Relativistic Mechanics

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1. Relativistic dynamics (R.P. Feynman)

We start with the work-energy theorem given by

$$\Delta E = W = \mathbf{F}\cdot d\mathbf{r}$$

or

$$\frac{\Delta E}{\Delta t} = \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} \qquad \text{or} \qquad \frac{dE}{dt} = \mathbf{F}\cdot\mathbf{u}$$

where ***F*** is the force and is given by

$$\mathbf{F} = \frac{d}{dt}[m(u)\mathbf{u}]$$

According to Einstein, E is described by

$$E = m(u)c^2$$

Then we get

$$\frac{d[m(u)c^2]}{dt} = \mathbf{u}\cdot\frac{d}{dt}[m(u)\mathbf{u}]$$

or

$$m(u)c^2\frac{dm(u)}{dt} = m(u)\mathbf{u}\cdot\frac{d}{dt}[m(u)\mathbf{u}]$$

or

$$\frac{c^2}{2}\frac{d}{dt}[m(u)]^2 = \frac{1}{2}\frac{d}{dt}[m(u)u]^2 .$$

This differential equation can be solved as

$$[m(u)c]^2 = [m(u)u]^2 + C$$

where C is a constant. When $u = 0$, we have

$$m(u=0)^2 c^2 = C$$

or

$$[m(u)]^2 (c^2 - u^2) = m(u=0)^2 c^2$$

Then we have

$$m(u) = \frac{m(0)}{\sqrt{1 - \frac{u^2}{c^2}}}$$

The energy and momentum are obtained as

$$E = m(u)c^2 = \frac{m(0)c^2}{\sqrt{1-\frac{u^2}{c^2}}}, \qquad \mathbf{p} = m(v)\mathbf{u} = \frac{m(0)\mathbf{u}}{\sqrt{1-\frac{u^2}{c^2}}}$$

or

$$\frac{E^2}{c^2} = [m(0)]^2 c^2 + p^2$$

For simplicity we use $m(0) = m_0$.

2. The derivation of *E*:

$$E = \int F dx = \int \frac{d}{dt}(mu) u dt = \int u d(mu)$$
$$= \int u d\left(\frac{m_0 u}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$
$$= m_0 \int u \left(1 - \frac{u^2}{c^2}\right)^{-3/2} du$$
$$= \frac{m_0 c^2}{\sqrt{1-\frac{u^2}{c^2}}}$$

In the limit $u \to 0$,

$$E = m_0c^2 + \frac{1}{2}m_0u^2 + \frac{3}{8}m_0\frac{u^4}{c^2} + ...$$

Note that

$$K = E - m_0c^2 \approx \frac{1}{2}m_0u^2$$

The relativistic kinetic energy is defined as

$$K = E - m_0c^2 = m_0c^2(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1).$$

3. Energy vs momentum relation

$$\frac{E^2}{c^2} = m_0{}^2c^2 + p^2$$

(a) When $p = 0$,

$$\frac{E}{c} = m_0c\,.$$

Point B:

$E/c = m_0c$ and $p = 0$.

Point A:

