

**Josephson effect**  
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**Brian David Josephson**, FRS (born 4 January 1940; Cardiff, Wales) is a Welsh physicist. He became a Nobel Prize laureate in 1973 for the prediction of the eponymous Josephson effect. As of late 2007, he was a retired professor at the University of Cambridge, where he is the head of the Mind–Matter Unification Project in the Theory of Condensed Matter (TCM) research group. He is also a fellow of Trinity College, Cambridge.



[http://en.wikipedia.org/wiki/Brian\\_David\\_Josephson](http://en.wikipedia.org/wiki/Brian_David_Josephson)

**1. DC Josephson junction**

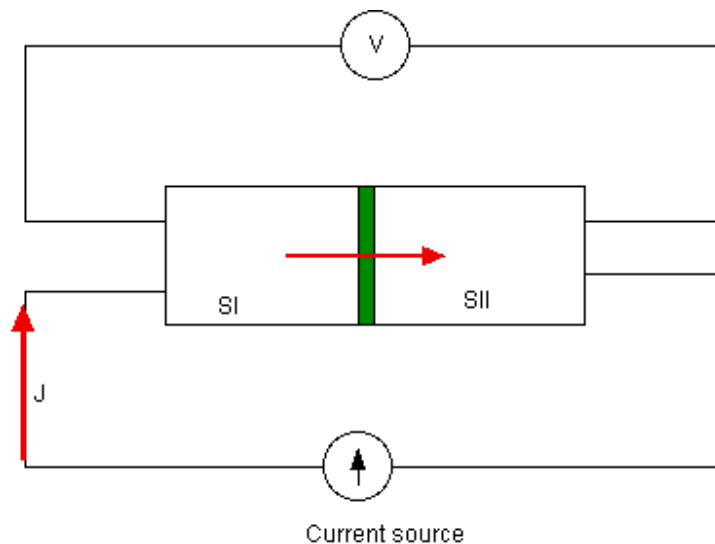


Fig. Schematic diagram for experiment of DC Josephson effect. Two superconductors SI and SII (the same metals) are separated by a very thin insulating layer (denoted

by green). A DC Josphson supercurrent (up to a maximum value  $I_c$ ) flows without dissipation through the insulating layer.

Let  $\psi_1$  be the probability amplitude of electron pairs on one side of a junction. Let  $\psi_2$  be the probability amplitude of electron pairs on the other side. For simplicity, let both superconductors be identical.

$$i\hbar \frac{\partial \psi_1}{\partial t} = \hbar T \psi_2,$$

$$i\hbar \frac{\partial \psi_2}{\partial t} = \hbar T \psi_1$$

where  $\hbar T$  is the effect of the electron-pair coupling or (transfer interaction across the insulator).  $T(1/s)$  is the measure of the leakage of  $\psi_1$  into the region 2, and of  $\psi_2$  into the region 1.

Let

$$\psi_1 = \sqrt{n_1} e^{i\theta_1},$$

$$\psi_2 = \sqrt{n_2} e^{i\theta_2}$$

where

$$|\psi_1|^2 = n_1$$

$$|\psi_2|^2 = n_2$$

Then we have

$$\frac{1}{2} \frac{\partial n_1}{\partial t} + in_1 \frac{\partial \theta_1}{\partial t} = -iT \sqrt{n_1 n_2} e^{i\delta}$$

$$\frac{1}{2} \frac{\partial n_2}{\partial t} + in_2 \frac{\partial \theta_2}{\partial t} = -iT \sqrt{n_1 n_2} e^{-i\delta}$$

where

$$\delta = \theta_2 - \theta_1.$$

Now equate the real and imaginary parts of Eqs.(3) and (4),

$$\frac{\partial n_1}{\partial t} = 2T\sqrt{n_1 n_2} \sin \delta$$

$$\frac{\partial n_2}{\partial t} = -2T\sqrt{n_1 n_2} \sin \delta$$

$$\frac{\partial \theta_1}{\partial t} = -T\sqrt{\frac{n_2}{n_1}} \cos \delta$$

$$\frac{\partial \theta_2}{\partial t} = -T\sqrt{\frac{n_1}{n_2}} \cos \delta$$

If  $n_1 \approx n_2$  as for identical superconductors 1 and 2, we have

$$\frac{\partial \theta_1}{\partial t} = \frac{\partial \theta_2}{\partial t} \quad \text{or} \quad \frac{\partial(\theta_2 - \theta_1)}{\partial t} = 0.$$

$$\frac{\partial n_1}{\partial t} = -\frac{\partial n_2}{\partial t}$$

The current flow from the superconductor  $S_1$  and to the superconductor  $S_2$  is proportional to  $\frac{\partial n_2}{\partial t}$ .  $J$  is the current of superconductor pairs across the junction

$$J = J_0 \sin(\theta_2 - \theta_1),$$

where  $J_0$  is proportional to  $T$  (transfer interaction).

$$I = I_0 \sin \phi. \tag{5}$$

## 2. AC Josephson effect

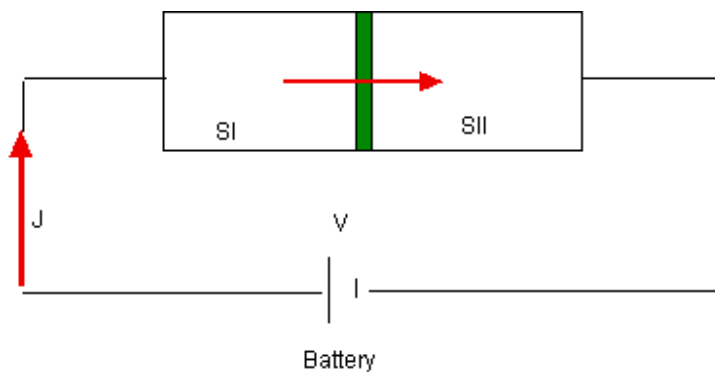


Fig. Schematic diagram for experiment of AC Josephson effect. A finite DC voltage is applied across both the ends.

