Relativistic Doppler effect

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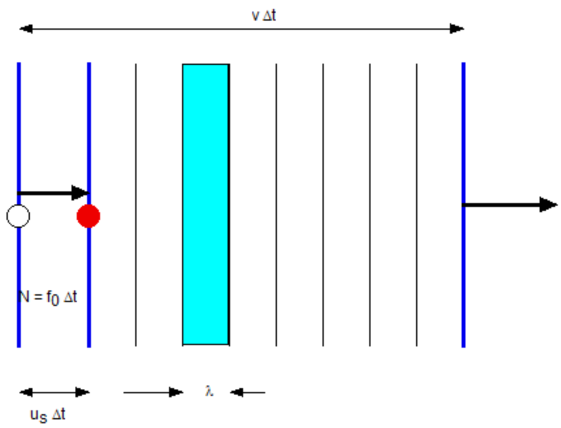
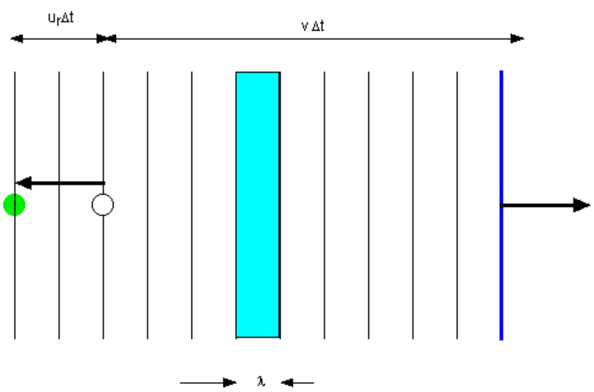
1. Doppler effect

The Doppler effect is the apparent change in frequency (or wavelength) that occurs because of motion of the source or observer of a wave. When the motion of the source or the observer is toward the other, the frequency appears to increase. When the motion of the source or observer is away from the other, the frequency appears to decrease



$$\lambda = \frac{(v-u_s)\Delta t}{N} = \frac{(v-u_s)\Delta t}{f_0\Delta t} = \frac{v-u_s}{f_0}$$



$$f\Delta t = \frac{(u_r + v)\Delta t}{\lambda}$$

$$f = \frac{(u_r + v)}{\lambda} = \left(\frac{v+u_r}{v-u_s}\right) f_0$$

u_s (>0) is the velocity of sender approaching the receiver.
u_r (>0) is the velocity of the receiver approaching the sender.
f_0 is the frequency of the sender and f is the frequency of the receiver.

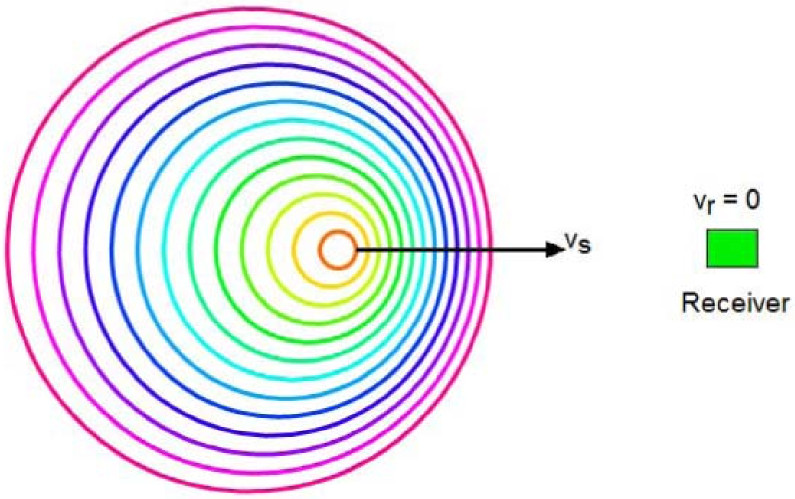


Fig. A receiver is stationary and a source is moving toward the receiver at the velocity v_s. v is the velocity of sound. $v_s < v$.

