

Maser
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A maser is a device that produces coherent electromagnetic waves through amplification due to stimulated emission. Historically the term came from the acronym "**microwave amplification by stimulated emission of radiation**", although modern masers emit over a broad portion of the electromagnetic spectrum. This has led some to replace "microwave" with "molecular" in the acronym, as suggested by Townes. When optical coherent oscillators were first developed, they were called optical masers, but it has become more common to refer to these as lasers.

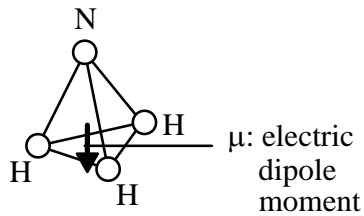
Theoretically, reflecting principles previously discussed by Joseph Weber at the June 1952 conference of the Institute of Radio Engineers^[1], the principle of the maser was described by Nikolay Basov and Alexander Prokhorov from Lebedev Institute of Physics at an All-Union Conference on Radio-Spectroscopy held by USSR Academy of Sciences in May 1952. They subsequently published their results in October 1954. Independently, Charles H. Townes, J. P. Gordon, and H. J. Zeiger built the first maser at Columbia University in 1953. The device used stimulated emission in a stream of energized ammonia molecules to produce amplification of microwaves at a frequency of 24 GHz. Townes later worked with Arthur L. Schawlow to describe the principle of the optical maser, or laser, which Theodore H. Maiman first demonstrated in 1960. For their research in this field Townes, Basov and Prokhorov were awarded the Nobel Prize in Physics in 1964.

Charles Hard Townes (born July 28, 1915) is an American Nobel Prize-winning physicist and educator. Townes is known for his work on the theory and application of the maser, on which he got the fundamental patent, and other work in quantum electronics connected with both maser and laser devices. He shared the Nobel Prize in Physics in 1964 with Nikolay Basov and Alexander Prokhorov.

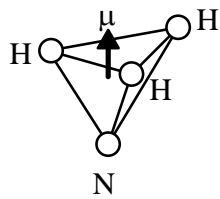
http://en.wikipedia.org/wiki/Charles_Hard_Townes



http://www.nobelprize.org/nobel_prizes/physics/laureates/1964/townes.jpg



$|1\rangle$ when the nitrogen is up.



$|2\rangle$ when the nitrogen is down.

We consider the parity operator $\hat{\pi}$, such that

$$\hat{\pi}|1\rangle = |2\rangle \quad \hat{\pi}|2\rangle = |1\rangle$$

Therefore the kets $|1\rangle$ and $|2\rangle$ are not the eigenkets of $\hat{\pi}$. Since

$$\hat{\pi} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$\hat{\pi}$ is regarded as the Pauli matrix $\hat{\sigma}_x$. The eigenkets of $\hat{\sigma}_x$ are $|\pm\rangle_x$.

$$\hat{\sigma}_x |\pm\rangle_x = \pm |\pm\rangle_x$$

with

$$|+\rangle_x = |\varphi_s^{(0)}\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle); \text{ symmetric state.}$$

$$|-\rangle_x = |\varphi_A^{(0)}\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle); \text{ antisymmetric state}$$

These two states are the eigenkets of $\hat{\pi}$.

We now consider the Hamiltonian \hat{H} .

The symmetry of two physical configuration suggests that

$$\langle 1|\hat{H}|1\rangle = \langle 2|\hat{H}|2\rangle = E_0$$

What about the off-diagonal elements? The vanishing of $\langle 2|\hat{H}|1\rangle$ would mean that a molecule initially in the state $|1\rangle$ would remain in that state. If $\langle 2|\hat{H}|1\rangle \neq 0$, there is a small amplitude for the system to mix between the two states.

$$\langle 2|\hat{H}|1\rangle = -A$$

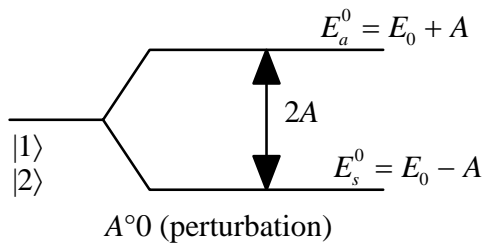
$$\hat{H} = \begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix} = E_0 \hat{1} - A \hat{\sigma}_x$$

This Hamiltonian commutes with the parity operator: $[\hat{H}, \hat{\pi}] = \hat{0}$.