Let a dc voltage  $V$  be applied across the junction. An electron pair experiences a potential energy difference  $qV$  on passing across the junction ( $q = -2e$ ). We can say that a pair on one side is at  $-eV$  and a pair on the other side is at  $eV$ .

$$i\hbar \frac{\partial \psi_1}{\partial t} = \hbar T \psi_2 - eV \psi_1$$

$$i\hbar \frac{\partial \psi_2}{\partial t} = \hbar T \psi_1 + eV \psi_2$$

or

$$\frac{1}{2} \frac{\partial n_1}{\partial t} + i n_1 \frac{\partial \theta_1}{\partial t} = \frac{ieV n_1}{\hbar} - iT \sqrt{n_1 n_2} e^{i\delta},$$

$$\frac{1}{2} \frac{\partial n_2}{\partial t} + i n_2 \frac{\partial \theta_2}{\partial t} = -\frac{ieV n_2}{\hbar} - iT \sqrt{n_1 n_2} e^{-i\delta}.$$

This equation breaks up into the real part and imaginary part,

$$\frac{\partial n_1}{\partial t} = 2T \sqrt{n_1 n_2} \sin \delta$$

$$\frac{\partial n_2}{\partial t} = -2T \sqrt{n_1 n_2} \sin \delta$$

$$\frac{\partial \theta_1}{\partial t} = \frac{eV}{\hbar} - T \sqrt{\frac{n_2}{n_1}} \cos \delta$$

$$\frac{\partial \theta_2}{\partial t} = -\frac{eV}{\hbar} - T \sqrt{\frac{n_1}{n_2}} \cos \delta$$

From these two equations with  $n_1 = n_2$ ,

$$\frac{\partial(\theta_2 - \theta_1)}{\partial t} = \frac{\partial \delta}{\partial t} = -\frac{2eV}{\hbar}.$$

$$J = J_0 \sin[\delta(t)].$$

with

$$\delta(t) = \delta(0) - \frac{2e}{\hbar} \int V dt.$$

When  $V = V_0 = \text{constant}$ , we have

$$\delta(t) = \delta(0) - \frac{2e}{\hbar} V_0 t.$$

$$J = J_0 \sin[\delta(0) - \frac{2e}{\hbar} V_0 t].$$

The current oscillates with frequency

$$\omega_0 = \frac{2e}{\hbar} V.$$

A DC voltage of  $1 \text{ eV}$  produces a frequency of  $483.5935 \text{ MHz}$ .

$$\dot{\phi} = \frac{2e}{\hbar} V.$$

**((Note))**

Suppose that  $V = V_0 = 1 \text{ V}$ . The corresponding frequency is estimated from the relation,

$$\frac{2eV_0}{\hbar} = 2\pi\nu_0,$$

or

$$\nu_0 = \frac{2eV_0}{2\pi\hbar} = \frac{2 \times (1.60219 \times 10^{-19}) \times 10^{-6}}{2\pi \times 1.05459 \times 10^{-27}} = 483.5935 \text{ MHz}.$$

### 3. ***I-V characteristic of Josephson tunneling junction***

We now consider the  $I-V$  characteristic of the Josephson tunneling junction where a insulating layer is sandwiched between two superconducting layers (the same type). A capacitor is formed by these two superconductors. In this type of Josephson junctions, one can see the quasiparticle  $I-V$  curve which is different with increasing voltage and decreasing voltage (hysteresis). There are two voltage states,  $0 \text{ V}$  and  $2\Delta/e$ , where  $\Delta$  is an energy gap of each superconductor. The  $I-V$  curve is characterized by (i) maximum Josephson tunneling current of Cooper pairs at  $V = 0$  and (ii) Quasi-particle tunneling current ( $V > 2\Delta/e$ ).

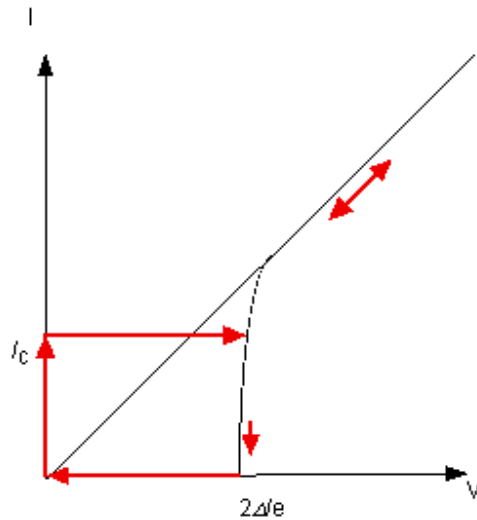


Fig. Schematic diagram of quasiparticle  $I$ - $V$  characteristic (usually observed in a S-I-S Josephson *tunneling-type*). Josephson current (up to a maximum value  $I_c$ ) flows at  $V = 0$ .  $\Delta$  is an energy gap of the superconductor. The DC Josephson supercurrent flows under  $V = 0$ . For  $V > 2\Delta/e$  the quasiparticle tunneling current is seen.

The strong nonlinearity in the quasiparticle  $I$ - $V$  curve of a tunneling junction is not appropriate to the application to the SQUID element. This nonlinearity can be removed by the use of thin normal film deposited across the electrodes. In this effective resistance is a parallel combination of the junction. The  $I$ - $V$  characteristic has no hysteresis. Such behavior is often observed in the bridge-type Josephson junction where two superconducting thin films are bridged by a very narrow superconducting thin film.

