Compton effect

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1. Overview

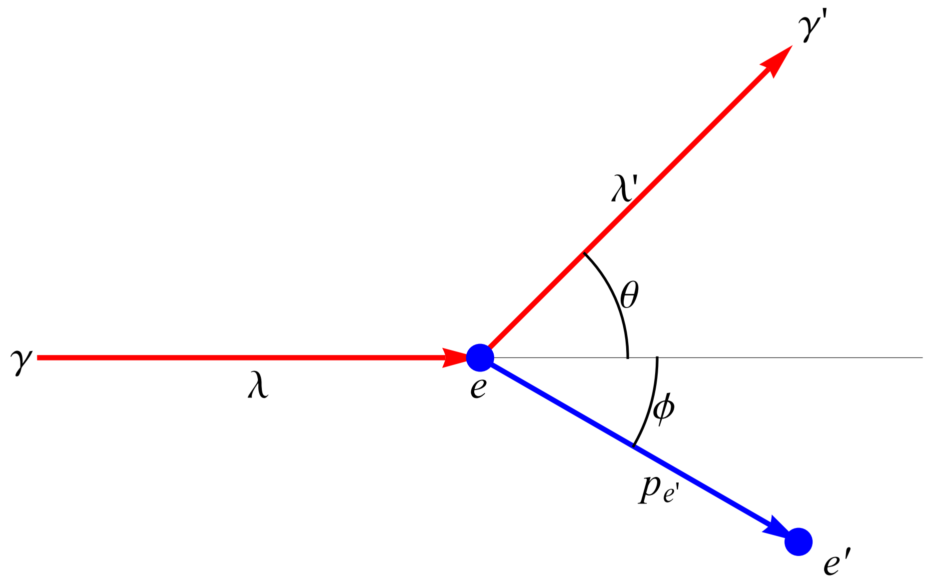
By the early 20th century, research into the interaction of X-rays with matter was well underway. It was known that when a beam of X-rays is directed at an atom, an electron is ejected and is scattered through an angle θ. Classical electromagnetism predicts that the wavelength of scattered rays should be equal to the initial wavelength;[3] however, multiple experiments found that the wavelength of the scattered rays was greater than the initial wavelength. In 1923, Compton published a paper in the *Physical Review* explaining the phenomenon. Using the notion of quantized radiation and the dynamics of special relativity, Compton derived the relationship between the shift in wavelength and the scattering angle:

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{mc}(1-\cos\theta)$$

λ_c is the Compton wavelength,

$$\lambda_c = \frac{h}{mc} = 2.4263102389 \times 10^{-12} \text{ m}$$

where h is the Planck's constant, m is the mass of electron, and c is the velocity of light.



2. Compton scattering

A photon γ with wavelength λ is directed at an electron e in an atom, which is at rest. The collision causes the electron to recoil, and a new photon γ' with wavelength λ' emerges at angle θ. Let e' denote the electron after the collision. From the energy conservation, we have

$$E_\gamma + E_e = E_{\gamma'} + E_{e'}$$

Compton postulated that photons carry momentum; thus from the conservation of momentum, the momenta of the particles should be related by

$$\mathbf{p}_\gamma = \mathbf{p}_{\gamma'} + \mathbf{p}_{e'}$$

assuming the initial momentum of the electron is zero. The photon energies are related to the frequencies by

$$E_\gamma = h\nu = \frac{hc}{\lambda}, \qquad E_\gamma = h\nu = \frac{hc}{\lambda'}$$

where h is the Planck constant, ν is the frequency of the incident photon γ and λ is the wavelength From the relativistic energy-momentum relation, the electron energies are

$$E_e = mc^2, \qquad E_{e'} = \sqrt{m^2c^4 + c^2 p_{e'}^{\ 2}}$$

Along with the conservation of energy, these relations imply that

$$h\nu + mc^2 = h\nu' + \sqrt{m^2c^4 + c^2 p_{e'}{}^2}$$

or

$$(h\nu + mc^2 - h\nu')^2 = m^2c^4 + c^2 p_{e'}{}^2$$

or

$$c^2 p_{e'}{}^2 = (h\nu + mc^2 - h\nu')^2 - m^2c^4 . \qquad (1)$$

From the conservation of momentum,

$$\mathbf{p}_{e'} = \mathbf{p}_{\gamma} - \mathbf{p}_{\gamma'}$$

The *x* component:

