Relativity of magnetic and electric fields

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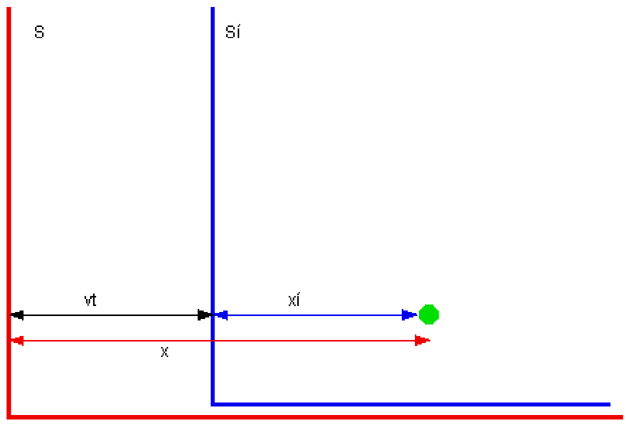
1. Lorentz transformation

1.1 Derivation of Lorentz transformation



We consider a Galilean transformation given by

$$x' = x - vt$$
$$x = x' + vt'$$
$$t' = t$$

$$\frac{dx'}{dt'} = \frac{dx}{dt'} - v\frac{dt}{dt'} = \frac{dx}{dt} - v$$
$$u' = u - v$$

We know that the velocity of light remains unchanged under a transformation (so-called the Lorentz transformation) satisfying the principle of relativity. This implies that the Lorentz transformation is not the same as the Galilean transformation.

Here we assume that

$$x' = \gamma(x - vt)$$
$$x = \gamma(x' + vt')$$

from the symmetry of transformation

What is the value of γ?

(i) The light is emitted at

$$t = t' = 0$$
$$x = x' = 0$$

(initially). The speed of light (in vacuum) is the same in all internal reference frames; it always has the value c.

$$\frac{x'}{t'} = \frac{x}{t} = c$$

((Mathematica))

Derivation of Lorentz transformation

```
eq1=x==γ (x'+v t');eq2=x'== γ (x- v t)
x' == (-t v + x) γ
eq3=Solve[{eq1,eq2},{x',t'}]//Simplify//Flatten
```

$$\left\{t' \to \frac{x\left(\frac{1}{\gamma}-\gamma\right)}{v} + t\gamma,\ x' \to (-t\,v + x)\,\gamma\right\}$$

```
eq4 = x'/t' == x/t /. eq3 // Simplify
```

$$\frac{v(-t\,v + x)\,\gamma^2}{x + t\,v\,\gamma^2 - x\,\gamma^2} == \frac{x}{t}$$

```
eq5=eq4/.{x→c t}//Simplify
```

$$-\frac{v(-c+v)\,\gamma^2}{c - c\,\gamma^2 + v\,\gamma^2} == c$$

```
eq6=Solve[eq5,γ]
```

$$\left\{\left\{\gamma \to -\frac{i\,c}{\sqrt{-c^2+v^2}}\right\},\left\{\gamma \to \frac{i\,c}{\sqrt{-c^2+v^2}}\right\}\right\}$$

```
T'=t'/.eq3/.eq6[[2]]//Simplify
```

$$\frac{i\,(c^2 t - v\,x)}{c\sqrt{-c^2+v^2}}$$

```
x'/.eq3
(-t v+x) γ
X'=x'/.eq3/.eq6[[2]]//Simplify
```

$$\frac{i\,c\,(-t\,v + x)}{\sqrt{-c^2+v^2}}$$

Then we have

$$x' = \gamma(x - vt) = \gamma(x - \beta ct)$$

$$t' = \gamma(t - \frac{\beta}{c}x)$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}},$$

with

$$\beta = \frac{v}{c}$$

Note that γ is expanded as

$$\gamma = 1 + \frac{\beta^2}{2} + \frac{3\beta^4}{8} + O(\beta^6)$$

in the limit of $\beta \to 0$.

For convenience, we introduce

$$x_4 = ict$$

or

$$-i\frac{x_4}{c} = t$$

Then we have

$$x_1' = \gamma(x_1 + i\beta x_4)$$
$$x_4' = \gamma(-i\beta x_1 + x_4)$$

or

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

or

$$x' = ax$$

$$x_1 = \gamma(x_1' - i\beta x_4')$$
$$x_4 = \gamma(i\beta x_1' + x_4')$$

or in the matrix form,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix}$$

$$x = a^{-1}x'$$

Note that $a^{-1} = a^T$

$$x_\mu = (a^{-1})_{\mu\nu} = (a^T)_{\mu\nu} = a_{\nu\mu}x'_\nu$$

1.2 Lorentz contraction

Imagine a stick moving to the right at the velocity *v*. Its rest length (that is, its length measured in S') is $\Delta x_1'$.

We measure the distance of the stick under the condition that $\Delta x_4 = 0$. Since

$$\Delta x_1' = \gamma(\Delta x_1 + i\beta\Delta x_4) = \gamma\Delta x_1$$

or

$$\Delta x_1 = \frac{1}{\gamma}\Delta x_1' = \sqrt{1-\beta^2}\,\Delta x_1' = \sqrt{1-\beta^2}\,\Delta x_0$$

The length of the stick measure in S (Δx_1) is shorter than that observed in S' (Δx_1, proper length)

1.3 Time dilation

We are watching one moving clock moving to the right at the velocity *v*.

$$\Delta x_4 = \gamma(i\beta\Delta x_1' + \Delta x_4')$$

with $\Delta x_1' = 0$. Then we have

$$\Delta x_4 = \gamma\Delta x_4' > \Delta x_4'.$$

or

$$\Delta t = \frac{1}{\sqrt{1-\beta^2}}\Delta t' = \frac{1}{\sqrt{1-\beta^2}}\Delta t_0 \geq \Delta t_0$$

The time in S (Δt) is longer than that observed in S' (Δt_0, proper time). The moving clocks run slow

1.4 Proper time

$$(dx_\mu')^2 = a_{\mu\lambda} a_{\mu\sigma} dx_\lambda dx_\sigma = \delta_{\lambda\sigma} dx_\lambda dx_\sigma = (dx_\mu)^2$$

We define the proper time as

$$(ds)^2 = c^2 (dt)^2 - (dx_1)^2 - (dx_2)^2 - (dx_3)^2 = c^2 (dt')^2 - (dx_1')^2 - (dx_2')^2 - (dx_3')^2$$

$$(ds)^2 = c^2 (dt)^2 \{1 - \frac{1}{c^2}[(\frac{dx_1}{dt})^2 + (\frac{dx_2}{dt})^2 + (\frac{dx_3}{dt})^2]\} = c^2 (dt)^2 [1 - \beta^2]$$

or

$$d\tau = \frac{ds}{c} = dt\sqrt{1-\beta^2}$$

where τ is a proper time.

1.5 Notation of four vector

Four vector notation

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ ib_0 \end{pmatrix}$$

b_μ (μ = 1, 2, 3, and 4)

where

b_1, b_2, b_3:	real
$b_4 = ib_0$	purely imaginary

((Note))

We use the Einstein convention, in which repeated indices are summed.

i, j, k (= 1 – 3)

μ, ν, λ (= 1, - 4)

The co-ordinate vector

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ ix_0 \end{pmatrix}$$

b_μ (μ = 1, 2, 3, and 4)

Under a Lorentz transformation, we have

$$x_\mu' = a_{\mu\nu} x_\nu$$

instead of

$$x_\mu' = \sum_\nu a_{\mu\nu} x_\nu$$

where

$$a_{\mu\nu} a_{\mu\lambda} = \delta_{\nu\lambda}$$

$$\left(a^{-1}a\right)_{\lambda\nu} = (a^{-1})_{\lambda\mu} a_{\mu\nu} = (a^T)_{\lambda\mu} a_{\mu\nu} = a_{\mu\lambda} a_{\mu\nu} = \delta_{\lambda\nu}$$

Note that

$$a^{-1} = a^T \qquad \text{(transpose matrix)}$$

(1)

$$x'_\mu x'_\mu = a_{\mu\nu} x_\nu a_{\mu\lambda} x_\lambda = a_{\mu\nu} a_{\mu\lambda} x_\nu x_\lambda = \delta_{\mu\lambda} x_\nu x_\lambda = x_\mu x_\mu$$

(2)

$$x_\mu' = a_{\mu\lambda} x_\lambda$$

$$a_{\mu\nu} x_\mu' = a_{\mu\nu} a_{\mu\lambda} x_\lambda = \delta_{\nu\lambda} x_\lambda = x_\nu$$

or

$$x_\nu = a_{\mu\nu} x_\mu' \quad \text{or} \quad x_\mu = a_{\nu\mu} x_\nu'$$

A four vector, by definition, transforms in the same way as x_μ under the Lorentz transformation.

$$\frac{\partial}{\partial x_\mu'} = \frac{\partial x_\nu}{\partial x_\mu'}\frac{\partial}{\partial x_\nu} = a_{\mu\nu}\frac{\partial}{\partial x_\nu}$$

where

$$x_\nu = a_{\mu\nu} x_\mu'$$

The scalar product $b \cdot c$ is defined by

$$b \cdot c = b_\mu c_\mu$$

It is invariant under the Lorentz transformation

$$b'_\mu c'_\mu = a_{\mu\nu} b_\nu a_{\mu\lambda} b_\lambda = a_{\mu\nu} a_{\mu\lambda} a_\nu b_\lambda = \delta_{\nu\lambda} a_\nu b_\lambda = a \cdot b$$

1.6 Four dimensional Laplacian operator

$$\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\mu} = \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \frac{\partial^2}{\partial x_3{}^2} + \frac{\partial^2}{\partial x_4{}^2} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

is invariant under the Lorentz transformation: Lorentz scalar

$$\frac{\partial}{\partial x_\mu'}\frac{\partial}{\partial x_\mu'} = a_{\mu\nu}\frac{\partial}{\partial x_\nu} a_{\mu\lambda}\frac{\partial}{\partial x_\lambda} = a_{\mu\nu} a_{\mu\lambda}\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\lambda} = \delta_{\nu\lambda}\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\lambda} = \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\nu} = \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\mu}$$

1.7 A tensor of second rank

A tensor of second rank, $t_{\mu\nu}$, transforms as

$$t_{\mu\nu}' = a_{\mu\lambda} a_{\nu\sigma} t_{\lambda\sigma}$$

1.8 A tensor of third rank

A tensor of third rank, $t_{\mu\nu\lambda}$, transforms as

$$t_{\mu\nu\lambda}' = a_{\mu\sigma} a_{\nu\rho} a_{\lambda\tau} t_{\sigma\rho\tau}$$

((Note))
We make no distinction between a covariant and contravariant vector. We do not define the metric tensor $g_{\mu\nu}$.

2. Velocity, acceleration, and force

2.1 Lorentz velocity transformation

$$x_\mu = (a^{-1})_{\mu\nu} x'_\nu$$

So we have

$$(a^{-1})_{\mu\nu} = a_{\nu\mu} = (a^T)_{\mu\nu}$$

$$x_1' = \gamma(x_1 - vt) \qquad x_1 = \gamma(x_1' + vt')$$
$$x_2' = x_2 \qquad x_2 = x_2'$$
$$x_3' = x_3 \qquad x_3 = x_3'$$
$$t' = \gamma(t - \frac{\beta}{c} x_1) \qquad t = \gamma(t' + \frac{\beta}{c} x_1')$$

Suppose that an object has velocity components as measured in *S*' and *S*.

$$u_1' = \frac{dx_1'}{dt'} = \frac{u_1 - v}{1 - \frac{\beta}{c} u_1} \qquad u_1 = \frac{dx_1}{dt'} = \frac{u_1' + v}{1 + \frac{\beta}{c} u_1'}$$
$$u_2' = \frac{dx_2'}{dt'} = \frac{1}{\gamma} \frac{u_2}{1 - \frac{\beta}{c} u_1} \qquad u_2 = \frac{dx_2}{dt} = \frac{1}{\gamma} \frac{u_2'}{1 + \frac{\beta}{c} u_1'}$$
$$u_3' = \frac{dx_3'}{dt'} = \frac{1}{\gamma} \frac{u_3}{1 - \frac{\beta}{c} u_1} \qquad u_3 = \frac{dx_3}{dt} = \frac{1}{\gamma} \frac{u_3'}{1 + \frac{\beta}{c} u_1'}$$

The Lorentz transformation of a velocity less than *c* never leads to a velocity greater than *c*. The relations reduce to the Galilean transformation for $v \ll c$.

Suppose that the particle is s photon, and $u_1 = c$ in the frame S. Then we have

$$u_1' = \frac{u_1 - v}{1 - \frac{\beta}{c} u_1} = \frac{c - v}{1 - \frac{\beta}{c} c} = \frac{c - v}{1 - \frac{v}{c}} = c$$

((Example))

$$u_1' = \frac{9}{10} c$$
$$v = \frac{9}{10} c$$

$$u_1 = \frac{u_1' + v}{1 + \frac{\beta}{c} u_1'} = \frac{\frac{9}{10} c + \frac{9}{10} c}{1 + \frac{81}{100}} = \frac{180}{181} c < c$$

whereas the Galilean transformation would have given

$$u_1 = u_1' + v = \frac{9}{10} c + \frac{9}{10} c = \frac{9}{5} c > c$$

2.2 Lorentz acceleration transformation

Similarly we have the acceleration components as measured in *S'* and *S*.

$$a_1 = \frac{du_1}{dt} = \frac{dt'}{dt}\frac{d}{dt'}\left(\frac{u_1'+v}{1+\frac{\beta}{c}u_1'}\right) = \frac{1}{\gamma}\frac{1}{(1+\frac{\beta}{c}u_1')}\frac{d}{dt'}\left(\frac{u_1'+v}{1+\frac{\beta}{c}u_1'}\right) = \frac{1}{\gamma}\frac{(1-\beta^2)a_1'}{(1+\frac{\beta}{c}u_1')^3} = \frac{1}{\gamma^3}\frac{a_1'}{(1+\frac{\beta}{c}u_1')^3}$$

$$a_2 = \frac{du_2}{dt} = \frac{1}{\gamma}\frac{dt'}{dt}\frac{d}{dt'}\left(\frac{u_2'}{1+\frac{\beta}{c}u_1'}\right) = \frac{1}{\gamma^2}\frac{1}{(1+\frac{\beta}{c}u_1')}\frac{d}{dt'}\left(\frac{u_2'}{1+\frac{\beta}{c}u_1'}\right) = \frac{1}{\gamma^2}\frac{a_2'}{(1+\frac{\beta}{c}u_1')^2} - \frac{1}{\gamma^2}\frac{a_1'a_2\frac{\beta}{c}}{(1+\frac{\beta}{c}u_1')^3}$$

$$a_3 = \frac{du_3}{dt} = \frac{1}{\gamma}\frac{dt'}{dt}\frac{d}{dt'}\left(\frac{u_3'}{1+\frac{\beta}{c}u_1'}\right) = \frac{1}{\gamma^2}\frac{1}{(1+\frac{\beta}{c}u_1')}\frac{d}{dt'}\left(\frac{u_3'}{1+\frac{\beta}{c}u_1'}\right) = \frac{1}{\gamma^2}\frac{a_3'}{(1+\frac{\beta}{c}u_1')^3} - \frac{1}{\gamma^2}\frac{a_1'a_3\frac{\beta}{c}}{(1+\frac{\beta}{c}u_1')^3}$$

where

$$\frac{dt'}{dt} = \frac{1}{\gamma}\frac{1}{(1+\frac{\beta}{c}u_1')}$$

The acceleration is a quantity of limited and questionable value in special relativity. Not only is it not an invariant, but the expressions for it are in general cumbersome, and moreover its different components transform in different ways.

2.3 Force F under the Lorentz transformation

Lorentz transformation

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(-\frac{v}{c^2}x_1 + t)$$

or

$$x = \gamma(x' + vt')$$
$$y = y'$$
$$z = z'$$
$$t = \gamma(\frac{v}{c^2} x_1' + t')$$

$$\frac{dp_1'}{dt'} = \frac{\gamma(dp_1 - \frac{\beta}{c} dE)}{\gamma(-\frac{v}{c^2} dx_1 + dt)} = \frac{\gamma(\frac{dp_1}{dt} - \frac{\beta}{c}\frac{dE}{dt})}{\gamma(-\frac{v}{c^2}\frac{dx_1}{dt} + 1)} = \frac{\frac{dp_1}{dt} - \frac{\beta}{c}\frac{dE}{dt}}{1 - \frac{v}{c^2}\frac{dx_1}{dt}} = \frac{F_1 - \frac{\beta}{c}\frac{dE}{dt}}{1 - \frac{v}{c^2} u_1}$$

Since $\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{u}$

$$F_1' = \frac{dp_1'}{dt'} = \frac{F_1 - \frac{\beta}{c}(\mathbf{u} \cdot \mathbf{F})}{1 - \frac{v}{c^2} u_1}$$

Similarly

$$F_2' = \frac{dp_2'}{dt'} = \frac{dp_2}{\gamma(-\frac{v}{c^2} dx_1 + dt)} = \frac{\frac{dp_2}{dt}}{\gamma(-\frac{v}{c^2}\frac{dx_1}{dt} + 1)} = \frac{F_2}{\gamma(1 - \frac{v}{c^2} u_1)}$$

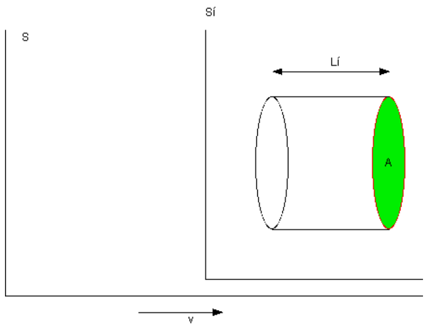
$$F_3' = \frac{dp_3'}{dt'} = \frac{dp_3}{\gamma(-\frac{v}{c^2} dx_1 + dt)} = \frac{\frac{dp_3}{dt}}{\gamma(-\frac{v}{c^2}\frac{dx_1}{dt} + 1)} = \frac{F_3}{\gamma(1 - \frac{v}{c^2} u_1)}$$

We consider one special case when the particle is instantaneously at rest in *S*. So that ***u*** = 0.

$$F_1' = F_1$$
$$F_2' = \frac{F_2}{\gamma}$$
$$F_3' = \frac{F_3}{\gamma}$$

The component of ***F*** parallel to the motion of S' is unchanged, whereas the components perpendicular are divided by γ.

3. Charge and current density
3.1 Charge density



We measure the distance of the cylinder under the condition that $\Delta x_4 = 0$. Since

$$\Delta x_1' = \gamma(\Delta x_1 + i\beta\Delta x_4) = \gamma\Delta x_1$$

or

$$\Delta x_1 = \frac{1}{\gamma}\Delta x_1' = \sqrt{1-\beta^2}\,\Delta x_1'$$

we have

$$L = \frac{1}{\gamma}L' = \sqrt{1-\beta^2}\,L'$$

but with the same area *A* (since dimension transverse to the motion are unchangeable. If we call ρ' (= ρ_0) the density of charges in the *S'* frame in which charges momentarily at rest, the total charge *Q* is the same in any system,

$$Q = \rho' L' A' = \rho_0 L' A' = \rho L A$$

or

$$\rho_0 L' = \rho L$$

or

$$\rho = \rho_0 \frac{L'}{L} = \gamma\rho_0$$

3.2 Current density J_μ

The current density J_μ is defined as

$$J_\mu = (\mathbf{J}, ic\rho) = (\rho\mathbf{u}, ic\rho)$$

where ***u*** is the velocity of the particle in the S frame. Evidently the charge density and current density go together to make a 4 vector.

$$J_\mu' = a_{\mu\nu} J_\nu$$

$$J_1' = \gamma(J_1 + i\beta J_4) = \gamma(J_1 - v\rho)$$

$$\rho' = \gamma(-\frac{\beta}{c} J_1 + \rho)$$

or

$$J_\mu = a_{\nu\mu} J_\nu'$$

$$J_1 = \gamma(J_1' - i\beta J_4') = \gamma(J_1' + v\rho')$$

$$\rho = \gamma(\frac{\beta}{c} J_1' + \rho')$$

Then we have

$$\rho = \gamma(\frac{\beta}{c} J_1' + \rho')$$

Note that

$$\rho = \gamma\rho'$$

when $J_1' = 0$.

