(u')\mathbf{u}'$$

This follows from the principle of relativity, according to which all systems of inertia must be treated on the same footing, so that any relation between physical quantities shall be form-invariant.

6.2 Conservation of momentum and definition of relativistic momentum

We now determine the function *f*. This function is uniquely determined when we require that the theorem of conservation of momentum shall hold in any system of inertia. Let S and S′ be two systems of inertia with the relative velocity ***v*** (along the *x* axis), and consider a collision between two identical particles *a* and *b*.

Let the frame S′ be the center of mass frame with two identical particles having initial velocities,

$$\mathbf{u}_a' = \mathbf{u},$$
$$\mathbf{u}_b' = -\mathbf{u},$$

along the *x* axis. The particles collide and scatter, emerging with final velocities

$$\mathbf{u}_c' = \mathbf{u}',$$
$$\mathbf{u}_d' = \mathbf{u}''$$

In the S' frame the conservation equations for momentum and energy are

$$\mathbf{p}_a' + \mathbf{p}_b' = \mathbf{p}_c' + \mathbf{p}_d'$$
$$E_a' + E_b' = E_c' + E_d'$$

or

$$m(u)\mathbf{u} - m(u)\mathbf{u} = m(u')\mathbf{u}' + m(u'')\mathbf{u}''$$
$$\varepsilon(u) + \varepsilon(u) = \varepsilon(u') + \varepsilon(u'')$$

Here we assume that

$$\mathbf{p} = m(u)\mathbf{u}$$
$$E = \varepsilon(u)$$

Because the particles are identical, it is necessary that $\varepsilon(u') = \varepsilon(u'')$. With the hypothesis of monotonic behavior of $\varepsilon(u)$, that $u' = u'' = u$. The first equation requires

$$\mathbf{u}'' = -\mathbf{u}'.$$

We now consider the collision in another frame *S* moving with a velocity $-\boldsymbol{u}$ in the *x* direction with respect to *S*'.

$$\mathbf{u}_a = \frac{2\mathbf{u}}{1 + \dfrac{u^2}{c^2}}$$

which is parallel to the *x* axis.

$$\mathbf{u}_b = 0$$

$$u_{c1} = \frac{u_1' + u}{1 + \dfrac{1}{c^2} u u_1'} = \frac{c\beta(1 + \cos\theta')}{1 + \beta^2 \cos\theta'}$$
$$u_{c2} = \frac{1}{\gamma} \frac{u_2'}{1 + \dfrac{1}{c^2} u u_1'} = \frac{c\beta \sin\theta'}{\gamma(1 + \beta^2 \cos\theta')}$$

$$u_{d1} = \frac{-u_1' + u}{1 - \dfrac{1}{c^2} u u_1'} = \frac{c\beta(1 - \cos\theta')}{1 - \beta^2 \cos\theta'}$$
$$u_{d2} = -\frac{1}{\gamma} \frac{u_2'}{1 - \dfrac{1}{c^2} u u_1'} = \frac{-c\beta \sin\theta'}{\gamma(1 - \beta^2 \cos\theta')}$$

where

$$\beta = \frac{u}{c}, \qquad \text{and} \qquad \gamma = \frac{1}{\sqrt{1 - \dfrac{u^2}{c^2}}}$$

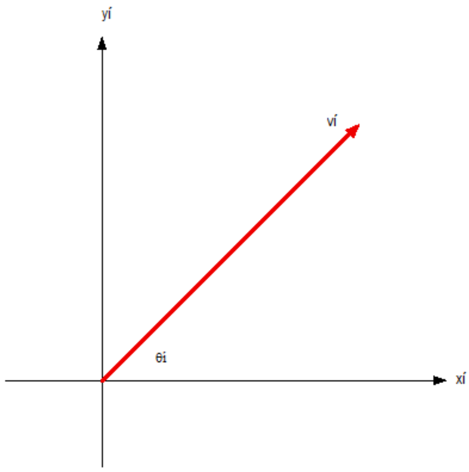


Fig. x and y component of the velocity $\boldsymbol{v}$' in the S' frame; v_1' and v_2'.

((Note)) Here we use the formula,

$$u_1 = \frac{dx_1}{dt'} = \frac{u_1'+v}{1+\dfrac{\beta}{c}u_1'}$$

$$u_2 = \frac{dx_2}{dt} = \frac{1}{\gamma}\frac{u_2'}{1+\dfrac{\beta}{c}u_1'}$$

$$u_3 = \frac{dx_3}{dt} = \frac{1}{\gamma}\frac{u_3'}{1+\dfrac{\beta}{c}u_1'}$$

The equations of conservation of momentum and energy in the frame S are rewritten as

$$m(u_a)\mathbf{u}_a + m(u_b)\mathbf{u}_b = m(u_c)\mathbf{u}_c + m(u_d)\mathbf{u}_d$$
$$\varepsilon(u_a) + \varepsilon(u_b) = \varepsilon(u_c) + \varepsilon(u_d)$$

The y component:

$$0 = m(u_c)u_{c2} + m(u_d)u_{d2}$$
$$= m(u_c)\frac{c\beta\sin\theta'}{\gamma(1+\beta^2\cos\theta')} - m(u_d)\frac{c\beta\sin\theta'}{\gamma(1-\beta^2\cos\theta')}$$

or

$$m(u_c) = \frac{(1+\beta^2\cos\theta')}{(1-\beta^2\cos\theta')}m(u_d)$$

This relation is valid for all θ and in particular $\theta = 0$.