$(p, E/c)$

The green straight line: the energy dispersion of the light

$$\frac{E}{c} = p$$

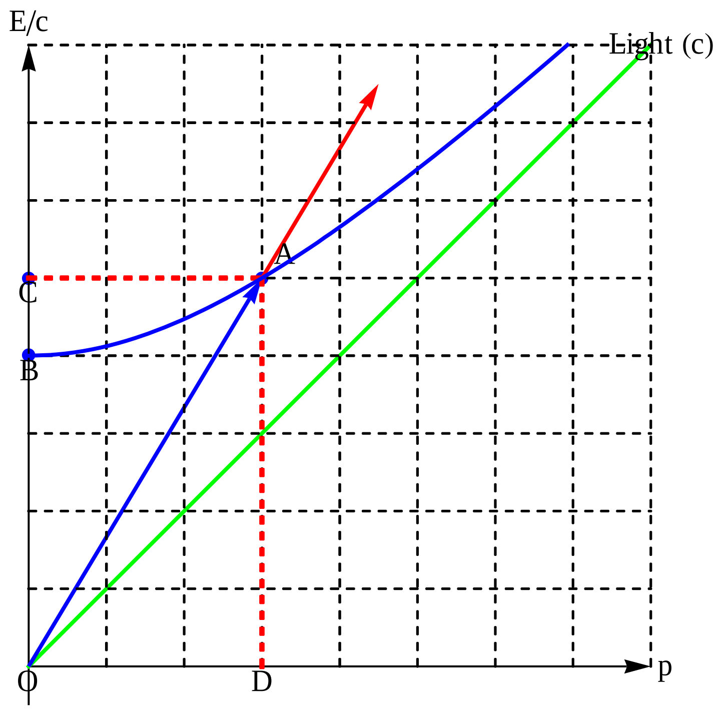

Fig.1 **Energy dispersion relation**

Momentum and energy conservation

$$P_{tot} = p_1 + p_2 = 0$$
$$E_{tot} = E(P_{tot}) = E(p_1) + E(p_2)$$

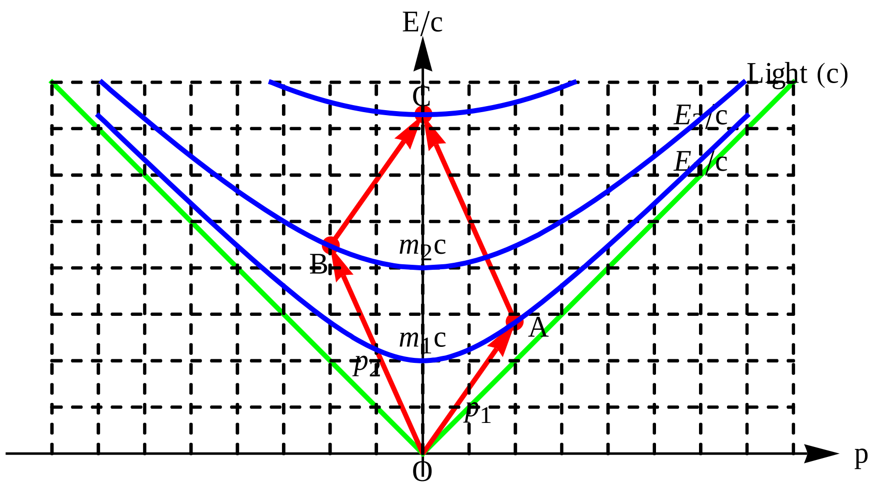

Fig.2

4. Addition of velocity in the Minkowski spacetime diagram

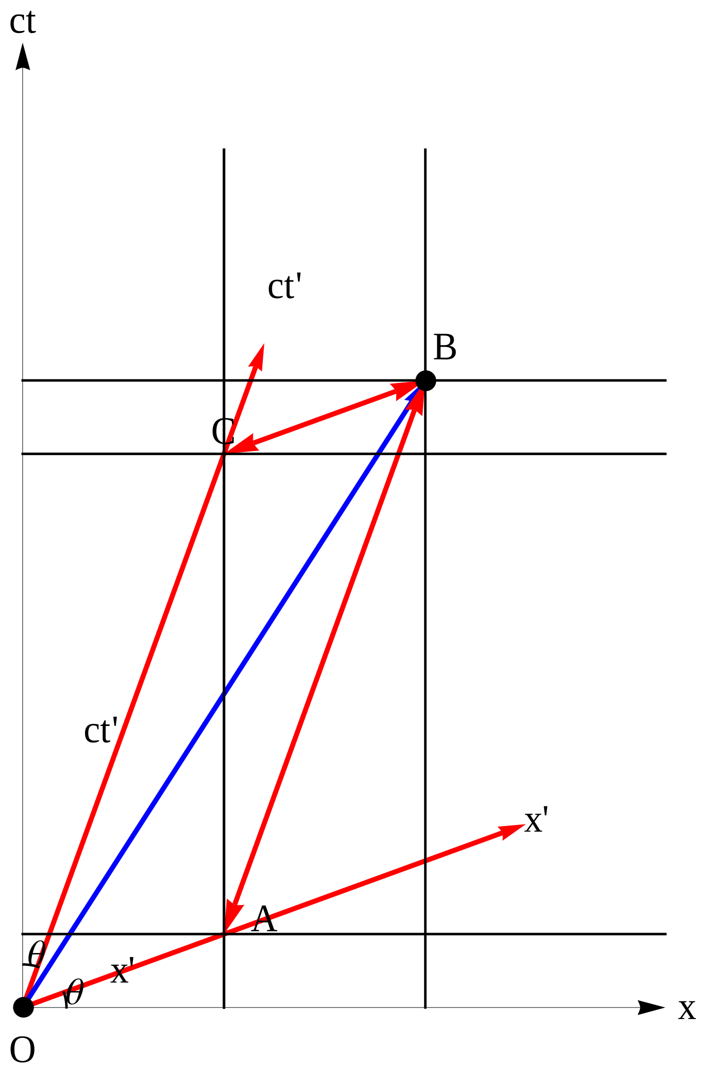

Fig.3 The derivation of the formula for adding velocities. The path OB corresponds to the velocity ($x'/ct' = u'/c$) in the S' frame. $\tan\theta = \dfrac{v}{c}$.

$$\left(\overrightarrow{OB}\right)_{S'} = (x'\cos\theta + ct'\sin\theta, x'\sin\theta + ct'\cos\theta)\ ,$$

Then the velocity in the S frame is obtained as

$$u_S = u = c\frac{k(x'\cos\theta + ct'\sin\theta)}{k(ct'\cos\theta + x'\sin\theta)} = c\frac{\dfrac{x'}{t'} + c\tan\theta}{c + \dfrac{x'}{t'}\tan\theta} = c\frac{u' + c\dfrac{v}{c}}{c + u'\dfrac{v}{c}} = \frac{u' + v}{1 + \dfrac{u'v}{c^2}}$$

where k is the scaling factor.

How about the velocity *u*' in the S' frame when the velocity *u* in the S frame is given? Solving the above equation with respect to a variable *u*', we have

$$u_{S'} = u' = \frac{u - v}{1 - \frac{uv}{c^2}}.$$

((**Note**))

The inverse transformation can be done without specific evaluation, by exchanging the primed and unprimed quantities and by reversing the sign of *v* wherever it occurs. This is often a labor-saving devices.

5. Example-1

The particle moves with a velocity of $v = c/2$, and the particle emits a proton forward with $u' = c/2$ in the S' frame. What velocity does the proton move in the rest frame (S frame)?

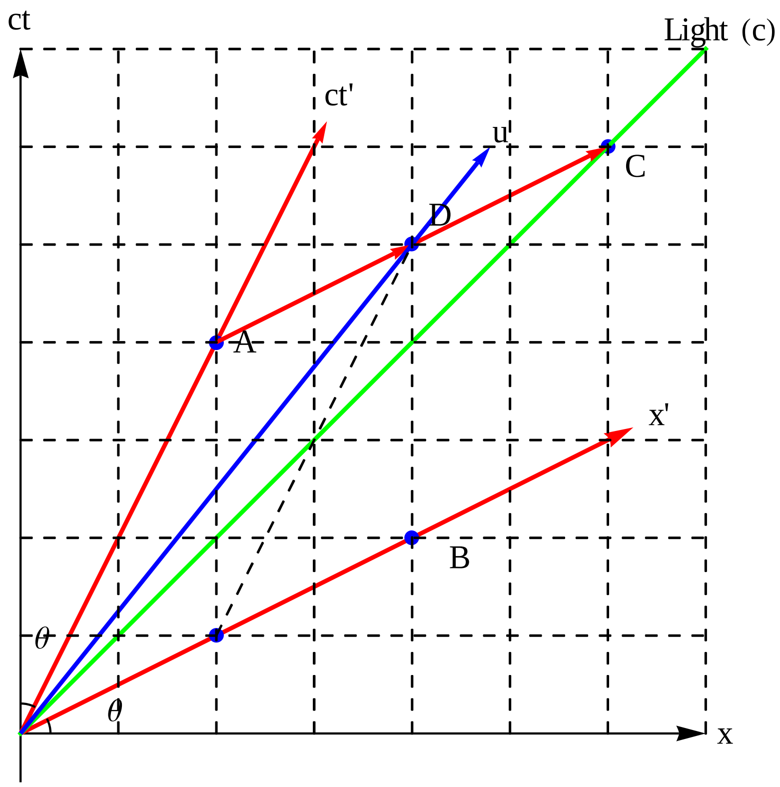

Fig.4 Adding velocities in the space-time diagram. The vector $\overrightarrow{OD}$ is the vector *u*' ($=\Delta x'/\Delta ct' = 1/2$) in the S' frame. In the S frame, this vector is described by *u* ($=\Delta x/\Delta ct = 4/5$).

The velocity of the proton is $4c/5$, since the coordinate of the point D is $x = 4$ units and $ct = 5$ units.