2. Relativistic Doppler effect

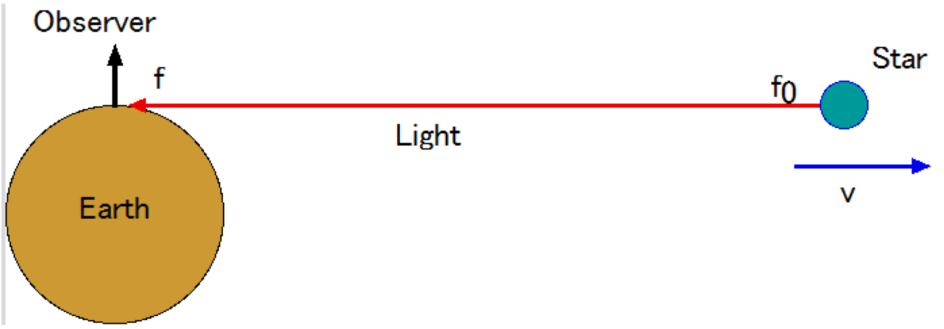


Fig.2

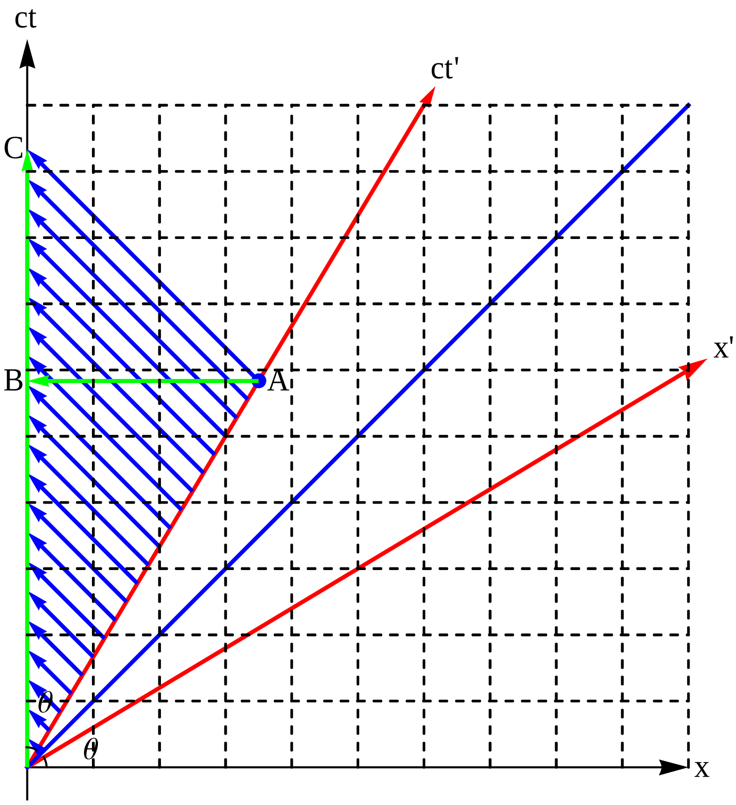


Fig.3 **Doppler effect**

We consider a moving light source that flashes with frequency f_0. The observer at rest in the S frame receives the light signal with a different frequency f (this is called the Doppler effect of light). First we calculate the distances in the S' frame;

$$\left(\overline{OC}\right)_{S'} = (\overline{OB}+\overline{BC})_{S'} = (\overline{OA})_{S'}\cos\theta + (\overline{AB})_{S'}\tan\frac{\pi}{4}$$

$$= ct'\cos\theta + ct'\sin\theta = ct'\frac{1+\frac{v}{c}}{\sqrt{1+\frac{v^2}{c^2}}}$$

Using the scaling factor

$$\left(\overline{OC}\right)_{S} = \sqrt{\frac{1+\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}}\left(\overline{OC}\right)_{S'}$$

we have

$$\left(\overline{OC}\right)_{S} = ct = \sqrt{\frac{1+\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}}\left(\overline{OC}\right)_{S'} = \sqrt{\frac{1+\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}}ct'\frac{1+\frac{v}{c}}{\sqrt{1+\frac{v^2}{c^2}}} = \sqrt{\frac{1+\frac{v}{c}}{1+\frac{v}{c}}}ct'$$

The frequency of the light observed at the S frame is related to that of the light emitted in the S' frame is

$$\left(\overline{OC}\right)_{S} = ct = \sqrt{\frac{1+\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}}\left(\overline{OC}\right)_{S'} = \sqrt{\frac{1+\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}}ct'\frac{1+\frac{v}{c}}{\sqrt{1+\frac{v^2}{c^2}}} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}ct'$$