When $\theta = 0$,

$$u_{c1} = \frac{2c\beta}{1+\beta^2} \qquad u_{d1} = 0$$

$$u_{c2} = 0 \qquad u_{d2} = 0$$

or

$$\mathbf{u}_c = \mathbf{u}_a, \qquad \mathbf{u}_d = 0$$

Then we have

$$m(u_c) = \frac{1+\beta^2}{1-\beta^2} m(u_d)$$

or

$$m(u_a) = \frac{1+\beta^2}{1-\beta^2} m(0) = m(0)\gamma(u_a) = m_0\gamma(u_a)$$

We note that

$$\mathbf{u}_a = \frac{2\mathbf{u}}{1+\frac{u^2}{c^2}}$$

$$\frac{1+\beta^2}{1-\beta^2} = \frac{1}{\sqrt{1-\frac{u_a^{\ 2}}{c^2}}} = \gamma(u_a)$$

The mass $m(u)$ is obtained as

$$m(u) = \frac{m_0}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}} = m_0\gamma(\mathbf{u})$$

The momentum of a particle of rest-mass m_0 and velocity $\boldsymbol{u}$ is

$$\mathbf{p} = \frac{m_0 \mathbf{u}}{\sqrt{1 - \frac{\mathbf{u}^2}{c^2}}}$$

6.2 Relativistic force and relativistic kinetic energy

We define the force ***F*** and the kinetic energy *K* as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

$$\frac{dK}{dt} = \mathbf{F} \cdot \mathbf{u}$$

where ***u*** is the velocity of the particle.

$$\frac{dK}{dt} = \mathbf{u} \cdot \mathbf{F} = \mathbf{u} \cdot \frac{d\mathbf{p}}{dt} = \mathbf{u} \cdot \frac{d}{dt} \frac{m_0 \mathbf{u}}{\sqrt{1 - \frac{\mathbf{u}^2}{c^2}}} = m_0 \mathbf{u} \cdot \frac{d\mathbf{u}}{dt} \frac{1}{\sqrt{1 - \frac{\mathbf{u}^2}{c^2}}} + m_0 \mathbf{u}^2 \frac{d}{dt} \frac{1}{\sqrt{1 - \frac{\mathbf{u}^2}{c^2}}}$$

or

$$\frac{dK}{dt} = m_0 \mathbf{u} \cdot \frac{d\mathbf{u}}{dt} \frac{1}{\left(1 - \frac{\mathbf{u}^2}{c^2}\right)^{1/2}} + m_0 \mathbf{u}^2 \frac{\mathbf{u} \cdot \frac{d\mathbf{u}}{dt}}{c^2 \left(1 - \frac{\mathbf{u}^2}{c^2}\right)^{3/2}}$$

or

$$\frac{dK}{dt} = m_0 \mathbf{u} \cdot \frac{d\mathbf{u}}{dt} [\frac{1 - \frac{\mathbf{u}^2}{c^2}}{\left(1 - \frac{\mathbf{u}^2}{c^2}\right)^{3/2}} + \frac{\frac{\mathbf{u}^2}{c^2}}{\left(1 - \frac{\mathbf{u}^2}{c^2}\right)^{3/2}}] = \frac{m_0 \mathbf{u} \cdot \frac{d\mathbf{u}}{dt}}{\left(1 - \frac{\mathbf{u}^2}{c^2}\right)^{3/2}} = \frac{\frac{m_0}{2} \frac{d\mathbf{u}^2}{dt}}{\left(1 - \frac{\mathbf{u}^2}{c^2}\right)^{3/2}}]$$

$$dK = \frac{m_0}{2} \frac{1}{\left(1 - \frac{\mathbf{u}^2}{c^2}\right)^{3/2}} d\mathbf{u}^2$$

$$K = \frac{m_0 c^2}{\sqrt{1 - \frac{\mathbf{u}^2}{c^2}}} + C$$

where C is a constant of integration. Since the kinetic energy may be taken as zero for $u = 0$, we have

$$C = -m_0c^2.$$

or

$$K = \frac{m_0c^2}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}} - m_0c^2 = m_0c^2[\gamma(\mathbf{u}) - 1]$$

Note that for $u/c \ll 1$,

$$K = \frac{m_0c^2}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}} - m_0c^2 \approx m_0c^2(1 + \frac{1}{2}\frac{\mathbf{u}^2}{c^2}) - m_0c^2 = \frac{1}{2}m_0\mathbf{u}^2,$$

which is in agreement with the classical result. The term m_0c^2 in the kinetic energy is independent of the velocity. It is called the rest energy, $E_0 = m_0c^2$, of the particle.

It is convenient to introduce the total energy E (= kinetic energy + rest energy) defined by

$$E = K + m_0c^2 = \frac{m_0c^2}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}} = m_0c^2\gamma(\mathbf{u})$$

E is the relativistic energy of a free particle. When a particle is at rest, $E_0 = m_0c^2$. This is Einstein's famous equation. This suggests that mass and energy are equivalent, and indeed this is the case.

It is useful to express E in terms of $\boldsymbol{p}$, as

$$E^2 = \mathbf{p}^2c^2 + m_0^2c^4.$$

since

$$\frac{E^2}{c^2} = m_0^2c^2[\gamma(\mathbf{u})]^2 = m_0^2c^2[\frac{1}{1-\frac{\mathbf{u}^2}{c^2}}] = m_0^2c^2[1 + \frac{\frac{\mathbf{u}^2}{c^2}}{1-\frac{\mathbf{u}^2}{c^2}}]$$

$$= m_0^2c^2 + \frac{m_0^2\mathbf{u}^2}{1-\frac{\mathbf{u}^2}{c^2}} = m_0^2c^2 + \mathbf{p}^2$$

The kinetic energy is simply expressed by

$$K = m_0 c^2[\gamma(\mathbf{u}) - 1]$$

7. Four-dimensional momentum

Here we define the four-dimensional momentum

$$p = m_0 \frac{d}{d\tau} x = m_0 \gamma(\mathbf{u}) \frac{dx}{dt} = (m_0\gamma(\mathbf{u})u_1, m_0\gamma(\mathbf{u})u_2, m_0\gamma(\mathbf{u})u_3, im_0 c\gamma(\mathbf{u})) = (\mathbf{p}, i\frac{E}{c})$$

where

$$d\tau = \frac{ds}{c} = dt\sqrt{1 - \frac{\mathbf{u}^2}{c^2}} = \frac{dt}{\gamma(\mathbf{u})},$$

This momentum is clearly a four-vector since dx_μ is a Lorentz four-vector and m_0 and $d\tau$ are Lorentz scalar. In fact, under the Lorentz transformation

$$p_\mu' = m_0 \frac{d}{d\tau} x_\mu' = m_0 a_{\mu\nu} \frac{d}{ds} x_\nu = a_{\mu\nu} p_\nu$$

$$(p_\mu')^2 = a_{\mu\nu} a_{\mu\lambda} p_\nu p_\lambda = \delta_{\nu\lambda} p_\nu p_\lambda = (p_\mu)^2 = \mathbf{p}^2 - \frac{E^2}{c^2} = -m_0{}^2 c^2$$

which is invariant under the Lorentz transformation.