In the S' frame, the motion of the proton is described by

$$\frac{\Delta x'}{\Delta(ct')} = \frac{c/2}{c} = \frac{1}{2}$$

The blue straight line passing through the point D corresponds to the motion of the proton. The velocity of the point D in the S frame is given by

$$\frac{\Delta x}{\Delta(ct)} = \frac{4}{5}$$

In other word, the proton moves with the velocity of 4*c*/5 in the S frame.

(a) When $u' = v = c/2$, we get

$$u_S = \frac{u+v}{1+\frac{uv}{c^2}} = \frac{4}{5}c$$

(b) When $u' = c$,

$$u_S = \frac{u+v}{1+\frac{uv}{c^2}} = \frac{c+v}{1+\frac{cv}{c^2}} = c$$

6. Example-2

Two particles (A and B) are moving in opposite directions as observed from a system S; at $u_A = 0.9c$ and $u_B = -0.9c$. What is the velocity of the particle B with respect to the particle A (S' frame)? Note that the S' frame is the frame where the particle A is at rest; $v = u_A = 0.9c$.

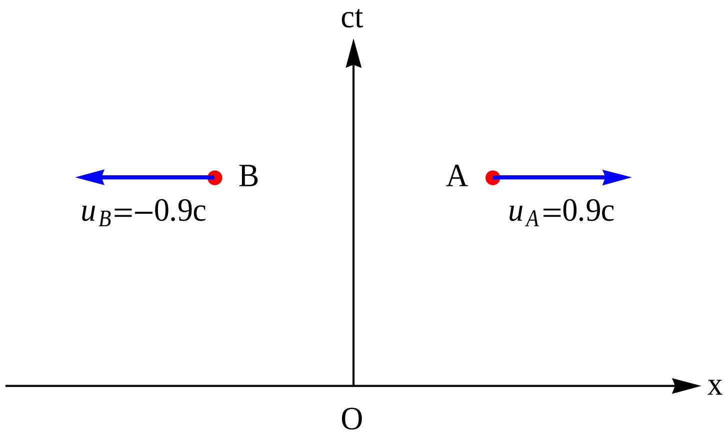

Fig.5 Two particles (A, B) moving with velocity u_A (= 0.9 c) and the velocity u_B (= -0.9 c) in the S frame.

For u_B= -0.9 c and $v = u_A = 0.9\ c$,

$$u_B' = \frac{u_B - v}{1 - \frac{u_B v}{c^2}} = \frac{-0.9c - 0.9c}{1 + 0.9^2} = -\frac{1.8c}{1.81}.$$

So the relative to the particle A, the particle B travels in the (-x') direction with speed 1.8 c/1.81.

((**Example**))

Three galaxies are aligned along an axis in order A, B, C. An observer in galaxy B is in the middle and observes that galaxies A and C are moving in opposite directions away from him, both with speeds 0.52c. What are the speeds of galaxies B and C as observed by someone in galaxy A?

In the K system;

$$u_x^A = -0.52c, \qquad u_x^B = 0, \qquad u_x^C = 0.52c$$

In the K' system

$$u_x^A{}' = \frac{u_x^A - v}{1 - \frac{v}{c^2}u_x^A} = 0 \qquad \text{or} \qquad v = u_x^A = -0.52c$$

$$u_x^B{}' = \frac{u_x^B - v}{1 - \frac{v}{c^2}u_x^B} = -v = 0.52c$$

$$u_x{}^C{}' = \frac{u_x{}^C - v}{1 - \frac{v}{c^2} u_x{}^C} = \frac{0.52c - (-0.52c)}{1 - \frac{(-0.52c)}{c^2} 0.52c} = \frac{1.04c}{1 + 0.27} = 0.8189c$$

((Note))
Without mistake, you may calculate the velocity of A, B, and C with respect to the observer at A.
(i) First calculate the relative velocity without taking into account of the special relativity.

For A; u_A - u_A
For B; u_B - u_A = u_B - v
For C: u_C - $\underline{u}_A$ = u_C - v

(ii) Next we use the formula: v = u_A = -0.52c.

$$\frac{u_A - v}{1 + \frac{1}{c^2} u_A(-v)} = 0$$

$$\frac{u_B - v}{1 + \frac{1}{c^2} u_B(-v)} = 0.52c$$

$$\frac{u_c - v}{1 + \frac{1}{c^2} u_c(-v)} = 0.8189$$

7. Addition law of velocity derived from the Lorentz transformation

$$x_1' = \gamma(x_1 - vt) \qquad x_1 = \gamma(x_1' + vt')$$
$$x_2' = x_2 \qquad x_2 = x_2'$$
$$x_3' = x_3 \qquad x_3 = x_3'$$
$$t' = \gamma(t - \frac{\beta}{c} x_1) \qquad t = \gamma(t' + \frac{\beta}{c} x_1')$$

Suppose that an object has velocity components as measured in *S'* and *S*.

$$u_1' = \frac{dx_1'}{dt'} = \frac{u_1 - v}{1 - \frac{\beta}{c} u_1} \qquad u_1 = \frac{dx_1}{dt} = \frac{u_1' + v}{1 + \frac{\beta}{c} u_1'}$$

$$u_2' = \frac{dx_2'}{dt'} = \frac{1}{\gamma} \frac{u_2}{1 - \frac{\beta}{c} u_1} \qquad u_2 = \frac{dx_2}{dt} = \frac{1}{\gamma} \frac{u_2'}{1 + \frac{\beta}{c} u_1'}$$

$$u_3' = \frac{dx_3'}{dt'} = \frac{1}{\gamma} \frac{u_3}{1 - \frac{\beta}{c} u_1} \qquad u_3 = \frac{dx_3}{dt} = \frac{1}{\gamma} \frac{u_3'}{1 + \frac{\beta}{c} u_1'}$$