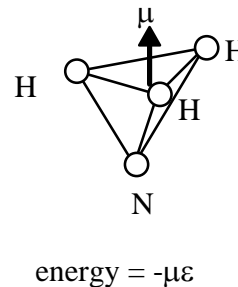
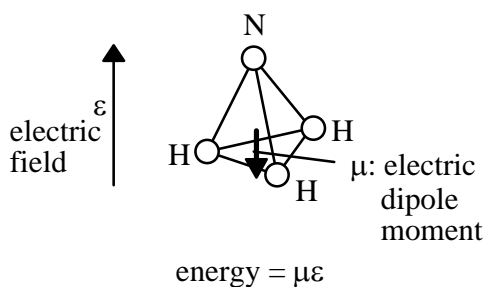
$$\hat{\sigma}_x |\pm\rangle_x = \pm |\pm\rangle_x$$

((Eigenvalue problem))

$$\hat{H} |\pm\rangle_x = (E_0 \hat{1} - A \hat{\sigma}_x) |\pm\rangle_x = (E_0 \mp A) |\pm\rangle_x$$



((Application of electric field))

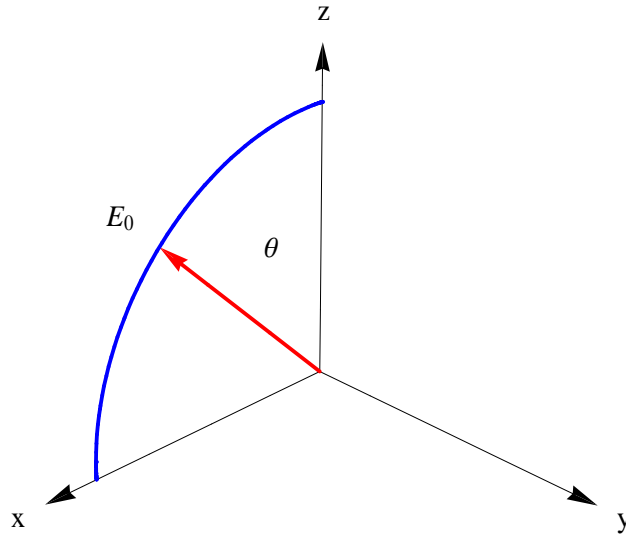


When the electric field is applied along the x axis (the axis of the electric dipole moment), the Hamiltonian is changed into

$$\hat{H} = \begin{pmatrix} E_0 + \mu\varepsilon & -A \\ -A & E_0 - \mu\varepsilon \end{pmatrix} = E_0 \hat{1} + \mu\varepsilon \hat{\sigma}_z - A \hat{\sigma}_x$$

The new Hamiltonian \hat{H} does not commute with the parity operator $\hat{\pi}$.

$$\hat{H} = E_0 \hat{1} + \sqrt{(\mu\varepsilon)^2 + A^2} \left(-\frac{A}{\sqrt{(\mu\varepsilon)^2 + A^2}} \hat{\sigma}_x + \frac{\mu\varepsilon}{\sqrt{(\mu\varepsilon)^2 + A^2}} \hat{\sigma}_z \right)$$



$$\mathbf{n} = \left(-\frac{A}{\sqrt{(\mu\varepsilon)^2 + A^2}}, 0, \frac{\mu\varepsilon}{\sqrt{(\mu\varepsilon)^2 + A^2}} \right)$$