$$p = (\mathbf{p}, i\frac{E}{c})$$

$$p_\mu' = a_{\mu\nu} p_\nu$$

$$\begin{aligned} p_1' &= \gamma(p_1 + i\beta p_4) \\ p_2' &= p_2 \\ p_3' &= p_3 \\ p_4' &= \gamma(-i\beta p_1 + p_4) \end{aligned} \quad \text{or} \quad \begin{aligned} p_1' &= \gamma(p_1 - \frac{\beta}{c}E) \\ p_2' &= p_2 \\ p_3' &= p_3 \\ E' &= \gamma(E - \beta c p_1) \end{aligned}$$

and

$$p_\mu' = a_{\mu\nu} p_\nu$$

$$p_1 = \gamma(p_1' - i\beta p_4')$$
$$p_2 = p_2'$$
$$p_3 = p_3'$$
$$p_4 = \gamma(i\beta p'_1 + p_4')$$

or

$$p_1 = \gamma(p_1' + \frac{\beta}{c}E')$$
$$p_2 = p_2'$$
$$p_3 = p_3'$$
$$E = \gamma(E' + \beta c p'_1)$$

The frame S′ is moving relative to S with velocity ***v*** in the direction of the positive *x*-axis. Here we note that

$$E = m_0 c^2 \gamma(\mathbf{u}) \qquad E' = m_0 c^2 \gamma(\mathbf{u}')$$
$$\mathbf{p} = m_0 \mathbf{u}\gamma(\mathbf{u}) \qquad \mathbf{p}' = m_0 \mathbf{u}'\gamma(\mathbf{u}')$$

((Velocity four-vector))

We define the velocity four-vector u_μ by

$$u_\mu = \frac{p_\mu}{m_0} = (\mathbf{u}\gamma(\mathbf{u}), ic\gamma(\mathbf{u}))$$

where m_0 is the rest mass, which is an invariant scalar in four dimensions.

8. Force

The force ***F*** is defined by

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

Here

$$\frac{dp_1}{dt} = \frac{dt'}{dt}\frac{dp_1}{dt'} = \frac{1}{\gamma(1 + \frac{\beta}{c}u_1')}[\frac{dp_1'}{dt'} + m_0 c\gamma\beta \frac{d}{dt'}\gamma(\mathbf{u}')]$$

$$\frac{dp_2}{dt} = \frac{dt'}{dt}\frac{dp_2}{dt'} = \frac{dp_2'}{dt'}\frac{1}{\gamma(1 + \frac{\beta}{c}u_1')}$$

$$\frac{dp_3}{dt} = \frac{dt'}{dt}\frac{dp_3}{dt'} = \frac{dp_3'}{dt'}\frac{1}{\gamma(1 + \frac{\beta}{c}u_1')}$$

When $u_1' = 0$ and $du_1'/dt' = 0$

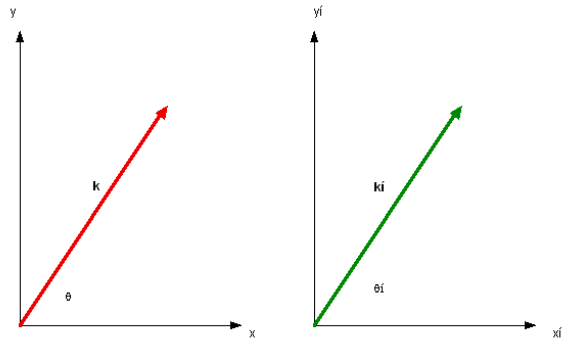
$$\frac{dp_1}{dt} = \frac{dt'}{dt}\frac{dp_1}{dt'} = \frac{1}{\gamma(1+\frac{\beta}{c}u_1')}[\frac{dp_1'}{dt'} + m_0 c\gamma\beta\frac{d}{dt'}\gamma(\mathbf{u}')]$$

$$\frac{dp_2}{dt} = \frac{1}{\gamma}\frac{dp_2'}{dt'}$$

$$\frac{dp_3}{dt} = \frac{1}{\gamma}\frac{dp_3'}{dt'}$$

9. Doppler shift and aberration

9.1



$$k_\mu = \frac{p}{\hbar} = (k, \frac{i\omega}{c})$$

where $\omega = ck$

$$k_\mu x_\mu = \mathbf{k}\cdot\mathbf{r} - \omega t \text{ =invariant under the Lorentz transformation.}$$

or

$$k_\mu x_\mu = kx\cos\theta + ky\sin\theta - \omega t$$

This should be equal to

$$k'_\mu x'_\mu = k'x'\cos\theta' + k'y'\sin\theta' - \omega' t'$$

Note that

$$x = \gamma(x' + vt')$$
$$t = \gamma(t' + \frac{v}{c^2}x')$$
$$\omega = ck$$
$$\omega' = ck'$$

Substituting these parameters into the invariant form, we have

$$t'(ck' - ck\gamma + kv\gamma\cos\theta) + x'(-\frac{kv\gamma}{c} + k\gamma\cos\theta - k'\cos\theta') + y'(k\sin\theta - k'\sin\theta') = 0$$

This should be satisfied for any *t'*, *x'*, and *y'*.

$$k\gamma(1 - \frac{v}{c}\cos\theta) = k'$$
$$k\gamma(\cos\theta - \frac{v}{c}) = k'\cos\theta'$$
$$k\sin\theta = k'\sin\theta'$$

or

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \frac{v}{c})}$$

9.2. Doppler shift

Since $k = \frac{2\pi}{\lambda}, k' = \frac{2\pi}{\lambda'}$

$$\frac{2\pi}{\lambda}\gamma(1 - \frac{v}{c}\cos\theta) = \frac{2\pi}{\lambda'}$$

or

$$\lambda' = \frac{\lambda}{\gamma(1 - \frac{v}{c}\cos\theta)}$$

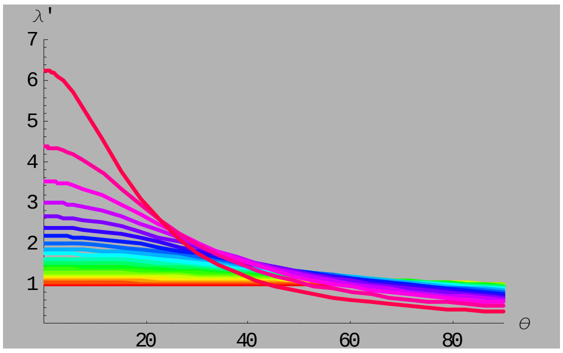


Fig. λ'/λ vs θ. $\beta = 0 - 0.99$ with $\Delta\beta = 0.05$.