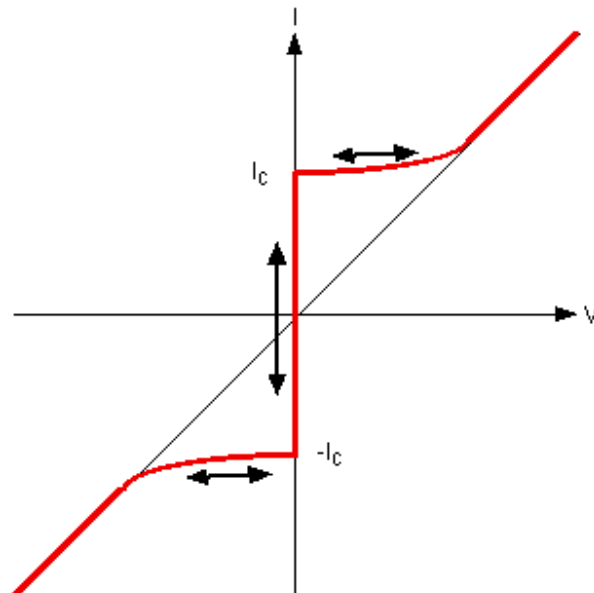


Fig. Schematic diagram of  $I$ - $V$  characteristic of a Josephson junction (usually observed in bridge-type junction), which is reversible on increasing and decreasing  $V$ . A Josephson supercurrent flows up to  $I_c$  at  $V = 0$ . A transition occurs from the  $V = 0$  state to a finite

voltage state for  $I > I_c$ . Above this voltage the  $I$ - $V$  characteristic exhibits an Ohm's law with a finite resistance of the Junction. The current has a oscillatory component of angular frequency  $\omega (= 2eV/\hbar)$  (the AC Josephson effect).

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#### 4. Flux quantization

We start with the current density

$$\mathbf{J}_s = \frac{q^* \hbar}{m^*} |\psi|^2 \left( \nabla \theta - \frac{q^*}{c\hbar} \mathbf{A} \right) = q^* |\psi|^2 \mathbf{v}_s.$$

Suppose that  $n_s = |\psi|^2 = \text{constant}$ . Then we have

$$\nabla \theta = \frac{m^*}{q^* \hbar n_s} \mathbf{J}_s + \frac{q^*}{c\hbar} \mathbf{A},$$

or

$$\oint \nabla \theta \cdot d\mathbf{l} = \frac{m^*}{q^* \hbar n_s} \oint \mathbf{J}_s \cdot d\mathbf{l} + \frac{q^*}{c\hbar} \oint \mathbf{A} \cdot d\mathbf{l}.$$

The path of integration can be taken inside the penetration depth where  $\mathbf{J}_s = 0$ .

$$\oint \nabla \theta \cdot d\mathbf{l} = \frac{q^*}{c\hbar} \oint \mathbf{A} \cdot d\mathbf{l} = \frac{q^*}{c\hbar} \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \frac{q^*}{c\hbar} \int \mathbf{B} \cdot d\mathbf{a} = \frac{q^*}{c\hbar} \Phi,$$

where  $\Phi$  is the magnetic flux. Then we find that

$$\Delta \theta = \theta_2 - \theta_1 = 2\pi n = \frac{q^*}{c\hbar} \Phi,$$

where  $n$  is an integer. The phase  $\theta$  of the wave function must be unique, or differ by a multiple of  $2\pi$  at each point,

$$\Phi = \frac{2\pi c\hbar}{|q^*|} n.$$

The flux is quantized. When  $|q^*| = 2|e|$ , we have a magnetic quantum fluxoid;

$$\Phi_0 = \frac{2\pi c\hbar}{2|e|} = \frac{ch}{2|e|} = 2.06783372 \times 10^{-7} \text{ Gauss cm}^2$$

((Note))

The current flows along the ring. However, this current flows only on the surface boundary (region from the surface to the penetration depth  $\lambda$ ). Inside of the system (region far from the surface boundary), there is no current since  $\nabla \times \mathbf{H} = 4\pi\mathbf{J}/c$  and  $\mathbf{H} = 0$ .

### 5. DC SQUID (double junctions): quantum mechanics

DC SQUID consists of two points contacts in parallel, forming a ring. Each contact forms a Josephson junctions of superconductor 1, insulating layer, and superconductor 2 ( $S_1$ -I- $S_2$ ). Suppose that a magnetic flux  $\Phi$  passes through the interior of the loop.

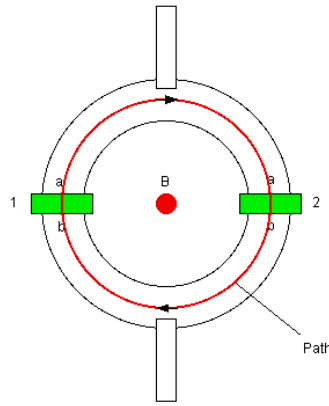


Fig. Schematic diagram of superconducting quantum interference device.  $\delta_1$  and  $\delta_2$  refer to two point-contact weak links. The rest of the circuit is strongly superconducting.

Here we have

$$\oint \nabla \theta \cdot d\mathbf{l} = \theta_{2a} - \theta_{1a} + \theta_{1b} - \theta_{2b}.$$

or

$$\theta_{2a} - \theta_{1a} + \theta_{1b} - \theta_{2b} = 2\pi \frac{\Phi}{\Phi_0}$$

or

$$\delta_1 - \delta_2 = 2\pi \frac{\Phi}{\Phi_0}$$

where  $\delta_1 (= \theta_{1b} - \theta_{1a})$  is the phase difference between the superconductors  $a$  and  $b$  through the junction 1 and  $\delta_2 (= \theta_{2b} - \theta_{2a})$  are is the phase difference between the superconductors  $a$  and  $b$  through the junction 2.

When  $\mathbf{B} = 0$  (or  $\Phi = 0$ ), we have  $\delta_1 - \delta_2 = 0$ . In general, we put the form

$$\delta_1 = \delta_0 + \frac{e}{\hbar c} \Phi, \quad \delta_2 = \delta_0 - \frac{e}{\hbar c} \Phi.$$



The total current is given by

$$\begin{aligned}
 I &= I_1 + I_2 = I_c[\sin(\delta_1) + \sin(\delta_2)] \\
 &= I_c[\sin(\delta_0 + \frac{e}{\hbar c}\Phi) + \sin(\delta_0 - \frac{e}{\hbar c}\Phi)] \\
 &= 2I_c \sin(\delta_0) \cos(\frac{e}{\hbar c}\Phi)
 \end{aligned}$$

or

$$I = 2I_c \sin(\delta_0) \cos(\pi \frac{\Phi}{\Phi_0}). \quad (30)$$

The current varies with  $\Phi$  and has a maximum of  $2I_c$  when  $\frac{e}{\hbar c}\Phi = s\pi$  ( $s$ : integers),

or

$$\Phi = \frac{\hbar c \pi}{e} s = \frac{\hbar c}{2e} s = \Phi_0 s. \quad (31)$$

The simple two point contact device corresponds to a two-slit interference pattern, for which the physically interesting quantity is the modulus of the amplitude rather than the square modulus, as it is for optical interference patterns.