$$\hat{H} = E_0 \hat{1} + \sqrt{(\mu\varepsilon)^2 + A^2} \hat{\sigma} \cdot \mathbf{n}$$

$$\hat{\sigma} \cdot \mathbf{n} |\pm\rangle_n = \pm |\pm\rangle_n$$

where

$$|+\rangle_n = \cos\frac{\theta}{2}|1\rangle + \sin\frac{\theta}{2}|2\rangle$$

and

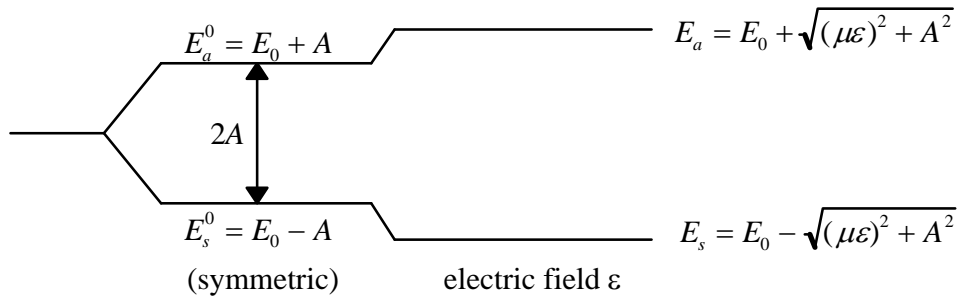
$$|-\rangle_n = -\sin\frac{\theta}{2}|1\rangle + \cos\frac{\theta}{2}|2\rangle$$

where

$$\sin\theta = -\frac{A}{\sqrt{(\mu\varepsilon)^2 + A^2}} \quad \cos\theta = \frac{\mu\varepsilon}{\sqrt{(\mu\varepsilon)^2 + A^2}}$$

Thus we have

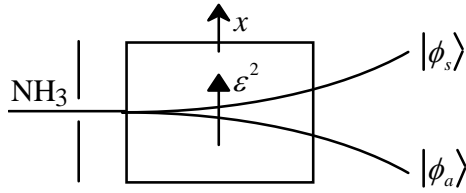
$$\hat{H}|\pm\rangle_n = (E_0 \pm \sqrt{(\mu\varepsilon)^2 + A^2})|\pm\rangle_n$$



In a weak electric field

$$E_s = E_0 - A\sqrt{1 + \frac{\mu^2\varepsilon^2}{A^2}} = E_0 - A - \frac{\mu^2\varepsilon^2}{2A^2} + \dots$$

$$E_a = E_0 + A\sqrt{1 + \frac{\mu^2\varepsilon^2}{A^2}} = E_0 + A + \frac{\mu^2\varepsilon^2}{2A^2} + \dots$$



Let us consider NH_3 in a region where ϵ is weak but where ϵ^2 has a strong gradient in the x -direction (i.e., along the axis of molecules).

$$\frac{d}{dx}(\epsilon^2) = \lambda$$

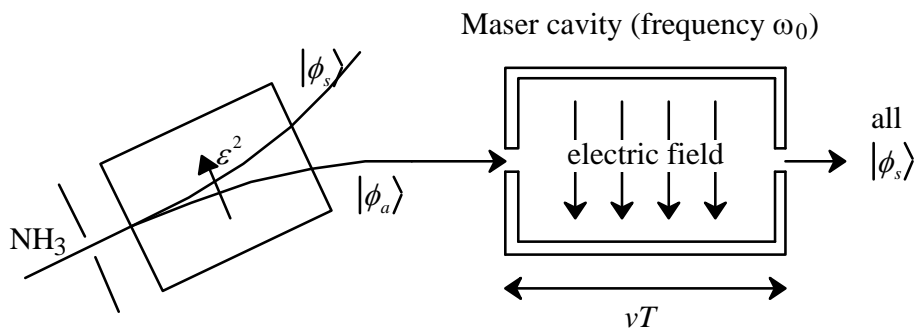
The molecules in the state $|\phi_s\rangle$ are subjected to a force parallel to the x axis:

$$F_s = -\frac{dE_s}{dx} = \frac{1}{2}\lambda \frac{\mu^2}{A}$$

Similarly, the molecules in the state $|\phi_a\rangle$ are subjected to an opposite force:

$$F_a = -\frac{dE_a}{dx} = -\frac{1}{2}\lambda \frac{\mu^2}{A}$$

This is the basis of the method which is used in the ammonia maser to sort the molecules and select those in the higher energy state.