9.3 longitudinal Doppler shift $\theta = 0$ (red shift)

We suppose that a source is located at the origin of the reference frame *S*. An observer moves relative to *S* at velocity *v*. So that he is at rest in S'.

$$\lambda' = \frac{\lambda}{\gamma(1-\frac{v}{c})} = \lambda\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

or

$$f' = f\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \qquad \text{(receding source)}$$

If *S'* moves toward S, rather than away from S, the signs in numerator and denominator of the radical would have been interchanged.

$$f' = f\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \qquad \text{(approaching source)}$$

((The red shift)) Wikipedia

The light from distant stars and more distant galaxies is not featureless, but has distinct spectral features characteristic of the atoms in the gases around the stars. When these spectra are examined, they are found to be shifted toward the red end of the spectrum. This shift is apparently a Doppler shift and indicates that essentially all of the galaxies are moving away from us. Using the results from the nearer ones, it becomes evident that the more distant galaxies are moving away from us faster. This is the kind of result one would expect for an expanding universe.

The building up of methods for measuring distance to stars and galaxies led Hubble to the fact that the red shift (recession speed) is proportional to distance. If this proportionality (called Hubble's Law) holds true, it can be used as a distance measuring tool itself.

The measured red shifts are usually stated in terms of a *z* parameter. The largest measured *z* values are associated with the quasars.

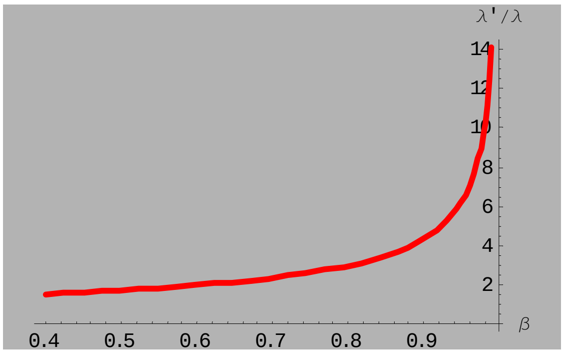


Fig. red shift: λ'/λ vs $\beta = v/c$

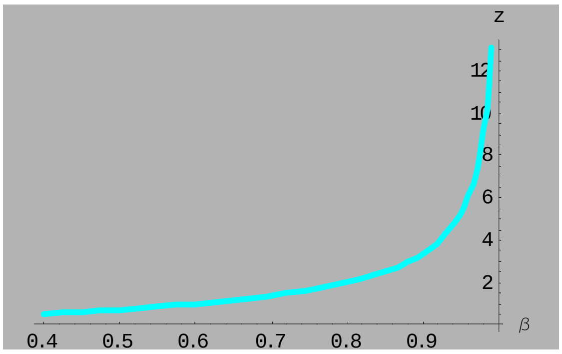


Fig. *z* parameter $z = \frac{\lambda'}{\lambda} - 1$ vs $\beta = v/c$

9.4 Relativistic aberration

From the equations above derived

$$k\gamma(1 - \frac{v}{c}\cos\theta) = k'$$

$$k\gamma(\cos\theta - \frac{v}{c}) = k'\cos\theta'$$

we have

$$\cos\theta' = \frac{\cos\theta - \frac{v}{c}}{1 - \frac{v}{c}\cos\theta}$$

For low velocity we can neglect v^2/c^2 and higher-order terms. Setting $\theta' = \theta + \Delta\theta$

$$\cos(\theta + \Delta\theta) = \cos\theta - \Delta\theta\sin\theta$$

and

$$\frac{\cos\theta - \frac{v}{c}}{1 - \frac{v}{c}\cos\theta} = \cos\theta - \beta\sin^2\theta$$

Then we have

$$\Delta\theta = \beta\sin\theta$$

9.5 Transverse Doppler shift

From the equations given by

$$k\gamma(1 - \frac{v}{c}\cos\theta) = k'$$

$$k\gamma(\cos\theta - \frac{v}{c}) = k'\cos\theta'$$

we have

$$k\gamma(1 - \frac{v^2}{c^2}) = k'(1 + \frac{v}{c}\cos\theta')$$

or

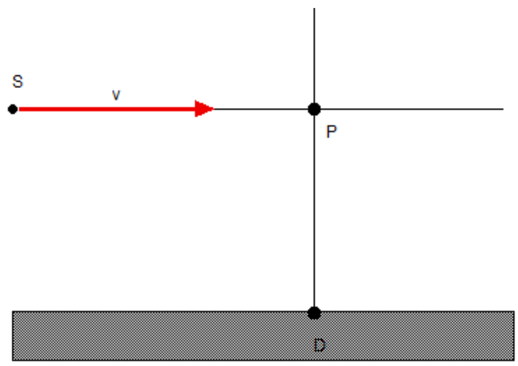
$$k\gamma^{-1} = k'(1 + \frac{v}{c}\cos\theta')$$

Since $\omega = 2\pi f = ck$ and $\omega' = 2\pi f' = ck'$

$$f' = \gamma^{-1} \frac{1}{(1+\frac{v}{c}\cos\theta')} f$$

When $\theta' = \pi/2$, $\cos\theta' = 0$;

$$f' = \gamma^{-1} f = \sqrt{1-\beta^2}\, f$$



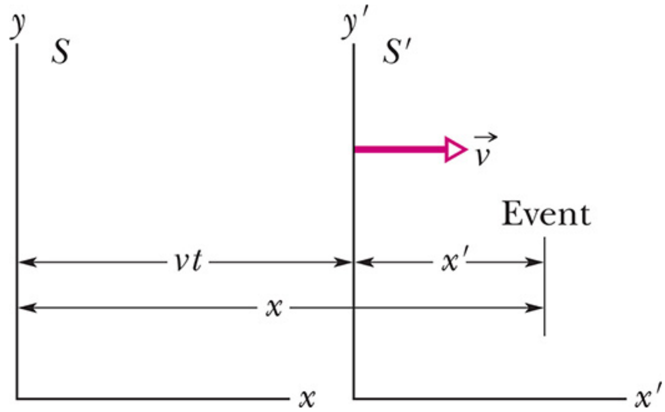
The source S moves past a detector D. When S reaches point P, the velocity o S is perpendicular to the line joining P and D. If the source is emitting the sound wave of frequency *f*, D detects that frequency (with no Doppler effect) when it intercepts the waves that were emitted at point P. However, if the source is emitting light waves, there is still a Doppler effect, called the transverse Doppler effect. In this case, the detected frequency of the light emitted when the source is at point P is