3.3 Invariance under the Lorentz transformation

We know that $J_\mu J_\mu$ is invariant under the Lorentz transformation

$$J'_\mu J'_\mu = a_{\mu\nu} J_\nu a_{\mu\lambda} J_\lambda = a_{\mu\nu} a_{\mu\lambda} J_\nu J_\lambda = \delta_{\mu\lambda} J_\nu J_\lambda = J_\mu J_\mu$$

or

$$J_\mu J_\mu = \mathbf{J}^2 - c^2\rho^2 = \mathbf{J}'^2 - c^2\rho'^2$$

Suppose that ***J'*** = 0 (or ***u'*** = 0) in the S' frame, where the point charge is at rest. $J = \rho\mathbf{u} = \rho\mathbf{v}$ (the frame S' moves at the velocity ***v*** relative to the frame S). Then we have

$$\rho^2 v^2 - c^2\rho^2 = 0 - c^2\rho'^2 = -c^2{\rho_0}^2$$

or

$$\rho\sqrt{1-\frac{v^2}{c^2}} = \rho_0\,, \qquad \text{or} \qquad \rho = \frac{\rho_0}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma\rho_0$$

4 Maxwell's equation field fensor

4.1 Four vectors for the vector potential and scalar potential

$$J_\mu = (\mathbf{J}, ic\rho)$$

$$A_\mu = (\mathbf{A}, i\frac{1}{c}\phi)$$

$$\partial_\mu = (\nabla, \frac{\partial}{\partial(ict)})$$

The equation of continuity;

$$\partial_\mu J_\mu = \nabla\cdot\mathbf{J} + (-i\frac{1}{c}\frac{\partial}{\partial t})(ic\rho) = \nabla\cdot\mathbf{J} + \frac{\partial}{\partial t}\rho = 0$$

Maxwell's equation;

$$\nabla\cdot\mathbf{D} = \rho$$

$$\nabla\cdot\mathbf{B} = 0$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}$$

where

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$$

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$$

$$\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A} - \nabla\phi$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

4.2. Gauge transformation

$$A_\mu = (A, i\frac{1}{c}\phi)$$

$$\mathbf{A}' = \mathbf{A} + \nabla\lambda\,,$$

$$\phi' = \phi - \frac{\partial\lambda}{\partial t}\,,$$

$$A'_\mu = A_\mu + \partial_\mu\lambda$$

((Note))

$$i\frac{1}{c}\phi' = i\frac{1}{c}\phi + \frac{\partial\lambda}{\partial(ict)}$$

$$A_4' = A_4 + \frac{\partial\lambda}{\partial x_4}$$

Lorentz gauge:

$$\frac{\partial A_\mu}{\partial x_\mu} = \partial_\mu A_\mu = \nabla\cdot\mathbf{A} + \frac{\partial}{\partial(ict)} i\frac{\phi}{c} = \nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0$$

4.3 Electromagnetic field tensor *F*

We define the field tensor as

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}$$

This tensor satisfies the Jacobi identity;

$$\frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} = 0$$

This equation holds automatically for the antisymmetric tensor

The magnetic field;

$$F_{12} = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = B_3$$

$$F_{23} = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} = B_1$$

$$F_{31} = \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} = B_2$$

<u>The electric field</u>;

$$F_{14} = \frac{\partial A_4}{\partial x_1} - \frac{\partial A_1}{\partial x_4} = \frac{E_1}{ic} = -\frac{i}{c}E_1$$

$$F_{24} = \frac{\partial A_4}{\partial x_2} - \frac{\partial A_2}{\partial x_4} = \frac{E_2}{ic} = -\frac{i}{c}E_2$$

$$F_{34} = \frac{\partial A_4}{\partial x_3} - \frac{\partial A_3}{\partial x_4} = \frac{E_3}{ic} = -\frac{i}{c}E_3$$

The field tensor is an anti-symmetric tensor of second rank and hence, has 6 independent components.

<u>Electromagnetic field tensor</u>;

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -\frac{i}{c}E_1 \\ -B_3 & 0 & B_1 & -\frac{i}{c}E_2 \\ B_2 & -B_1 & 0 & -\frac{i}{c}E_3 \\ \frac{i}{c}E_1 & \frac{i}{c}E_2 & \frac{i}{c}E_3 & 0 \end{pmatrix}$$

We show that

$$F_{\mu\nu}' = \frac{\partial A_\nu'}{\partial x_\mu'} - \frac{\partial A_\mu'}{\partial x_\nu'} = a_{\mu\sigma}a_{\nu\tau}F_{\sigma\tau}$$

$$\frac{\partial}{\partial x_\mu'} = a_{\mu\nu}\frac{\partial}{\partial x_\nu}, \qquad \text{and} \qquad A_\mu' = a_{\mu\lambda}A_\lambda$$

$$F_{\mu\nu}{}' = \frac{\partial A_\nu{}'}{\partial x_\mu{}'} - \frac{\partial A_\mu{}'}{\partial x_\nu{}'} = a_{\mu\sigma}\frac{\partial A_\nu{}'}{\partial x_\sigma} - a_{\nu\tau}\frac{\partial A_\mu{}'}{\partial x_\tau} = a_{\mu\sigma}\frac{\partial(a_{\nu\tau}A_\tau)}{\partial x_\sigma} - a_{\nu\tau}\frac{\partial(a_{\mu\sigma}A_\sigma)}{\partial x_\tau}$$

or

$$F_{\mu\nu}{}' = a_{\mu\sigma}a_{\nu\tau}\left(\frac{\partial A_\tau}{\partial x_\sigma} - \frac{\partial A_\sigma}{\partial x_\tau}\right) = a_{\mu\sigma}a_{\nu\tau}F_{\sigma\tau}$$

4.4 Maxwell’s equation (1)

The Maxwell's equation is given by

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 J_\mu$$

$$\frac{\partial F_{1\mu}}{\partial x_\mu} = \frac{\partial F_{11}}{\partial x_1} + \frac{\partial F_{12}}{\partial x_2} + \frac{\partial F_{13}}{\partial x_3} + \frac{\partial F_{14}}{\partial x_4} = \frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3} - \frac{\frac{i}{c}}{ic}\frac{\partial E_1}{\partial t} = (\nabla\times\mathbf{B})_1 - \frac{1}{c^2}\frac{\partial E_1}{\partial t}$$

$$(\nabla\times\mathbf{B})_1 = \varepsilon_0\mu_0\frac{\partial E_1}{\partial t} + \mu_0 J_1 = \mu_0(J_1 + \varepsilon_0 E_1)$$

$$\frac{\partial F_{2\mu}}{\partial x_\mu} = \frac{\partial F_{21}}{\partial x_1} + \frac{\partial F_{22}}{\partial x_2} + \frac{\partial F_{23}}{\partial x_3} + \frac{\partial F_{24}}{\partial x_4} = -\frac{\partial B_3}{\partial x_1} + \frac{\partial B_1}{\partial x_3} - \frac{\frac{i}{c}}{ic}\frac{\partial E_2}{\partial t} = (\nabla\times\mathbf{B})_2 - \frac{1}{c^2}\frac{\partial E_2}{\partial t}$$

$$(\nabla\times\mathbf{B})_2 = \varepsilon_0\mu_0\frac{\partial E_2}{\partial t} + \mu_0 J_2 = \mu_0(J_2 + \varepsilon_0 E_2)$$

$$\frac{\partial F_{3\mu}}{\partial x_\mu} = \frac{\partial F_{31}}{\partial x_1} + \frac{\partial F_{32}}{\partial x_2} + \frac{\partial F_{33}}{\partial x_3} + \frac{\partial F_{34}}{\partial x_4} = \frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2} - \frac{\frac{i}{c}}{ic}\frac{\partial E_3}{\partial t} = (\nabla\times\mathbf{B})_3 - \frac{1}{c^2}\frac{\partial E_3}{\partial t}$$

$$(\nabla\times\mathbf{B})_3 = \varepsilon_0\mu_0\frac{\partial E_3}{\partial t} + \mu_0 J_3 = \mu_0(J_3 + \varepsilon_0 E_3)$$

$$\frac{\partial F_{4\mu}}{\partial x_\mu} = \frac{\partial F_{41}}{\partial x_1} + \frac{\partial F_{42}}{\partial x_2} + \frac{\partial F_{43}}{\partial x_3} + \frac{\partial F_{44}}{\partial x_4} = \frac{i}{c}\frac{\partial E_1}{\partial x_1} + \frac{i}{c}\frac{\partial E_2}{\partial x_2} + \frac{i}{c}\frac{\partial E_3}{\partial x_3} = \frac{i}{c}\nabla\cdot\mathbf{E} = \mu_0 J_4$$

$$\frac{i}{c}\nabla\cdot\mathbf{E} = \mu_0 ic\rho$$

$$\nabla\cdot\mathbf{E} = \mu_0 c^2\rho = \frac{\rho}{\varepsilon_0}$$

((**Note**))

$$\mathbf{B} = \nabla\times\mathbf{A} = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ A_1 & A_2 & A_3 \end{vmatrix} = (\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}, \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}, \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2})$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} = (-\frac{\partial\phi}{\partial x_1} - ic\frac{\partial A_1}{\partial x_4}, -\frac{\partial\phi}{\partial x_2} - ic\frac{\partial A_2}{\partial x_4}, -\frac{\partial\phi}{\partial x_3} - ic\frac{\partial A_3}{\partial x_4})$$

or

$$\mathbf{E} = ic(\frac{\partial A_4}{\partial x_1} - \frac{\partial A_1}{\partial x_4}, \frac{\partial A_4}{\partial x_2} - \frac{\partial A_2}{\partial x_4}, \frac{\partial A_4}{\partial x_3} - \frac{\partial A_3}{\partial x_4})$$

where

$$\phi = \frac{c}{i}A_4 .$$

4.5 Invariants of the field

$F_{\mu\nu}F_{\mu\nu}$ is invariant under the Lorentz transformation

$$F_{\mu\nu}'F_{\mu\nu}' = a_{\mu\lambda}a_{\nu\rho}a_{\mu\sigma}a_{\nu\tau}F_{\lambda\rho}F_{\sigma\tau} = \delta_{\lambda\sigma}\delta_{\rho\tau}F_{\lambda\rho}F_{\sigma\tau} = F_{\lambda\rho}F_{\lambda\rho} = F_{\mu\nu}F_{\mu\nu}$$

$$F_{\mu\nu}F_{\mu\nu} = 2[B_1^2 + B_2^2 + B_3^2 - \frac{1}{c^2}(E_1^2 + E_2^2 + E_3^2)] = invariant$$

A further invariant is obtained by contraction of the field tensor with the "completely anti-symmetric unit tensor of fourth rank" defined by

$\varepsilon_{\kappa\lambda\mu\nu} =$

0 if two indices are equal,

1 if $(\kappa\lambda\mu\nu)$ is an even permutation of (1234), and
-1 if $(\kappa\lambda\mu\nu)$ is an odd permutation of (1234).

(Levi-Civita tensor)

One may be convinced easily that $\varepsilon_{\kappa\lambda\mu\nu}$ is a tensor of rank 4 because

$$\varepsilon_{\kappa'\lambda'\mu'\nu'}' = a_{\kappa'\kappa} a_{\lambda'\lambda} a_{\mu'\mu} a_{\nu'\nu} \varepsilon_{\kappa\lambda\mu\nu}$$

Now we consider

$$\varepsilon_{\kappa\lambda\mu\nu} F_{\kappa\lambda} F_{\mu\nu} = \varepsilon_{1234} F_{12} F_{34} + \varepsilon_{1324} F_{13} F_{24} + ... = -\frac{8i}{c} \mathbf{E} \cdot \mathbf{B}$$

So the scalar product $\mathbf{E} \cdot \mathbf{B}$ is Lorentz invariant,

4.6 Equation of continuity

$$F_{\mu\nu} = -F_{\nu\mu}$$

$$F_{\mu\nu} = \frac{F_{\mu\nu} + F_{\mu\nu}}{2} = \frac{F_{\mu\nu} - F_{\nu\mu}}{2}$$

$$\frac{\partial}{\partial x_\mu} \left(\frac{\partial F_{\mu\nu}}{\partial x_\nu} \right) = \frac{1}{2} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu} (F_{\mu\nu} - F_{\nu\mu}) = \frac{1}{2} \left(\frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu} F_{\mu\nu} - \frac{\partial}{\partial x_\nu} \frac{\partial}{\partial x_\mu} F_{\nu\mu} \right) = 0$$

Since

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 J_\mu$$

we have

$$\frac{\partial}{\partial x_\mu} J_\mu = 0$$

4.7 Maxwell's equation using dual tensor

Using the electromagnetic tensor

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & B_3 & -B_2 & -\frac{i}{c}E_1 \\ -B_3 & 0 & B_1 & -\frac{i}{c}E_2 \\ B_2 & -B_1 & 0 & -\frac{i}{c}E_3 \\ \frac{i}{c}E_1 & \frac{i}{c}E_2 & \frac{i}{c}E_3 & 0 \end{pmatrix}$$

the dual tensor $G_{\mu\nu}$ is defined as

$$G_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}F_{\lambda\sigma}$$

or

$$(G_{\mu\nu}) = \begin{pmatrix} 0 & F_{34} & F_{42} & F_{23} \\ -F_{34} & 0 & F_{14} & F_{31} \\ -F_{42} & -F_{14} & 0 & F_{12} \\ -F_{23} & -F_{31} & -F_{12} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\frac{i}{c}E_3 & \frac{i}{c}E_2 & B_1 \\ \frac{i}{c}E_3 & 0 & -\frac{i}{c}E_1 & B_2 \\ -\frac{i}{c}E_2 & \frac{i}{c}E_1 & 0 & B_3 \\ -B_1 & -B_2 & -B_3 & 0 \end{pmatrix}$$

Note that

$$F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}G_{\lambda\sigma} .$$

Using the Jacobi identity

$$\frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} = 0$$

we get

$$3\frac{\partial G_{\mu\nu}}{\partial x_\nu} = \varepsilon_{\mu\nu\lambda\sigma}\frac{\partial F_{\lambda\sigma}}{\partial x_\nu} + \varepsilon_{\mu\lambda\sigma\nu}\frac{\partial F_{\sigma\nu}}{\partial x_\lambda} + \varepsilon_{\mu\sigma\nu\lambda}\frac{\partial F_{\nu\lambda}}{\partial x_\sigma}$$
$$= \varepsilon_{\mu\nu\lambda\sigma}\left(\frac{\partial F_{\lambda\sigma}}{\partial x_\nu} + \frac{\partial F_{\sigma\nu}}{\partial x_\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x_\sigma}\right) = 0$$

since

$$\varepsilon_{\mu\nu\lambda\sigma} = \varepsilon_{\mu\lambda\sigma\nu} = \varepsilon_{\mu\sigma\nu\lambda}.$$

Then we have the Maxwell's equation,

$$\frac{\partial G_{\mu\nu}}{\partial x_\nu} = 0.$$

(a)

$$\frac{\partial G_{1\nu}}{\partial x_\nu} = \frac{\partial G_{11}}{\partial x_1} + \frac{\partial G_{12}}{\partial x_2} + \frac{\partial G_{13}}{\partial x_3} + \frac{\partial G_{14}}{\partial x_4} = -\frac{i}{c}\frac{\partial E_3}{\partial y} + \frac{i}{c}\frac{\partial E_2}{\partial z} + \frac{\partial B_1}{ic\partial t} = 0$$

or

$$(\nabla \times \mathbf{E})_1 = -\frac{\partial}{\partial t}B_1$$

(b)

$$\frac{\partial G_{2\nu}}{\partial x_\nu} = \frac{\partial G_{21}}{\partial x_1} + \frac{\partial G_{22}}{\partial x_2} + \frac{\partial G_{23}}{\partial x_3} + \frac{\partial G_{24}}{\partial x_4} = \frac{i}{c}\frac{\partial E_3}{\partial x} - \frac{i}{c}\frac{\partial E_1}{\partial z} + \frac{\partial B_2}{ic\partial t} = 0$$

or

$$(\nabla \times \mathbf{E})_2 = -\frac{\partial}{\partial t}B_2$$

(c)

$$\frac{\partial G_{3\nu}}{\partial x_\nu} = \frac{\partial G_{31}}{\partial x_1} + \frac{\partial G_{32}}{\partial x_2} + \frac{\partial G_{33}}{\partial x_3} + \frac{\partial G_{34}}{\partial x_4} = -\frac{i}{c}\frac{\partial E_2}{\partial x} + \frac{i}{c}\frac{\partial E_1}{\partial y} + \frac{\partial B_3}{ic\partial t} = 0$$

or

$$(\nabla \times \mathbf{E})_3 = -\frac{\partial}{\partial t}B_3$$

(d)

$$\frac{\partial G_{4\nu}}{\partial x_\nu} = \frac{\partial G_{41}}{\partial x_1} + \frac{\partial G_{42}}{\partial x_2} + \frac{\partial G_{43}}{\partial x_3} + \frac{\partial G_{44}}{\partial x_4} = -\frac{\partial B_1}{\partial x} - \frac{\partial B_2}{\partial y} - \frac{\partial B_3}{\partial z} = 0$$

or

$$\nabla \cdot \mathbf{B} = 0$$

Note

$$G_{\mu\nu} F_{\mu\nu} = -\frac{4i}{c}(B_1 E_1 + B_2 E_2 + B_3 E_3) = -\frac{4i}{c}\mathbf{E} \cdot \mathbf{B}$$

4.8 Summary

The Maxwell's equation can be expressed by

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 J_\mu ,$$

and

$$\frac{\partial G_{\mu\nu}}{\partial x_\nu} = 0$$

using the tensors *F* and *G*.

5. Vector potential under the Lorentz transformation

$$A_\mu = (\mathbf{A}, i\frac{1}{c}\phi)$$

$$A_\mu' = (\mathbf{A}', i\frac{1}{c}\phi')$$

$$A_\mu' = a_{\mu\nu} A_\nu$$

$$A_\mu = (a^{-1})_{\mu\nu} A_\nu'$$

$$A_1' = \frac{cA_1 - \beta\phi}{c\sqrt{1-\beta^2}} \qquad A_1' = \frac{cA_1 - \beta\phi}{c\sqrt{1-\beta^2}}$$

$$A_2' = A_2 \qquad A_2' = A_2$$

$$A_3' = A_3 \qquad A_3' = A_3$$

$$A_4' = -\frac{i}{c}\frac{(c\beta A_1 - \phi)}{\sqrt{1-\beta^2}} \qquad \phi' = \frac{\phi - c\beta A_1}{\sqrt{1-\beta^2}}$$

and

$$A_1 = \frac{cA_1' + \beta\phi'}{c\sqrt{1-\beta^2}} \qquad A_1 = \frac{cA_1' + \beta\phi'}{c\sqrt{1-\beta^2}}$$

$$A_2 = A_2' \qquad A_2 = A_2'$$

$$A_3 = A_3' \qquad A_3 = A_3'$$

$$A_4 = \frac{i}{c}\frac{(c\beta A'_1 + \phi')}{\sqrt{1-\beta^2}} \qquad \phi = \frac{(c\beta A'_1 + \phi')}{\sqrt{1-\beta^2}}$$

6. *E* and *B* under the Lorentz transformation

6.1 Transformation

$$F'_{\mu\nu} = a_{\mu\lambda} a_{\nu\sigma} F_{\lambda\sigma}$$

$$a_{\mu\xi} a_{\nu\eta} F'_{\mu\nu} = a_{\mu\xi} a_{\nu\eta} a_{\mu\lambda} a_{\nu\sigma} F_{\lambda\sigma} = a_{\mu\xi} a_{\mu\lambda} a_{\nu\eta} a_{\nu\sigma} F_{\lambda\sigma} = \delta_{\xi\lambda}\delta_{\eta\sigma} F_{\lambda\sigma} = F_{\xi\eta}$$

or

$$F_{\mu\nu} = a_{\lambda\mu} a_{\sigma\nu} F'_{\lambda\sigma} = (a^T)_{\mu\lambda} (a^T)_{\nu\sigma} F'_{\lambda\sigma} = (a^{-1})_{\mu\lambda} (a^{-1})_{\nu\sigma} F'_{\lambda\sigma}$$

$$E_1' = E_1 \qquad E_1' = E_1$$

$$E_2' = \gamma(E_2 - c\beta B_3) \qquad E_2' = \gamma(\mathbf{E} + \mathbf{v}\times\mathbf{B})_2$$

$$E_3' = \gamma(E_3 + c\beta B_2) \qquad E_3' = \gamma(\mathbf{E} + \mathbf{v}\times\mathbf{B})_3$$

$$B_1' = B_1 \qquad B_1' = B_1$$

$$B_2' = \gamma(B_2 + \frac{\beta}{c} E_3) \qquad B_2' = \gamma(\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E})_2$$

$$B_3' = \gamma(B_3 - \frac{\beta}{c} E_2) \qquad B_3' = \gamma(\mathbf{B} - \frac{1}{c}\mathbf{v}\times\mathbf{E})_3$$

$$E_1 = E_1' \qquad E_1 = E_1'$$
$$E_2 = \gamma(E_2' + c\beta B_3') \qquad E_2 = \gamma(\mathbf{E}' - \mathbf{v}\times\mathbf{B}')_2$$
$$E_3 = \gamma(E_3' - c\beta B_2') \qquad E_3 = \gamma(\mathbf{E}' - \mathbf{v}\times\mathbf{B}')_3$$

$$B_1 = B_1' \qquad B_1 = B_1'$$
$$B_2 = \gamma(B_2' - \frac{\beta}{c}E_3') \qquad B_2 = \gamma(\mathbf{B}' + \frac{1}{c^2}\mathbf{v}\times\mathbf{E}')_2$$
$$B_3 = \gamma(B_3' + \frac{\beta}{c}E_2') \qquad B_3 = \gamma(\mathbf{B}' + \frac{1}{c^2}\mathbf{v}\times\mathbf{E}')_3$$

6.2 Choice of the frame S' which has pure electric or pure magnetic fields

From the Sec.3.5, we find that

(1) $\mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2$ = invariant under the Lorentz transformation

(2) $\mathbf{E}\cdot\mathbf{B}$ = invariant under the Lorentz transformation

Here we assume that $\mathbf{E}\cdot\mathbf{B}$=0 and $\mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2 \neq 0$

Then one can find a frame S' in which (***E***' = 0 and ***B***' ≠ 0) [pure magnetic field], or (***B***' = 0 and ***E***' ≠ 0) [pure electric field]. The proof is given in the following.