((Formula of the Doppler effect))

$$\frac{f}{f_0} = \frac{ct'}{ct} = \frac{1}{\kappa} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} < 1$$

$$t = \kappa t'$$

with

$$\kappa = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} > 1$$

3. Derivation of relativistic Doppler effect II

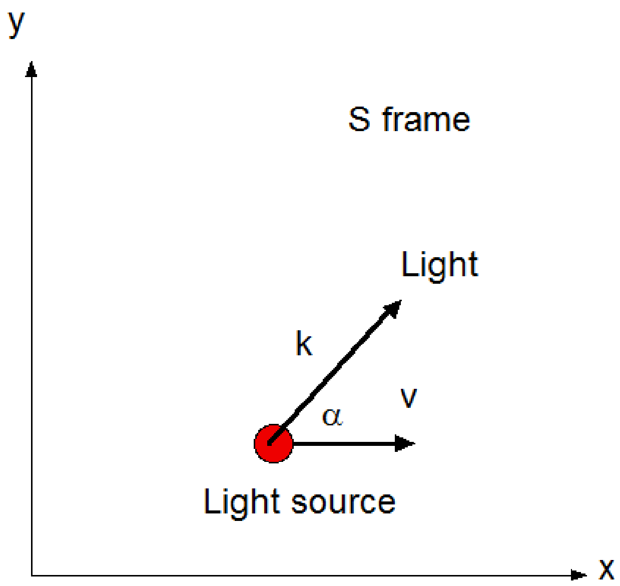


Fig.4 Derivation of the Doppler effect

The light source moves at the velocity *v* along the *x* axis. The light is emitted from the light source. The angle between the direction of the light and the *x* axis is α. We assume that the angular frequency for the observer in the S frame is ω and that the angular frequency for the light source in the S' frame is $\omega' = \omega_0$. Then we get the relation

$$\frac{E'}{c} = \frac{\frac{E}{c} - p\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$$

where $E = \hbar\omega$, $E' = \hbar\omega' = \hbar\omega_0$,and $p_x = \hbar k \cos\alpha$.

$$\frac{\hbar\omega'}{c} = \frac{\frac{\hbar\omega}{c} - \hbar k\cos\alpha\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\frac{\hbar\omega}{c} - \hbar\omega\cos\alpha\frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

or

$$\omega' = \frac{\omega(1-\frac{v}{c}\cos\alpha)}{\sqrt{1-\frac{v^2}{c^2}}}$$

or

$$\omega = \omega'\frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}\cos\alpha} = \omega_0\frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}\cos\alpha}$$

4. Consideration

We consider the physics meaning of the above equation using a figure below.

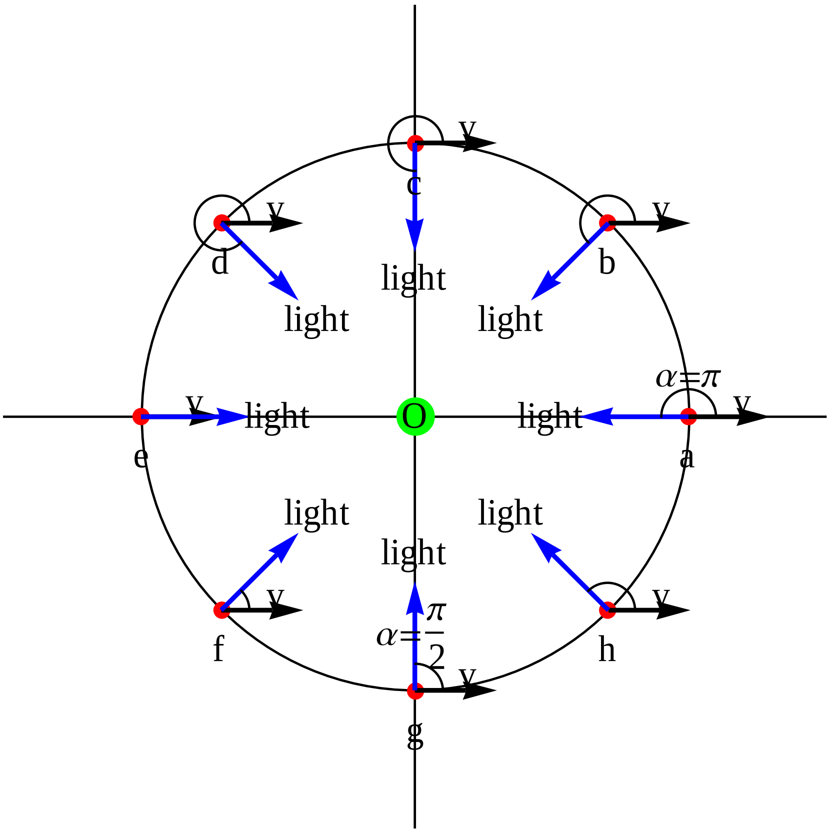


Fig. Observer at the origin. The star moves at the velocity v along the *x* axis. The light is emitted from the star.