$$f' = f\sqrt{1-\frac{v^2}{c^2}}$$

10. Typical problems

10.1 Problem 37-23

In Fig., observer S detects two flashes of light. A big flash occurs at $x_1 = 1200$ m and, 5.00 μs later, small flash occurs at $x_2 = 480$ m. As detected by observer S', the two flashes occur at a single coordinate *x*'. (a) What is the speed parameter of S', and (b) is S' moving in the positive or negative direction of the *x* axis? To S', (c) which flash occurs first and (d) what is the time interval between the flashes?



((**Solution**))

x_1 =1200 m, t_1 = 0 μs
x_2 =480 m, t_1 = 5 μs

The Lorentz transformation,

$$x_1' = \gamma(x_1 - vt_1)$$
$$x_2' = \gamma(x_2 - vt_2)$$

(a) Since $x_1' = x_2'$

$$x_1 - x_2 = v(t_1 - t_2)$$
$$v = \frac{x_1 - x_2}{t_1 - t_2} = \frac{1200 - 480}{0\mu s - 5\mu s} = -1.44\times 10^8 m/s$$

(b) Negative direction

(c)

$$\beta = \frac{|v|}{c} = \frac{1.44\times 10^8}{2.99792458\times 10^8} = 0.480$$

where $c = 2.99792458 \times 10^8$ m/s

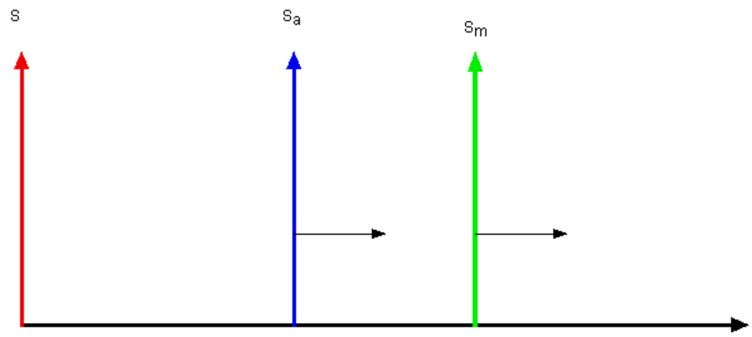
$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = 1.13990$$

$$t_1' = \gamma(t_1 - \frac{v}{c^2}x_1) = 2.1921\mu s$$
$$t_2' = \gamma(t_2 - \frac{v}{c^2}x_2) = 6.57753\mu s$$
$$t_2' - t_1' = 4.3854\mu s$$

10.2 Problem 37-31 Armada of spaceship

An armada of spaceships that is 1.00 ly long (in its rest frame) moves with speed 0.800 *c* relative to ground station in frame S. A messanger travels from the rear of the armada to the front with a speed of 0.950 *c* relative to S. How long does the trip take as measured (a) in the messanger's rest frame, (b) in the armada's rest frame, and (c) by a observer in the frame S?

((**Solution**))



(a)

The velocity of Armada relative to the messenger's rest frame (S_m)

$$u_1' = \frac{u_1 - v}{1 - \frac{\beta}{c}u_1} = \frac{u_1 - 0.95c}{1 - 0.95\frac{u_1}{c}} = \frac{0.80c - 0.95c}{1 - 0.80 \times 0.95} = -0.625c$$

where $v = 0.95\ c$ (the velocity of the Messenger in the frame S) and $\beta = 0.95$. $u_1 = 0.8\ c$ (the velocity of Armada in the frame-S). The length of the Armada as measured in S_m is

$$L_1 = L_0\sqrt{1 - \frac{u_1'^2}{c^2}} = 1.00\sqrt{1 - (-0.625)^2}(ly) = 0.781(ly)$$

The time for the trip is

$$t = \frac{0.781(ly)}{0.625c} = \frac{0.781c(y)}{0.625c} = 1.25(y)$$

(b)

The velocity of messenger relative to the Armada's rest frame (S_a)

$$u_1' = \frac{u_1 - v}{1 - \frac{\beta}{c}u_1} = \frac{u_1 - 0.8c}{1 - 0.80\frac{u_1}{c}} = \frac{0.95c - 0.8c}{1 - 0.80 \times 0.95} = 0.625c$$

where $v = 0.8\ c$ (the velocity of Armada in the frame-S) and $\beta = 0.95$. $u_1 = 0.95\ c$ (the velocity of the Messenger in the frame S).

The length of the Armada in the rest frame of S_a is 1.00 ly. Then the time for the trip is

$$t = \frac{1.00(ly)}{0.625c} = \frac{1.00c(y)}{0.625c} = 1.60(y)$$

(c)
The length of Armada in the S frame is

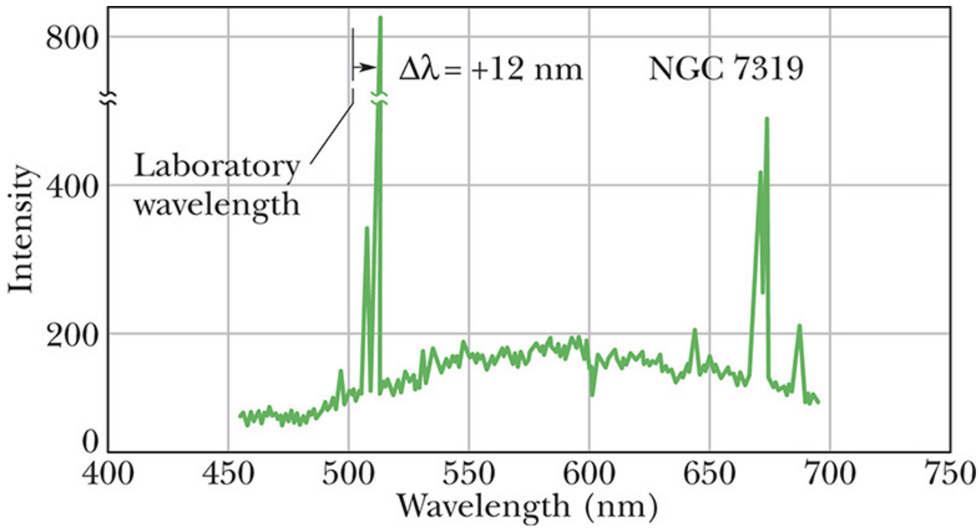
$$L_1 = L_0\sqrt{1-\frac{v^2}{c^2}} = 1.00\sqrt{1-0.8^2}(ly) = 0.6(ly)$$

where $v = 0.8\ c$ (the velocity of Armada in the frame-S).
The time for the trip is