8. The conservation of momentum and mass of a moving object

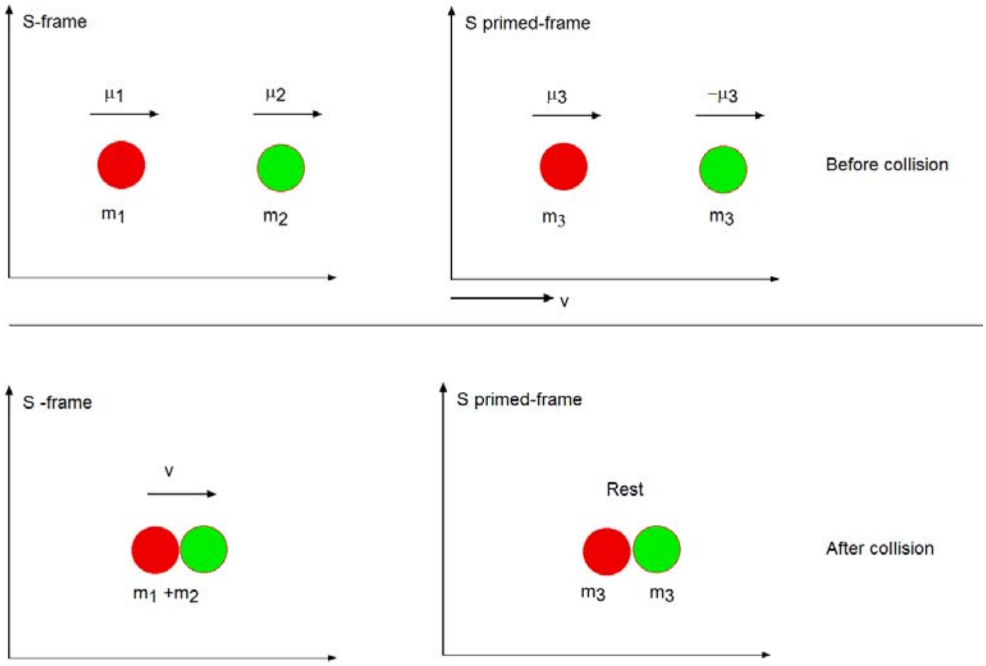

Fig.6 Momentum conservation before and after collision. The S and S' frame.

The example is the inelastic collision of two balls of equal mass in the S' frame. They roll together with velocities μ_3 and $-\mu_3$ in this system and stick together. We describe this collision in the S and S' frames. The momentum conservation is

$$m_3\mu_3 + m_3(-\mu_3) = 0 \qquad \text{in the S' frame}$$

$$m_1\mu_1 + m_1\mu_2 = (m_1 + m_2)v \qquad \text{in the S frame}$$

where the S' frame moves along the positive *x* direction at the velocity *v*. The velocities μ_1 and μ_2 in the S frame are

$$\mu_1 = \frac{\mu_3 + v}{1 + \frac{\mu_3 v}{c^2}}, \qquad \mu_2 = \frac{-\mu_3 + v}{1 - \frac{\mu_3 v}{c^2}}.$$

From the momentum conservation law, we get

$$m_1(\mu_1 - v) = m_2(v - \mu_2)$$

or

$$\frac{m_1}{m_2} = \frac{v - \mu_2}{\mu_1 - v} = \frac{1 + \frac{\mu_3 v}{c^2}}{1 - \frac{\mu_3 v}{c^2}}.$$

since

$$\sqrt{1 - \frac{\mu_2{}^2}{c^2}} = \frac{\sqrt{(c-v)(c+v)(c-\mu_3)(c+\mu_3)}}{c^2(1 - \frac{\mu_3 v}{c^2})},$$

and

$$\sqrt{1 - \frac{\mu_1{}^2}{c^2}} = \frac{\sqrt{(c-v)(c+v)(c-\mu_3)(c+\mu_3)}}{c^2(1 + \frac{\mu_3 v}{c^2})}.$$

we get

$$m_1\sqrt{1 - \frac{\mu_1{}^2}{c^2}} = m_2\sqrt{1 - \frac{\mu_2{}^2}{c^2}} = m_0$$

when the ball is at rest, its mass is m_0. The mass in motion must be

$$m(u) = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}$$

where u is the velocity of the system.

((**Mathematica**))

```
eq1 = {μ1 → (μ3 + v)/(1 + μ3 v/c^2), μ2 → (-μ3 + v)/(1 - μ3 v/c^2)};

eq2 = 1 - μ2^2/c^2 /. eq1 // Factor
```

$$\frac{(c-v)\,(c+v)\,(c-\mu 3)\,(c+\mu 3)}{\left(c^2 - v\,\mu 3\right)^2}$$

```
eq3 = 1 - μ1^2/c^2 /. eq1 // Factor
```

$$\frac{(c-v)\,(c+v)\,(c-\mu 3)\,(c+\mu 3)}{\left(c^2 + v\,\mu 3\right)^2}$$

```
seq1 = Sqrt[eq2/eq3] // Simplify[#, {c^2 + v μ3 > 0, c^2 - v μ3 > 0}] &
```

$$\frac{c^2 + v\,\mu 3}{c^2 - v\,\mu 3}$$

9. Lorentz transformation for *E* and *p*

u is the velocity in the S frame, while u' is the velocity in the S' frame.

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

$$E' = \frac{m_0 c^2}{\sqrt{1 - \frac{u'^2}{c^2}}}, \qquad p' = \frac{m_0 u'}{\sqrt{1 - \frac{u'^2}{c^2}}}$$

Noting that

$$\sqrt{1 - \frac{u'^2}{c^2}} = \frac{\sqrt{(1 - \frac{u^2}{c^2})(1 - \frac{v^2}{c^2})}}{1 - \frac{uv}{c^2}},$$

we get

$$E' = \frac{m_0 c^2}{\sqrt{1 - \frac{u'^2}{c^2}}} = \frac{m_0 c^2 (1 - \frac{uv}{c^2})}{\sqrt{1 - \frac{u^2}{c^2}}\sqrt{1 - \frac{v^2}{c^2}}} = \frac{E - pv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

or

$$\frac{E'}{c} = \frac{\frac{E}{c} - p\frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

which we recognize as being exactly of the same form as

$$ct' = \frac{ct - \frac{v}{c}x}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Similarly,

$$p' = \frac{m_0 u'}{\sqrt{1 - \frac{u'^2}{c^2}}} = \frac{m_0 (1 - \frac{uv}{c^2}) \frac{u - v}{1 - \frac{uv}{c^2}}}{\sqrt{1 - \frac{u^2}{c^2}}\sqrt{1 - \frac{v^2}{c^2}}}$$

$$= \frac{m_0 (u - v)}{\sqrt{1 - \frac{u^2}{c^2}}\sqrt{1 - \frac{v^2}{c^2}}} = \frac{p - E\frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

or

$$p' = \frac{p - \frac{E}{c}\frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

which we recognize as being exactly of the same form as

$$x' = \frac{x - ct\frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Thus the transformations for the new energy (E'/c) and momentum (p') in terms of the old energy (E/c) and momentum (p) are exactly the same as the transformation for ct' in terms of x and ct, and x' in terms of x and ct.