## 6 Analogy of the diffraction with double slits and single slit

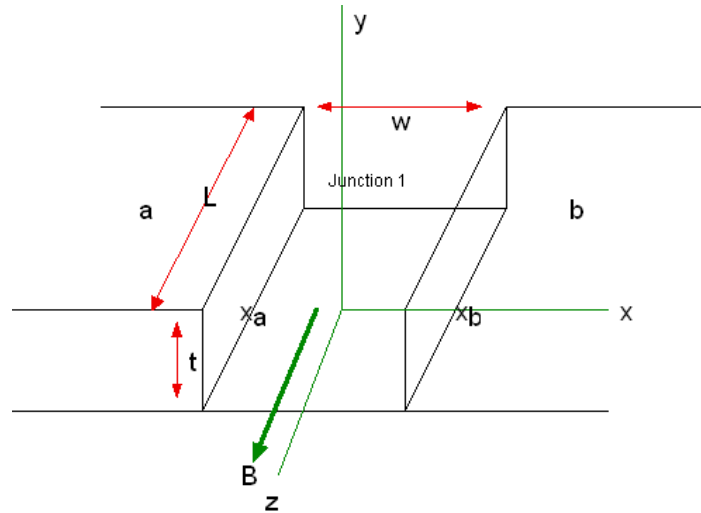


Fig. Diffraction effect of Josephson junction. A magnetic field  $B$  along the  $z$  direction, which is penetrated into the junction (in the normal phase).

We consider a junction (1) of rectangular cross section with magnetic field  $\mathbf{B}$  applied in the plane of the junction, normal to an edge of width  $w$ ,

$$J = J_0 \sin[\delta_1 + \frac{q}{\hbar c} \int_1^2 \mathbf{A} \cdot d\mathbf{l}],$$

with  $q = -2e$ . We use the vector potential  $\mathbf{A}$  given by

$$\mathbf{A} = \frac{1}{2}(\mathbf{B} \times \mathbf{r}) = (-\frac{By}{2}, \frac{Bx}{2}, 0),$$

$$\mathbf{A}' = \mathbf{A} + \nabla\chi = (-By, 0, 0),$$

where

$$\chi = -\frac{Bxy}{2}.$$

Then we have

$$J = J_0 \sin[\delta_1 + \frac{q}{\hbar c} \int_{x_a}^{x_b} (-By) dx] = J_0 \sin[\delta_1 - \frac{qB}{\hbar c} yW],$$

$$dI_1 = J L dy = J_0 L \sin[\delta_1 - \frac{qB}{\hbar c} yW] dy,$$

or

$$I_1 = J_0 L \int_{-t/2}^{t/2} \sin[\delta_1 - \frac{qB}{\hbar c} yW] dy.$$

Then we have

$$I_1 = J_0 L \frac{2\hbar c}{BWq} \sin(\delta_1) \sin(\frac{qB}{\hbar c} \frac{t}{2} W).$$

Here we introduce the total magnetic flux passing through the area  $Wt$  ( $\Phi_W = BWt$ ),  $I_c = J_0 Lt$ , and

$$\frac{\Phi_W}{\Phi_0} = \frac{BWt}{2\pi\hbar c} = \frac{eBWt}{\pi\hbar c}, \quad \text{or} \quad \pi \frac{\Phi_W}{\Phi_0} = \frac{eBWt}{\hbar c}.$$

Therefore we have

$$I_1 = I_c \sin(\delta_1) \frac{\sin\left(\frac{\pi\Phi_W}{\Phi_0}\right)}{\left(\frac{\pi\Phi_W}{\Phi_0}\right)}.$$

The total current is given by

$$I = I_1 + I_2 = I_c [\sin(\delta_1) + \sin(\delta_2)] \frac{\sin\left(\frac{\pi\Phi_W}{\Phi_0}\right)}{\left(\frac{\pi\Phi_W}{\Phi_0}\right)},$$

or

$$I = I_1 + I_2 = 2I_c \sin(\delta_0) \cos\left(\pi \frac{\Phi}{\Phi_0}\right) \frac{\sin\left(\frac{\pi\Phi_W}{\Phi_0}\right)}{\left(\frac{\pi\Phi_W}{\Phi_0}\right)}. \quad (32)$$

The short period variation is produced by interference from the two Josephson junctions, while the long period variation is a diffraction effect and arises from the finite dimensions of each junction. The interference pattern of  $|I|^2$  is very similar to the intensity of the Young's double slits experiment. If the slits have finite width, the intensity must be multiplied by the diffraction pattern of a single slit, and for large angles the oscillations die out.

**((Young's double slit experiment))**

We consider the Young's double slits (the slits are separated by  $d$ ). Each slit has a finite width  $a$ .

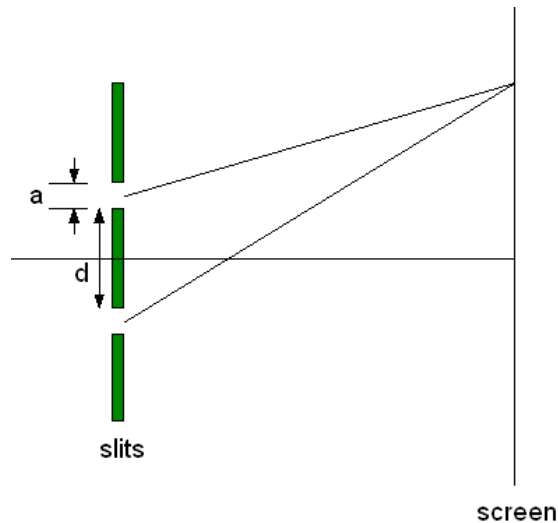


Fig. Geometric construction for describing the Young's double-slit experiment (not to scale).