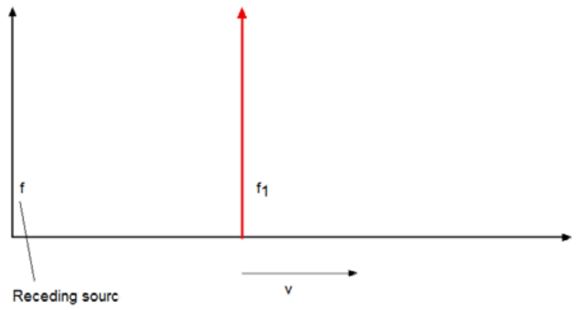
$$t = \frac{0.60(ly)}{(0.95-0.80)c} = \frac{0.6(y)}{0.15c} = 4(y)$$

10.3 Problem 37-36

Figure is a graph of intensity versus wavelength for light reaching Earth from galaxy NGC 7319, which is about 3 x 10^8 light-years away. The most intense light is emitted by the oxygen in NGC 7319. In a laboratory that emission is at wavelength $\lambda = 513$ nm, but on the light from NGC7319 it has been shifted to 525 nm due to the Doppler effect (all the emissions from NGC7319 have been shifted). (a) What is the radial speed of NGC 7319 relative to Earth? (b) Is the relative motion toward or away from our planet?



((**Solution**))



$$f' = f\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \qquad \text{(receding)}$$

$$f' = f\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \qquad \text{(approaching)}$$

We now consider the case of receding.

$$c = \lambda f = \lambda' f'$$

Then we have

$$\frac{\lambda'}{\lambda} = \frac{f}{f'} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} = \sqrt{\frac{1+\beta}{1-\beta}} > 1$$

Oxygen:

λ = 513 nm
λ' = 525 nm

$$\frac{\lambda'}{\lambda} = \sqrt{\frac{1+\beta}{1-\beta}} = \frac{525}{513} = 1.0234$$

$$\beta = 0.023$$

$$v = 6.94 \times 10^6 m/s$$

This indicates the expanding universe.

10.4 Problem 37-52 (SP-37) Muon

If m_0 is a particle's rest mass, p is its momentum magnitude, and K is its kinetic energy, show that

$$m_0 = \frac{(pc)^2 - K^2}{2Kc^2}$$

(b) For low particle speeds, show that the right side of the equation reduces to m. (c) If a particle has K = 55.0 MeV when p = 121 MeV/c, what is the ratio m/m_e of its mass to the electron mass.

(a) ((**Solution**))

(a)

$$K = m_0c^2\gamma - m_0c^2 = m_0c^2(\gamma - 1)$$

$$E_0 = m_0c^2$$

$$E = m_0c^2\gamma = K + m_0c^2$$

$$\mathbf{p} = \frac{m_0\mathbf{v}}{\sqrt{1-\beta^2}} = m_0\gamma\mathbf{v}$$

Then

$$\begin{aligned}\mathbf{p}^2c^2 - K^2 &= m_0{}^2c^2\gamma^2v^2 - m_0{}^2c^4(\gamma-1)^2 \\ &= m_0{}^2c^4\gamma^2\frac{\gamma^2-1}{\gamma^2} - m_0{}^2c^4(\gamma-1)^2 \\ &= m_0{}^2c^4[\gamma^2 - 1 - (\gamma-1)^2] \\ &= 2m_0{}^2c^4(\gamma-1)\end{aligned}$$

$$\frac{\mathbf{p}^2c^2 - K^2}{2Kc^2} = \frac{2m_0{}^2c^4(\gamma-1)}{2m_0c^4(\gamma-1)} = m_0$$

(b) For $v \ll c$

$$K = m_0c^2(\gamma-1) \approx m_0c^2\left(1 + \frac{1}{2}\beta^2 - 1\right) = \frac{1}{2}m_0v^2$$

$$\mathbf{p} = \frac{m_0\mathbf{v}}{\sqrt{1-\beta^2}} \approx m_0\mathbf{v}\left(1 + \frac{1}{2}\beta^2\right)$$

$$\frac{\mathbf{p}^2c^2 - K^2}{2Kc^2} = \frac{c^2{m_0}^2v^2(1+\frac{1}{2}\beta^2)^2 - \frac{1}{4}{m_0}^2v^4}{2c^2\frac{1}{2}m_0v^2}$$

$$\approx \frac{c^2{m_0}^2v^2(1+\beta^2) - \frac{1}{4}{m_0}^2v^4}{2c^2\frac{1}{2}m_0v^2}$$

$$\approx m_0$$

(c) $K = 55$ MeV, $p = 121$ MeV/c

$$\frac{m_0}{m_e} = 206.654$$

So the particle is muon.