$$p_{e'} \cos\phi + \frac{h}{\lambda'}\cos\theta = \frac{h}{\lambda}$$

The *y* component:

$$p_{e'} \sin\phi - \frac{h}{\lambda'}\sin\theta = 0$$

3. Derivation of formula

Then by making use of the scalar product, we have

$$p_{e'}{}^2 = (\mathbf{p}_{\gamma} - \mathbf{p}_{\gamma'}) \cdot (\mathbf{p}_{\gamma} - \mathbf{p}_{\gamma'}) = p_{\gamma}{}^2 + p_{\gamma'}{}^2 - 2p_{\gamma} p_{\gamma'} \cos\theta$$

or

$$\begin{aligned} p_{e'}{}^2 c^2 &= p_{\gamma}{}^2 c^2 + p_{\gamma'}{}^2 c^2 - 2c^2 p_{\gamma} p_{\gamma'} \cos\theta \\ &= (h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu')\cos\theta \end{aligned} \qquad (2)$$

From Eq. (1) and Eq. (2),

$$(h\nu + mc^2 - h\nu')^2 - m^2c^4 = (h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu')\cos\theta$$

or

$$mc^2 h\nu - mc^2 h\nu' = (h\nu)(h\nu')(1-\cos\theta)$$

or

$$\frac{(\nu-\nu')}{\nu\nu'} = \frac{1}{\nu'} - \frac{1}{\nu} = \frac{h}{mc^2}(1-\cos\theta)$$

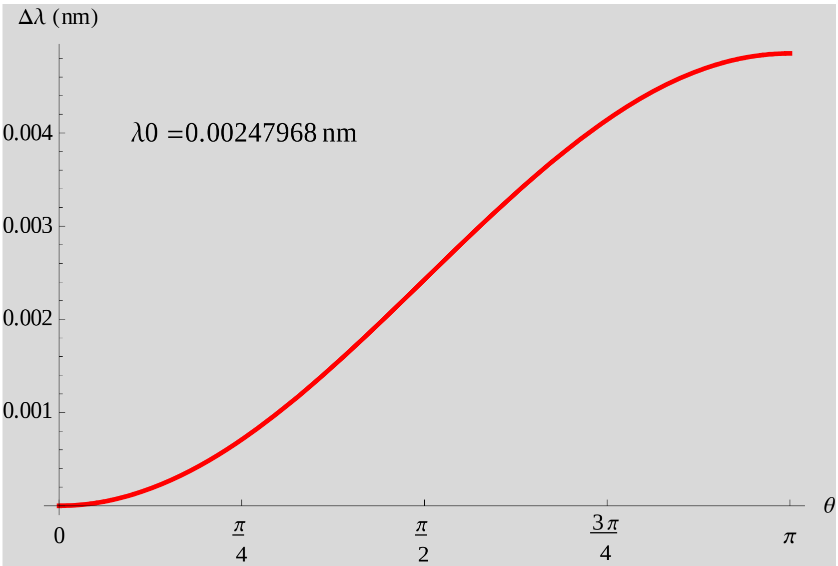
Then we get

$$\Delta\lambda = \lambda' - \lambda = \lambda_c(1-\cos\theta)$$

where $\lambda_c = \dfrac{h}{mc}$ is the Compton wavelength. The CODATA 2006 value for the Compton wavelength of the electron is