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10. Four dimensional vector

Here we use the proper time $d\tau$, measured in the moving frame with a velocity **u** (the velocity of the particle)

$$d\tau = \sqrt{1 - \frac{u^2}{c^2}}dt \, .$$

In this frame the particle is at rest.

$$p = \frac{m_0 u}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}\frac{dx}{dt} = m_0 \frac{dx}{d\tau}$$

$$\frac{E}{c} = \frac{m_0 c}{\sqrt{1 - \frac{u^2}{c^2}}} = m_0 c\frac{dt}{d\tau} = m_0 \frac{d(ct)}{d\tau}$$

In other words, the four momentum is defined by

$$p_i = m_0 \frac{dx_i}{d\tau}$$

In other words, the four momentum is defined by

$$p_\mu = m_0 \frac{dx_i}{d\tau} = (\mathbf{p}, i\frac{E}{c})$$

Thus p_i is the four dimensional vector just lime x_i, since the proper time is relativistically invariant.

$$x_\mu = (\mathbf{r}, ict)$$

The co-ordinate vector

$$p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}$$

Under a Lorentz transformation, we have

$$p_\mu' = a_{\mu\nu} p_\nu$$

$$a_{\mu\nu} a_{\mu\lambda} = \delta_{\nu\lambda}$$

Then we have

$$p'_\mu \; p'_\mu = a_{\mu\nu} p_\nu a_{\mu\lambda} p_\lambda = a_{\mu\nu} a_{\mu\lambda} p_\nu p_\lambda = \delta_{\mu\lambda} p_\nu p_\lambda = p_\mu p_\mu$$

We note that

$$p_\mu p_\mu = \mathbf{p}^2 - \frac{E^2}{c^2} = \frac{{m_0}^2 u^2}{1-\frac{u^2}{c^2}} - \frac{{m_0}^2 c^2}{1-\frac{u^2}{c^2}} = -\frac{{m_0}^2 c^2}{1-\frac{u^2}{c^2}}(1-\frac{u^2}{c^2}) = -{m_0}^2 c^2$$

In other words, we have

$$p'_\mu \; p'_\mu = p_\mu p_\mu = --{m_0}^2 c^2$$

$p_\mu p_\mu$ is invariant and it must have the same value in every frame.

10. Zero rest mass

A case of special interest arises when the rest mass m_0 is zero.

$$\frac{E^2}{c^2} - \mathbf{p}^2 = {m_0}^2 c^4 = 0$$

or

$$E = cp$$

The interesting point is that the theory of relativity implies that a particle of zero rest mass have a nonzero energy and momentum. If its velocity is u, the energy and momentum are

$$p = \frac{m_0 u}{\sqrt{1 - \frac{u^2}{c^2}}}, \qquad \frac{E}{c} = \frac{m_0 c}{\sqrt{1 - \frac{u^2}{c^2}}}.$$

if u/c is fixed at a value less than unity, E and p approaches zero as m_0 approaches zero. But if we let u/c approach unity, while m_0 approaches zero, in such a way that

$$\frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}} = R$$

where R is constant. Then we obtain

$$E = Rc^2, \qquad p = Rc, \qquad E^2 = c^2 p^2$$

Therefore, a body can have nonzero energy and momentum , even thou its rest mass is zero, if and only if it is moving at the speed of light ($u = c$) [Bohm].

11. Doppler effect

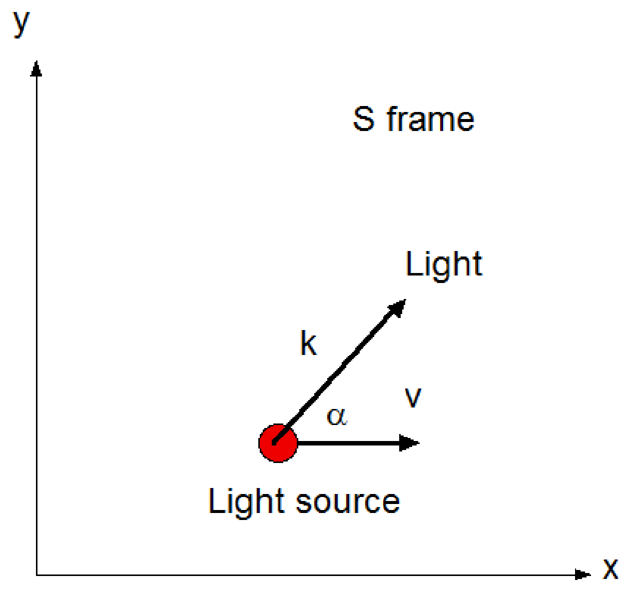


Fig.7 Derivation of the Doppler effect

The light source moves at the velocity v along the x axis. The light is emitted from the light source. The angle between the direction of the light and the x axis is α. We assume that the angular frequency for the observer in the S frame ia w and that the angular frequency for the light source in the S' frame is $\omega' = \omega_0$. Then we get the relation

$$\frac{E'}{c} = \frac{\frac{E}{c} - p\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$$

where $E = \hbar\omega$, $E' = \hbar\omega' = \hbar\omega_0$, and $p_x = \hbar k\cos\alpha$.

$$\frac{\hbar\omega'}{c} = \frac{\frac{\hbar\omega}{c} - \hbar k\cos\alpha\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\frac{\hbar\omega}{c} - \hbar\omega\cos\alpha\frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

or

$$\omega' = \frac{\omega(1-\frac{v}{c}\cos\alpha)}{\sqrt{1-\frac{v^2}{c^2}}}$$

or

$$\omega = \omega'\frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}\cos\alpha} = \omega_0\frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}\cos\alpha}$$

When $\alpha = \pi$, we have

$$\omega = \omega_0\frac{\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{v}{c}} = \omega_0\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} < \omega_0$$