**((double slits))**

$E$  is the electric field of a light with the wavelength  $\lambda$ .  $d$  is the separation distance between the centers of the slits.

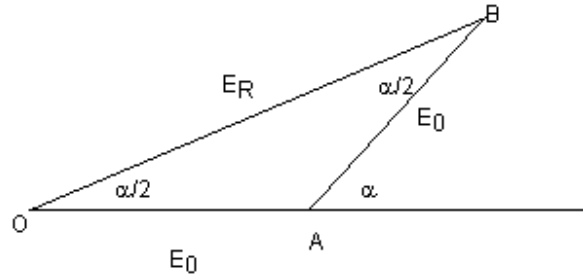


Fig. A reconstruction of the resultant phasor  $E_R$  which is the combination of two phasors ( $E_0$ ).

$$E_R = 2E_0 \cos \frac{\alpha}{2}.$$

The intensity:

$$I \propto E_R^2 = 4E_0^2 \cos^2 \frac{\alpha}{2} = 2E_0^2 (1 + \cos \alpha).],$$

where the phase difference  $\alpha$  is given by

$$\alpha = \frac{2\pi d}{\lambda} \sin \theta.$$

**((single slit))**

We assume that each slit has a finite width  $a$ .

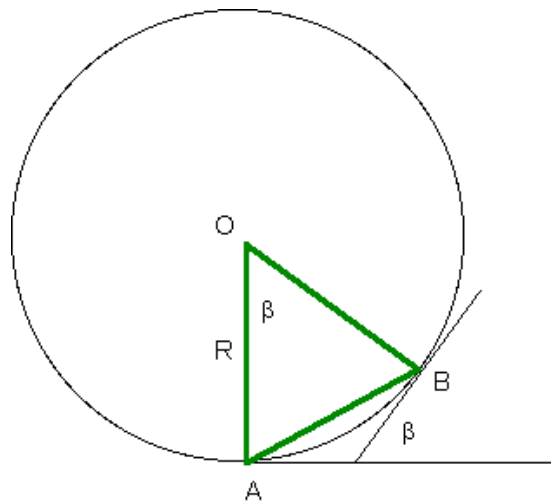


Fig. Phaser diagram for a large number of coherent sources. All the ends of phasors lie on the circular arc of radius  $R$ . The resultant electric field magnitude  $E_R$  equals the length of the chord.

$$E_0 = R\beta,$$

$$E_R = 2R \sin \frac{\beta}{2} = 2 \frac{E_0}{\beta} \sin \frac{\beta}{2} = E_0 \frac{\sin \frac{\beta}{2}}{\frac{\beta}{2}}.$$

where the phase difference  $\beta$  is given by  $\beta = \frac{2\pi a}{\lambda} \sin \theta$ . Then the resultant intensity  $I$  for the double slits (the distance  $d$ ) (each slit has a finite width  $a$ ) is given by

$$I = I_m \cos^2 \frac{\alpha}{2} \left( \frac{\sin \frac{\beta}{2}}{\frac{\beta}{2}} \right)^2 = \frac{I_m}{2} (1 + \cos \alpha) \left( \frac{\sin \frac{\beta}{2}}{\frac{\beta}{2}} \right)^2.$$

The maximum of  $I_B$  is

$$I_{\max} = 2I_c \left| \cos \left( \pi \frac{\Phi_{\text{ext}}}{\Phi_0} \right) \right|. \quad (40)$$

The critical current is a periodic function of the external magnetic flux.

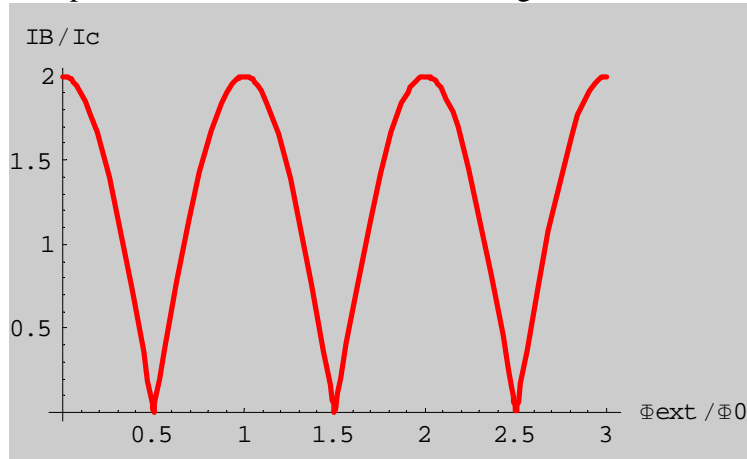


Fig. Ideal case for the  $I_B/I_c$  vs  $\Phi_{\text{ext}}/\Phi_0$  curve in the DC SQUID, where  $I_B$  is the maximum supercurrent.  $I_B = 2 I_c$  when  $\Phi_{\text{ext}}/\Phi_0 = n$  (integer) and  $I_B = 0$  for  $\Phi_{\text{ext}}/\Phi_0 = n + 1/2$ .