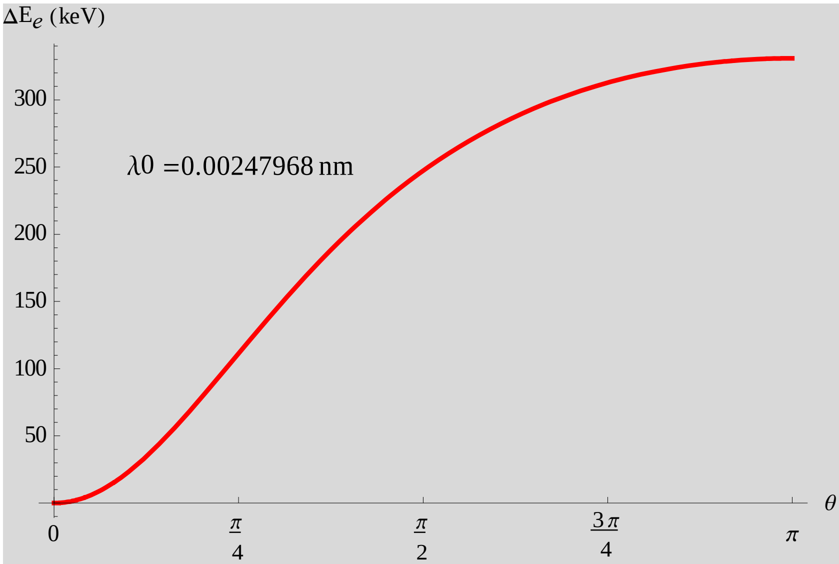
$$\lambda_c = \frac{h}{mc} = 2.4263102175\pm33\times10^{-12}\ \text{m} = 2.4263102175\pm33\times10^{-2}\ \text{A}$$

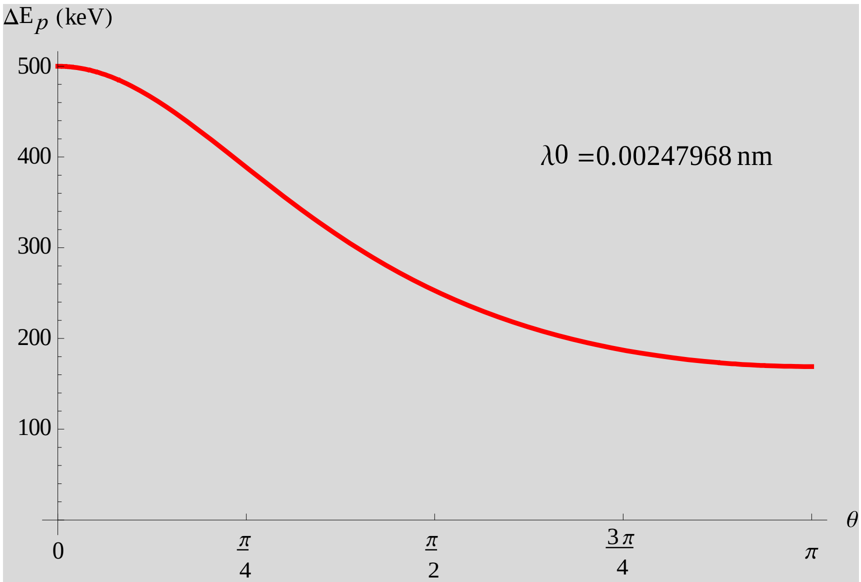
Note that the scattered photon always has a long wavelength than the incident photon.