12. Doppler effect (Minkowski spacetime diagram)

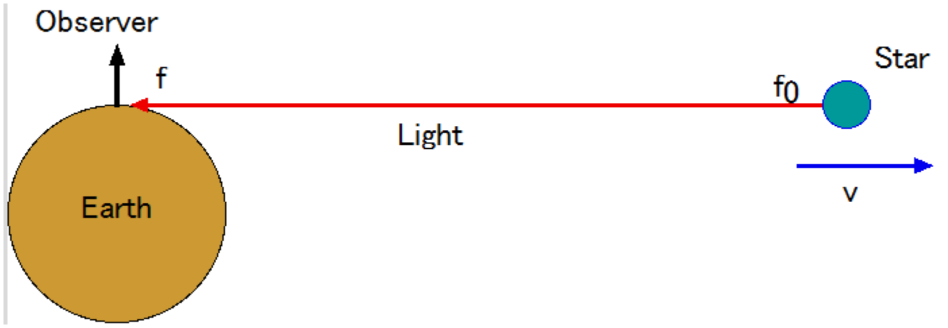


Fig.8

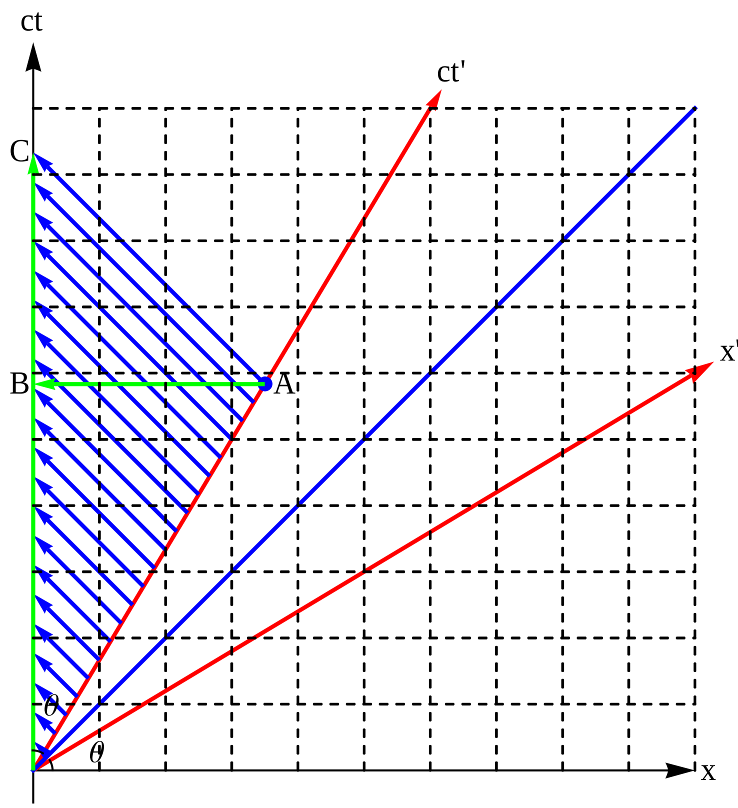


Fig.9 **Doppler effect-I**

We consider a moving light source that flashes with frequency f_0. The observer at rest in the S frame receives the light signal with a different frequency f (this is called the Doppler effect of light). First we calculate the distances in the S' frame;

$$\left(\overline{OC}\right)_{S'} = (\overline{OB} + \overline{BC})_{S'} = (\overline{OA})_{S'}\cos\theta + (\overline{AB})_{S'}\tan\frac{\pi}{4}$$

$$= ct'\cos\theta + ct'\sin\theta = ct'\frac{1+\frac{v}{c}}{\sqrt{1+\frac{v^2}{c^2}}}$$

Using the scaling factor

$$\left(\overline{OC}\right)_S = \sqrt{\frac{1+\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}}\left(\overline{OC}\right)_{S'}$$

The frequency of the light observed at the S frame is related to that of the light emitted in the S' frame is

$$\left(\overline{OC}\right)_S = ct = \sqrt{\frac{1+\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}}\left(\overline{OC}\right)_{S'} = \sqrt{\frac{1+\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}}ct'\frac{1+\frac{v}{c}}{\sqrt{1+\frac{v^2}{c^2}}} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}ct'$$

((Formula of the Doppler effect))

$$\frac{f}{f_0} = \frac{ct'}{ct} = \frac{1}{\kappa} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} < 1$$

$$t = \kappa t'$$

with

$$\kappa = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} > 1$$

13. Doppler effect-II

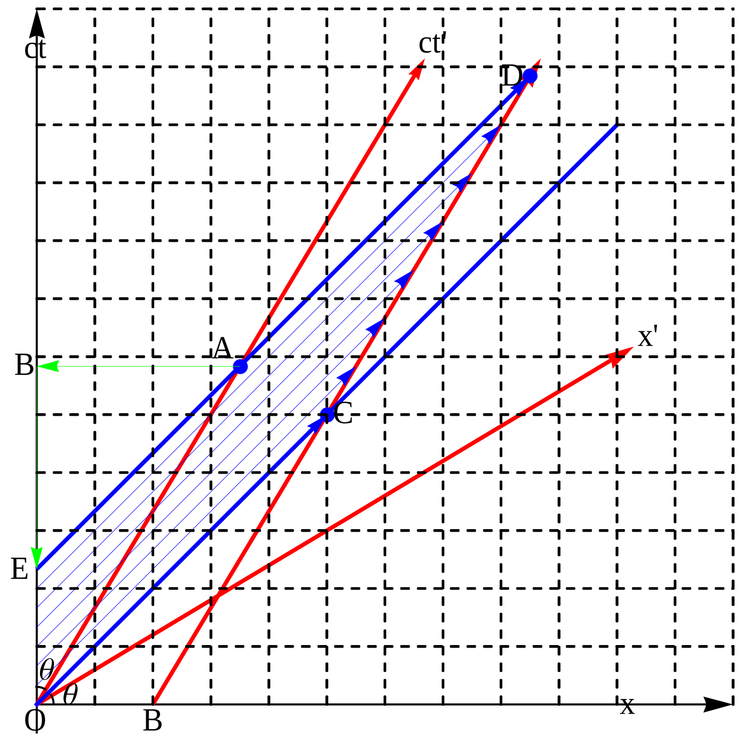


Fig.10 Doppler effect-II

We suppose that a source is located at the origin of the reference frame S, and that an observer moves relative to the frame S at the velocity v, so that the observer is at rest in the frame S'. Each emitted pulse travels with speed c. Suppose a first pulse is sent out at $t = 0$ when the observer is at the position $x = x_0$, and suppose the $(n+1)$-th pulse is sent out at $t = n\tau$. This will have covered n periods of vibration, so that the measured frequency of the source in S is $\nu = \dfrac{1}{\tau}$.