We consider the general case (but still  $L = 0$  and  $\beta_J \gg 1$ )

$$\beta_J \frac{d\phi_1}{d\tau} + \sin \phi_1 = \frac{\kappa_B}{2} - \kappa_s, \quad (41)$$

$$\beta_J \frac{d\phi_2}{d\tau} + \sin \phi_2 = \frac{\kappa_B}{2} + \kappa_s, \quad (42)$$

$$\phi_2 = \phi_1 - 2\pi \frac{\Phi_{ext}}{\Phi_0}. \quad (44)$$

From the addition of Eqs.(41) and (42) with the help of the relation Eq.(43), we have

$$\beta_J \frac{d}{d\tau} \left( \phi_1 - \pi \frac{\Phi_{ext}}{\Phi_0} \right) + \cos\left(\pi \frac{\Phi_{ext}}{\Phi_0}\right) \sin\left(\phi_1 - \pi \frac{\Phi_{ext}}{\Phi_0}\right) = \frac{\kappa_B}{2}.$$

When we introduce a new parameter

$$\varphi_1 = \phi_1 - \pi \frac{\Phi_{ext}}{\Phi_0},$$

we have the final form

$$\beta_J \frac{d\varphi_1}{d\tau} + \cos\left(\pi \frac{\Phi_{ext}}{\Phi_0}\right) \sin \varphi_1 = \frac{\kappa_B}{2}.$$

We are interested in the DC current-voltage characteristic so we need to determine the time averaged voltage

$$V = \left\langle \frac{\hbar}{2e} \frac{d\phi_1}{dt} \right\rangle = \left\langle I_c R \beta_J \frac{d\phi_1}{d\tau} \right\rangle = \frac{I_c R \beta_J}{T} \int_0^T \frac{d\phi_1}{d\tau} d\tau = I_c R \beta_J \frac{2\pi}{\tau}.$$

Here

$$\tau = \int_0^{2\pi} \frac{d\phi_1}{\frac{d\phi_1}{d\tau}} = \int_0^{2\pi} \frac{\beta_J d\phi_1}{\frac{\kappa_B}{2} - \cos\left(\pi \frac{\Phi_{ext}}{\Phi_0}\right) \sin \varphi_1} = \frac{\beta_J}{\cos\left(\pi \frac{\Phi_{ext}}{\Phi_0}\right)} \int_0^{2\pi} \frac{d\phi_1}{\kappa' - \sin \varphi_1} = \frac{1}{\cos\left(\pi \frac{\Phi_{ext}}{\Phi_0}\right)} \tau',$$

$$\kappa' = \frac{\kappa_B}{2 \cos\left(\pi \frac{\Phi_{ext}}{\Phi_0}\right)},$$

where

$$\tau' = \int_0^{2\pi} \frac{d\phi}{\kappa' - \sin \phi} = \int_0^{\pi} \frac{d\phi}{\kappa' - \sin \phi} + \int_{\pi}^{2\pi} \frac{d\phi}{\kappa' + \sin \phi} = \int_0^{\pi} \left[ \frac{1}{\kappa' - \sin \phi} + \frac{1}{\kappa' + \sin \phi} \right] d\phi,$$

or

$$\tau' = \frac{2\pi}{\sqrt{\kappa'^2 - 1}} \text{ for } \kappa > 1,$$

$$\tau' = 0 \quad \text{for } \kappa < 1.$$

Then we have

$$V = 2\pi I_c R \beta_J \frac{1}{\tau} = \frac{2\pi I_c R}{\tau'} \cos\left(\pi \frac{\Phi_{ext}}{\Phi_0}\right) = I_c R \sqrt{\kappa'^2 - 1} \cos\left(\pi \frac{\Phi_{ext}}{\Phi_0}\right),$$

or

$$\frac{V}{I_c R} = \left| \cos\left(\pi \frac{\Phi_{ext}}{\Phi_0}\right) \right| \sqrt{\kappa'^2 - 1},$$

or

$$|\kappa'| = \sqrt{1 + \left( \frac{V}{I_c R \cos\left(\pi \frac{\Phi_{ext}}{\Phi_0}\right)} \right)^2} = \frac{1}{2 \left| \cos\left(\pi \frac{\Phi_{ext}}{\Phi_0}\right) \right|} \frac{I_B}{I_c},$$

or

$$\frac{I_B}{2I_c} = \sqrt{\cos^2\left(\pi \frac{\Phi_{ext}}{\Phi_0}\right) + \left(\frac{V}{RI_c}\right)^2},$$

or

$$\frac{\langle V \rangle}{RI_c} = \sqrt{\left(\frac{I_B}{2I_c}\right)^2 - \cos^2\left(\pi \frac{\Phi_{ext}}{\Phi_0}\right)},$$

or

$$\langle V \rangle = \frac{R}{2} \sqrt{I_B^2 - 4I_c^2 \cos^2\left(\pi \frac{\Phi_{ext}}{\Phi_0}\right)}. \quad (45)$$

When this equation for the voltage is compared with that for one Josephson junction with  $\beta_J \gg 1$

$$\langle V \rangle = R\sqrt{I^2 - I_c^2}.$$

We find that the critical current is  $2I_c \left| \cos\left(\pi \frac{\Phi_{ext}}{\Phi_0}\right) \right|$ . This means that the critical current is  $2I_c$  for  $\Phi_{ext}/\Phi_0 = n$  (integer) and zero for  $\Phi_{ext}/\Phi_0 = n+1/2$ . In other words, the critical current is a periodic function of  $\Phi$  with a period  $\Phi_0$ . However, the actual critical current does not oscillate between 0 and  $2I_c$  because of the finite self-inductance  $L$ . In the above model,  $L$  (or  $\beta = 0$ ) is assumed to be zero. The critical current varies between  $2I_c$  and finite value depending on the value of  $\beta$  (see the detail in Sec.7.1).

When the total current  $I_B$  is constant, the voltage across the DC SQUID periodically changes with the external magnetic flux. This is the phenomenon one exploits to create the most sensitive magnetic field detection.