APPENDIX

A. Relativistic-covariant Lagrangian formalism

A.1 Lagrangian *L*

We start with the proper time $d\tau$ (Lorentz scalar)

$$d\tau = \frac{ds}{c} = dt\sqrt{1-\frac{\mathbf{u}^2}{c^2}}$$

where τ is a proper time and $\boldsymbol{u}$ is the velocity of the particle in the frame *S*. The integral $\int_a^b ds$ taken between a given pair of world points has its maximum value if it is taken along the straight line joining two points.

$$S = -\alpha\int_a^b ds = -\alpha c\int_{t_a}^{t_b} dt\sqrt{1-\frac{\mathbf{u}^2}{c^2}} = \int_{t_a}^{t_b} L dt$$

where

$$L = -\alpha c\sqrt{1-\frac{\mathbf{u}^2}{c^2}}$$

Nonrelativistic case

$$L = -\alpha c(1 - \frac{\mathbf{u}^2}{c^2})^{1/2} = -\alpha c(1 - \frac{\mathbf{u}^2}{2c^2}) = \frac{\alpha}{2c}\mathbf{u}^2 - \alpha c$$

In the classical mechanics,

$$\frac{\alpha}{2c} = \frac{m_0}{2} \qquad \text{or} \qquad \alpha = m_0 c$$

Therefore the Lagrangian L is given by

$$L = -m_0 c^2 (1 - \frac{\mathbf{u}^2}{c^2})^{1/2}$$

The momentum $\boldsymbol{p}$ is defined by

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{u}} = \frac{m_0 \mathbf{u}}{\sqrt{1 - \frac{\mathbf{u}^2}{c^2}}} = m_0 \mathbf{u} \gamma(\mathbf{u}) = m_0 \frac{d\mathbf{r}}{d\tau} = m_0 \frac{d\mathbf{r}}{dt}\frac{dt}{d\tau}$$

A.2 Hamiltonian

The Hamiltonian H is defined by

$$H = \mathbf{p} \cdot \mathbf{u} - L = \gamma(\mathbf{u}) m_0 \mathbf{u}^2 + m_0 c^2 \frac{1}{\gamma(\mathbf{u})} = \frac{\gamma(\mathbf{u})^2 m_0 \mathbf{u}^2 + m_0 c^2}{\gamma(\mathbf{u})} = \gamma(\mathbf{u}) m_0 c^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{\mathbf{u}^2}{c^2}}} = E$$

or

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{\mathbf{u}^2}{c^2}}}$$

We have

$$\frac{E^2}{c^2} = \frac{m_0^{\ 2} c^2}{1 - \frac{\mathbf{u}^2}{c^2}} = \frac{m_0^{\ 2} c^2 (1 - \frac{\mathbf{u}^2}{c^2}) + m_0^{\ 2} \mathbf{u}^2}{1 - \frac{\mathbf{u}^2}{c^2}} = m_0^{\ 2} c^2 + \mathbf{p}^2$$

B. Muon-decay experiment: time dilation (Senior laboratory, Binghamton University)

B.1. Introduction

The muon is one of nature's fundamental "building blocks of matter" and acts in many ways as if it were an unstable heavy electron, for reasons no one fully understands. Discovered in 1937 by C.W. Anderson and S.H. Neddermeyer when they exposed a cloud chamber to cosmic rays, its finite lifetime was first demonstrated in 1941 by F. Rasetti. The instrument described here permits you to measure the charge averaged mean muon lifetime in plastic scintillator, to measure the relative flux of muons as a function of height above sea-level.

The muon is unstable and decays with a proper lifetime $\tau = 2.197$ μs. This particle is in many respect a "heavy" electron. The mass of the muon is 206.77 times more massive than electron. The muon was first detected in cosmic rays-radiation that comes to Earth from outside the solar system. These rays are very energetic, and typically, the muons in them have a speed such that $v/c \approx 0.99$, or $\gamma = 7.1$.

We may first ask how far a muon with v/c of 0.99 will travel over its lifetime. We should point out that the muons that arrive here from outer space are born close to Earth in the decays of other particles that are part of the cosmic radiation. If the relativity were not a factor, such a muon would travel a distance τv, on the average, before decaying, where τ is the lifetime and v is the speed of the muon relative to the Earth. Ignoring relativity, with $v/c = 0.99$, we have an average path length of

$$d = v\tau = 653m$$

But if we take time dilation into account, this equation must be multiplied by γ, giving

$$d = \gamma vt = 7.1 \times 653m = 4.77km\,.$$

Thus a muon detector could register muons at a position farther from the point of their creation than a nonrelativistic treatment. It suggests a distance $d = \gamma v\tau$, rather than the distance $v\tau$ that would be expected if there were no time dilation.

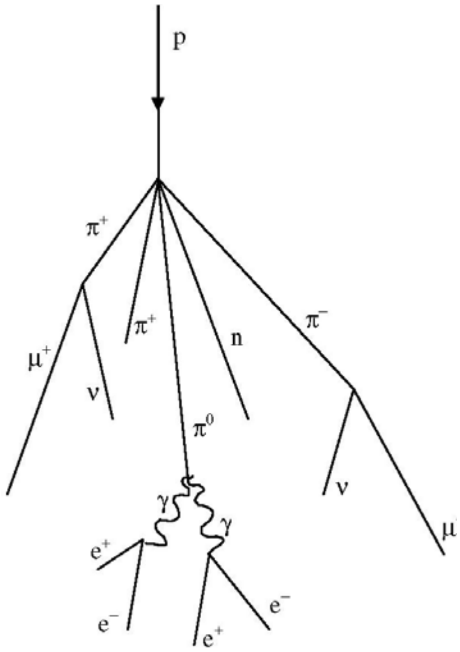
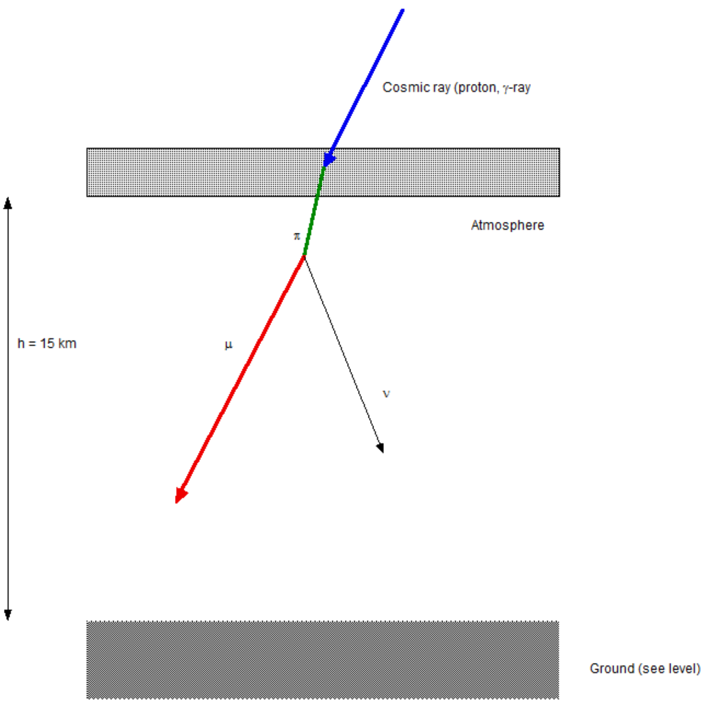


Fig. Cosmic ray cascade induced by a cosmic ray proton striking an air molecule nucleus.