(a) When $\alpha = \pi$ **(longitudinal Doppler effect)**

$$\omega = \omega_0 \frac{\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{v}{c}} = \omega_0 \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} < \omega_0$$

(receding)

The observed wavelength becomes longer (red shift).

(e) When $\alpha = 0$ (**longitudinal Doppler effect**)

$$\omega = \omega' \frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}} = \omega_0 \frac{\sqrt{(1+\frac{v}{c})(1-\frac{v}{c})}}{\sqrt{(1-\frac{v}{c})^2}} = \omega_0 \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} > \omega_0$$

(approaching)

The observed wavelength becomes shorter (blue-shift).

(c) When $\alpha = 3\pi/2$ (**transverse Doppler effect),** we have

$$\omega = \omega' \sqrt{1-\frac{v^2}{c^2}} = \omega_0 \sqrt{1-\frac{v^2}{c^2}}$$

(g) When $\alpha = \pi/2$ (**transverse Doppler effect),** we have

$$\omega = \omega' \sqrt{1-\frac{v^2}{c^2}} = \omega_0 \sqrt{1-\frac{v^2}{c^2}} < \omega_0$$

(b) When $\alpha = 5\pi/4$**,** we have

$$\omega = \omega' \frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}\cos(\frac{5\pi}{4})} = \omega_0 \frac{\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{v}{c\sqrt{2}}}$$

(f) When $\alpha = \pi/4$**,** we have

$$\omega = \omega' \frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}\cos\frac{\pi}{4}} = \omega_0 \frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c\sqrt{2}}}$$

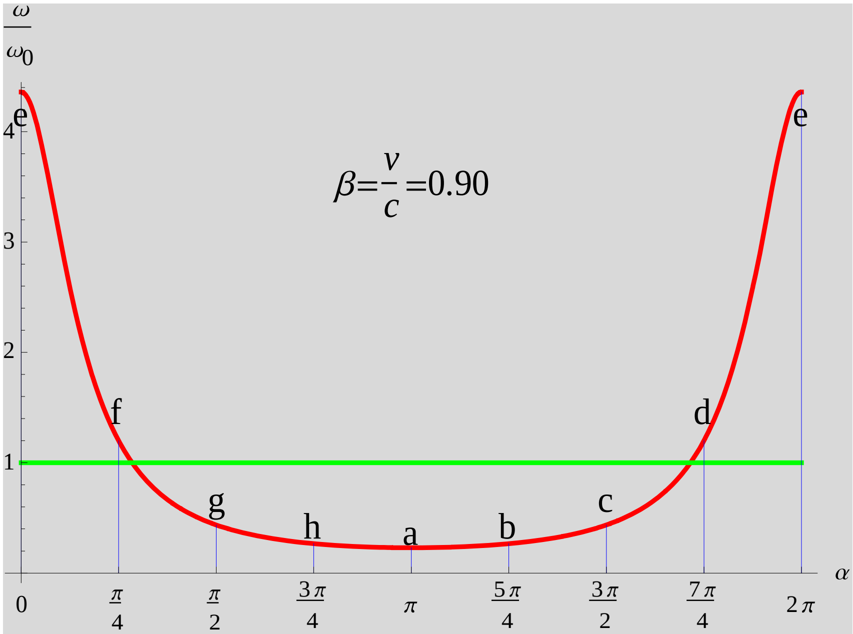
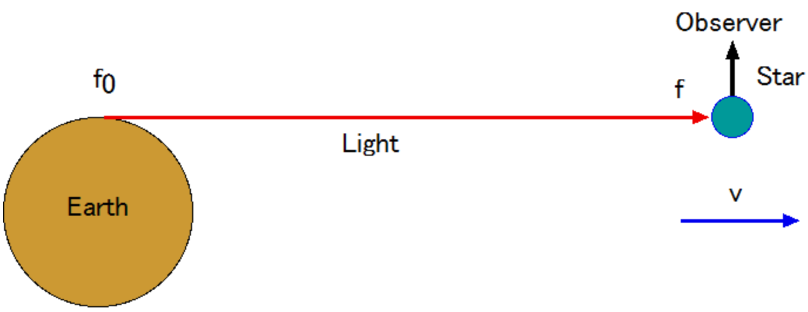


Fig. Plot of $\frac{\omega}{\omega_0}=\frac{\sqrt{1-\beta^2}}{1-\beta\cos\alpha}$ with $\beta = v/c = 0.90$. The green line denotes $\omega = \omega_0$.

a	longitudinal Doppler effect (receding)
e	longitudinal Doppler effect (approaching)
c and g	transverse Doppler effect.