In the ammonia maser, the beam with molecules in the state $|\phi_a^{(0)}\rangle$ and with the higher energy is sent through a resonant cavity.
Here we note that

$$|\varphi_s^{(0)}\rangle = |+\rangle_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \hat{U}|1\rangle \quad \text{for } (E_0 - A)$$

$$|\varphi_a^{(0)}\rangle = |-\rangle_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \hat{U}|2\rangle \quad \text{for } (E_0 + A)$$

where

$$\hat{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \hat{U}^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The Hamiltonian \hat{H} is given by

$$\hat{H}(t) = \begin{pmatrix} E_0 + \mu\mathcal{E}(t) & -A \\ -A & E_0 - \mu\mathcal{E}(t) \end{pmatrix}$$

in the presence of a time-dependent electric field \mathcal{E} . We assume that

$$|\psi(t)\rangle = c_1(t)|1\rangle + c_2(t)|2\rangle = c_s(t)|\varphi_s^{(0)}\rangle + c_a(t)|\varphi_a^{(0)}\rangle$$

or

$$c_1(t)|1\rangle + c_2(t)|2\rangle = c_s(t)\hat{U}|1\rangle + c_a(t)\hat{U}|2\rangle$$

or

$$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} c_s(t) \\ c_a(t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_s(t) \\ c_a(t) \end{pmatrix}$$

We set up the Schrödinger equation

$$i\hbar \frac{d}{dt} \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} E_0 + \mu\mathcal{E}(t) & -A \\ -A & E_0 - \mu\mathcal{E}(t) \end{pmatrix} \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$$

This can be rewritten as

$$i\hbar \frac{d}{dt} \begin{pmatrix} c_s(t) \\ c_a(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}^{-1} \begin{pmatrix} E_0 + \mu\mathcal{E}(t) & -A \\ -A & E_0 - \mu\mathcal{E}(t) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} c_s(t) \\ c_a(t) \end{pmatrix}$$

$$= \begin{pmatrix} E_0 - A & \mu\mathcal{E}(t) \\ \mu\mathcal{E}(t) & A + E_0 \end{pmatrix} \begin{pmatrix} c_s(t) \\ c_a(t) \end{pmatrix}$$

First we write

$$c_s(t) = \gamma_s(t) e^{-\frac{i(E_0 - A)t}{\hbar}}$$

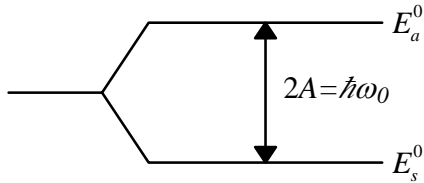
$$c_a(t) = \gamma_a(t) e^{-\frac{i(E_0 + A)t}{\hbar}}$$

$$i\hbar \frac{d\gamma_s(t)}{dt} = \mu\mathcal{E}(t) e^{-i\omega_0 t} \gamma_a(t)$$

$$i\hbar \frac{d\gamma_a(t)}{dt} = \mu\mathcal{E}(t) e^{i\omega_0 t} \gamma_s(t)$$

where

$$E_a^{(0)} - E_s^{(0)} = \hbar\omega_0 = 2A$$



We consider the case:

$$\mathcal{E}(t) = 2\varepsilon_0 \cos\omega t = \varepsilon_0 (e^{i\omega t} + e^{-i\omega t})$$