In the above figure, we assume that

$$\tan\theta = \frac{v}{c}.$$

$$\left(\overline{OA}\right)_{S'} = \left(\overline{OD}\right)_{S'} = n\tau'$$

$$\left(\overline{OB}\right)_{S'} = \left(\overline{OA}\right)_{S'}\cos\theta, \qquad \left(\overline{AB}\right)_{S'} = \left(\overline{EB}\right)_{S'} = \left(\overline{OA}\right)_{S'}\sin\theta$$

Then we have

$$\left(\overline{OE}\right)_S = k\left(\overline{OE}\right)_{S'} = k\left(\overline{OA}\right)_{S'}(\cos\theta - \sin\theta)$$

$$= n\tau' \frac{1-\frac{v}{c}}{\sqrt{1+\frac{v^2}{c^2}}} \frac{\sqrt{1+\frac{v^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}} = n\tau' \frac{1-\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = n\tau' \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} = n\tau$$

Then we have

$$\tau = \tau' \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

or

$$f' = \frac{1}{\tau'} = \frac{1}{\tau}\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} = f\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} < f$$

The wave lengths in the S and S' are defined by

$\lambda = c\tau$, and $\lambda' = c\tau'$

Then the ratio of the wavelength is given by

$$\frac{\lambda'}{\lambda} = \frac{c\tau'}{c\tau} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} > 1 \qquad \text{(red shift)}$$

14. Twin paradox

There are twin: Mary (the Moving twin) and Frank (the Fixed twin). We imagine that each person sends equally spaced time signals of their own proper times to the other, The cumulative counts of time signals for the whole trip are then compared. Suppose that each person is transmitting *f* pulses per unit time. As Mary travels away from Frank, each observer will receive the other's signal at the reduced rate

$$f' = f\sqrt{\frac{1-\beta}{1+\beta}}$$

But for how long? Here is the asymmetry. As soon as Mary reverses, she begins to receive signals from Frank at the enhanced rate

$$f'' = f\sqrt{\frac{1+\beta}{1-\beta}}$$

With Frank it is quite different. The last signal sent by Mary before she reverses does not reach Frank until a time *L*/*c* later. Thus for much more than one-half the total time Frank is recording the Mary's signals at the lower rate *f*'. Only in the latter stages does Frank receive pulses at the higher rate *f*''.

Note that each observer receives as many signals as the other sends between start and finish of trip. Frank is able to infer from his observations that it took place at the midmoment of the journey time as measured by Mary, since equal numbers of signal received by Frank at the two different rates f' and f'' (French).

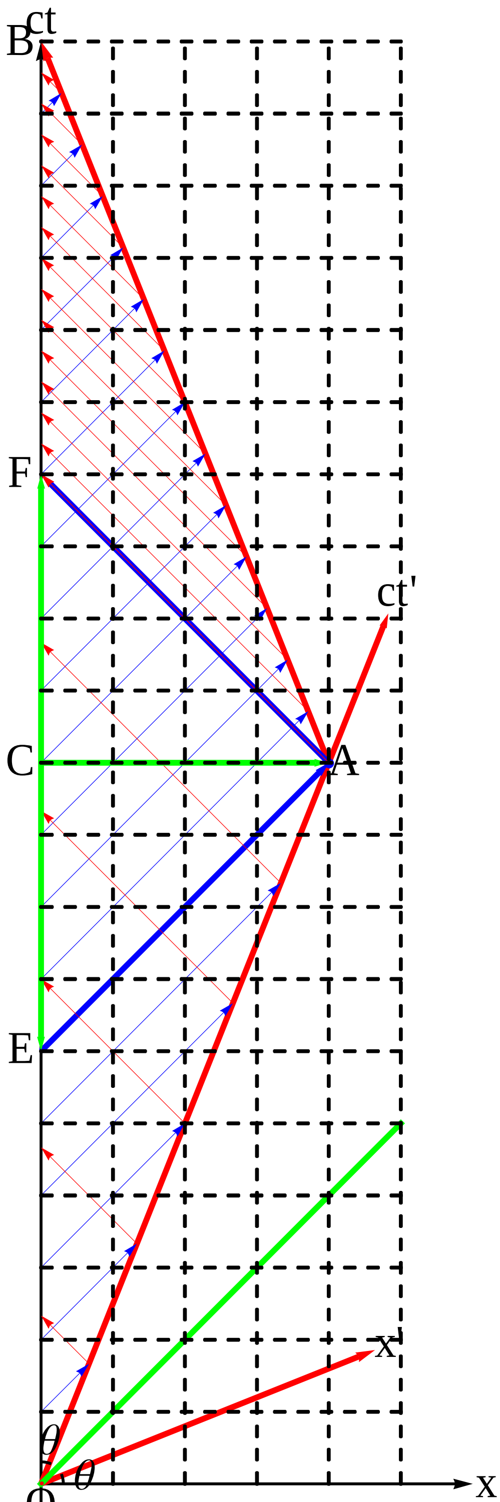


Fig.7. **Twin paradox**

In the S frame,

$$(\overline{CA})_S = L\,, \qquad\qquad (\overline{OC})_S = vT$$

In the S' frame,

$$(\overline{OA})_{S'} = cT'\,, \qquad\qquad \left(\overline{CA}\right)_{S'} = (\overline{CE})_{S'} = (\overline{CF})_{S'} = cT'\sin\theta = \frac{cT'\frac{v}{c}}{\sqrt{1+\frac{v^2}{c^2}}}$$

$$L = \left(\overline{CA}\right)_S = k(\overline{CA})_{S'} = \frac{cT'\frac{v}{c}}{\sqrt{1+\frac{v^2}{c^2}}}\frac{\sqrt{1+\frac{v^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{cT'\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = cT\frac{v}{c} = vT$$

$$\left(\overline{OC}\right)_{S'} = (\overline{OA})_{S'}\cos\theta = \frac{cT'}{\sqrt{1+\frac{v^2}{c^2}}}$$

$$\left(\overline{OC}\right)_S = cT = k(\overline{OC})_{S'} = \frac{cT'}{\sqrt{1+\frac{v^2}{c^2}}}\frac{\sqrt{1+\frac{v^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{cT'}{\sqrt{1-\frac{v^2}{c^2}}}$$

or

$$cT = \frac{cT'}{\sqrt{1-\frac{v^2}{c^2}}} > cT'.$$

The time (in the S frame) of detecting one of twin arrives at the turnaround: T_F

$$cT_F = \left(\overline{OF}\right)_S = k(\overline{OF})_{S'} = k[(\overline{OF})_{S'} + (\overline{CF})_{S'}]$$

$$= \left(\frac{cT'}{\sqrt{1+\frac{v^2}{c^2}}} + \frac{cT'\frac{v}{c}}{\sqrt{1+\frac{v^2}{c^2}}}\right)\frac{\sqrt{1+\frac{v^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{cT'(1+\frac{v}{c})}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= cT(1+\frac{v}{c}) = vT + cT = L(1+\frac{c}{v})$$

or

$$T_F = \frac{L}{v} + \frac{L}{c}.$$

The time T_E in the S frame:

$$T_E = \frac{L}{v} - \frac{L}{c}.$$

((Note))