This figure is obtained from the Instruction manual of Mr. SQUID.<sup>9</sup>

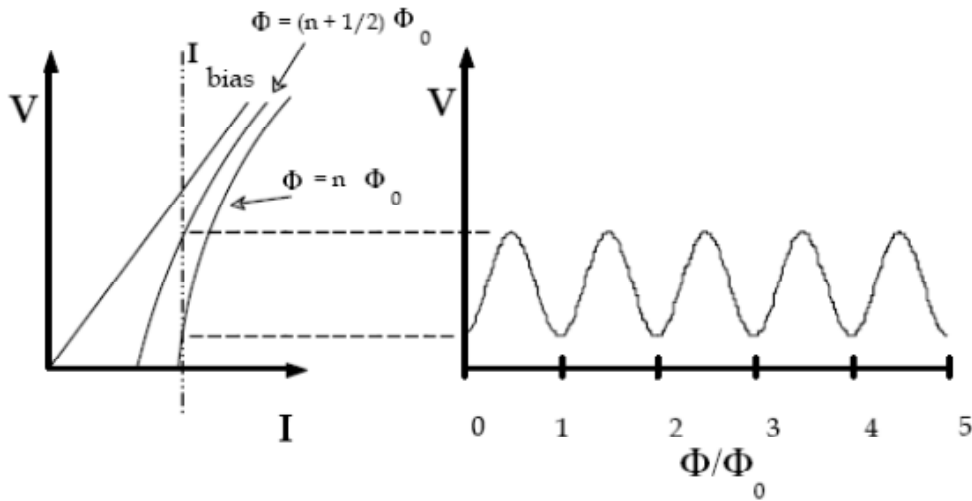


Fig. Detected voltage vs the magnetic flux  $\Phi/\Phi_0$ . The current  $I_B$  is kept at fixed value which is a little larger than  $2I_c$ . The detected voltage shows a maximum for  $\Phi = (n+1/2)\Phi_0$ , and a minimum for  $\Phi = n\Phi_0$ . The detected voltage is a periodic function of  $\Phi$  with a period of  $\Phi_0$ . (This figure is copied from the User Guide of Mr SQUID<sup>9</sup>).



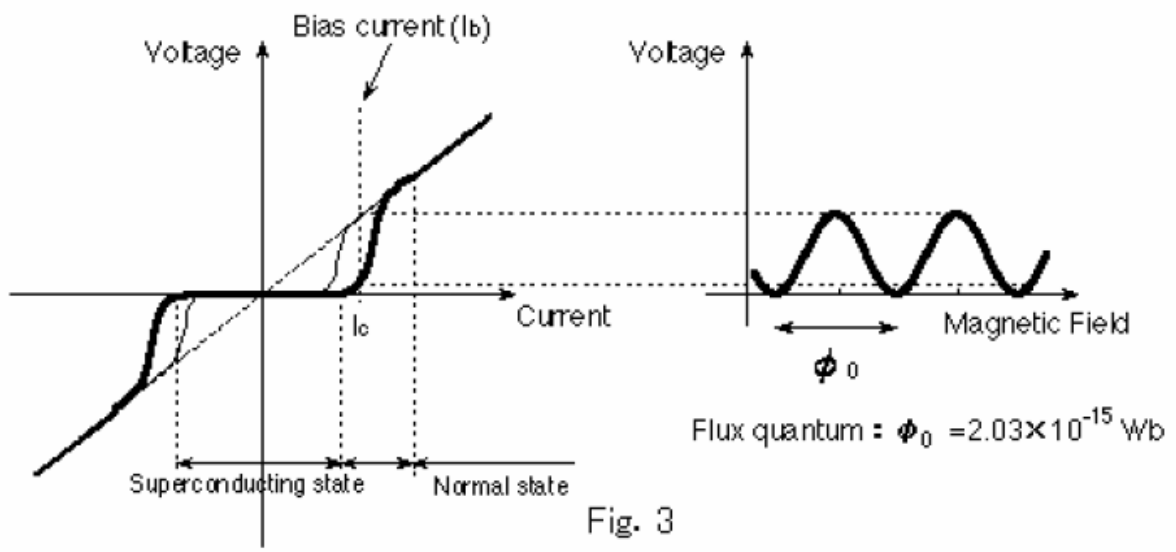
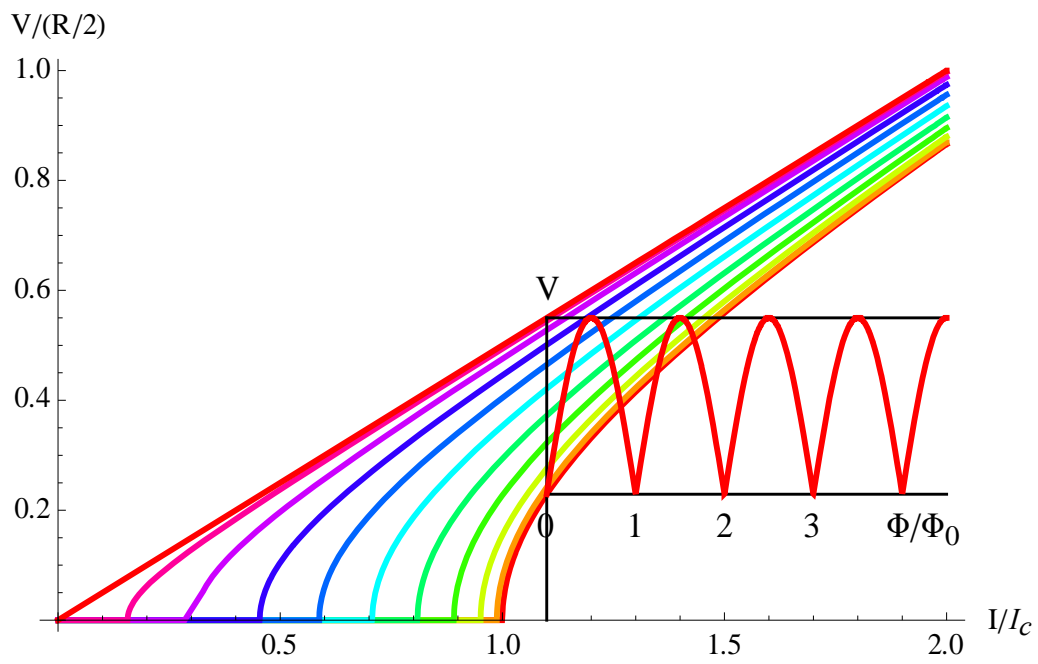


Fig. 3



((Mr. SQUID))

We use Mr SQUID for the measurement on the properties of the Josephson effect.