(a) Pure magnetic field (*E*' = 0)

We assume that ***E***' = 0. From the Lorentz transformation, we have

$$E_1' = E_1 = 0 \qquad\qquad E_1 = 0$$
$$E_2' = \gamma(E_2 - c\beta B_3) = 0 \qquad \text{or} \qquad E_2 = c\beta B_3 = vB_3$$
$$E_3' = \gamma(E_3 + c\beta B_2) = 0 \qquad\qquad E_3 = -c\beta B_2 = -vB_2$$

The condition $\mathbf{E}\cdot\mathbf{B}$=0 is satisfied since

$$\mathbf{E}\cdot\mathbf{B} = E_1B_1 + E_2B_2 + E_3B_3 = vB_2B_3 - vB_2B_3 = 0$$

The condition $\mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2 = \mathbf{B}'^2 - \frac{1}{c^2}\mathbf{E}'^2 \neq 0$ can be rewritten as

$$\mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2 = \mathbf{B}'^2 > 0$$

This implies that one can find the frame where $\mathbf{B}'^2 \neq 0$ and ***E***' = 0.

((Note))
From the relation

$$E_1 = 0$$
$$E_2 = c\beta B_3 = vB_3$$
$$E_3 = -c\beta B_2 = -vB_2$$

we get

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}$$

(b) Pure electric field (*B'* = 0)

Next we assume that ***B'*** = 0. Then we have

$$B_1' = B_1 = 0 \qquad\qquad B_1 = 0$$
$$B_2' = \gamma(B_2 + \frac{\beta}{c}E_3) = 0, \qquad \text{or} \qquad B_2 = -\frac{v}{c^2}E_3$$
$$B_3' = \gamma(B_3 - \frac{\beta}{c}E_2) = 0 \qquad\qquad B_3 = \frac{v}{c^2}E_2$$

The condition $\mathbf{E} \cdot \mathbf{B}$=0 is satisfied since

$$\mathbf{E} \cdot \mathbf{B} = E_1B_1 + E_2B_2 + E_3B_3 = -\frac{v}{c^2}E_2E_3 + \frac{v}{c^2}E_2E_3 = 0$$

The condition $\mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2 = \mathbf{B}'^2 - \frac{1}{c^2}\mathbf{E}'^2 \neq 0$ can be rewritten as

$$\mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2 = -\frac{1}{c^2}\mathbf{E}'^2 < 0$$

This implies that one can find the frame where $\mathbf{E}'^2 \neq 0$ and ***B'*** = 0.

((**Note**))
From the relation

$$B_1 = 0$$
$$B_2 = -\frac{v}{c^2}E_3$$
$$B_3 = \frac{v}{c^2}E_2$$

we get

$$\mathbf{B} = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E})$$

7. Energy-momentum tensor and Maxwell's stress

7.1 force density

We define the vector of the force density as f_μ

$$F_{\mu\nu} J_\nu = f_\mu$$

Here we have

$$f_i = \rho E_i + (\mathbf{J} \times \mathbf{B})_i$$

where

$$\mathbf{J} \times \mathbf{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ J_1 & J_2 & J_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$\begin{cases} f_1 = \rho E_1 + (J_2 B_3 - J_3 B_2) \\ f_2 = \rho E_2 + (J_3 B_1 - J_1 B_3) \\ f_3 = \rho E_3 + (J_1 B_2 - J_2 B_1) \end{cases}$$

$$\begin{aligned} f_1 &= F_{1\nu} J_\nu = F_{11} J_1 + F_{12} J_2 + F_{13} J_3 + F_{14} J_4 \\ &= B_3 J_2 - B_2 J_3 - \frac{i}{c} E_1 (ic\rho) \\ &= (\mathbf{B} \times \mathbf{J})_1 + \rho E_1 \\ f_2 &= (\mathbf{B} \times \mathbf{J})_2 + \rho E_2 \\ f_3 &= (\mathbf{B} \times \mathbf{J})_3 + \rho E_3 \\ f_4 &= F_{4\nu} J_\nu = \frac{i}{c} E_1 J_1 + \frac{i}{c} E_2 J_2 + \frac{i}{c} E_3 J_3 \\ &= \frac{i}{c} (\mathbf{E} \cdot \mathbf{J}) = i \left(\frac{\mathbf{E} \cdot \mathbf{J}}{c} \right) \end{aligned}$$

7.2 Maxwell's equation

The Maxwell's equation is given by

$$\frac{\partial F_{\nu\lambda}}{\partial x_\lambda} = \mu_0 J_\nu$$

The current density:

$$J_\mu = (\mathbf{J}, ic\rho)$$

$$f_\mu = F_{\mu\nu} J_\nu = \frac{1}{\mu_0} F_{\mu\nu} \frac{\partial F_{\nu\lambda}}{\partial x_\lambda}$$

$$\mu_0 f_\mu = F_{\mu\nu} \frac{\partial F_{\nu\lambda}}{\partial x_\lambda}$$

The left-hand side can be split into two terms,

$$\mu_0 f_\mu = \frac{\partial}{\partial x_\lambda}(F_{\mu\nu} F_{\nu\lambda}) - F_{\nu\lambda} \frac{\partial}{\partial x_\lambda} F_{\mu\nu}$$

The second term:

$$F_{\nu\lambda} \frac{\partial}{\partial x_\lambda} F_{\mu\nu} = \frac{1}{2} F_{\nu\lambda} \frac{\partial}{\partial x_\lambda} F_{\mu\nu} + \frac{1}{2} F_{\lambda\nu} \frac{\partial}{\partial x_\nu} F_{\mu\lambda} = \frac{1}{2} F_{\nu\lambda} \frac{\partial}{\partial x_\lambda} F_{\mu\nu} + \frac{1}{2} F_{\nu\lambda} \frac{\partial}{\partial x_\nu} F_{\lambda\mu}$$

or

$$F_{\nu\lambda} \frac{\partial}{\partial x_\lambda} F_{\mu\nu} = \frac{1}{2} F_{\nu\lambda} \left(\frac{\partial}{\partial x_\lambda} F_{\mu\nu} + \frac{\partial}{\partial x_\nu} F_{\lambda\mu}\right) = -\frac{1}{2} F_{\nu\lambda} \frac{\partial}{\partial x_\mu} F_{\nu\lambda} = -\frac{1}{4} \frac{\partial}{\partial x_\mu}(F_{\nu\lambda} F_{\nu\lambda})$$

Here we use the Jacobi identity;

$$\frac{\partial}{\partial x_\lambda} F_{\mu\nu} + \frac{\partial}{\partial x_\mu} F_{\nu\lambda} + \frac{\partial}{\partial x_\nu} F_{\lambda\mu} = 0 \qquad \text{(Jacobi identity)}$$

Then we have

$$F_{\nu\lambda} \frac{\partial}{\partial x_\lambda} F_{\mu\nu} = -\frac{1}{4} \delta_{\mu\lambda} \frac{\partial}{\partial x_\lambda}(F_{\sigma\tau} F_{\sigma\tau})$$

The force density is rewritten as

$$f_\mu = \frac{1}{\mu_0} \frac{\partial}{\partial x_\lambda}(F_{\mu\nu} F_{\nu\lambda} + \frac{1}{4} \delta_{\mu\lambda} F_{\sigma\tau} F_{\sigma\tau}) = \frac{\partial T_{\mu\lambda}}{\partial x_\lambda}$$

with the symmetric energy-momentum tensor (Maxwell's stress tensor)

$$T_{\mu\nu} = \frac{1}{\mu_0}(F_{\mu\lambda}F_{\lambda\nu} + \frac{1}{4}\delta_{\mu\lambda}F_{\sigma\tau}F_{\sigma\tau})$$

$$Tr[T_{\mu\nu}] = T_{\mu\mu} = \frac{1}{\mu_0}(F_{\mu\lambda}F_{\lambda\mu} + \frac{1}{4}F_{\sigma\tau}F_{\sigma\tau}) = 0$$

7.3 Conservation law

$$\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{J}$$

$$u = \frac{1}{2}(\varepsilon_0\mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2)$$

$$\varepsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t} + \mathbf{f} = (\nabla\cdot\overleftrightarrow{\mathbf{T}})$$

where

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) \qquad \text{: pointing vector}$$

$$\mathbf{G} = \varepsilon_0\mu_0\mathbf{S} = \frac{1}{c^2}\mathbf{S} \qquad \text{: momentum of the field}$$

$$\mathbf{f} = \rho\mathbf{E} + (\mathbf{J}\times\mathbf{B})$$

$$T_{ij} = (\varepsilon_0 E_i E_j + \frac{1}{\mu_0}B_i B_j) - \frac{1}{2}\delta_{ij}(\varepsilon_0\mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2)$$

or

$$\mu_0 T_{ij} = (\frac{1}{c^2}E_i E_j + B_i B_j) - \frac{1}{2}\delta_{ij}(\frac{1}{c^2}\mathbf{E}^2 + \mathbf{B}^2)$$

where $c^2 = \frac{1}{\varepsilon_0\mu_0}$

$$(J_\mu) = (\mathbf{J}, ic\rho)$$

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & B_3 & -B_2 & -\frac{i}{c}E_1 \\ -B_3 & 0 & B_1 & -\frac{i}{c}E_2 \\ B_2 & -B_1 & 0 & -\frac{i}{c}E_3 \\ \frac{i}{c}E_1 & \frac{i}{c}E_2 & \frac{i}{c}E_3 & 0 \end{pmatrix}$$

$$\mu_0 T_{\mu\nu} = F_{\mu\alpha}F_{\alpha\nu} + \frac{1}{4}\delta_{\mu\nu}F_{\alpha\beta}F_{\alpha\beta}$$

where

$$F_{\alpha\beta}F_{\alpha\beta} = 2[(B_1^2 + B_2^2 + B_3^2) - \frac{1}{c^2}(E_1^2 + E_2^2 + E_3^2)]$$

$$F_{1\alpha}F_{\alpha 2} = \frac{1}{c^2}E_1E_2 + B_1B_2$$

$$F_{2\alpha}F_{\alpha 3} = \frac{1}{c^2}E_2E_3 + B_2B_3$$

$$F_{1\alpha}F_{\alpha 3} = \frac{1}{c^2}E_3E_1 + B_3B_1$$

$$F_{1\alpha}F_{\alpha 4} = -\frac{i}{c}(E_2B_3 - B_2E_3) = -\frac{i\mu_0}{c}S_1$$

$$F_{2\alpha}F_{\alpha 4} = -\frac{i}{c}(E_3B_1 - B_1E_3) = -\frac{i\mu_0}{c}S_2$$

$$F_{3\alpha}F_{\alpha 4} = -\frac{i}{c}(E_1B_2 - B_2E_1) = -\frac{i\mu_0}{c}S_3$$

$$F_{1\alpha}F_{\alpha 1} = \frac{1}{c^2}E_1^2 - (B_2{}^2 + B_3{}^2)$$

$$F_{2\alpha}F_{\alpha 2} = \frac{1}{c^2}E_2^2 - (B_3{}^2 + B_1{}^2)$$

$$F_{3\alpha}F_{\alpha 3} = \frac{1}{c^2}E_3^2 - (B_1{}^2 + B_2{}^2)$$

$$F_{4\alpha}F_{\alpha 4} = \frac{1}{c^2}(E_1^2 + E_2^2 + E_3^2)$$

The Maxwell's stress tensor is given by

$$\mu_0 T_{11} = \frac{1}{2c^2}(E_1^{\ 2} - E_2^{\ 2} - E_3^{\ 2}) + \frac{1}{2}(B_1^{\ 2} - B_2^{\ 2} - B_3^{\ 2})$$

$$\mu_0 T_{12} = \frac{1}{c^2} E_1 E_2 + B_1 B_2$$

$$\mu_0 T_{13} = \frac{1}{c^2} E_3 E_1 + B_3 B_1$$

$$\mu_0 T_{14} = -\frac{i}{c}(E_2 B_3 - B_2 E_3) = -\frac{i\mu_0}{c} S_1 = -i\mu_0 c G_1$$

$$\mu_0 T_{22} = \frac{1}{2c^2}(-E_1^{\ 2} + E_2^{\ 2} - E_3^{\ 2}) + \frac{1}{2}(-B_1^{\ 2} + B_2^{\ 2} - B_3^{\ 2})$$

$$\mu_0 T_{23} = \frac{1}{c^2} E_2 E_3 + B_2 B_3$$

$$\mu_0 T_{24} = -\frac{i}{c}(E_3 B_1 - B_1 E_3) = -\frac{i\mu_0}{c} S_2 = -i\mu_0 c G_2$$

$$\mu_0 T_{33} = \frac{1}{2c^2}(-E_1^{\ 2} - E_2^{\ 2} + E_3^{\ 2}) + \frac{1}{2}(-B_1^{\ 2} - B_2^{\ 2} + B_3^{\ 2})$$

$$\mu_0 T_{34} = -\frac{i}{c}(E_1 B_2 - B_2 E_1) = -\frac{i\mu_0}{c} S_3 = -i\mu_0 c G_3$$

$$\mu_0 T_{44} = \frac{1}{2}(B_1^{\ 2} + B_2^{\ 2} + B_3^{\ 2}) + \frac{1}{2c^2}(E_1^{\ 2} + E_2^{\ 2} + E_3^{\ 2}) = \mu_0 u$$

Explicitly, the elements of *T* are

$$(T_{\mu\nu}) = \begin{pmatrix} T_{11} & T_{12} & T_{13} & -icG_1 \\ T_{21} & T_{22} & T_{23} & -icG_2 \\ T_{31} & T_{33} & T_{34} & -icG_3 \\ -icG_1 & -icG_2 & -icG_3 & u \end{pmatrix}$$

$$T_{11} + T_{22} + T_{33} = -u$$

8. Lorentz force

8.1 Origin of the Lorentz force

Consider a particle of charge q moving with velocity $\boldsymbol{v}$ (along the x axis) with respect to the reference frame S in a region with electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$.

In the frame S, the Lorentz force on this charge is given by

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = (qE_1, q(E_2 - vB_3), q(E_3 + vB_2))$$

In the frame S’, the Lorentz force is given by

$$\mathbf{F}' = q\mathbf{E}' = (qE_1', qE_2', qE_3')$$

where q is a relativistic invariant and is at rest.

The fields in S and S' are related by

$$E_1' = E_1$$
$$E_2' = \gamma(E_2 - vB_3)$$
$$E_3' = \gamma(E_3 + vB_2)$$

Then we have

$$F_1' = qE_1' = qE_1$$
$$F_2' = qE_2' = q\gamma(E_2 - vB_3)$$
$$F_3' = qE_3' = q\gamma(E_3 + vB_2)$$

What is the relation between ***F*** and ***F***'?

$$F_1' = qE_1' = qE_1 = F_1$$
$$F_2' = qE_2' = q\gamma(E_2 - vB_3) = \gamma F_2$$
$$F_3' = qE_3' = q\gamma(E_3 + vB_2) = \gamma F_3$$

or

$$F_1 = F_1'$$
$$\gamma F_2 = F_2'$$
$$\gamma F_3 = F_3'$$

8.2 force density and charge density

$$\mathbf{f} = \rho\mathbf{E} + (\mathbf{J} \times \mathbf{B})$$

We choose the frame S' in which the system with the charge density is at rest.
We now calculate the force density vector

$$\mathbf{f}' = \rho'\mathbf{E}'$$

when $\mathbf{J}' = 0$ (the system is at rest).