In the above figure, we assume that

$$\tan\theta = \frac{v}{c}.$$

$$\left(\overline{OA}\right)_{S'} = n\tau'$$

$$\left(\overline{OC}\right)_{S'} = \left(\overline{OA}\right)_{S'}\cos\theta, \qquad \left(\overline{AC}\right)_{S'} = \left(\overline{EC}\right)_{S'} = \left(\overline{CF}\right)_{S'} = \left(\overline{CA}\right)_{S'}\sin\theta$$

Then we have

$$\left(\overline{OE}\right)_S = k\left(\overline{OE}\right)_{S'} = k\left(\overline{OA}\right)_{S'}(\cos\theta - \sin\theta)$$

$$= n\tau'\frac{1-\frac{v}{c}}{\sqrt{1+\frac{v^2}{c^2}}}\frac{\sqrt{1+\frac{v^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}} = n\tau'\frac{1-\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = n\tau'\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} = n\tau$$

Similarly we have

$$\left(\overline{BF}\right)_S = \left(\overline{OE}\right)_S = n\tau_1' \frac{1+\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = n\tau_1' \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} = n\tau_1$$

$$f'' = \frac{1}{\tau_1} = \frac{1}{\tau_1'}\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} = f\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

APPENDIX
From Panofski-Philips (Classical Electricity and Magnetism)

Suppose that $p_x = 0$ in the S frame;

$$\frac{E}{c} = m_0 c\,.$$

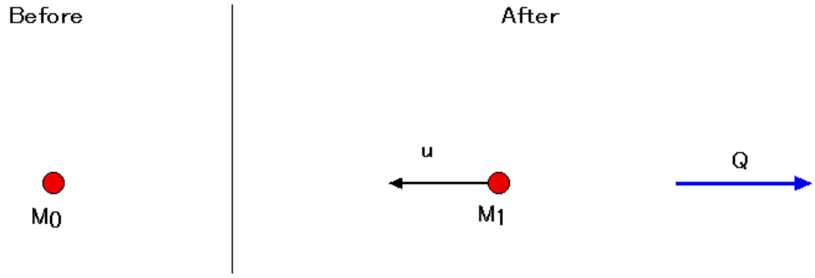
Then in the S' frame, we observe a momentum

$$p_x' = \frac{p_x - \frac{E}{c}\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{-\frac{E}{c}\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = -\frac{vE'}{c^2}$$

where

$$\frac{E'}{c} = \frac{\frac{E}{c} - p_x\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\frac{E}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$$

and



Energy conservation

$$E_1 + Q = M_0 c^2, \qquad \text{or} \qquad E_1 = -Q + M_0 c^2$$

Momentum conservation

$$-p_1 + \frac{Q}{c} = 0, \qquad \text{or} \qquad p_1 = \frac{Q}{c}$$

we use the relation given by

$$E_1^{\ 2} - p_1^{\ 2} c^2 = (-Q + M_0 c^2)^2 - c^2 \frac{Q^2}{c^2} M_1 c^4 = M_1^{\ 2} c^4$$

We define the energy difference between the initial and final atoms, taken at rest,

$$\Delta E = M_0 c^2 - M_1 c^2$$

Then we get an equation for Q as

$$-(\Delta E)^2 + 2c^2 M_0 (-Q + \Delta E) = 0 .$$

leading to

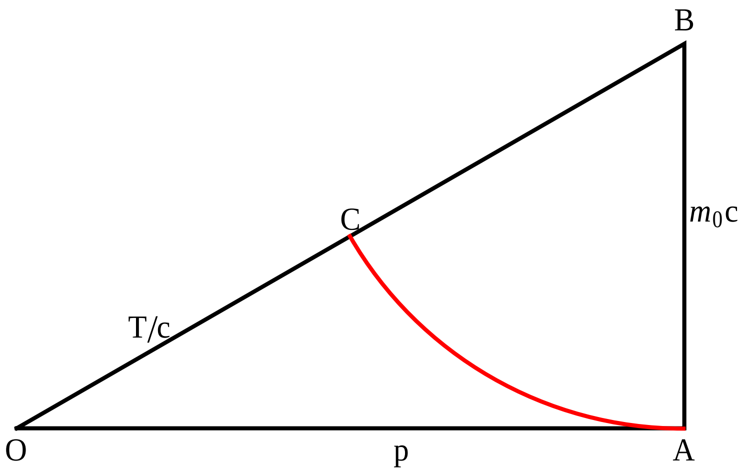
$$Q = \Delta E \left(1 - \frac{\Delta E}{2M_0 c^2}\right)$$

((Example))

For heavy atom, the recoil is a small effect.
We consider ^{198}Hg (M_0 = 198 a.m.u .= 3.28 x 10^{-25} kg), which emits photons with an energy Q = 412 keV = 6.601 x 10^{-14} J. We calculate

$$\frac{\Delta E}{2M_0 c^2} = 1.12 \text{ x } 10^{-6}$$

This is small, but not really negligible in our quantum world.



$\overline{OA} = p$, $\overline{AB} = \overline{BC} = m_0 c$,

$\overline{OB} = E / c$ $\overline{OC} = T / c$

$$\frac{E^2}{c^2} = p^2 + m_0{}^2 c^2 \qquad \frac{E^2 - m_0{}^2 c^4}{c^2} = p^2$$

$$\frac{E}{c} = m_0 c + \frac{T}{c}, \qquad E = m_0 c^2 + T$$

$$\frac{u}{c} = \frac{cp}{E}$$

where T is the a relativistic kinetic energy.

$$T = \frac{m_0 c^2}{\sqrt{1 - \frac{u^2}{c^2}}} - m_0 c^2 = E - m_0 c^2$$

(a) Find the mass of a proton in MeV. ($m_0 c^2$)
(b) Proton has a relativistic kinetic energy of T =2.00 GeV. Find the energy, momentum, and speed of proton

```
Clear["Global`*"];

Physconst = {me → 9.1093821545 × 10^-31, u → 1.660538782 × 10^-27,
   eV → 1.602176487 × 10^-19, qe → 1.602176487 × 10^-19,
   c → 2.99792458 × 10^8, Mp → 1.672621637 × 10^-27,
   h → 6.62606896 × 10^-34, ℏ → 1.05457162853 × 10^-34,
   GeV → 10^9 1.602176487 × 10^-19, MeV → 10^6 1.602176487 × 10^-19};

rule1 = {T → 2000};
```

$$f1 = \frac{Mp\, c^2}{MeV} \;/.\; Physconst$$

938.272

```
E1 = (f1 + T) /. rule1
```

2938.27

$$p1 = \frac{1}{c}\sqrt{E1^2 - f1^2}$$

$$\frac{2784.44}{c}$$

$$\frac{c\, p1}{E1} \;//\; Simplify$$

0.947644

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