5. Doppler effect from the Minkowski space-time diargam



We suppose that a source is located at the origin of the reference frame S, and that an observer moves relative to the frame S at the velocity v, so that the observer is at rest in the frame S'. Each emitted pulse travels with speed c. Suppose a first pulse is sent out at $t = 0$ when the observer is at the position $x = x_0$, and suppose the (n+1)-th pulse is sent out at $t =$

$n\tau$. This will have covered n periods of vibration, so that the measured frequency of the source in S is $\nu = \dfrac{1}{\tau}$.

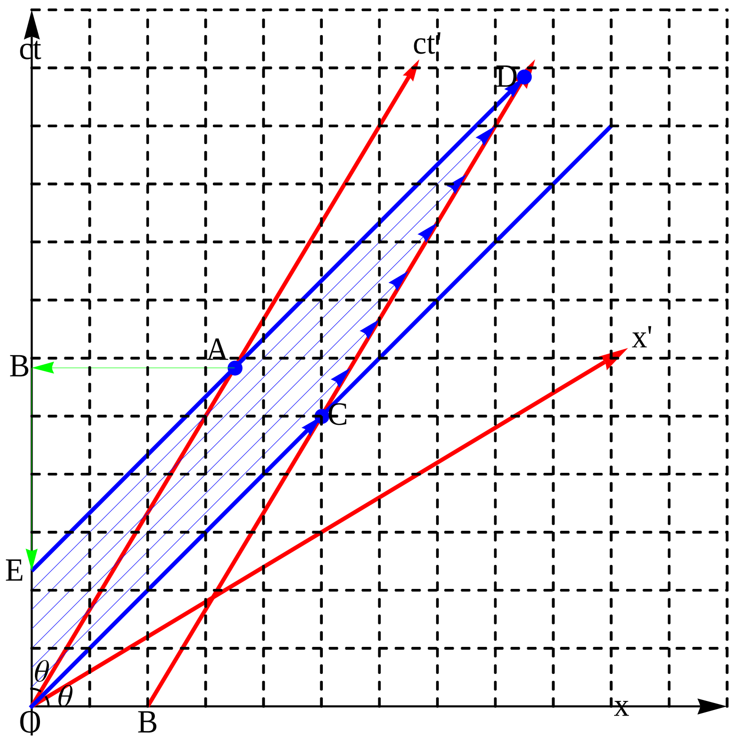


Fig.5

In the above figure, we assume that

$$\tan\theta = \frac{v}{c}.$$

$$\left(\overline{OA}\right)_{S'} = \left(\overline{CD}\right)_{S'} = n\tau'$$

$$\left(\overline{OB}\right)_{S'} = \left(\overline{OA}\right)_{S'}\cos\theta, \qquad \left(\overline{AB}\right)_{S'} = \left(\overline{EB}\right)_{S'} = \left(\overline{OA}\right)_{S'}\sin\theta$$

Then we have

$$(\overline{OE})_S = k(\overline{OE})_{S'} = k(\overline{OA})_{S'}(\cos\theta - \sin\theta)$$

$$= n\tau' \frac{1-\frac{v}{c}}{\sqrt{1+\frac{v^2}{c^2}}} \frac{\sqrt{1+\frac{v^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}} = n\tau' \frac{1-\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = n\tau' \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} = n\tau$$

Then we have

$$\tau = \tau' \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

or

$$f' = \frac{1}{\tau'} = \frac{1}{\tau} \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} = f \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} < f$$

The wave lengths in the S and S' are defined by

$\lambda = c\tau$, and $\lambda' = c\tau'$

Then the ratio of the wavelength is given by

$$\frac{\lambda'}{\lambda} = \frac{c\tau'}{c\tau} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} > 1 \qquad \text{(red shift)}$$

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6. Twin paradox

There are twin: Mary (the Moving twin) and Frank (the Fixed twin). We imagine that each person sends equally spaced time signals of their own proper times to the other, The cumulative counts of time signals for the whole trip are then compared. Suppose that each person is transmitting f pulses per unit time. As Mary travels away from Frank, each observer will receive the other's signal at the reduced rate

$$f' = f\sqrt{\frac{1-\beta}{1+\beta}}$$

But for how long? Here is the asymmetry. As soon as Mary reverses, she begins to receive signals from Frank at the enhanced rate

$$f'' = f\sqrt{\frac{1+\beta}{1-\beta}}$$

With Frank it is quite different. The last signal sent by Mary before she reverses does not reach Frank until a time L/c later. Thus for much more than one-half the total time Frank is recording the Mary's signals at the lower rate f'. Only in the latter stages does Frank receive pulses at the higher rate f''.