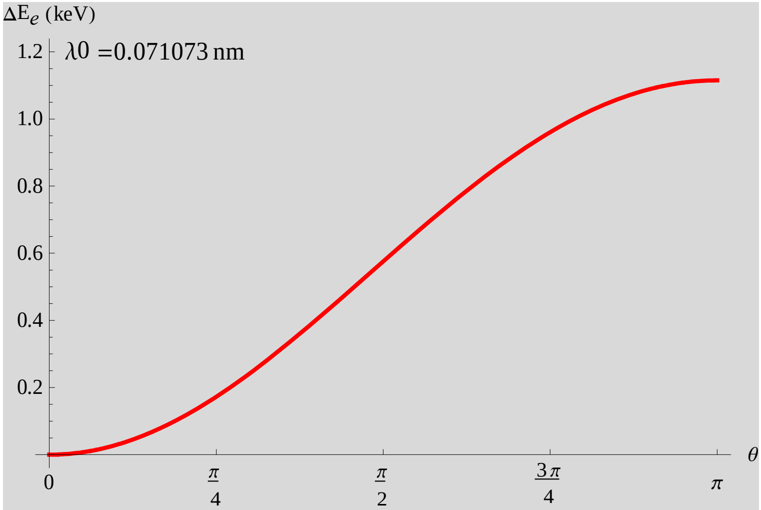


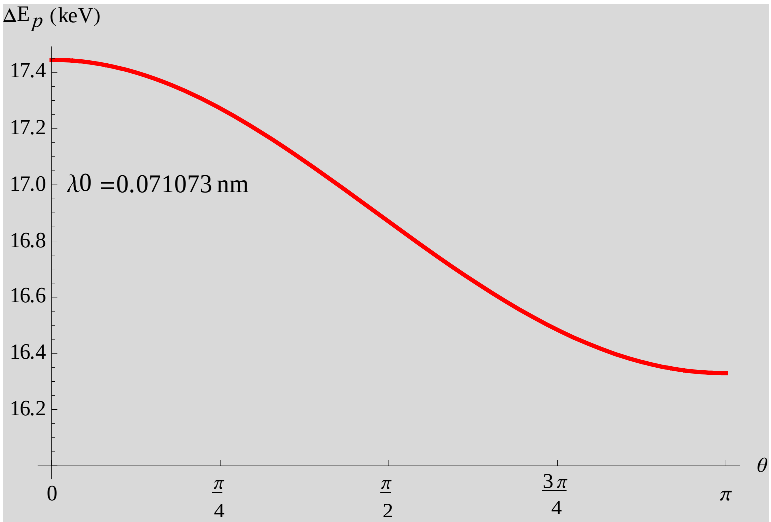
((Note)) The momentum conservation along the y direction

$$\frac{h}{\lambda'}\sin\theta = p_{e'}\sin\phi$$









4. Determination of angles θ and ϕ

We start with the momentum conservation law,

$$p_{e'} \cos\phi = \frac{h}{\lambda} - \frac{h}{\lambda'} \cos\theta \,, \tag{3}$$

$$p_{e'} \sin\phi = \frac{h}{\lambda'} \sin\theta \,, \tag{4}$$

Dividing Eq.(4) by Eq.(3);

$$\tan\phi = \frac{\frac{h}{\lambda'}\sin\theta}{\frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta}$$

Using the relation

$$\lambda' = \lambda + \lambda_c(1 - \cos\theta)$$

we get

$$\tan\phi = \frac{\frac{h}{\lambda'}\sin\theta}{\frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta} = \frac{\lambda\sin\theta}{\lambda' - \lambda\cos\theta}$$

$$= \frac{\lambda\sin\theta}{\lambda(1-\cos\theta) + \lambda_c(1-\cos\theta)}$$

$$= \frac{\lambda\sin\theta}{(\lambda+\lambda_c)(1-\cos\theta)}$$

$$= \frac{2\lambda\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2(\lambda+\lambda_c)\sin^2\frac{\theta}{2}}$$

$$= \frac{\lambda\cos\frac{\theta}{2}}{(\lambda+\lambda_c)\sin\frac{\theta}{2}} = \frac{\lambda}{(\lambda+\lambda_c)\tan\frac{\theta}{2}}$$

The recoil angle of electron ϕ is

$$\cot\phi = (1 + \frac{\lambda_c}{\lambda})\tan\frac{\theta}{2}$$

5. Energy of scattered electron

$$h\nu + mc^2 = h\nu' + \sqrt{m^2c^4 + c^2 p_{e'}^{\ 2}}$$

$$
\begin{aligned}
E' &= \sqrt{m^2c^4 + c^2 p_{e'}{}^2} \\
&= h(\nu - \nu') + mc^2 \\
&= hc\left(\frac{\lambda' - \lambda}{\lambda\lambda'}\right) + mc^2 \\
&= hc\left(\frac{\Delta\lambda}{\lambda\lambda'}\right) + mc^2
\end{aligned}
$$

Then the kinetic energy of the scattered electron is

$$
\begin{aligned}
K' &= E' - mc^2 \\
&= hc\left(\frac{\lambda' - \lambda}{\lambda\lambda'}\right) \\
&= hc\left(\frac{\Delta\lambda}{\lambda\lambda'}\right)
\end{aligned}
$$