Then we have

$$i\hbar \frac{d\gamma_s(t)}{dt} = \mu\varepsilon_0 [e^{i(\omega - \omega_0)t} + e^{-i(\omega + \omega_0)t}] \gamma_a(t)$$

$$i\hbar \frac{d\gamma_a(t)}{dt} = \mu\varepsilon_0 [e^{i(\omega + \omega_0)t} + e^{-i(\omega - \omega_0)t}] \gamma_s(t)$$

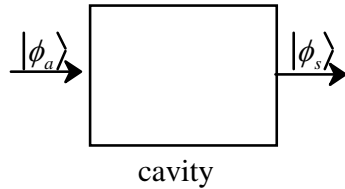
((Rotating wave approximation))

The terms with $(\omega + \omega_0)$ oscillate very rapidly about an average value of zero and, therefore do not contribute very much on the average to the rate of change of γ .

$$i\hbar \frac{d\gamma_s(t)}{dt} = \mu\epsilon_0 e^{i(\omega - \omega_0)t} \gamma_a(t)$$

$$i\hbar \frac{d\gamma_a(t)}{dt} = \mu\epsilon_0 e^{-i(\omega - \omega_0)t} \gamma_s(t)$$

Let us suppose that it takes the time T to go through the cavity. If we make the cavity just long enough so that $\mu\epsilon_0 T / \hbar = \pi / 2$, then a molecules which enters in the upper state $|\phi_a^{(0)}\rangle$ will certainly leave it in the lower state $|\phi_s^{(0)}\rangle$.



In other words, its energy is decreased, and the loss of energy cannot go anywhere else but into the machinery which generate the field.

In summary, the molecules enter the cavity, the cavity field-oscillating at exactly the right frequency-induces transition from the upper to the lower states, and the energy released is fed into the oscillatory field. The molecular energy is converted into the energy of an external electromagnetic field.

We define that

$$\Delta = \omega - \omega_0, \quad \Gamma_0 = \frac{\mu\epsilon_0}{\hbar}$$

$$\Omega = \sqrt{\frac{\Delta^2}{4} + \Gamma_0^2}$$

Then we get

$$i \frac{d\gamma_s(t)}{dt} = \frac{\mu\epsilon_0}{\hbar} e^{i\Delta t} \gamma_a(t) = \Gamma_0 e^{i\Delta t} \gamma_a(t)$$

$$i \frac{d\gamma_a(t)}{dt} = \frac{\mu\mathcal{E}_0}{\hbar} e^{-i\Delta t} \gamma_s(t) = \Gamma_0 e^{-i\Delta t} \gamma_s(t)$$

with the initial condition

$$\gamma_a(0) = 1, \quad \gamma_s(0) = 0$$

The solution is as follows.

$$\gamma_s(t) = -\frac{ie^{-i\Delta t} \Gamma_0 \sin(\Omega t)}{\Omega}$$

$$\gamma_a(t) = \frac{e^{-i(\Delta+2\Omega)t} [-\Delta + 2\Omega + e^{i2\Omega t} (\Delta + 2\Omega)]}{4\Omega},$$

The probabilities are given by

$$\begin{aligned} P_s(t) &= \gamma_s(t)^* \gamma_s(t) \\ &= \frac{4\Gamma_0^2 \sin^2\left(\frac{t}{2}\sqrt{4\Gamma_0^2 + \Delta^2}\right)}{4\Gamma_0^2 + \Delta^2} \\ &= \frac{2\Gamma_0^2 [1 - \cos(t\sqrt{4\Gamma_0^2 + \Delta^2})]}{4\Gamma_0^2 + \Delta^2} \end{aligned}$$

$$P_a(t) = \gamma_a(t)^* \gamma_a(t) = \frac{2\Gamma_0^2 + \Delta^2 + 2\Gamma_0^2 \cos(t\sqrt{4\Gamma_0^2 + \Delta^2})}{4\Gamma_0^2 + \Delta^2}$$