Note that each observer receives as many signals as the other sends between start and finish of trip. Frank is able to infer from his observations that it took place at the midmoment of the journey time as measured by Mary, since equal numbers of signal received by Frank at the two different rates f' and f'' (See French, Special relativity).

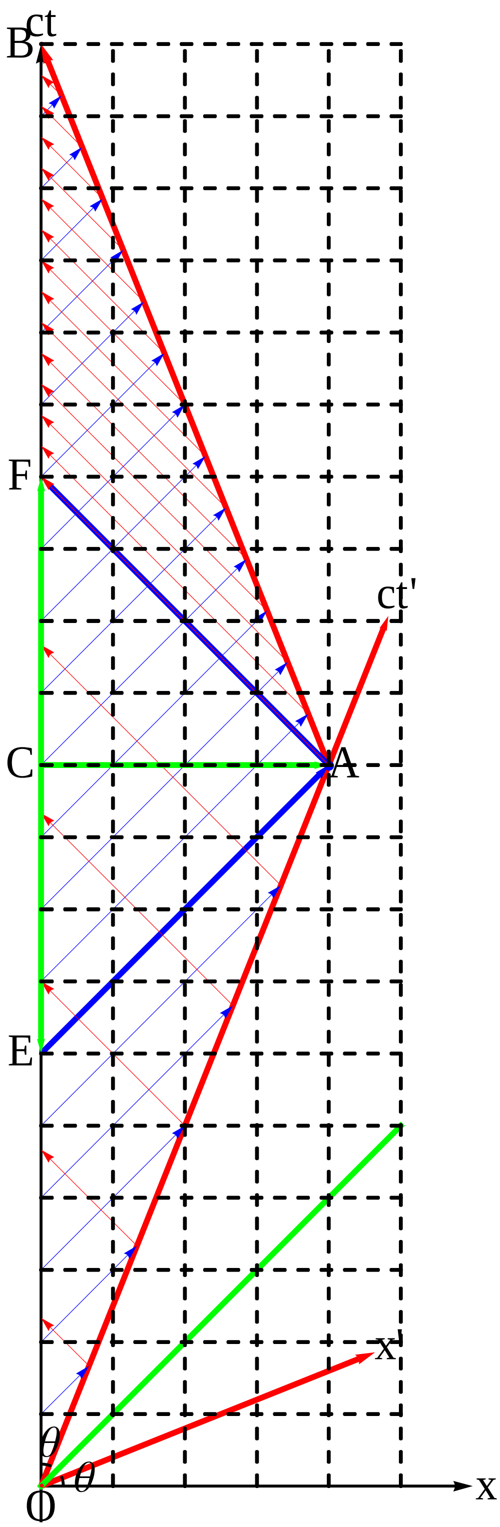

Fig.7

In the S frame,

$$(\overline{CA})_S = L\,, \qquad\qquad (\overline{OC})_S = vT$$

In the S' frame,

$$(\overline{OA})_{S'} = cT'\,, \qquad\qquad \left(\overline{CA}\right)_{S'} = (\overline{CE})_{S'} = (\overline{CF})_{S'} = cT'\sin\theta = \frac{cT'\frac{v}{c}}{\sqrt{1+\frac{v^2}{c^2}}}$$

$$L = \left(\overline{CA}\right)_S = k(\overline{CA})_{S'} = \frac{cT'\frac{v}{c}}{\sqrt{1+\frac{v^2}{c^2}}}\frac{\sqrt{1+\frac{v^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{cT'\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = cT\frac{v}{c} = vT$$

$$\left(\overline{OC}\right)_{S'} = (\overline{OA})_{S'}\cos\theta = \frac{cT'}{\sqrt{1+\frac{v^2}{c^2}}}$$

$$\left(\overline{OC}\right)_S = cT = k(\overline{OC})_{S'} = \frac{cT'}{\sqrt{1+\frac{v^2}{c^2}}}\frac{\sqrt{1+\frac{v^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{cT'}{\sqrt{1-\frac{v^2}{c^2}}}$$

or

$$cT = \frac{cT'}{\sqrt{1-\frac{v^2}{c^2}}} > cT'.$$

The time (in the S frame) of detecting one of twin arrives at the turnaround: T_F