The mean production height in the atmosphere of the muons detected at sea-level is approximately 15 km. Travelling at the speed of light, the transit time from production point to sea-level is then 50 μsec (= 15 x 10^3 m/(3x10^8 m/s)). Since the lifetime of at-rest muons (τ_U = 2.19 μs) is more than a factor of 20 smaller, the appearance of an appreciable sea-level muon flux is qualitative evidence for the time dilation effect of special relativity.

B2. Measurement of the lifetime of muons

To measure the muon's lifetime, we are interested in only those muons that enter, slow, stop and then decay inside the plastic scintillator. Figure 1 summarizes this process. Such muons have a total energy of only about 160 MeV as they enter the tube. As a muon slows to a stop, the excited scintillator emits light that is detected by a photomultiplier tube (PMT), eventually producing a logic signal that triggers a timing clock. *A stopped muon,* after a bit, decays into an electron, a neutrino and an anti-neutrino. Since the electron mass is so much smaller that the muon mass, $m_\mu/m_e \sim 210$, the electron tends to be very energetic and to produce scintillator light essentially all along its path length. The neutrino and anti-neutrino also share some of the muon's total energy but they entirely escape detection. This second burst of scintillator light is also seen by the PMT and used to trigger the timing clock. The distribution of time intervals between successive clock triggers for a set of muon decays is the physically interesting quantity used to measure the muon lifetime.

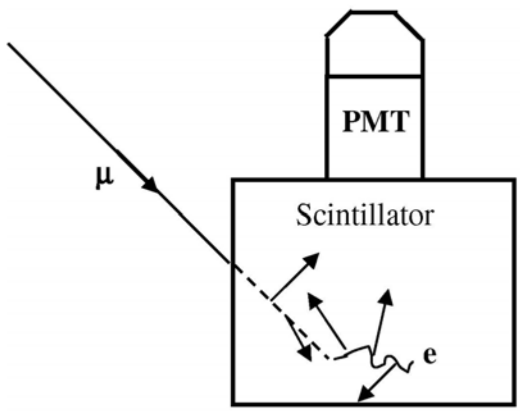
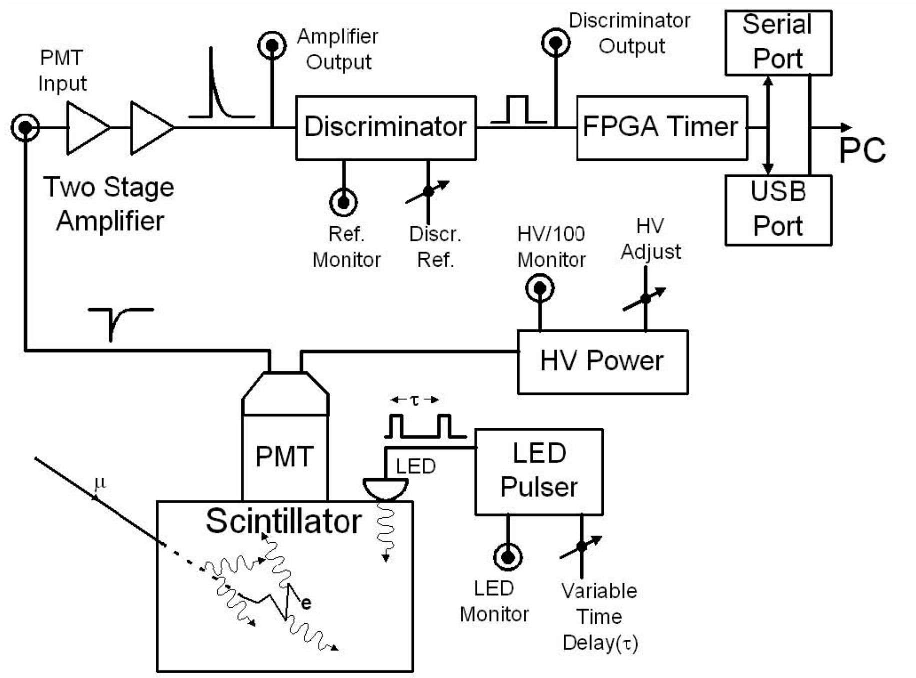


Fig.1 Schematic showing the generation of the two light pulses (short arrows) used in determining the muon lifetime. One light pulse is from the slowing muon (dotted line) and the other is from its decay into an electron or positron (wavey line).



B3. Experimental result

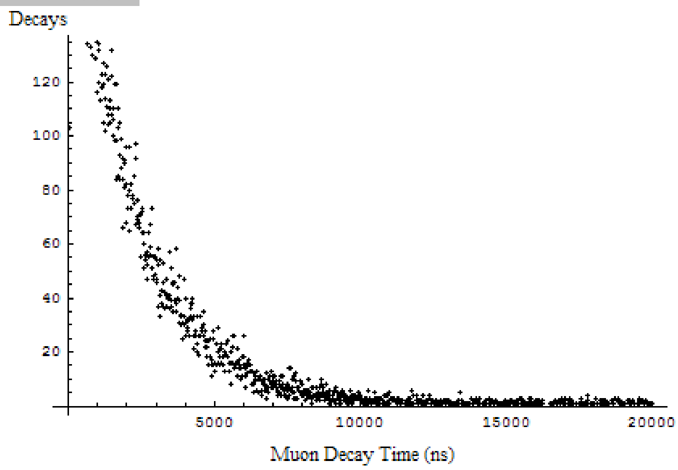


Fig. Data were taken in the Senior Laboratory (SUNY-Binghamton) using the apparatus purchased from Teachspin

The result for the mean muon lifetime of 2.11 ± 0.03 μs. However, this result is still less than the free space value of the mean muon lifetime,

$\tau_U = 2.19703 \pm 0.00004$ μs.

The halflife is given by

$\tau_{1/2} = \ln 2\ \tau_U = 1.52$ μs.

from the definition.