We note the Lorentz transformation of 4-dimensional vector, current density and charge density

$$J_\mu = (\mathbf{J}, ic\rho)$$

$$J_1 = \gamma(J_1' - i\beta J_4') = \gamma(J_1' + v\rho')$$
$$\rho = \gamma(\frac{\beta}{c} J_1' + \rho')$$

Then we have

$$\rho = \gamma\rho'$$
$$J_1 = \gamma v\rho' = \rho v$$

The Lorentz transformation of ***E*** and ***B***,

$$E_1' = E_1$$
$$E_2' = \gamma(E_2 - vB_3)$$
$$E_3' = \gamma(E_3 + vB_2)$$

Then we have

$$\gamma\mathbf{f}' = (\rho'\gamma E_1, \rho'\gamma^2(E_2 - vB_3), \rho'\gamma^2(E_3 + vB_2))$$

or

$$\gamma\mathbf{f}' = (\rho E_1, \rho\gamma(E_2 - vB_3), \rho\gamma(E_3 + vB_2))$$

In the frame *S*, the Lorentz force is given by

$$\mathbf{f} = \rho[\mathbf{E} + (\mathbf{v}\times\mathbf{B})] = (\rho E_1, \rho(E_2 - vB_3), \rho(E_3 + vB_2))$$

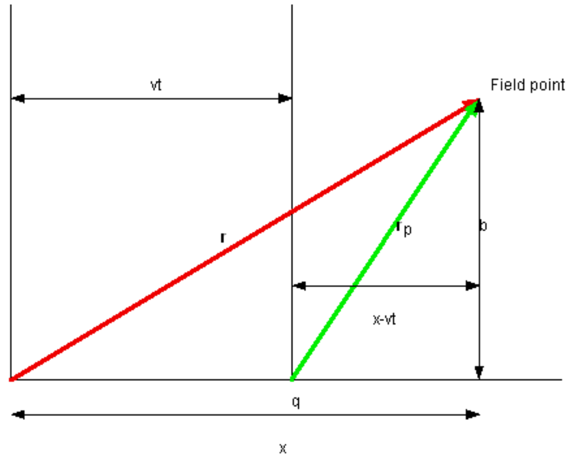
Thus we have

$$\gamma f_1' = f_1$$
$$f_2' = f_2$$
$$f_3' = f_3$$

9. Lienard-Wiechert potential

9.1 Lienard-Wiechert potential

We now consider the Lienard-Wiechert potential



In the *S'* frame:

$$\phi' = \frac{q}{4\pi\varepsilon_0}\frac{1}{r'}$$

$$\mathbf{A}' = 0$$

$$A_1 = \frac{A_1' + \frac{v}{c^2}\phi'}{\sqrt{1-\beta^2}} \qquad\qquad A_1 = \frac{\frac{v}{c^2}\phi'}{\sqrt{1-\beta^2}}$$

$$A_2 = A_2' \qquad \text{or} \qquad A_2 = 0$$

$$A_3 = A_3' \qquad\qquad A_3 = 0$$

$$\phi = \frac{vA_1' + \phi'}{\sqrt{1-\beta^2}} \qquad\qquad \phi = \frac{\phi'}{\sqrt{1-\beta^2}} = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{1-\beta^2}}\frac{1}{r'}$$

Then we get

$$\phi = \frac{\phi'}{\sqrt{1-\beta^2}} = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{1-\beta^2}}\frac{1}{\sqrt{x_1'^2 + x_2'^2 + x_3'^2}}$$

where

$$x_1' = \gamma(x_1 - vt)$$
$$x_2' = x_2$$
$$x_3' = x_3$$

The scalar potential ϕ is given by

$$\phi = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{1-\beta^2}}\frac{1}{\sqrt{\gamma^2(x_1-vt)^2+x_2{}^2+x_3{}^2}} = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{(x_1-vt)^2+(1-\beta^2)(x_2{}^2+x_3{}^2)}}$$

or

$$\phi = \frac{q}{4\pi\varepsilon_0}\frac{1}{R^*}$$

with

$$R^* = \sqrt{(x_1-vt)^2+(1-\beta^2)(x_2{}^2+x_3{}^2)}$$

Similarly we have for the vector potential

$$\mathbf{A} = (A_1, 0, 0)$$

with

$$A_1 = \frac{\frac{v}{c^2}\phi'}{\sqrt{1-\beta^2}} = \frac{v}{c^2}\frac{q}{4\pi\varepsilon_0}\frac{1}{R^*} = \frac{qv\mu_0}{4\pi}\frac{1}{R^*}$$

The electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ are given by

$$\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A} - \nabla\phi \qquad \text{and} \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}(1-\beta^2)\frac{\mathbf{R}}{R^{*3}}$$

$$\mathbf{B} = \frac{\mathbf{v}}{c^2}\times\mathbf{E}$$

where

$$\mathbf{R} = (x - vt, y, z)$$

For a slow moving charge ($v<<c$), we can take for $\boldsymbol{E}$ the Coulomb field. Then w have

$$\mathbf{B} = \frac{\mathbf{v}}{c^2}\times\mathbf{E} = \frac{q\mathbf{v}\times\mathbf{r}}{4\pi\varepsilon_0 c^2 r^2} = \frac{\mu_0}{4\pi}\frac{q\mathbf{v}\times\mathbf{r}}{r^2}$$

((Mathematica-10))

Lienard-Wiechert potential

```
<<Calculus`VectorAnalysis`
SetCoordinates[Cartesian[x,y,z]]
Cartesian[x,y,z]
```

$R = \sqrt{(x - v t)^2 + (1-\beta^2)(y^2+z^2)}$

$\sqrt{(-t v + x)^2 + (y^2+z^2)(1-\beta^2)}$

$\phi = \frac{q}{4\pi \epsilon0}\frac{1}{R}$

$\frac{q}{4\pi\sqrt{(-t v+x)^2+(y^2+z^2)(1-\beta^2)}\,\epsilon0}$

$A1 = \frac{q v \mu0}{4\pi}\frac{1}{R}$

$\frac{q v \mu0}{4\pi\sqrt{(-t v+x)^2+(y^2+z^2)(1-\beta^2)}}$

```
A={A1,0,0}
```

$\left\{\frac{q v \mu0}{4\pi\sqrt{(-t v+x)^2+(y^2+z^2)(1-\beta^2)}}, 0, 0\right\}$

```
B1=Curl[A]//FullSimplify
```

$\left\{0, \frac{q v z(-1+\beta^2)\mu0}{4\pi((-t v+x)^2-(y^2+z^2)(-1+\beta^2))^{3/2}}, -\frac{q v y(-1+\beta^2)\mu0}{4\pi((-t v+x)^2-(y^2+z^2)(-1+\beta^2))^{3/2}}\right\}$

Electric field in the frame S

E1 = -Grad[ϕ] - D[A, t] /. {μ0 → 1/(ϵ0 c²)} // FullSimplify

$\left\{\frac{q(-c^2+v^2)(t v - x)}{4 c^2\pi((-t v+x)^2-(y^2+z^2)(-1+\beta^2))^{3/2}\epsilon0}, -\frac{q y(-1+\beta^2)}{4\pi((-t v+x)^2-(y^2+z^2)(-1+\beta^2))^{3/2}\epsilon0}, -\frac{q z(-1+\beta^2)}{4\pi((-t v+x)^2-(y^2+z^2)(-1+\beta^2))^{3/2}\epsilon0}\right\}$

```
V1={v,0,0}
{v,0,0}
```

$eq1 = \frac{1}{c^2}$ **Cross[V1, E1] // Simplify**

$\left\{0, \frac{q v z(-1+\beta^2)}{4 c^2\pi((-t v+x)^2-(y^2+z^2)(-1+\beta^2))^{3/2}\epsilon0}, -\frac{q v y(-1+\beta^2)}{4 c^2\pi((-t v+x)^2-(y^2+z^2)(-1+\beta^2))^{3/2}\epsilon0}\right\}$

eq1 - B1 /. {μ0 → $\frac{1}{c^2\epsilon0}$} // Simplify

```
{0,0,0}
```

9.2 Distribution of the electric field

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}(1-\beta^2)\frac{\mathbf{R}_p}{R^{*3}}$$

where

$$R^* = \sqrt{(x-vt)^2 + (1-\beta^2)(y^2+z^2)}$$

$$\mathbf{R}^* = (x-vt, \sqrt{1-\beta^2}\,y, \sqrt{1-\beta^2}\,z)$$

$$\mathbf{R}_p = (x-vt, y, z)$$

$\boldsymbol{R}_p$ is the relative coordinate of the field point and the charge point. The electric field is along the position vector $\boldsymbol{R}_p$. $\boldsymbol{R}_p$ is a vector from the instantaneous location of the charge in S to the point where $\boldsymbol{E}$ is measured in *S*.

((Mathematica-11))
The electric field of a charge moving with the constant speed with v ($\beta = v/c$) on the unit circle of the real space

Lienard-Wiechert problem; field for a uniformly moving charge

```
<<Graphics`PlotField`
E1X[θ_, β_] := x (1 - β^2)/(Sqrt[x^2 + (1 - β^2) y^2])^3 /. {x → Cos[θ], y → Sin[θ]} // Simplify
E1Y[θ_, β_] := y (1 - β^2)/(Sqrt[x^2 + (1 - β^2) y^2])^3 /. {x → Cos[θ], y → Sin[θ]} // Simplify
E1X[θ,β]
```

$$\frac{(1-\beta^2)\,\mathrm{Cos}[\theta]}{(\mathrm{Cos}[\theta]^2 - (-1+\beta^2)\,\mathrm{Sin}[\theta]^2)^{3/2}}$$

```
E1Y[θ,β]
```

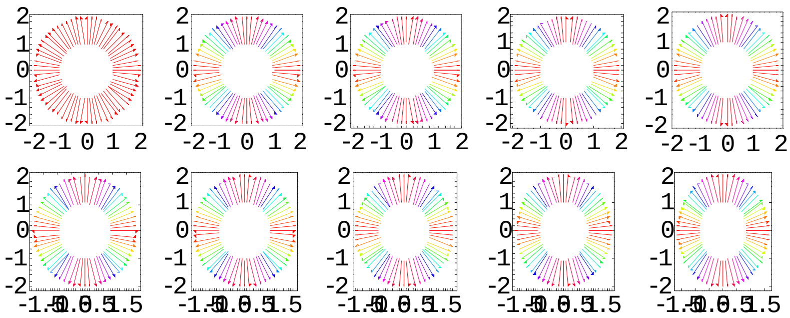
$$\frac{(1-\beta^2)\,\mathrm{Sin}[\theta]}{(\mathrm{Cos}[\theta]^2 - (-1+\beta^2)\,\mathrm{Sin}[\theta]^2)^{3/2}}$$

```
s1[β_]:=Table[{{E1X[θ,β],E1Y[θ,β]},{E1X[θ,β],E1Y[θ,β]}
},{θ,0,2 π,π/32}]

s2[β_]:=ListPlotVectorField[ Evaluate[s1[β]]//N//Chop,C
olorFunction→Hue,
AspectRatio→Automatic,ScaleFactor→1,Frame→True,PlotPoi
nts→20,AxesOrigin→{0,0},DeFaultColor→Hue[0.6],DisplayF
unction→Identity]
General::spell1 : Possible spelling error: new symbol
   name "DeFaultColor" is similar to existing symbol "DefaultColor". More…
```

β = 0, 0.04, 0.08, 0.12, 0.16
β = 0.20, 0.24, 0.28, 0.32, 0.36

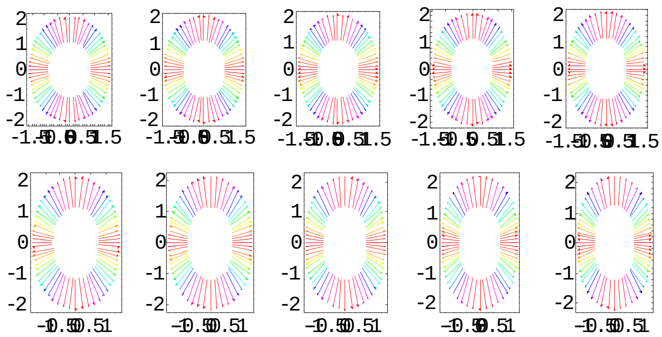
```
ps1=Evaluate[Table[s2[β],{β,0,0.36,0.04}]];Show[GraphicsArray[Partition[ps1,5]],DisplayFunction→$DisplayFunction]
```



-GraphicsArray-

β = 0.4, 0.42, 0.44, 0.46, 0.48
β = 0.50, 0.52, 0.54, 0.56, 0.58

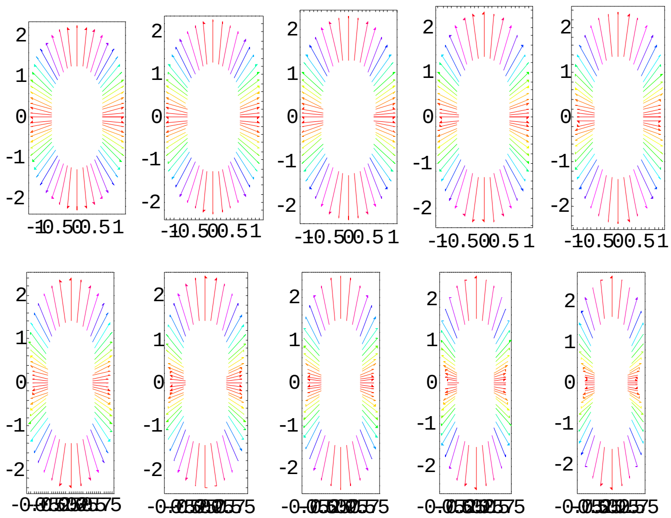
```
ps2=Evaluate[Table[s2[β],{β,0.4,0.58,0.02}]];Show[GraphicsArray[Partition[ps2,5]],DisplayFunction→$DisplayFunction]
```



-GraphicsArray-

β=0.6, 0.62, 0.64, 0.66, 0.68
β = 0.70, 0.72, 0.74, 0.76, 0.78

```
ps3=Evaluate[Table[s2[β],{β,0.6,0.78,0.02}]];Show[GraphicsArray[Partition[ps3,5]],DisplayFunction→$DisplayFunction]
```

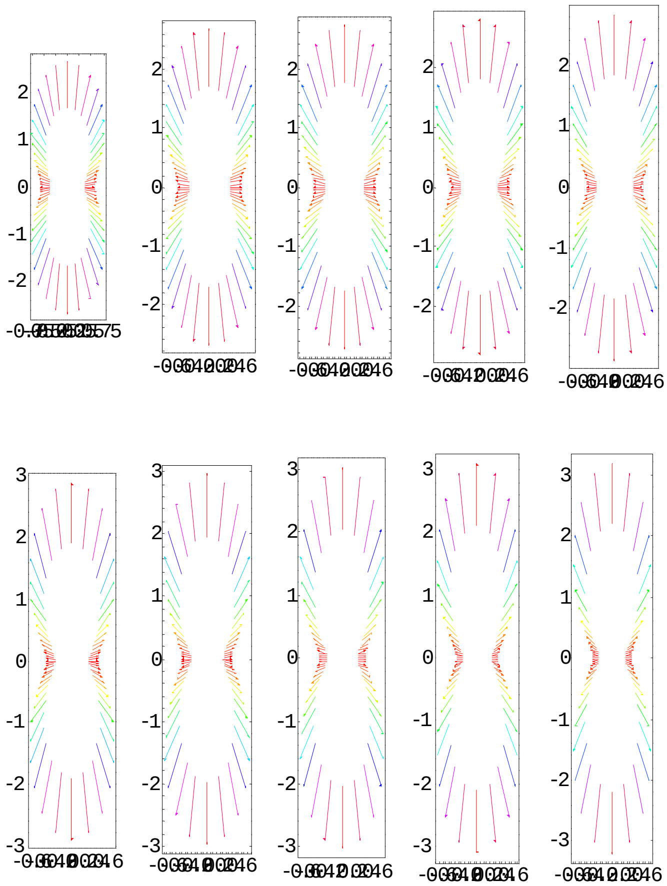


-GraphicsArray-

β = 0.80, 0.91, 0.82, 0.83, 0.84,

β = 0.85, 0.86, 0.87, 0.88, 0.89

```
ps4=Evaluate[Table[s2[β],{β,0.8,0.89,0.01}]];Show[GraphicsArray[Partition[ps4,5]],DisplayFunction→$DisplayFunction]
```

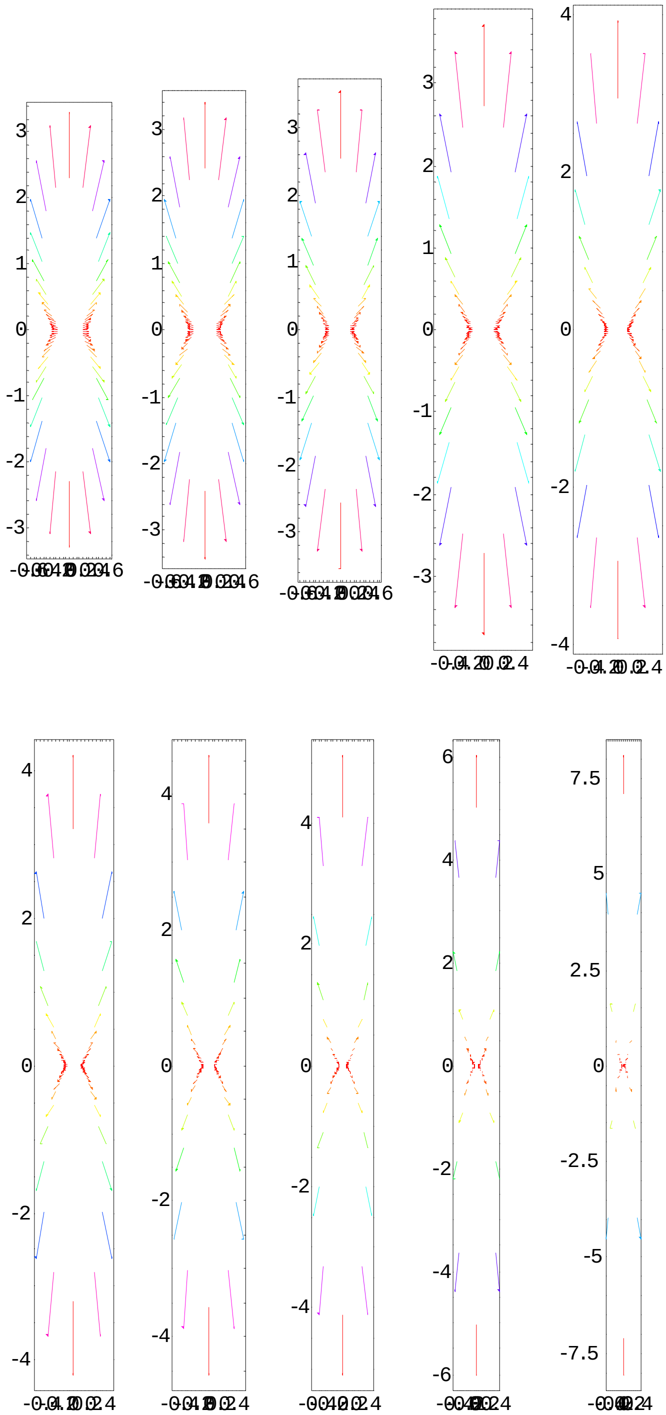


-GraphicsArray-

β=0.9, 0.91, 0.92, 0.93, 0.94

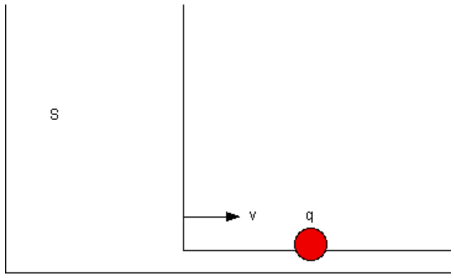
β=0.95, 0.96, 0.97, 0.98, 0.99

```
ps5=Evaluate[Table[s2[β],{β,0.9,0.99,0.01}]];Show[GraphicsArray[Partition[ps5,5]],DisplayFunction→$DisplayFunction]
```



-GraphicsArray-

10. Relativity of Electric field and magnetic field



$E = 0$ and $B \neq 0$

We consider the charge q moving along the x axis in the presence of the magnetic field $\boldsymbol{B}$ (the frame S). In *the frame S, there is only an external magnetic field* $\boldsymbol{B}$. Thus the magnetic force on the charge is given by

$$\mathbf{F}_{\perp} = q(\mathbf{v} \times \mathbf{B})$$

Suppose that there is no electric field ($\boldsymbol{E}$ = 0) in the frame S $(\mathbf{B} \neq 0)$. The $\boldsymbol{E}$' and $\boldsymbol{B}$' in the frame S' are related to those in the frame S as

$$E_1' = E_1 = 0$$
$$E_2' = \gamma(E_2 - c\beta B_3) = -\gamma v B_3$$
$$E_3' = \gamma(E_3 + c\beta B_2) = \gamma v B_2$$

$$B_1' = B_1$$
$$B_2' = \gamma(B_2 + \frac{\beta}{c}E_3) = \gamma B_2$$
$$B_3' = \gamma(B_3 - \frac{\beta}{c}E_2) = \gamma B_3$$

or

$$\mathbf{E}' = \gamma(\mathbf{v} \times \mathbf{B}) = \mathbf{v} \times \mathbf{B}' \qquad (1)$$

Then the force (electric force) on the charge q in the frame S' is

$$\mathbf{F}'_{\perp} = q\mathbf{E}' = q\gamma(\mathbf{v} \times \mathbf{B})$$

since the charge q is the same for any frame and the particle is at rest in the frame S'. There is no force due to $\boldsymbol{B}$' since the particle is at rest in the frame S'. $\mathbf{F}'_{\perp}$ is the force of $\boldsymbol{F}$' in a direction perpendicular to the velocity $\boldsymbol{v}$. Thus we have

$$\mathbf{F}_{\perp} = \frac{1}{\gamma}\mathbf{F}_{\perp}'$$

11. Derivation of the Biot Savart law

<u>***B***'= 0 and ***E***'$\neq$ 0.</u>

We consider that the magnetic field ***B'****=0 in the frame S'*. In the frame *S'*, there is only an external electric field ***E'*** (the point charge is at rest). The ***E*** and ***B*** in the frame *S* are related to those in the frame *S'* as

$$E_1 = E_1' \qquad B_1 = 0$$

$$E_2 = \gamma E_2' \qquad B_2 = -\frac{\gamma}{c^2} v E_3'$$

$$E_3 = \gamma E_3' \qquad B_3 = \frac{\gamma}{c}\beta E_2' = \frac{\gamma}{c^2} v E_2'$$

or

$$\mathbf{B} = \frac{\gamma}{c^2}(\mathbf{v}\times\mathbf{E}') = \frac{1}{c^2}(\mathbf{v}\times\mathbf{E}), \qquad (2)$$

Using the result from the Lienard-Wiechert potential (β<<1) (see Sec.8)

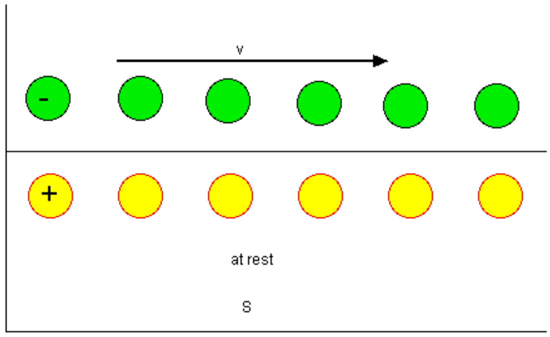
$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}(1-\beta^2)\frac{\mathbf{R}}{R^{*3}} \approx \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3}$$

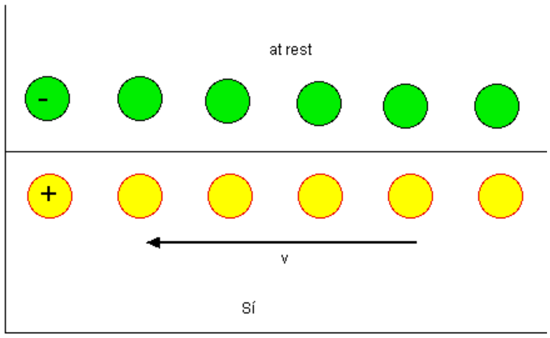
$$\mathbf{B} = \frac{1}{c^2}(\mathbf{v}\times\mathbf{E}) = \frac{1}{c^2}\frac{q}{4\pi\varepsilon_0}\frac{\mathbf{v}\times\mathbf{r}}{r^3} = \frac{\mu_0}{4\pi}\frac{q\mathbf{v}\times\mathbf{r}}{r^3}$$

which is the application of the Biot-Savart law to a point charge.