$$
\begin{aligned}
cT_F &= \left(\overline{OF}\right)_S = k(\overline{OF})_{S'} = k[(\overline{OF})_{S'} + (\overline{CF})_{S'}] \\
&= \left(\frac{cT'}{\sqrt{1+\frac{v^2}{c^2}}} + \frac{cT'\frac{v}{c}}{\sqrt{1+\frac{v^2}{c^2}}}\right)\frac{\sqrt{1+\frac{v^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}} \\
&= \frac{cT'(1+\frac{v}{c})}{\sqrt{1-\frac{v^2}{c^2}}} \\
&= cT(1+\frac{v}{c}) = vT + cT = L(1+\frac{c}{v})
\end{aligned}
$$

or

$$T_F = \frac{L}{v} + \frac{L}{c}.$$

The time T_E in the S frame:

$$T_E = \frac{L}{v} - \frac{L}{c}.$$

((**Note**))
In the above figure, we assume that

$$\tan\theta = \frac{v}{c}.$$

$$\left(\overline{OA}\right)_{S'} = n\tau'$$

$$\left(\overline{OC}\right)_{S'} = \left(\overline{OA}\right)_{S'}\cos\theta, \qquad \left(\overline{AC}\right)_{S'} = \left(\overline{EC}\right)_{S'} = \left(\overline{CF}\right)_{S'} = \left(\overline{CA}\right)_{S'}\sin\theta$$

Then we have

$$(\overline{OE})_S = k(\overline{OE})_{S'} = k(\overline{OA})_{S'}(\cos\theta - \sin\theta)$$

$$= n\tau' \frac{1-\frac{v}{c}}{\sqrt{1+\frac{v^2}{c^2}}} \frac{\sqrt{1+\frac{v^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}} = n\tau' \frac{1-\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = n\tau' \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} = n\tau$$

Similarly we have

$$(\overline{BF})_S = (\overline{OE})_S = n\tau_1' \frac{1+\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = n\tau_1' \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} = n\tau_1$$

$$f'' = \frac{1}{\tau_1} = \frac{1}{\tau_1'}\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} = f\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

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7. Red shift and Hubble's law (special relativity)

We suppose that a source is located at the origin of the reference frame *S*. An observer moves relative to *S* at velocity *v*. So that he is at rest in S' (in Fig we use S_1 instead of S' for convenience). According to the special relativity, we obtain the Doppler effect for the light as

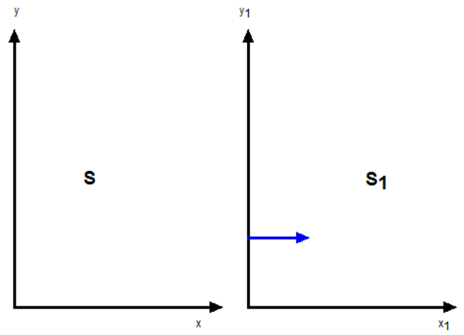
$$\lambda' = \frac{\lambda}{\gamma(1-\frac{v}{c})} = \lambda\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \approx \lambda(1+\frac{v}{c}+\frac{v^2}{2c^2}+...)$$

and

$$f' = f\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

where c is the speed of light and $\lambda f = \lambda' f' = c$. This means that a spectral line that normally has a wavelength λ is observed at a longer wavelength λ'. Note that

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

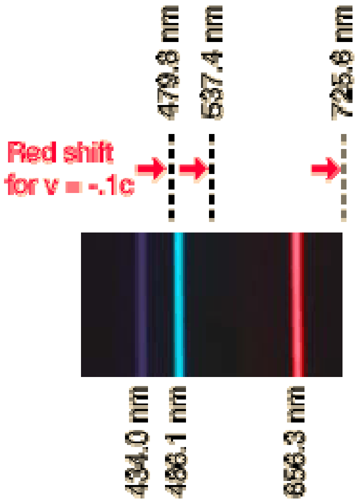


The spectral line is shifted by an amount of $\Delta\lambda = \lambda' - \lambda$. The red shift of the galaxy (usually noted by z) is given by