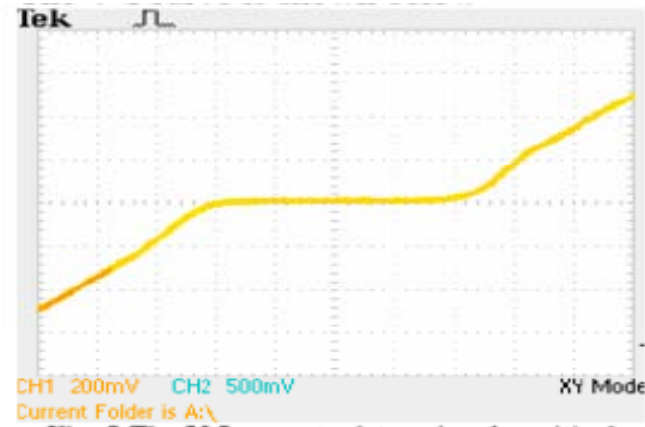


Fig. 5 The V-I curve to determine the critical current

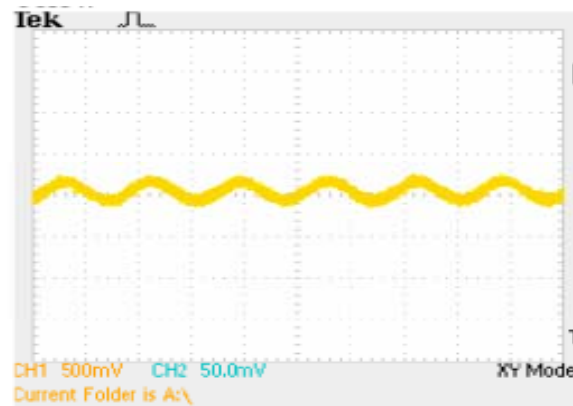


Fig. 8 The V - Φ curve to determine the modulation depth

### APPENDIX-1

#### Current density for the superconductors

We consider the current density for the superconductor.  $\psi$  is the order parameter of the superconductor and  $m^*$  and  $q^*$  are the mass and charge of the Cooper pairs. The current density is invariant under the gauge transformation.

$$\mathbf{J}_s = \frac{q^*}{m^*} \text{Re}[\langle \psi | \hat{\mathbf{p}} - \frac{q^*}{c} \mathbf{A} | \psi \rangle],$$

This can be rewritten as

$$\begin{aligned}
\mathbf{J}_s &= \frac{q^*}{m^*} \text{Re}[\psi^* (\frac{\hbar}{i} \nabla \psi - \frac{q^*}{c} \mathbf{A} \psi)] \\
&= \frac{q^*}{2m^*} [(\psi^* \frac{\hbar}{i} \nabla \psi - \frac{q^*}{c} \mathbf{A} \psi^* \psi) + (-\psi \frac{\hbar}{i} \nabla \psi^* - \frac{q^*}{c} \mathbf{A} \psi^* \psi)] \\
&= \frac{q^* \hbar}{2m^* i} (\psi^* \nabla \psi - \psi \nabla \psi^*) - \frac{q^{*2} |\psi|^2}{m^* c} \mathbf{A}
\end{aligned}$$

The density is also gauge independent.

$$\rho_s = \langle \mathbf{r} | \psi \rangle^2.$$

Now we assume that

$$\psi(\mathbf{r}) = |\psi(\mathbf{r})| e^{i\theta(\mathbf{r})}.$$

We note that

$$\begin{aligned}
\psi^* (\mathbf{p} - \frac{q^*}{c} \mathbf{A}) \psi(\mathbf{r}) &= \psi^* \frac{\hbar}{i} \nabla [e^{i\theta(\mathbf{r})} |\psi(\mathbf{r})|] - \frac{q^*}{c} \mathbf{A} |\psi(\mathbf{r})|^2 \\
&= |\psi(\mathbf{r})| e^{-i\theta(\mathbf{r})} \frac{\hbar}{i} [i |\psi(\mathbf{r})| e^{i\theta(\mathbf{r})} \nabla \theta(\mathbf{r}) + e^{i\theta(\mathbf{r})} \nabla |\psi(\mathbf{r})|] - \frac{q^*}{c} \mathbf{A} |\psi(\mathbf{r})|^2. \\
&= \hbar |\psi(\mathbf{r})|^2 [\nabla \theta(\mathbf{r}) - \frac{q^*}{c \hbar} \mathbf{A}] - i \hbar |\psi(\mathbf{r})| \nabla |\psi(\mathbf{r})|
\end{aligned}$$

The last term is pure imaginary. Then the current density is obtained as

$$\mathbf{J}_s = \frac{q^* \hbar}{m^*} |\psi|^2 (\nabla \theta - \frac{q^*}{c \hbar} \mathbf{A}) = q^* |\psi|^2 \mathbf{v}_s$$

or

$$\hbar \nabla \theta = \frac{q^*}{c} \mathbf{A} + m^* \mathbf{v}_s.$$

Since

$$\boldsymbol{\pi} = m^* \mathbf{v}_s = \mathbf{p} - \frac{q^*}{c} \mathbf{A}$$

we have

$$\mathbf{p} = \frac{q^*}{c} \mathbf{A} + m^* \mathbf{v}_s = \hbar \nabla \theta$$

Note that  $\mathbf{J}_s$  (or  $\mathbf{v}_s$ ) is gauge-invariant. Under the gauge transformation, the wave function is transformed as

$$\psi'(\mathbf{r}) = \exp\left(\frac{iq^* \chi}{\hbar c}\right) \psi(\mathbf{r}).$$

This implies that

$$\theta \rightarrow \theta' = \theta + \frac{q^* \chi}{\hbar c},$$

Since  $\mathbf{A}' = \mathbf{A} + \nabla \chi$ , we have

$$\begin{aligned} \mathbf{J}_s' &= \hbar \left( \nabla \theta' - \frac{q^*}{c \hbar} \mathbf{A}' \right) \\ &= \hbar \left[ \nabla \left( \theta + \frac{q^* \chi}{\hbar c} \right) - \frac{q^*}{c \hbar} (\mathbf{A} + \nabla \chi) \right] \\ &= \hbar \left( \nabla \theta - \frac{q^*}{c \hbar} \mathbf{A} \right) \end{aligned}$$

So the current density is invariant under the gauge transformation.