12. Ampere's law (Feynman 13-9)

We consider that the electrons located on the linear chain (the line density $-\lambda_0$) moves at the velocity ***v***. At the same time there are positive ions located on the same chain (the line density λ_0). We now consider the frame S' which moves at the velocity ***v***.





((Formula))

$$\rho = \gamma\rho'$$

where ρ for the frame where the particle moves at the velocity v along the x axis, and ρ' for the frame where the particle is at rest.

We assume that

(1) The line densities of electrons and positive ions are given by $-\lambda_0$ and $+\lambda_0$ in the frame S.

(2) The line densities of electrons and positive ions are given by λ_- and λ_+ in the frame *S*'

$$(-\lambda_0) = \gamma(-\lambda_-) \qquad \text{or} \qquad \lambda_- = \frac{1}{\gamma}\lambda_0 = \sqrt{1-\frac{v^2}{c^2}}\lambda_0 \qquad \text{for electrons}$$

$$\lambda_+ = \gamma\lambda_0 \qquad \text{or} \qquad \lambda_+ = \gamma\lambda_0 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\lambda_0 \qquad \text{for ions}$$

The net line charge density in the frame S' is

$$\lambda' = \lambda_+ - \lambda_- = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\lambda_0 - \sqrt{1-\frac{v^2}{c^2}}\lambda_0 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{v^2}{c^2}\lambda_0 = \gamma\frac{v^2}{c^2}\lambda_0$$

((Note))

This relation can be also derived from the Lorentz transformation of the 4-dimensional current density

$$J_\mu = (\mathbf{J}, ic\rho) = (\rho\mathbf{v}, ic\rho)$$

$$J_\mu' = a_{\mu\nu}J_\nu$$

$$J_1' = \gamma(J_1 + i\beta J_4) = \gamma(J_1 - v\rho)$$
$$\rho' = \gamma(-\frac{\beta}{c}J_1 + \rho)$$

$$J_\mu = a_{\nu\mu}J_\nu'$$

$$J_1 = \gamma(J_1' - i\beta J_4') = \gamma(J_1' + v\rho')$$
$$\rho = \gamma(\frac{\beta}{c}J_1' + \rho')$$

Here we define

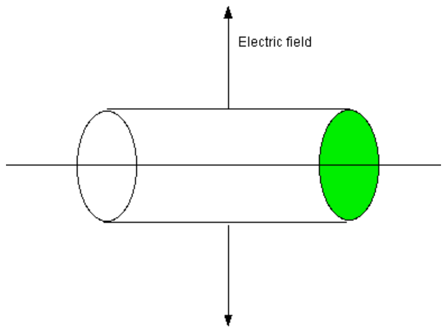
$$\lambda = A\rho$$
$$\lambda' = A\rho'$$

A is the same for the S and S', since the plane of A is perpendicular to $\boldsymbol{v}$.

$$\lambda' = \gamma(-\frac{\beta}{c}J_1 + \lambda) = \gamma\frac{\beta}{c}\lambda_0 v = \gamma\frac{v^2}{c^2}\lambda_0$$

where $I_1 = AJ_1 = (-\lambda_0)v$ and $\lambda = 0$.

So the positive line density produces an electric field $\boldsymbol{E}$'. We use the Gauss's law.



The electric field ***E***' at the distance *s* from the axis of the cylinder,

$$E'(2\pi sh) = \frac{1}{\varepsilon_0}(h\lambda')$$

where *s* is the radius of the Gaussian surface (cylinder).

or

$$E' = \frac{\lambda'}{2\pi\varepsilon_0 s} = \frac{1}{2\pi\varepsilon_0 s}\gamma\frac{v^2}{c^2}\lambda_0$$

So there is an electrical force on *q* in *S*';

$$F_\perp' = qE' = \frac{q}{2\pi\varepsilon_0 s}\gamma\frac{v^2}{c^2}\lambda_0 .$$

But if there is a force on the test charge *q* in *S*', there must be one in *S*. In fact, one can calculate it by using the transformation rules for forces. Since *q* is at rest *S*' and $F_\perp$ is perpendicular to the *x* axis. Then we have

$$F_\perp' = \gamma F_\perp \qquad \text{or} \qquad F_\perp = \frac{1}{\gamma}F_\perp'$$

Using this result we have

$$F_\perp = \frac{1}{\gamma}F_\perp' = \frac{q}{2\pi\varepsilon_0 s}\frac{v^2}{c^2}\lambda_0 = \frac{q\mu_0}{2\pi s}v(v\lambda_0) = q\frac{\mu_0(v\lambda_0)}{2\pi s}v$$

where $B = \frac{\mu_0(v\lambda_0)}{2\pi s}$ is a magnetic field due to the line current density $v\lambda_0$ (Ampere's law). The force has a form as $F = qvB$.

13. Derivation of the Ampere's law from relativity

We analyze the fields and currents as viewed from two frames. *S* where the ions are at rest. *S'* where the electrons are, on the average, at rest.

$$J_\mu = (\mathbf{J}, ic\rho)$$

$$J_\mu = (a^{-1})_{\mu\nu} J_\nu' = a_{\nu\mu} J_\nu'$$

Multiplying the cross-sectional area (A) of the wires, we obtain the following transformation for currents and linear charge densities.

$$I_\mu = AJ_\mu = (A\mathbf{J}, icA\rho) = (\mathbf{I}, ic\lambda)$$

$$I_\mu = a_{\nu\mu} I_\nu'$$

$$I_1 = \gamma(I_1' - i\beta I_4') = \gamma(I_1' + v\lambda')$$

$$I_4 = ic\lambda = \gamma(i\beta I_1' + I_4') = \gamma(i\frac{v}{c} I_1' + ic\lambda')$$

or

$$I_\pm = \gamma(I_\pm' + v\lambda_\pm')$$

$$\lambda_\pm = \gamma(\frac{v}{c^2} I_\pm' + \lambda_\pm')$$

where $\lambda = A\rho$, the subscript 1 is neglected and the plus and minus subscript refer to the ions and the electrons, respectively.

In *S'* we know that $I_-' = 0$ since the electrons are at rest.

$$\lambda_- = \gamma(\frac{v}{c^2} I_-' + \lambda_-') = \gamma\lambda_-'$$

In *S* the net charge per unit length must vanish.

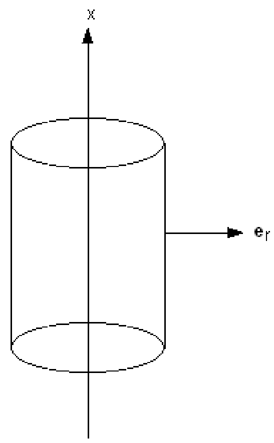
$$0 = \lambda_+ + \lambda_- = \lambda_+ + \gamma\lambda_-'$$

or

$$\lambda_-' = -\frac{\lambda_+}{\gamma}$$

The fields in *S*’ due to λ_-' are

$$\mathbf{E}_-' = \frac{\lambda_-'}{2\pi\varepsilon_0 r'}\mathbf{e}_r$$
$$\mathbf{B}_-' = 0$$



The fields in *S* due to λ_+ are

$$\mathbf{E}_+ = \frac{\lambda_+}{2\pi\varepsilon_0 r}\mathbf{e}_r$$
$$\mathbf{B}_+ = 0$$

We now consider the field transformation for

$$\mathbf{E}_-' = \frac{\lambda_-'}{2\pi\varepsilon_0 r'}\mathbf{e}_r$$
$$\mathbf{B}_-' = 0$$

noting that $\hat{r} = \hat{r}'$ (perpendicular to the *x* axis), we find that the fields in *S* are

$$E_1 = E_1' = 0 \qquad B_1 = B_1' = 0$$
$$E_2 = \gamma(E_2' + c\beta B_3') = \gamma E_2' \qquad B_2 = \frac{\gamma}{c}(cB_2' - \beta E_3') = -\frac{\gamma v}{c^2}E_3'$$
$$E_3 = \gamma E_3' \qquad B_3 = \frac{\gamma}{c}(\beta E_2' + cB_3') = \frac{\gamma v}{c^2}E_2'$$

or

$$\mathbf{E}_- = \gamma \frac{\mathbf{e}_r}{2\pi\varepsilon_0 r} \lambda_-',$$

$$\mathbf{B}_- = \frac{\gamma}{c^2}(\mathbf{v} \times \mathbf{E}_-')$$

Then the total fields in the frame S are

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\mathbf{e}_r}{2\pi\varepsilon_0 r}(\lambda_+ + \gamma\lambda_-') = 0$$

$$\mathbf{B} = \mathbf{B}_+ + \mathbf{B}_- = \mathbf{B}_- = \frac{\gamma}{c^2}(\mathbf{v} \times \mathbf{E}_-') = \frac{\gamma}{c^2}(\mathbf{v} \times \mathbf{e}_r)\frac{\lambda_-'}{2\pi\varepsilon_0 r'} = \frac{\gamma\lambda_-' v}{2\pi\varepsilon_0 c^2}\frac{(\mathbf{e}_x \times \mathbf{e}_r)}{r}$$

Since $I_- = \gamma(I_-' + v\lambda_-')$ and $I_-' = 0$, we have

$$I_- = \gamma v \lambda_-'$$

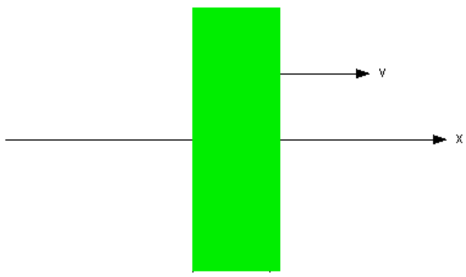
Using $\mathbf{e}_x \times \mathbf{e}_r = \mathbf{e}_\phi$, we obtain

$$\mathbf{B} = \frac{\mu_0 I_-}{2\pi r}\mathbf{e}_\phi$$
$$\mathbf{E} = 0$$

We see that a magnetic field due to current flow is a relativistic effect.

14. Capacitance moving along the *x* axis with a uniform velocity

14.1 The capacitance moves along the *x* direction which is parallel to the electric field of the capacitance.



In the frame S' where the charges are at rest.

$$E_1' = \frac{\sigma'}{\varepsilon_0}$$
$$E_2' = 0$$
$$E_3' = 0$$

and

$$B_1' = 0$$
$$B_2' = 0$$
$$B_3' = 0$$

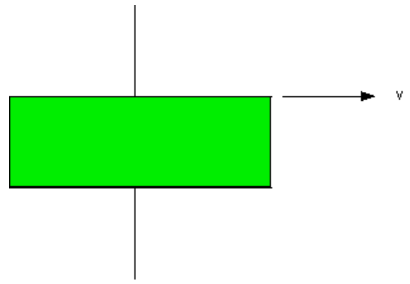
$$E_1 = E_1' \qquad B_1 = 0$$
$$E_2 = 0 \qquad B_2 = 0$$
$$E_3 = 0 \qquad B_3 = 0$$

Thus we have

$$E_1 = E_1'$$

where $\sigma' = \sigma$

14.2 The capacitance moves along the *x* direction which is perpendicular to the electric field of the capacitance.



In the frame S' where the charges are at rest.

$$E_1' = 0 \qquad B_1' = 0$$
$$E_2' = 0 \qquad B_2' = 0$$
$$E_3' = \frac{\sigma'}{\varepsilon_0} \qquad B_3' = 0$$

where

$$\sigma = \gamma\sigma'$$

$$E_1 = E_1' = 0$$
$$E_2 = \gamma(E_2' + c\beta B_3') = 0$$
$$E_3 = \gamma(-c\beta B_2' + E_3') = \gamma E_3' = \gamma \frac{\sigma'}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

$$B_1 = B_1' = 0$$
$$B_2 = \gamma(B_2' - \frac{\beta}{c} E_3') = \frac{-\beta E_3'}{c\sqrt{1-\beta^2}} = -\gamma \frac{v}{c^2} \frac{\sigma'}{\varepsilon_0} = -\frac{v}{c^2} \frac{\sigma}{\varepsilon_0}$$
$$B_3 = \gamma(\frac{\beta}{c} E_2' + B_3') = 0$$

15. Relativistic-covariant Lagrangian formalism

15.1 Lagrangian *L* (simple case)

Proper time

$$(dx_\mu')^2 = a_{\mu\lambda} a_{\mu\sigma} dx_\lambda dx_\sigma = \delta_{\lambda\sigma} dx_\lambda dx_\sigma = (dx_\mu)^2$$

We define the proper time as

$$(ds)^2 = c^2 (dt)^2 - (dx_1)^2 - (dx_2)^2 - (dx_3)^2 = c^2 (dt')^2 - (dx_1')^2 - (dx_2')^2 - (dx_3')^2$$

$$(ds)^2 = c^2 (dt)^2 \{1 - \frac{1}{c^2}[(\frac{dx_1}{dt})^2 + (\frac{dx_2}{dt})^2 + (\frac{dx_3}{dt})^2]\} = c^2 (dt)^2 (1 - \frac{\mathbf{u}^2}{c^2})$$

or

$$d\tau = \frac{ds}{c} = dt\sqrt{1 - \frac{\mathbf{u}^2}{c^2}}$$

where τ is a proper time and ***u*** is the velocity of the particle in the frame *S*.

The integral $\int_a^b ds$ taken between a given pair of world points has its maximum value if it is taken along the straight line joining two points.

$$S = -\alpha \int_a^b ds = -\alpha c \int_{t_a}^{t_b} dt \sqrt{1 - \frac{\mathbf{u}^2}{c^2}} = \int_{t_a}^{t_b} L dt$$

where

$$L = -\alpha c\sqrt{1-\frac{\mathbf{u}^2}{c^2}}$$

Nonrelativistic case

$$L = -\alpha c(1-\frac{\mathbf{u}^2}{c^2})^{1/2} = -\alpha c(1-\frac{\mathbf{u}^2}{2c^2}) = \frac{\alpha}{2c}\mathbf{u}^2 - \alpha c$$

In the classical mechanics,

$$\frac{\alpha}{2c} = \frac{m}{2} \qquad \text{or} \qquad \alpha = mc$$

Therefore the Lagrangian L is given by

$$L = -mc^2(1-\frac{\mathbf{u}^2}{c^2})^{1/2}$$

The momentum $\boldsymbol{p}$ is defined by

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{u}} = \frac{m\mathbf{u}}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}} = m\mathbf{u}\gamma(\mathbf{u}) = m\frac{d\mathbf{r}}{d\tau} = m\frac{d\mathbf{r}}{dt}\frac{dt}{d\tau}$$

((Note))
This momentum coincides with the components of four-vector momentum p_μ defined by

$$p_\mu = m\frac{dx_\mu}{d\tau}$$

15.2 Hamiltonian

The Hamiltonian H is defined by

$$H = \mathbf{p}\cdot\mathbf{u} - L = \gamma(\mathbf{u})m\mathbf{u}^2 + mc^2\frac{1}{\gamma(\mathbf{u})} = \frac{\gamma(\mathbf{u})^2 m\mathbf{u}^2 + mc^2}{\gamma(\mathbf{u})} = \gamma(\mathbf{u})mc^2 = \frac{mc^2}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}} = E$$

or

$$E = \frac{mc^2}{\sqrt{1 - \frac{\mathbf{u}^2}{c^2}}}$$

We have

$$\frac{E^2}{c^2} = \frac{m^2c^2}{1 - \frac{\mathbf{u}^2}{c^2}} = \frac{m^2c^2(1 - \frac{\mathbf{u}^2}{c^2}) + m^2\mathbf{u}^2}{1 - \frac{\mathbf{u}^2}{c^2}} = m^2c^2 + \mathbf{p}^2$$

15.3 Lagrangian form in the presence of an electromagnetic field

The action function for a charge in an electromagnetic field

$$S = \int_a^b (-mcds + qA_\mu dx_\mu)$$

where the second term is invariant under the Lorentz transformation.

$$A_\mu = (\mathbf{A}, i\frac{1}{c}\phi), \qquad \text{and} \qquad dx_\mu = (dx_1, dx_2, dx_3, icdt)$$

Then we have

$$S = \int_a^b (-mcds + qA_\mu dx_\mu) = \int_a^b [-mc^2\sqrt{1 - \frac{\mathbf{u}^2}{c^2}} + q(\mathbf{A}\cdot\mathbf{u} - \phi)]dt$$

The integrand in the Lagrangian function of a charge (q) in the electromagnetic field,

$$L = -mc^2\sqrt{1 - \frac{\mathbf{u}^2}{c^2}} + q(\mathbf{A}\cdot\mathbf{u} - \phi)$$

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{u}} = \frac{m\mathbf{u}}{\sqrt{1 - \frac{\mathbf{u}^2}{c^2}}} + q\mathbf{A}$$

where

$$A_\mu = (\mathbf{A}, i\frac{1}{c}\phi)$$

The Hamiltonian H is given by

$$H = \mathbf{p}\cdot\mathbf{u} - L = \frac{m\mathbf{u}^2}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}} + e\mathbf{A}\cdot\mathbf{u} - (-mc^2\sqrt{1-\frac{\mathbf{u}^2}{c^2}} + q\mathbf{A}\cdot\mathbf{u} - q\phi)$$

or

$$H = \mathbf{p}\cdot\mathbf{u} - L = \frac{m\mathbf{u}^2}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}} + e\mathbf{A}\cdot\mathbf{u} - (-mc^2\sqrt{1-\frac{\mathbf{u}^2}{c^2}} + q\mathbf{A}\cdot\mathbf{u} - q\phi)$$

$$H = \frac{mc^2}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}} + q\phi$$

or

$$\left(\frac{H - q\phi}{c}\right)^2 = \frac{m^2c^2(1-\frac{\mathbf{u}^2}{c^2}) + m^2\mathbf{u}^2}{1-\frac{\mathbf{u}^2}{c^2}} = m^2c^2 + (\mathbf{p} - q\mathbf{A})^2$$

15.4 Expression for the Lagrangian in terms of 4-dimensional velocity

Here we use dτ instead of dt in the expression of Lagrangian.

$$ds = cd\tau$$

η_μ is a four-dimensional velocity defined by

$$\eta_\mu = \frac{dx_\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx_\mu}{dt} = (\gamma(\mathbf{u})u_1, \gamma(\mathbf{u})u_2, \gamma(\mathbf{u})u_3, ic\gamma(\mathbf{u}))$$

$$A_\mu\eta_\mu = A_1\eta_1 + A_2\eta_2 + A_3\eta_3 + A_4\eta_4 = \gamma(\mathbf{u})(\mathbf{u}\cdot\mathbf{A} - \phi)$$

since

$$A_\mu = (\mathbf{A}, i\frac{1}{c}\phi), \qquad \eta_4 = \frac{dt}{d\tau}\frac{dx_4}{dt} = ic\frac{dt}{d\tau}$$

$$S = \int_a^b (-mcds + qA_\mu dx_\mu) = \int_a^b (-mc^2 + qA_\mu\cdot\eta_\mu)d\tau$$

$$L = -mc^2 + qA_\mu \eta_\mu$$

15.5 Lagrangian and Hamiltonian in terms of the field tensor $F_{\mu\nu}$

$$F_{\mu\nu}F_{\mu\nu} = 2({B_1}^2 + {B_2}^2 + {B_3}^2) - \frac{2}{c^2}({E_1}^2 + {E_2}^2 + {E_3}^2)$$

This is invariant under the Lorentz transformation.