Note that

$$P_s(t) + P_a(t) = 1$$

((**Special case**))

When $\Delta = 0$ and $\Omega = \Gamma_0$, we have

$$\gamma_a(t) = \cos\left(\frac{t\Gamma_0}{2}\right)$$

and

$$\gamma_s(t) = -i \sin\left(\frac{\Gamma_0 t}{2}\right)$$

((Mathematica))

```
Clear["Global`*"]; expr_* := expr /. {Complex[a_, b_] :=> Complex[a, -b]};
```

$$\Delta = \omega - \omega_0, \quad \sqrt{\Gamma_0^2 + \frac{\Delta^2}{4}} = \Omega, \quad \Gamma_0 = \frac{\mu s_0}{\hbar}$$

$$\text{eq1} = i \text{D}[\gamma_a[t], t] == \Gamma_0 \text{Exp}[-i \Delta t] \gamma_s[t];$$

$$\text{eq2} = i \text{D}[\gamma_s[t], t] == \Gamma_0 \text{Exp}[i \Delta t] \gamma_a[t];$$

```
s11 = DSolve[{eq1, eq2, \gamma_a[0] == 1, \gamma_s[0] == 0}, {\gamma_a[t], \gamma_s[t]}, t] //  
Simplify[#, {\Delta > 0, \Gamma_0 > 0}] &;
```

$$\text{s12} = \text{s11} /. \left\{ \sqrt{4 \Gamma_0^2 + \Delta^2} \rightarrow 2 \Omega, \frac{1}{\sqrt{4 \Gamma_0^2 + \Delta^2}} \rightarrow \frac{1}{2 \Omega} \right\} // \text{Simplify};$$

```
\gamma_a[t_] = \gamma_a[t] /. s12[[1]] // FullSimplify
```

$$\frac{e^{-\frac{1}{2} i t (\Delta + 2 \Omega)} (-\Delta + 2 \Omega + e^{2 i t \Omega} (\Delta + 2 \Omega))}{4 \Omega}$$

```
\gamma_s[t_] = \gamma_s[t] /. s12[[1]] // FullSimplify
```

$$-\frac{i e^{\frac{i t \Delta}{2}} \Gamma_0 \text{Sin}[t \Omega]}{\Omega}$$

Pa = $\gamma_a[t]$ * $\gamma_a[t]$ // FullSimplify

$$\frac{\Delta^2 + 4 \Omega^2 - (\Delta^2 - 4 \Omega^2) \cos[2 t \Omega]}{8 \Omega^2}$$

Pa1 = Pa /. { $\Omega \rightarrow \frac{\sqrt{4 \Gamma_0^2 + \Delta^2}}{2}$ } // Simplify

$$\frac{2 \Gamma_0^2 + \Delta^2 + 2 \Gamma_0^2 \cos\left[t \sqrt{4 \Gamma_0^2 + \Delta^2}\right]}{4 \Gamma_0^2 + \Delta^2}$$

Ps = $\gamma_s[t]$ * $\gamma_s[t]$ // FullSimplify

$$\frac{\Gamma_0^2 \sin[t \Omega]^2}{\Omega^2}$$

Ps1 = Ps /. { $\Omega \rightarrow \frac{\sqrt{4 \Gamma_0^2 + \Delta^2}}{2}$ } // Simplify

$$\frac{4 \Gamma_0^2 \sin\left[\frac{1}{2} t \sqrt{4 \Gamma_0^2 + \Delta^2}\right]^2}{4 \Gamma_0^2 + \Delta^2}$$

Pa1 + Ps1 // Simplify

1

Pa2 = Pa1 /. $\Delta \rightarrow 0$ // Simplify[#, $\Gamma_0 > 0$] &

$$\cos[t \Gamma_0]^2$$

Ps2 = Ps1 /. $\Delta \rightarrow 0$ // Simplify[#, $\Gamma_0 > 0$] &

$$\sin[t \Gamma_0]^2$$