$$z = \frac{\lambda' - \lambda}{\lambda} \approx \frac{v}{c}$$

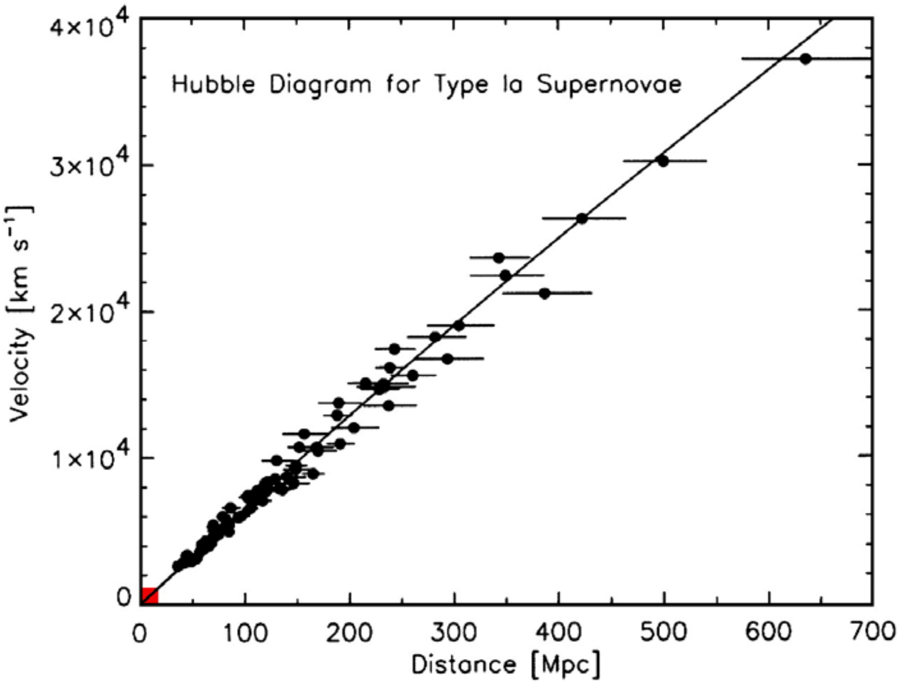
(a) The red shift

The light from distant stars and more distant galaxies is not featureless, but has distinct spectral features characteristic of the atoms in the gases around the stars. When these spectra are examined, they are found to be shifted toward the red end of the spectrum. This shift is apparently a Doppler shift and indicates that essentially all of the galaxies are moving away from us. The measured red shifts are usually stated in terms of a z parameter. The largest measured z values are associated with the quasars.



Hydrogen red-shift example

(b) Hubble's law



The relationship between the distances to galaxies and the red shift is one of the most important astronomical discoveries of the twentieth century. This relation tells us that we are living in an expanding universe. In 1929, Hubble published this discovery. According to the Hubble's law, the recessional velocity v of a galaxy is related to its distance r from the Earth by

$v = H_0 r,$

where H_0 is constant commonly called the Hubble constant and H_0 = 75 km s^{-1} Mpc^{-1}. Here Mpc is a megaparsecs (parsec, 1 pc = 3.262 ly; light year, 1 ly = 9.462 x 10^{15} m = 63,240 AU).

pc = 3.262 x 9.462 x 10^{15} m = 3.0857 x 10^{16} m.
1 Mpc = 3.0857 x 10^{22} m.= 3.0857 x 10^{19} km

Since $v = zc$ and $v = H_0 r$, then H_0r = zc. Thus the distance to a galaxy is related to its red shift by

$$r = \frac{zc}{H_0}$$

(c) Big Bang

How long ago did the Big Bang take place?

$$T_0 = \frac{r}{v} = \frac{1}{H_0}$$

where T_0 is the same for all galaxies.

T_0 = 1/(75 km s^{-1} MPc^{-1})) = $\frac{1}{75}\frac{MPc \cdot s}{km} = \frac{1}{75} \cdot \frac{3.09 \times 10^{19}}{3.156 \times 10^{7}} year = 1.3 \times 10^{10}$ years=13 billion years

Note that 1 year=365 x 24 x 60 x 60 = 3.156x10^{7} sec. The age of the solar system is 4.5 billion.years.

(d). Dicke and Peebles (1960)

Early universe had been at least as hot as the Sun center, where He is currently produced. The hot early universe must therefore have been filled with many high-energy, short-wavelength photons, which formed a radiation field with that can be given by Planck's blackbody law. The universe has expanded so much since those ancient times that all those short-wavelength photons have their wavelengths stretched by a tremendous factor. As a result, they have becomes low-energy, long-wavelength photons.

The temperature of this cosmic radiation field is now quite low, only a few degrees above 0 K.

(e). Arno Penzias and Robert Wilson

No matter where in the sky they pointed their antenna, they detected faint background noise.

They had discovered the cooled-down cosmic background radiation left over from the hot Big Bang.

(f). Cosmic Background

Left over from the hot Big Bang.

T = 2.726 K, cosmic microwave background

$$\lambda_{\max} = \frac{0.0029}{T[K]}[m] \text{ (Wien's displacement law)}$$

When T = 2.726 K, $\lambda_{\max} = 1.06[mm]$.

The spectrum of the cosmic microwave background is given by COBE (Cosmic Background Explorer).