We may try the Lagrangian density

$$L = -\frac{1}{4\mu_0} F_{\mu\nu}F_{\mu\nu} + J_\mu A_\mu$$

By regarding each component of A_μ as an independent field, we find that the Lagrange equation

$$\frac{\partial L}{\partial A_\mu} = \frac{\partial}{\partial x_\nu}[\frac{\partial L}{\partial(\frac{\partial A_\mu}{\partial x_\nu})}]$$

is equivalent to

$$\frac{\partial F_{\mu\nu}}{\partial x_\mu} = \mu_0 J_\mu .$$

The Hamiltonian density H_{em} for the free Maxwell field can be evaluated as follows.

$$L_{em} = -\frac{1}{4\mu_0} F_{\mu\nu}F_{\mu\nu}$$

$$H_{em} = \frac{\partial L_{em}}{\partial\left(\frac{\partial A_\mu}{\partial x_4}\right)}\frac{\partial A_\mu}{\partial x_4} - L_{em} = -\frac{F_{4\mu}}{\mu_0}(F_{4\mu} + \frac{\partial A_4}{\partial x_\mu}) - \frac{1}{2\mu_0}(\mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2)$$

or

$$H_{em} = \frac{1}{2}\varepsilon_0 \mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2 - \varepsilon_0 \mathbf{E}\cdot\nabla\phi$$

$$\int H_{em} d\mathbf{r} = \frac{1}{2}\int (\varepsilon_0 \mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2) d\mathbf{r} - \int \varepsilon_0 (\mathbf{E}\cdot\nabla\phi) d\mathbf{r} = \frac{1}{2}\int (\varepsilon_0 \mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2) dr$$

((Note))

$$\int (\mathbf{E}\cdot\nabla\phi) d\mathbf{r} = \int [\nabla\cdot(\mathbf{E}\phi) - \phi\nabla\cdot\mathbf{E}] d\mathbf{r} = \int \nabla\cdot(\mathbf{E}\phi) d\mathbf{r} = \int (\mathbf{E}\phi)\cdot d\mathbf{a} = 0$$

where $\mathbf{E}\phi$ vanishes sufficiently rapidly at infinity.

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} = 0 \text{ (in this case).}$$

16. Relativistic form of Newton's law

16.1 Relativistic force

We define the force ***F*** and the kinetic energy *T* as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

$$\frac{dT}{dt} = A = \mathbf{F}\cdot\mathbf{u}$$

where ***u*** is the velocity of the particle.

$$\frac{dT}{dt} = \mathbf{u}\cdot\mathbf{F} = \mathbf{u}\cdot\frac{d\mathbf{p}}{dt} = \mathbf{u}\cdot\frac{d}{dt}\frac{m\mathbf{u}}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}} = m\mathbf{u}\cdot\frac{d\mathbf{u}}{dt}\frac{1}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}} + m\mathbf{u}^2\frac{d}{dt}\frac{1}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}}$$

or

$$A = \frac{dT}{dt} = m\mathbf{u}\cdot\frac{d\mathbf{u}}{dt}\frac{1}{\left(1-\frac{\mathbf{u}^2}{c^2}\right)^{1/2}} + m\mathbf{u}^2\frac{\mathbf{u}\cdot\frac{d\mathbf{u}}{dt}}{c^2\left(1-\frac{\mathbf{u}^2}{c^2}\right)^{3/2}}$$

or

$$A = \frac{dT}{dt} = m\mathbf{u}\cdot\frac{d\mathbf{u}}{dt}\left[\frac{1-\frac{\mathbf{u}^2}{c^2}}{\left(1-\frac{\mathbf{u}^2}{c^2}\right)^{3/2}} + \frac{\frac{\mathbf{u}^2}{c^2}}{\left(1-\frac{\mathbf{u}^2}{c^2}\right)^{3/2}}\right] = \frac{m\mathbf{u}\cdot\frac{d\mathbf{u}}{dt}}{\left(1-\frac{\mathbf{u}^2}{c^2}\right)^{3/2}} = \frac{\frac{m}{2}\frac{d\mathbf{u}^2}{dt}}{\left(1-\frac{\mathbf{u}^2}{c^2}\right)^{3/2}}]$$

$$dT = \frac{m}{2}\frac{1}{\left(1-\frac{\mathbf{u}^2}{c^2}\right)^{3/2}}d\mathbf{u}^2$$

$$T = \frac{mc^2}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}} + C$$

where C is a constant of integration. Since the kinetic energy may be taken as zero for u = 0.

16.2 Relativistic energy

Then we have C = =mc^2.

$$T = \frac{mc^2}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}} - mc^2$$

It is convenient to introduce a quantity E defined by

$$E = T + mc^2 = \frac{mc^2}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}} = \gamma(\mathbf{u})mc^2$$

E is the energy of a free particle

17. Four-dimensional momentum

17.1. Definition

$$ds = cdt\sqrt{1-\frac{\mathbf{u}^2}{c^2}} = \frac{cdt}{\gamma(\mathbf{u})}$$

$$d\tau = \frac{ds}{c} = \frac{dt}{\gamma(\mathbf{u})}$$

where τ is a proper time.

$$\gamma(\mathbf{u}) = \frac{1}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}}$$

Here we define the four-dimensional momentum

$$p = m\frac{d}{d\tau}x = m\gamma(\mathbf{u})\frac{dx_\mu}{dt} = (m\gamma(\mathbf{u})u_1, m\gamma(\mathbf{u})u_2, m\gamma(\mathbf{u})u_3, imc\gamma(\mathbf{u})) = (\mathbf{p}, i\frac{E}{c})$$

This is exactly the same as the expressions of ***p*** obtained from the Lagrangian.

$$E = mc^2\gamma(\mathbf{u}) = \frac{mc^2}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}}$$

17.2 Lorentz transformation

This momentum is clearly a four-vector since dx_μ is a Lorentz four-vector and *m* and dτ are Lorentz scalar. In fact, under the Lorentz transformation

$$p_\mu' = m\frac{d}{d\tau}x_\mu' = ma_{\mu\nu}\frac{d}{ds}x_\nu = a_{\mu\nu}p_\nu$$

$$(p_\mu')^2 = a_{\mu\nu}a_{\mu\lambda}p_\nu p_\lambda = \delta_{\nu\lambda}p_\nu p_\lambda = (p_\mu)^2$$

So this is invariant under the Lorentz transformation.

$$p = (\mathbf{p}, i\frac{E}{c})$$

$$p_1' = \frac{p_1 + i\beta p_4}{\sqrt{1-\beta^2}}$$
$$p_2' = p_2$$
$$p_3' = p_3$$
$$p_4' = \frac{-i\beta p_1 + p_4}{\sqrt{1-\beta^2}}$$

or

$$p_1' = \frac{p_1 - \frac{\beta}{c}E}{\sqrt{1-\beta^2}}$$
$$p_2' = p_2$$
$$p_3' = p_3$$
$$E' = \frac{E - \beta c p_1}{\sqrt{1-\beta^2}}$$

and

$$p_1 = \frac{p_1' - i\beta p_4'}{\sqrt{1-\beta^2}}$$
$$p_2 = p_2'$$
$$p_3 = p_3'$$
$$p_4 = \frac{i\beta p_1' + p_4'}{\sqrt{1-\beta^2}}$$

or

$$p_1 = \frac{p_1' + \frac{\beta}{c}E'}{\sqrt{1-\beta^2}}$$
$$p_2 = p_2'$$
$$p_3 = p_3'$$
$$E = \frac{E' + \beta c p_1'}{\sqrt{1-\beta^2}}$$

18. Four-dimensional velocity (or proper 4-velocity)

$$ds = cdt\sqrt{1-\frac{\mathbf{u}^2}{c^2}} = \frac{cdt}{\gamma(\mathbf{u})}$$

$$d\tau = \frac{ds}{c} = \frac{dt}{\gamma(\mathbf{u})}$$

where τ is a proper time. $\gamma(\mathbf{u}) = \frac{1}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}}$

Here we define the four-dimensional velocity by

$$\eta_\mu = \frac{dx_\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx_\mu}{dt} = (\gamma(\mathbf{u})u_1, \gamma(\mathbf{u})u_2, \gamma(\mathbf{u})u_3, ic\gamma(\mathbf{u}))$$

where u_i is the 3-velocity

$$u_i = \frac{dx_i}{dt} \qquad (i = 1, 2, 3)$$

$$\eta_\mu' = a_{\mu\nu}\eta_\nu$$

$$\begin{pmatrix} \eta_1' \\ \eta_2' \\ \eta_3' \\ \eta_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix}$$

or

$$\eta_\mu = a_{\nu\mu}\eta_\nu'$$

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \eta_1' \\ \eta_2' \\ \eta_3' \\ \eta_4' \end{pmatrix}$$

with

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

21. Force

$$\frac{dp_1}{dt} = \frac{dt}{dt'}\frac{dp_1}{dt} = \frac{1}{\gamma(1 + \frac{\beta}{c}u_1')}[\frac{dp_1'}{dt'} + mc\gamma\beta\frac{d}{dt'}\gamma(\mathbf{u}')]$$

$$\frac{dp_2}{dt} = \frac{dt'}{dt}\frac{dp_2}{dt'} = \frac{dp_2'}{dt'}\frac{1}{\gamma(1 + \frac{\beta}{c}u_1')}$$

$$\frac{dp_3}{dt} = \frac{dt'}{dt}\frac{dp_3}{dt'} = \frac{dp_3'}{dt'}\frac{1}{\gamma(1 + \frac{\beta}{c}u_1')}$$

When $u_1' = 0$ and $du_1'/dt' = 0$

$$\frac{dp_1}{dt} = \frac{dt'}{dt}\frac{dp_1}{dt'} = \frac{1}{\gamma(1 + \frac{\beta}{c}u_1')}[\frac{dp_1'}{dt'} + mc\gamma\beta\frac{d}{dt'}\gamma(\mathbf{u}')]$$

$$\frac{dp_2}{dt} = \frac{1}{\gamma}\frac{dp_2'}{dt'}$$

$$\frac{dp_3}{dt} = \frac{1}{\gamma}\frac{dp_3'}{dt'}$$

22. Minkowski force

We define the Minkowski force as

$$K_\mu = \frac{dp_\mu}{d\tau}$$

This is a 4-dimesional vector. The spatial components of K_μ are related to the ordinary force by

$$\mathbf{K} = \frac{d\mathbf{p}}{d\tau} = \frac{dt}{d\tau}\frac{d\mathbf{p}}{dt} = \frac{1}{\sqrt{1-\mathbf{u}^2}}\frac{d\mathbf{p}}{dt} = \frac{1}{\sqrt{1-\mathbf{u}^2}}\mathbf{F}$$

The 4th componenet

$$K_4 = \frac{dp_4}{d\tau} = \frac{dt}{d\tau}\frac{dp_4}{dt} = \frac{1}{c}\frac{dE}{dt}$$

23 Lorentz force in the relativistic mechanics

23.1

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = q[\mathbf{E} + (\mathbf{v}\times\mathbf{B})]$$

holds in an arbitrary frame *S*. This expression is the correct relativistic form for Newton's second law. The momentum form is more fundamental.

The four-dimensional momentum is given by

$$\mathbf{p} = m\frac{\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}$$

$$E_{kin} = \frac{mc^2}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}$$

$$\frac{E^2}{c^2} = \frac{m^2c^2}{1-\frac{\mathbf{v}^2}{c^2}} = \frac{m^2c^2(1-\frac{\mathbf{v}^2}{c^2}) + m^2\mathbf{v}^2}{1-\frac{\mathbf{v}^2}{c^2}} = m^2c^2 + \mathbf{p}^2$$

or

$$E_{kin} = c(m^2c^2 + \mathbf{p}^2)^{1/2}$$

The final form of the equation of motion is given by

$$\frac{d}{dt}\mathbf{p} = q[\mathbf{E} + (\mathbf{v}\times\mathbf{B})] \qquad (1)$$

$$\mathbf{p} = \frac{\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}},$$

$$\frac{dE_{kin}}{dt} = \mathbf{F}\cdot\mathbf{v} = q(\mathbf{v}\cdot\mathbf{E})$$

where

$$E_{kin} = \frac{mc^2}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} = c\sqrt{m^2c^2+\mathbf{p}^2} \qquad (2)$$

24. Cyclotron motion: a particle in a uniform magnetic field along the *z* axis.

We now consider the case of $\boldsymbol{E} = 0$.

$$\frac{d}{dt}E_{kin} = 0$$

Thus we have $\gamma(\mathbf{v}) = \frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} = \frac{E_{kin}}{mc^2}$ = constant

The momentum:

$$\mathbf{p} = \frac{E_{kin}\mathbf{v}}{c^2}$$

The equation of motion

$$\frac{d}{dt}\mathbf{v} = \frac{c^2}{E_{kin}}q(\mathbf{v}\times\mathbf{B})$$

or

$$\dot{v}_x = \frac{c^2 qB}{E_{kin}} v_y$$
$$\dot{v}_y = -\frac{c^2 qB}{E_{kin}} v_x$$
$$\dot{v}_z = 0$$

We use the complex plane for the solution.

$$\frac{d}{dt}(v_x + iv_y) = -\frac{ic^2 qB}{E_{kin}}(v_x + iv_y)$$

or

$$(v_x + iv_y) = (v_x{}^0 + iv_y{}^0)\exp[-\frac{ic^2 qBt}{E_{kin}}] = v\exp[-i(\omega t + \alpha)]$$

where

$$\omega = \frac{c^2 qB}{E_{kin}}$$

$$v_x{}^0 + iv_y{}^0 = ve^{-i\alpha}$$

Then we have

$$v_x = \frac{dx}{dt} = v\cos(\omega t + \alpha)$$
$$v_x = \frac{dy}{dt} = -v\sin(\omega t + \alpha)$$

or

$$v_x{}^2 + v_y{}^2 = v^2 = cons\tan t$$

$$x = \frac{v}{\omega}\sin(\omega t + \alpha) + x_1$$
$$y = \frac{v}{\omega}\cos(\omega t + \alpha) + y_1$$

This equation describes a cyclotron motion (circular motion with radius *R*).

$$R = \frac{v}{\omega} = \frac{vE_{kin}}{c^2 qB} = \frac{p}{qB}$$

where w is the angular frequency,

$$\omega = \frac{c^2 qB}{E_{kin}}$$

or

$$BR = \frac{1}{q} p$$

The radius has a maximum when $\frac{v}{c} = \frac{1}{\sqrt{2}}$

In summary

$$x = \frac{\sqrt{p_{0x}{}^2 + p_{0y}{}^2}}{qB} \sin(\frac{c^2 qB}{E_{kin}} t + \alpha)$$
$$y = \frac{\sqrt{p_{0x}{}^2 + p_{0y}{}^2}}{qB} \cos(\frac{c^2 qB}{E_{kin}} t + \alpha)$$

25. The motion of the particle under an electric field ($\mathbf{E} = -\nabla\phi$)

$$\frac{d}{dt} E_{kin} = q(\mathbf{v} \cdot \mathbf{E}) = -qv \cdot \nabla\phi = -q \frac{d}{dt} \phi$$

or

$$\frac{d}{dt}(E_{kin} + q\phi) = 0$$

or

$$E_{kin} + q\phi = \text{constant}$$

We now consider the capacitance consisting of two parallel planes. Suppose that the particle with charge q on the one plate moves to the other plate. The initial velocity is equal to zero. What is the velocity of the particle arriving at the other plate?

$$E_{kin} + q\phi_2 = mc^2 + q\phi_1$$

When $\phi = \phi_1 - \phi_2$,

$$\frac{1}{1 - \frac{\mathbf{v}^2}{c^2}} = (1 + \frac{q\phi}{mc^2})^2$$

or

$$v = c[1 - \frac{1}{(1 + \frac{q\phi}{mc^2})^2}]^{1/2}$$

26. Equation of motion under a constant electric field

We assume that $\boldsymbol{E}$ is along the y axis. The initial momentum $\boldsymbol{p}_0$ is in the (x, y) plane. The particle is at the origin at $t = 0$.

$$\frac{d}{dt}\mathbf{p} = q\mathbf{E} \qquad (1)$$

$$\mathbf{p} = \mathbf{p}_0 + q\mathbf{E}t$$

or

$$\mathbf{p} = (p_{0x}, qEt + p_{0y}, 0)$$

$$E_{kin} = c(m^2c^2 + \mathbf{p}^2)^{1/2} = [m^2c^4 + c^2(p_{0x}{}^2 + p_{0y}{}^2) + c^2(q^2E^2t^2 + 2p_{0y}qEt)]^{1/2}$$

or

$$E_{kin} = [(E_{kin}{}^0)^2 + c^2(q^2E^2t^2 + 2p_{0y}qEt)]^{1/2}$$

where $E_{kin}{}^0$ is the kinetic energy at the beginning of the motion ($t = 0$).

$$E_{kin}{}^0 = \sqrt{m^2c^4 + c^2(p_{0x}{}^2 + p_{0y}{}^2)}$$

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} = m\mathbf{v}\frac{E_{kin}}{mc^2} = \frac{E_{kin}\mathbf{v}}{c^2}$$

Thus we have

$$\mathbf{v} = \frac{c^2}{E_{kin}}\mathbf{p} = \frac{c^2}{E_{kin}}(p_{0x}, qEt + p_{0y}, 0) = \frac{c^2}{[(E_{kin}{}^0)^2 + c^2(q^2E^2t^2 + 2p_{0y}qEt)]^{1/2}}(p_{0x}, qEt + p_{0y}, 0)$$

or

$$\frac{dx}{dt} = \frac{c^2 p_{0x}}{[(E_{kin}{}^0)^2 + c^2(q^2E^2t^2 + 2p_{0y}qEt)]^{1/2}}$$

$$\frac{dy}{dt} = \frac{c^2(p_{0y} + qEt)}{[(E_{kin}{}^0)^2 + c^2(q^2E^2t^2 + 2p_{0y}qEt)]^{1/2}}$$

$$\frac{dz}{dt} = 0$$

Solving these differential equations (we use the Mathematica),

$$y = \frac{1}{qE}[\sqrt{(E_{kin}{}^0)^2 + c^2q^2E^2t^2 + 2p_{0y}c^2Eqt} - E_{kin}{}^0]$$

or

$$y = \frac{c}{qE}[\sqrt{(p_{0y} + qEt)^2 + m^2c^2 + p_{0x}{}^2} - \frac{E_{kin}{}^0}{c}]$$

$$x = \frac{cp_{0x}}{qE}\ln\frac{p_{0y} + qEt + \sqrt{(p_{0y} + qEt)^2 + m^2c^2 + p_{0x}{}^2}}{p_{0y} + \frac{E_{kin}{}^0}{c}}$$

$$z = 0$$

We now consider the special case when $p_{0y} = 0$.

$$\frac{qEy}{c} = \sqrt{(qEt)^2 + m^2c^2 + p_{0x}{}^2} - \sqrt{m^2c^2 + p_{0x}{}^2}$$

$$x = \frac{cp_{0x}}{qE}\ln\frac{qEt + \sqrt{(qEt)^2 + m^2c^2 + p_{0x}{}^2}}{\sqrt{m^2c^2 + p_{0x}{}^2}}$$

or

$$\exp(\frac{qEx}{cp_{0x}}) = \frac{qEt + \sqrt{(qEt)^2 + m^2c^2 + p_{0x}{}^2}}{\sqrt{m^2c^2 + p_{0x}{}^2}}$$

$$\exp(-\frac{qEx}{cp_{0y}}) = \frac{\sqrt{m^2c^2 + p_{0x}{}^2}}{qEt + \sqrt{(qEt)^2 + m^2c^2 + p_{0x}{}^2}} = \frac{-qEt + \sqrt{(qEt)^2 + m^2c^2 + p_{0x}{}^2}}{\sqrt{m^2c^2 + p_{0x}{}^2}}$$

$$\cosh(\frac{qEx}{cp_{0x}}) = \frac{\sqrt{(qEt)^2 + m^2c^2 + p_{0x}{}^2}}{\sqrt{m^2c^2 + p_{0x}{}^2}}$$

$$\frac{qEy}{c} + \sqrt{m^2c^2 + p_{0x}{}^2} = \sqrt{(qEt)^2 + m^2c^2 + p_{0x}{}^2} = \sqrt{m^2c^2 + p_{0x}{}^2}\cosh(\frac{qEx}{cp_{0x}})$$

or

$$y = \frac{E_{kin}{}^0}{qE}[\cosh(\frac{qEx}{cp_{0x}}) - 1] = \frac{E_{kin}{}^0}{qE}[\cosh(\frac{qEx}{cp_{0x}}) - 1] = \frac{c}{qE}\sqrt{m^2c^2 + p_{0x}{}^2}[\cosh(\frac{qEx}{cp_{0x}}) - 1]$$

Thus in a uniform electric field, a charge q moves along a catenary curve.

((Mathematica-13))

Zimmerman

2D motion of a relativistic particle in a uniform electric field

```
Y1 = {px0/m, q E0 t/m, 0}
{px0/m, E0 q t/m, 0}
eq1 = ξ^2/(1 - ξ^2/c^2) == Y1.Y1
ξ^2/(1 - ξ^2/c^2) == px0^2/m^2 + E0^2 q^2 t^2/m^2
eq2=Solve[eq1,ξ]//Simplify
{{ξ -> -Sqrt[-c^2 (px0^2 + E0^2 q^2 t^2)]/Sqrt[-c^2 m^2 - px0^2 - E0^2 q^2 t^2]}, {ξ -> Sqrt[-c^2 (px0^2 + E0^2 q^2 t^2)]/Sqrt[-c^2 m^2 - px0^2 - E0^2 q^2 t^2]}}
```

eq3 = $\sqrt{1-\frac{\xi^2}{c^2}}$ /. **eq2**[[**2**]] // **Simplify**

$$\sqrt{\frac{c^2 m^2}{c^2 m^2 + px0^2 + E0^2 q^2 t^2}}$$

V={x'[t],y'[t],z'[t]}

{x′[t], y′[t], z′[t]}

eq4=eq3 Y1

$$\left\{\frac{px0\sqrt{\frac{c^2 m^2}{c^2 m^2+px0^2+E0^2 q^2 t^2}}}{m}, \frac{E0\, q\, t\sqrt{\frac{c^2 m^2}{c^2 m^2+px0^2+E0^2 q^2 t^2}}}{m}, 0\right\}$$

eq5=Table[V[[i]]==eq4[[i]],{i,1,3}]

$$\left\{x'[t] == \frac{px0\sqrt{\frac{c^2 m^2}{c^2 m^2+px0^2+E0^2 q^2 t^2}}}{m},\; y'[t] == \frac{E0\, q\, t\sqrt{\frac{c^2 m^2}{c^2 m^2+px0^2+E0^2 q^2 t^2}}}{m},\; z'[t] == 0\right\}$$

eq6=DSolve[{eq5,{x[0]==0,y[0]==0,z[0]==0}},{x[t],y[t],z[t]},t]//Simplify

$$\Bigg\{\Bigg\{x[t] \to \frac{1}{E0\, m\, q}\Bigg(-px0\sqrt{\frac{c^2 m^2}{c^2 m^2 + px0^2}}\sqrt{c^2 m^2 + px0^2}\,\mathrm{Log}\Big[2\sqrt{c^2 m^2 + px0^2}\Big] + px0\sqrt{\frac{c^2 m^2}{c^2 m^2 + px0^2 + E0^2 q^2 t^2}}\sqrt{c^2 m^2 + px0^2 + E0^2 q^2 t^2}\,\mathrm{Log}\Big[2\Big(E0\, q\, t + \sqrt{c^2 m^2 + px0^2 + E0^2 q^2 t^2}\Big)\Big]\Bigg),$$

$$y[t] \to \frac{c^2 m\left(\sqrt{\frac{c^2 m^2}{c^2 m^2+px0^2}} - \sqrt{\frac{c^2 m^2}{c^2 m^2+px0^2+E0^2 q^2 t^2}}\right)}{E0\sqrt{\frac{c^2 m^2}{c^2 m^2+px0^2}}\, q\sqrt{\frac{c^2 m^2}{c^2 m^2+px0^2+E0^2 q^2 t^2}}},\; z[t] \to 0\Bigg\}\Bigg\}$$

rule1={m→1,q→1,E0→0.1,px0→0.1,c→1}

{m→1,q→1,E0→0.1,px0→0.1,c→1}

x[t_]=x[t]/.eq6[[1,1]]/.rule1//Simplify

$$-0.698122 + 10.\sqrt{1.01 + 0.01\, t^2}\sqrt{\frac{1}{101. + 1.\, t^2}}\,\mathrm{Log}\Big[2\Big(0.1\, t + \sqrt{1.01 + 0.01\, t^2}\Big)\Big]$$

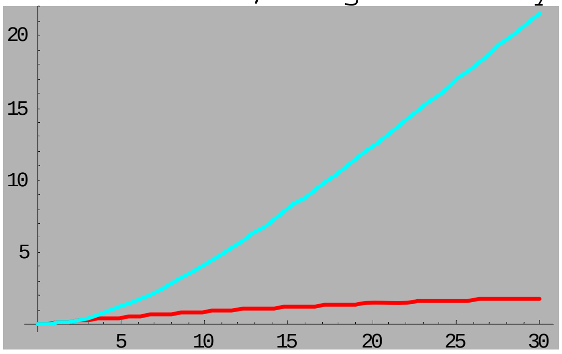
y[t_]=y[t]/.eq6[[1,2]]/.rule1//Simplify

$$\frac{1. - 10.0499\sqrt{\frac{1}{101. + 1.\, t^2}}}{\sqrt{\frac{1}{101. + 1.\, t^2}}}$$

z[t_]=z[t]/.eq6[[1,3]]/.rule1//Simplify

0

```
pl1=Plot[{x[t],y[t]},{t,0,30},PlotStyle→Table[Hue[0.5 i],{i,0,1}], Prolog→AbsoluteThickness[2], PlotPoints→50,Background→GrayLevel[0.7]]
```



-Graphics-

Nonrelativistic motion

```
xnon[t_] = px0 t / m /. rule1
0.1 t
ynon[t_] = q E0 t^2 / (2 m) /. rule1
0.05 t^2
```

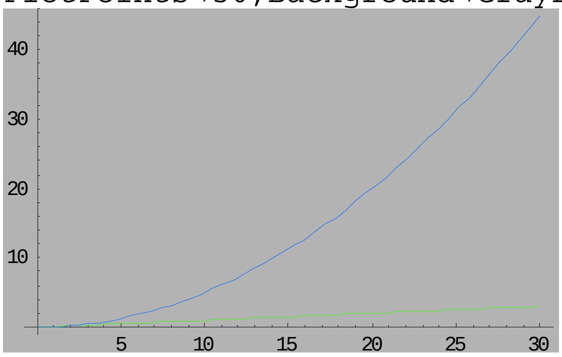
$xnon[t\_] = \frac{px0\, t}{m}$ /. rule1

0.1 t

$ynon[t\_] = \frac{q\, E0\, t^2}{2\, m}$ /. rule1

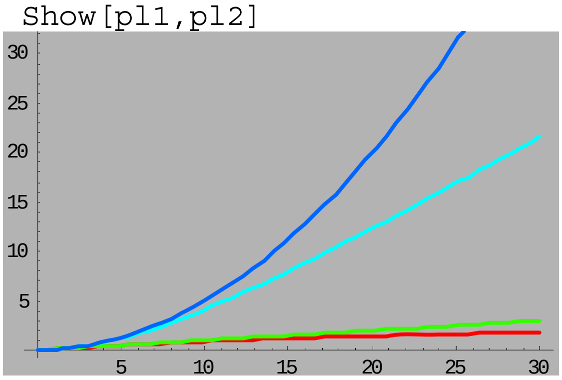
$0.05\, t^2$

```
pl2=Plot[{xnon[t],ynon[t]},{t,0,30},PlotStyle→Table[Hue[0.3 i],{i,1,2}], Prolog→AbsoluteThickness[1.6], PlotPoints→50,Background→GrayLevel[0.7]]
```



-Graphics-

```
Show[pl1,pl2]
```



-Graphics-

28. A particle in a uniform electric field and a magnetic field

Let the electric field ***E*** be parallel to the *y* axis and the magnetic field ***B*** parallel to the *z* axis. At $t = 0$ the particle is at the point (0,0,0) and has a momentum $\boldsymbol{p}_0$.

Lorentz invariant:

$$\frac{d}{dt}\mathbf{p} = \mathbf{F} = q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})] \tag{1}$$

According to the Lorentz invariance, we have

$$\mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2 = \mathbf{B}'^2 - \frac{1}{c^2}\mathbf{E}'^2$$
$$\mathbf{E} \cdot \mathbf{B} = \mathbf{E}' \cdot \mathbf{B}'$$

Since $\mathbf{E} \cdot \mathbf{B} = 0$, we have $\mathbf{E}' \cdot \mathbf{B}' = 0$

(1)

We assume a frame that $\mathbf{B}' = 0$.

In this case, we have

$$\mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2 = -\frac{1}{c^2}\mathbf{E}'^2 < 0$$

or

$$\mathbf{B}^2 < \frac{1}{c^2}\mathbf{E}^2$$

or

$$B_3 < \frac{1}{c}E_2$$

This is a condition for ***E*** and ***B***. Using the Lorentz transformation, we have

$$E_1' = E_1 = 0$$
$$E_2' = \gamma(E_2 - c\beta B_3)$$
$$E_3' = \gamma(E_3 + c\beta B_2) = 0$$

$$B_1' = B_1 = 0$$
$$B_2' = \gamma(B_2 + \frac{\beta}{c}E_3) = 0$$
$$B_3' = \gamma(B_3 - \frac{\beta}{c}E_2) = 0$$

We choose $B_3' = 0$

$$B_3' = B_3 - \frac{\beta}{c} E_2 = 0$$

or

$$v = c^2 \frac{B_3}{E_2} < c$$

In this case,

$$\mathbf{B}' = 0$$
$$E_2' = \gamma(E_2 - c\beta B_3) = \frac{1}{\gamma} E_2 = \frac{E}{\gamma} = E'$$
$$E_1' = 0$$
$$E_3' = 0$$

The frame *S'* move relative to the frame *S* with a velocity *v* along the *x* axis. We know the equation of motion for the particle in a uniform electric field $\mathbf{E}'$ along the *y* axis.

$$x' = \frac{cp_{0x}'}{qE'} \ln \frac{p_{0y}' + qE't' + \sqrt{(p_{0y}' + qE't')^2 + m^2c^2 + p_{0x}'^2}}{p_{0y}' + \frac{E_{kin}^{0}{}'}{c}}$$

$$y' = \frac{c}{qE'}[\sqrt{(p_{0y}' + qE't')^2 + m^2c^2 + p_{0x}'^2} - \frac{E_{kin}^{0}{}'}{c}]$$

$$E_{kin}'^{0} = c(m^2c^2 + p_{0x}'^2 + p_{0y}'^2)^{1/2}$$

with $v = c^2 \frac{B_3}{E_2} < c$

The Lorentz transformation between $p_\mu^0 = (\mathbf{p}_0, i\frac{E_{kin}^0}{c})$ and $p_\mu^0{}' = (\mathbf{p}_0', i\frac{E_{kin}^0{}'}{c})$ is given by

$$p_{01}'=\gamma(p_{01}-\frac{\beta}{c}E_{kin}{}^{0})$$
$$p_{02}'=p_{02}$$
$$p_{03}'=p_{03}$$
$$E_{kin}{}^{0}{}'=\gamma(E_{kin}{}^{0}-\beta c p_{01})$$

The required equations of motion for the particle in the frame S is obtained using the Lorentz transformation.

$$x_1=\gamma(x_1'-i\beta x_4')$$
$$x_2=x_2'$$
$$x_3=x_3'$$
$$x_4=\gamma(i\beta x_1'+x_4')$$

$$x=\gamma(x'+vt')$$
$$y=y'$$
$$z=z'$$
$$t=\gamma(\frac{\beta}{c}x'+t')$$

(2)

We assume a frame *S*' that $\mathbf{E}'=0$.

In this case, we have

$$\mathbf{B}^2-\frac{1}{c^2}\mathbf{E}^2=\mathbf{B}'^2>0$$

or

$$\mathbf{B}^2>\frac{1}{c^2}\mathbf{E}^2$$

or

$$B_3>\frac{1}{c}E_2$$

This is a condition for ***E*** and ***B***. Using the Lorentz transformation, we have

$$E_1'=E_1=0$$
$$E_2'=\gamma(E_2-c\beta B_3)$$
$$E_3'=\gamma(E_3+c\beta B_2)=0$$

$$B_1'=B_1=0$$
$$B_2'=\gamma(B_2+\frac{\beta}{c}E_3)=0$$
$$B_3'=\gamma(B_3-\frac{\beta}{c}E_2)$$

We choose $E_2'=0$

$$E_2'=\gamma(E_2-c\beta B_3)=0$$

or

$$v=\frac{E_2}{B_3}$$

In this case,

$$\mathbf{E}'=0$$
$$B_3'=\gamma(B_3-\frac{\beta}{c}vB_3)=\frac{1}{\gamma}B_3$$
$$B_1'=0$$
$$B_2'=0$$

The frame S' move relative to the frame S with a velocity v ($=E_2/B_3<c$) along the x axis. We know the equation of motion for the particle in a uniform electric field $\mathbf{B}'$ along the z' axis.

$$x'=\frac{\sqrt{p_{0x}'^2+p_{0y}'^2}}{qB'}\sin(\frac{c^2qB'}{E_{kin}'}t'+\alpha')$$
$$y'=\frac{\sqrt{p_{0x}'^2+p_{0y}'^2}}{qB'}\cos(\frac{c^2qB'}{E_{kin}'}t'+\alpha')$$

with $v=E_2/B_3<c$.

The Lorentz transformation between $p_\mu{}^0=(\mathbf{p}_0,i\frac{E_{kin}{}^0}{c})$ and $p_\mu{}^0{}'=(\mathbf{p}_0',i\frac{E_{kin}{}^0{}'}{c})$ is given by

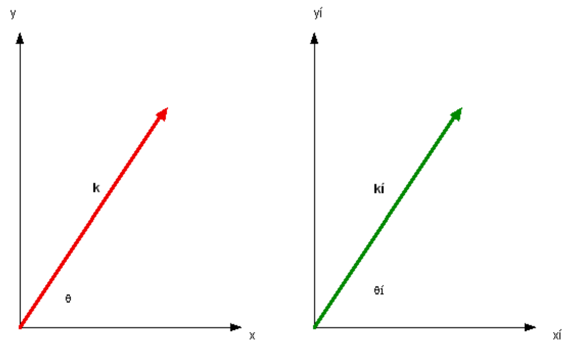
$$p_{01}'=\gamma(p_{01}-\frac{\beta}{c}E_{kin}{}^0)$$
$$p_{02}'=p_{02}$$
$$p_{03}'=p_{03}$$
$$E_{kin}{}^0{}'=\gamma(E_{kin}{}^0-\beta cp_{01})$$

The required equations of motion for the particle in the frame S is obtained using the Lorentz transformation.

$$x_1 = \gamma(x_1' - i\beta x_4') \qquad x = \gamma(x' + vt')$$
$$x_2 = x_2' \qquad y = y'$$
$$x_3 = x_3' \qquad z = z'$$
$$x_4 = \gamma(i\beta x_1' + x_4') \qquad t = \gamma(\frac{\beta}{c}x' + t')$$

19. Doppler shift and aberration

19.1.



$$k_\mu = \frac{p}{\hbar} = (k, \frac{i\omega}{c})$$

where $\omega = ck$

$$k_\mu x_\mu = \mathbf{k}\cdot\mathbf{r} - \omega t = \text{invariant under the Lorentz transformation.}$$

or

$$k_\mu x_\mu = kx\cos\theta + ky\sin\theta - \omega t$$

This should be equal to

$$k'_\mu x'_\mu = k'x'\cos\theta' + k'y'\sin\theta' - \omega' t'$$

Note that

$$x = \gamma(x' + vt')$$
$$t = \gamma(t' + \frac{v}{c^2}x')$$
$$\omega = ck$$
$$\omega' = ck'$$

Substituting these parameters into the invariant form, we have

$$t'(ck'-ck\gamma+kv\gamma\cos\theta)+x'(-\frac{kv\gamma}{c}+k\gamma\cos\theta-k'\cos\theta')+y'(k\sin\theta-k'\sin\theta')=0$$

This should be satisfied for any *t'*, *x'*, and *y'*.

$$k\gamma(c-v\cos\theta)=ck'$$

$$k\gamma(\cos\theta-\frac{v}{c})=k'\cos\theta'$$

$$k\sin\theta=k'\sin\theta'$$

19.2. Doppler shift

Since $k=\frac{2\pi}{\lambda}, k'=\frac{2\pi}{\lambda'}$

$$\frac{2\pi}{\lambda}\gamma(1-\frac{v}{c}\cos\theta)=\frac{2\pi}{\lambda'}$$

or

$$\lambda'=\frac{\lambda}{\gamma(1-\frac{v}{c}\cos\theta)}$$

19.3 Derivation of the formula using Mathematica

((Mathematica-14))

```
eq1=(k (x Cos[θ]+y Sin[θ])- ω t -k1 (x1 Cos[θ1]+y1
Sin[θ1])+ ω1 t1)/.{ω→c k,ω1→c k1}//Simplify
 -c k t+c k1 t1+k x Cos[θ]-k1 x1 Cos[θ1]+k y Sin[θ]-k1
y1 Sin[θ1]
 rule1 = {x→γ (x1 + v t1), t→γ (t1 + v/c^2 x1), y→y1}
 {x→(t1 v+x1) γ, t→(t1 + v x1/c^2) γ, y→y1}
 eq2=eq1/.rule1//Simplify
 c k1 t1 - c k (t1 + v x1/c^2) γ + k (t1 v + x1) γ Cos[θ] -
  k1 x1 Cos[θ1] + k y1 Sin[θ] - k1 y1 Sin[θ1]
 Collect[eq2,{x1,y1,t1}]
 t1 (c k1 - c k γ + k v γ Cos[θ]) +
  x1 (-k v γ/c + k γ Cos[θ] - k1 Cos[θ1]) + y1 (k Sin[θ] - k1 Sin[θ1])
```

```
rule2 = {k → 2π/λ, k1 → 2π/λ1}
```

$$\left\{k \to \frac{2\pi}{\lambda},\ k1 \to \frac{2\pi}{\lambda 1}\right\}$$

```
eq3=(c k1-c k γ+k v γ Cos[θ])/.rule2//Simplify
```

$$\frac{2\pi\,(c\,(\lambda - \gamma\,\lambda 1) + v\,\gamma\,\lambda 1\,\mathrm{Cos}[\theta])}{\lambda\,\lambda 1}$$

```
eq4 = -k v γ/c + k γ Cos[θ] - k1 Cos[θ1] /. rule2 // Simplify
```

$$-\frac{2\pi\,(v\,\gamma\,\lambda 1 - c\,\gamma\,\lambda 1\,\mathrm{Cos}[\theta] + c\,\lambda\,\mathrm{Cos}[\theta 1])}{c\,\lambda\,\lambda 1}$$

```
eq5=k Sin[θ]-k1 Sin[θ1]/.rule2//Simplify
```

$$\frac{2\pi\,\mathrm{Sin}[\theta]}{\lambda} - \frac{2\pi\,\mathrm{Sin}[\theta 1]}{\lambda 1}$$

```
eq31=Solve[eq3==0,λ1]//Simplify
```

$$\left\{\left\{\lambda 1 \to \frac{c\,\lambda}{c\,\gamma - v\,\gamma\,\mathrm{Cos}[\theta]}\right\}\right\}$$

```
λ1[θ_] = λ1 /. eq31[[1]] /. {γ → 1/Sqrt[1 - v^2/c^2]} // Simplify
```

$$\frac{c\,\sqrt{1 - \frac{v^2}{c^2}}\,\lambda}{c - v\,\mathrm{Cos}[\theta]}$$

Longitudinal Doppler shift

```
λ1[0]
```

$$\frac{c\,\sqrt{1 - \frac{v^2}{c^2}}\,\lambda}{c - v}$$

```
Series[λ1[0],{v,0,4}]
```

$$\lambda + \frac{\lambda\,v}{c} + \frac{\lambda\,v^2}{2\,c^2} + \frac{\lambda\,v^3}{2\,c^3} + \frac{3\,\lambda\,v^4}{8\,c^4} + O[v]^5$$

Transverse Doppler shift

```
λ1[π/2]
```

$$\sqrt{1 - \frac{v^2}{c^2}}\,\lambda$$

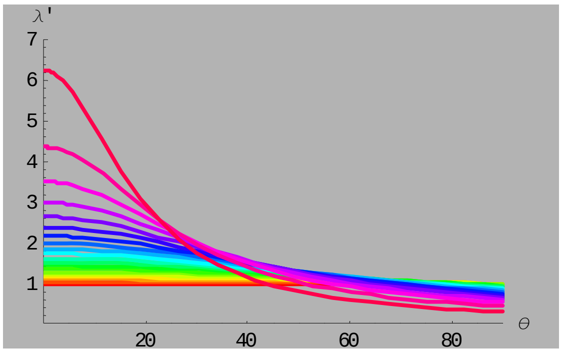
```
Series[λ1[π/2], {v, 0, 4}]
```

$$\lambda - \frac{\lambda\,v^2}{2\,c^2} - \frac{\lambda\,v^4}{8\,c^4} + O[v]^5$$

```
f1 = λ1[θ]/λ /. {v → c β} /. θ → π α/180 // FullSimplify
```

$$\frac{\sqrt{1 - \beta^2}}{1 - \beta\,\mathrm{Cos}\left[\frac{\pi\,\alpha}{180}\right]}$$

```
Plot[Evaluate[Table[f1,{β,0,0.99,0.05}]],{α,0,90},PlotS
tyle→Table[Hue[0.05
i],{i,0,20}],Prolog→AbsoluteThickness[2],PlotRange→{{0,
90},{0,7}},Background→GrayLevel[0.7],AxesLabel→{"θ","λ
'"}]
```



-Graphics-

19.4. longitudinal Doppler shift $\theta = 0$ (red shift)

We suppose that a source is located at the origin of the reference frame *S*. An observer moves relative to *S* at velocity *v*. So that he is at rest in S'.

$$\lambda' = \frac{\lambda}{\gamma(1-\frac{v}{c})} = \lambda\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

If *S*' moves toward S, rather than away from S, the signs in numerator and denominator of the radical would have been unterchanged.

((The red shift)) Wikipedia

The light from distant stars and more distant galaxies is not featureless, but has distinct spectral features characteristic of the atoms in the gases around the stars. When these spectra are examined, they are found to be shifted toward the red end of the spectrum. This shift is apparently a Doppler shift and indicates that essentially all of the galaxies are moving away from us. Using the results from the nearer ones, it becomes evident that the more distant galaxies are moving away from us faster. This is the kind of result one would expect for an expanding universe.

The building up of methods for measuring distance to stars and galaxies led Hubble to the fact that the red shift (recession speed) is proportional to distance. If this proportionality (called Hubble's Law) holds true, it can be used as a distance measuring tool itself.

The measured red shifts are usually stated in terms of a z parameter. The largest measured z values are associated with the quasars.

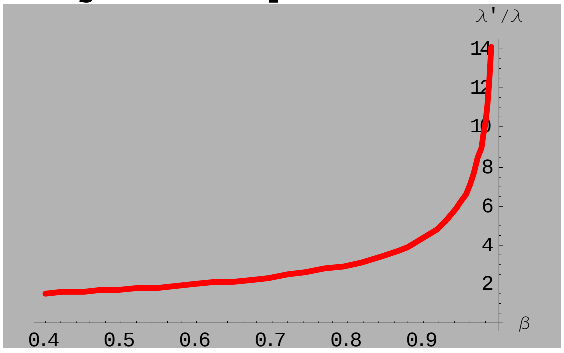
((Mathematica-16))

Red shift: λ'/λ vs $\beta = v/c$

eq1 = $\sqrt{\frac{1+\beta}{1-\beta}}$

$$\sqrt{\frac{1+\beta}{1-\beta}}$$

```
Plot[eq1,{β,0.4,0.99},
PlotStyle→{Hue[0],Thickness[0.015]},
Background→GrayLevel[0.7],AxesLabel→{"β","λ'/λ"}]
```



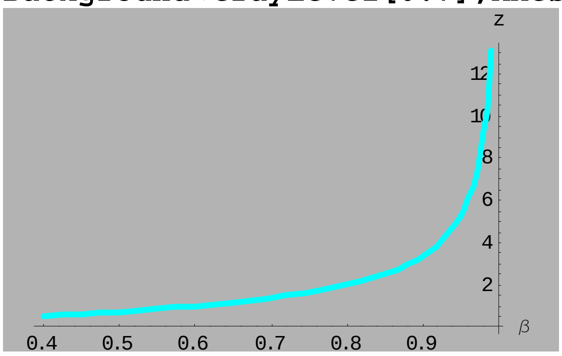
-Graphics-

z-parameter = (λ'-λ)/λ=(λ'/λ)-1

```
z=eq1-1
```

$$-1+\sqrt{\frac{1+\beta}{1-\beta}}$$

```
Plot[z,{β,0.4,0.99},
PlotStyle→{Hue[0.5],Thickness[0.015]},
Background→GrayLevel[0.7],AxesLabel→{"β","z"}]
```



-Graphics-

19.5. Transverse Doppler shift ($\theta = \frac{\pi}{2}$)

$$\lambda' = \frac{\lambda}{\gamma} = \lambda\sqrt{1-\frac{v^2}{c^2}}$$

19.6. Aberration

From these equations

$$k\gamma(1-\frac{v}{c}\cos\theta)=k'$$

$$k\gamma(\cos\theta-\frac{v}{c})=k'\cos\theta'$$

we have

$$\cos\theta'=\frac{\cos\theta-\frac{v}{c}}{1-\frac{v}{c}\cos\theta}$$

For low velocity we can neglect v^2/c^2 and higher-order terms. Setting $\theta'=\theta+\Delta\theta$

$$\cos(\theta+\Delta\theta)=\cos\theta-\Delta\theta\sin\theta$$

and

$$\frac{\cos\theta-\frac{v}{c}}{1-\frac{v}{c}\cos\theta}=\cos\theta-\beta\sin^2\theta$$

Then we have

$$\Delta\theta=\beta\sin\theta$$

((Mathematica-17))

Aberration

```
eq1=Cos[θ+Δθ]
Cos[Δθ+θ]
Series[eq1,{Δθ,0,1}]
Cos[θ] - Sin[θ] Δθ + O[Δθ]^2
eq11=Cos[θ]-Sin[θ] Δθ
Cos[θ]-Δθ Sin[θ]
eq2 = (Cos[θ] - β)/(1 - β Cos[θ])
(-β + Cos[θ])/(1 - β Cos[θ])
Series[eq2,{β,0,1}]
Cos[θ] + (-1 + Cos[θ]^2) β + O[β]^2
eq22 = Cos[θ] + (-1 + Cos[θ]^2) β
Cos[θ] + β (-1 + Cos[θ]^2)
eq3=eq11==eq22
Cos[θ] - Δθ Sin[θ] == Cos[θ] + β (-1 + Cos[θ]^2)
eq31=Solve[eq3,Δθ]//Simplify
```

```
General::spell1 : Possible spelling error: new
   symbol name "eq31" is similar to existing symbol "eq1". More…
{{ΔΘ→β Sin[Θ]}}
```

REFERENCES

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Appendix

A-1 Capacitance

The capacitor moves at the constant speed in the *x* direction.
In the S' system

$$E_2' = \frac{\sigma_0}{\varepsilon_0}$$
$$E_3' = 0$$
$$E_1' = 0$$

From the Lorentz transformation, the electric field and the magnetic field in the S system are given by

$$E_1 = E_1' = 0$$
$$E_2 = \gamma(E_2' + c\beta B_3') = \gamma E_2' = \gamma \frac{\sigma_0}{\varepsilon_0}$$
$$E_3 = \gamma(E_3' - c\beta B_2') = 0$$

$$B_1 = B_1' = 0$$

$$B_2 = \gamma(B_2' - \frac{\beta}{c} E_3') = 0$$

$$B_3 = \gamma(B_3' + \frac{\beta}{c} E_2') = \gamma \frac{\beta}{c} E_2' = \gamma \frac{\beta}{c} \frac{\sigma_0}{\varepsilon_0}$$

Since E_2 is expressed by

$$E_2 = \frac{\sigma}{\varepsilon_0} = \gamma \frac{\sigma_0}{\varepsilon_0}$$

or

$\sigma = \gamma \sigma_0$

A-2 Faraday's law

1. A conducting rod moving through a uniform magnetic field

We consider a metal rod (conductor) which moves at a constant velocity (***v***) in a direction perpendicular to its length. Pervading the space through which the rod moves there is a uniform magnetic field ***B*** (//z) constant in time. There is no electric field in the reference frame F.

The rod contains charged particles that will move if a force is applied to them. Any charged that is carried along with the rod, such as the particle of charge q moves through the magnetic field **B** and thus experience a force.

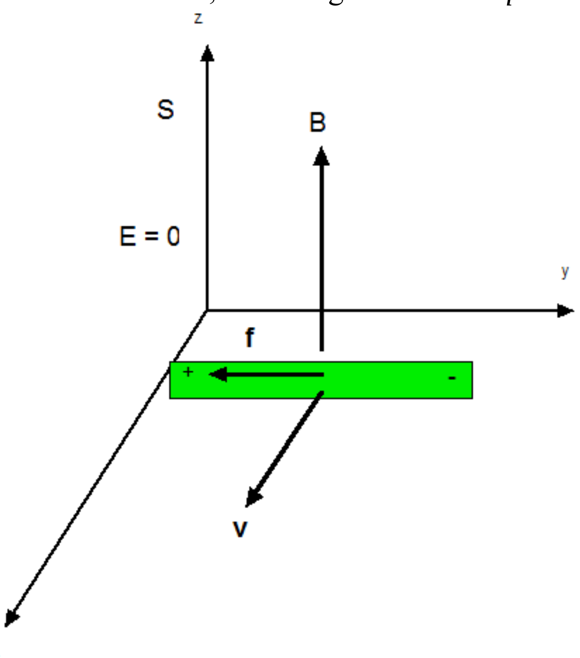
$$\mathbf{f} = q(\mathbf{v} \times \mathbf{B})$$

The direction of the force is dependent on the sign of the charge q.

When the rod is moving at constant speed and things have settled to a steady state, the force ***f*** must be balanced, at every point inside the rod, by an equal and opposite force. This can only arise from an electric field in the rod. The electric field develops in the following way. The force ***f*** pushes negative charges toward one end of the rod, leaving the other end positively charges. This goes on until these separated charges themselves cause an electric field ***E*** such that, everywhere in the interior of the rod,

$$q\mathbf{E} + \mathbf{f} = 0,$$

Then the motion of charge relative to the rod ceases. This charge distribution causes an electric field outside the rod, as well as inside. Inside the rod, there has developed an electric field $\mathbf{E} = \mathbf{v} \times \mathbf{B}$, exerting a force $q\boldsymbol{E}$ which just balances the force



$q(\mathbf{v} \times \mathbf{B})$.

Let us observe the system from a frame F' that moves with the rod. What is the magnetic field **B**' and the electric field **E**'? Note that there is no electric field ($\boldsymbol{E}$ = 0) in the frame F ($\mathbf{B} \neq 0$). The $\boldsymbol{E}$' and $\boldsymbol{B}$' in the frame F' are related to those in the frame F as

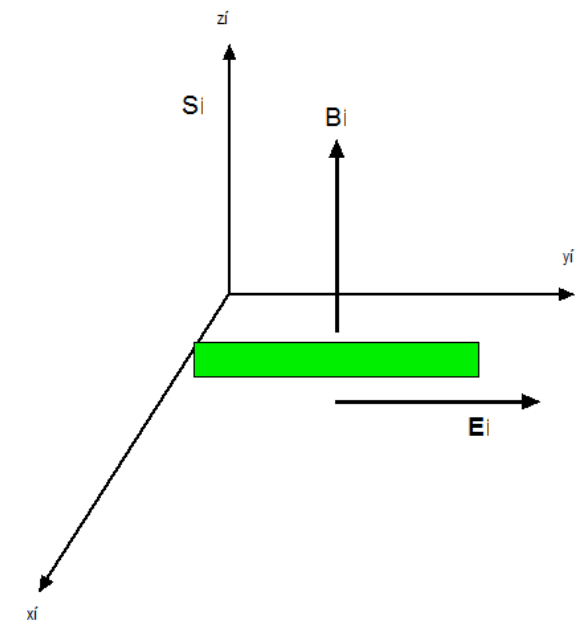
$$E_1' = E_1 = 0$$
$$E_2' = \gamma(E_2 - c\beta B_3) = -\gamma v B_3 \quad ,$$
$$E_3' = \gamma(E_3 + c\beta B_2) = \gamma v B_2 = 0$$

$$B_1' = B_1 = 0$$
$$B_2' = \gamma(B_2 + \frac{\beta}{c} E_3) = \gamma B_2 = 0$$
$$B_3' = \gamma(B_3 - \frac{\beta}{c} E_2) = \gamma B_3$$

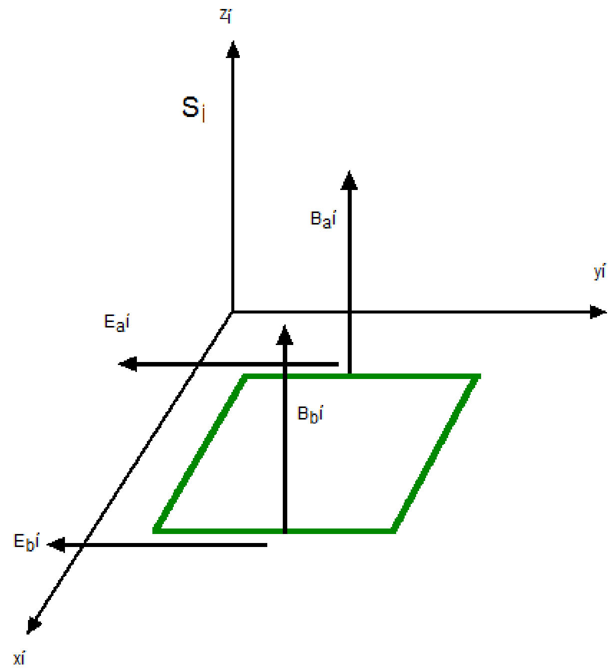
or

$$\mathbf{E}' = \gamma(\mathbf{v} \times \mathbf{B}) = \mathbf{v} \times \mathbf{B}'$$

where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}} \approx 1$ for $v \ll c$. The magnetic field **B**' (= $\gamma\boldsymbol{B}$) is almost equal to $\boldsymbol{B}$. The electric field $\boldsymbol{E}$' has only a component along the y' axis (the same as y xis). The presence of the magnetic field $\boldsymbol{B}$' has no influence on the static charge distribution.



2. A loop moving through a nonuniform magnetic field



F denotes the force which acts on a charge q that rides along with the loop. We evaluate the line integral of ***F***, taken around the whole loop. On the two sides of the loop which lie parallel to the direction of motion, ***F*** is perpendicular to the path element d***s***. So there is no contribution to the line integral. Taking account of the contributions from the other two sides, each of length *w*, we have

$$W_{net} = \oint \mathbf{F} \cdot d\mathbf{s} = \oint q(\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{s} = qv(B_a - B_b)w$$

where

$$\mathbf{F}_a = q(\mathbf{v} \times \mathbf{B}_a)$$
$$\mathbf{F}_b = q(\mathbf{v} \times \mathbf{B}_b)$$

From the definition, we get

$$W_{net} = qV = q\oint \mathbf{E} \cdot d\mathbf{s} = qv(B_a - B_b)w$$

or

$$V = \oint \mathbf{E} \cdot d\mathbf{s} = v(B_a - B_b)w$$

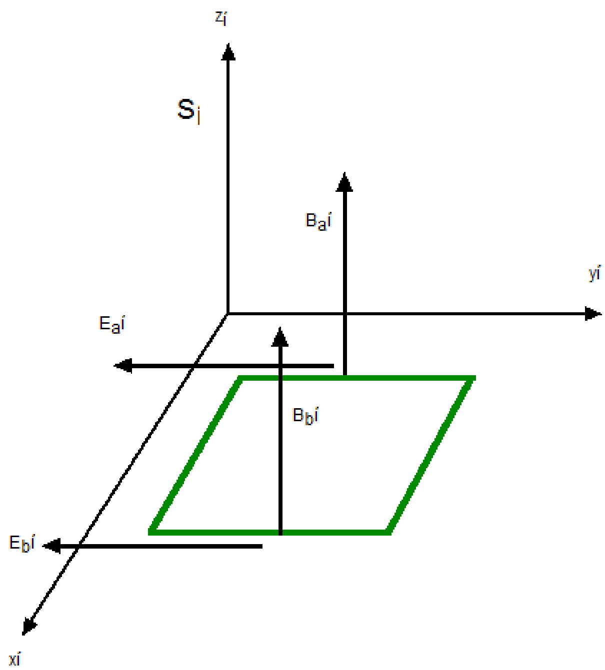
The electromotive force V is related in a very simple way to the rate of change of magnetic flux through the loop. The magnetic flux through a loop is the surface integral of ***B*** over a surface which has the loop for its boundary.

$$\Delta\Phi = B_b vw\Delta t - B_a vw\Delta t = -(B_a - B_b)vw\Delta t$$

or

$$V = \oint \mathbf{E} \cdot d\mathbf{s} = \oint (\nabla \times \mathbf{E}) \cdot d\mathbf{a} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \mathbf{B} \cdot d\mathbf{a}$$
$$= (B_a - B_b)vw$$

We now consider the frame F' attached to the loop.



From the Lorentz transformation of ***E*** and ***B***,

$$E_1' = E_1 = 0 \qquad B_1' = B_1 = 0$$
$$E_2' = \gamma(E_2 - vB_3) = -\gamma v B_3 \quad , \qquad B_2' = \gamma(B_2 + \frac{\beta}{c}E_3) = \gamma B_2 = 0$$
$$E_3' = \gamma(E_3 + vB_2) = \gamma v B_2 = 0 \qquad B_3' = \gamma(B_3 - \frac{\beta}{c}E_2) = \gamma B_3$$

we have

$$E_a' = -\gamma v B_a \approx -vB_a \qquad E_b' = -\gamma v B_b = -vB_b$$
$$B_a' = \gamma B_a \approx B_a \quad , \qquad B_b' = \gamma B_b \approx B_b$$

For observers in the frame S', ***E***$_a$' and ***E***$_b$' are genuine electric field. It is not an electrostatic field. The integral of ***E***' around the loop, which is the electromotive force *V*, is given by

$$V' =' \oint \mathbf{E} \cdot d\mathbf{s} = vB_a w - vB_b w = vw(B_a - B_b)$$

which is the same as that obtained for the frame S.