Rutherford scattering

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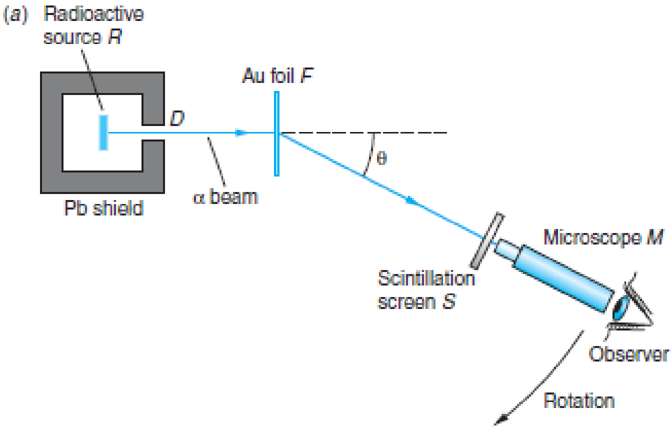
Ernest Rutherford, 1st Baron Rutherford of Nelson OM, FRS (30 August 1871 – 19 October 1937) was a New Zealand-born British chemist and physicist who became known as the father of nuclear physics. In early work he discovered the concept of radioactive half life, proved that radioactivity involved the transmutation of one chemical element to another, and also differentiated and named alpha and beta radiation. This work was done at McGill University in Canada. It is the basis for the Nobel Prize in Chemistry he was awarded in 1908 "for his investigations into the disintegration of the elements, and the chemistry of radioactive substances".

http://en.wikipedia.org/wiki/Ernest_Rutherford

1. Rutherford scattering experiment



P.A. Tipler and R.A. Llewellyn, Modern Physics 5-th edition (Fig.4.4)

Rutherford scattering is the scattering of α-particle (light-particle with charge 2*e*) by a nucleus (heavy particle with charge *Ze*). The mass of nucleus is much larger than that of the α-particle. Thus the nucleus remains unmoved before and after collision. There is a repulsive Coulomb interaction between the nucleus and the α particle, leading to the hyperbolic orbit of the α-particle. The potential energy of the interaction is given by

$$U = \frac{2Ze^2}{4\pi\varepsilon_0 r^2}$$

The boundary conditions can be specified by the kinetic energy K and the angular momentum L of the α-particles, or by the initial velocity v_0 and impact parameter b,

$$K = \frac{1}{2}m{v_0}^2, \quad \text{and} \qquad L = mv_0 b$$

where m is the mass of the α-particle.

((Note)) α particle is He nucleus consisting of two protons and two neutrons (He^{2+})

((**Ewald's sphere**))

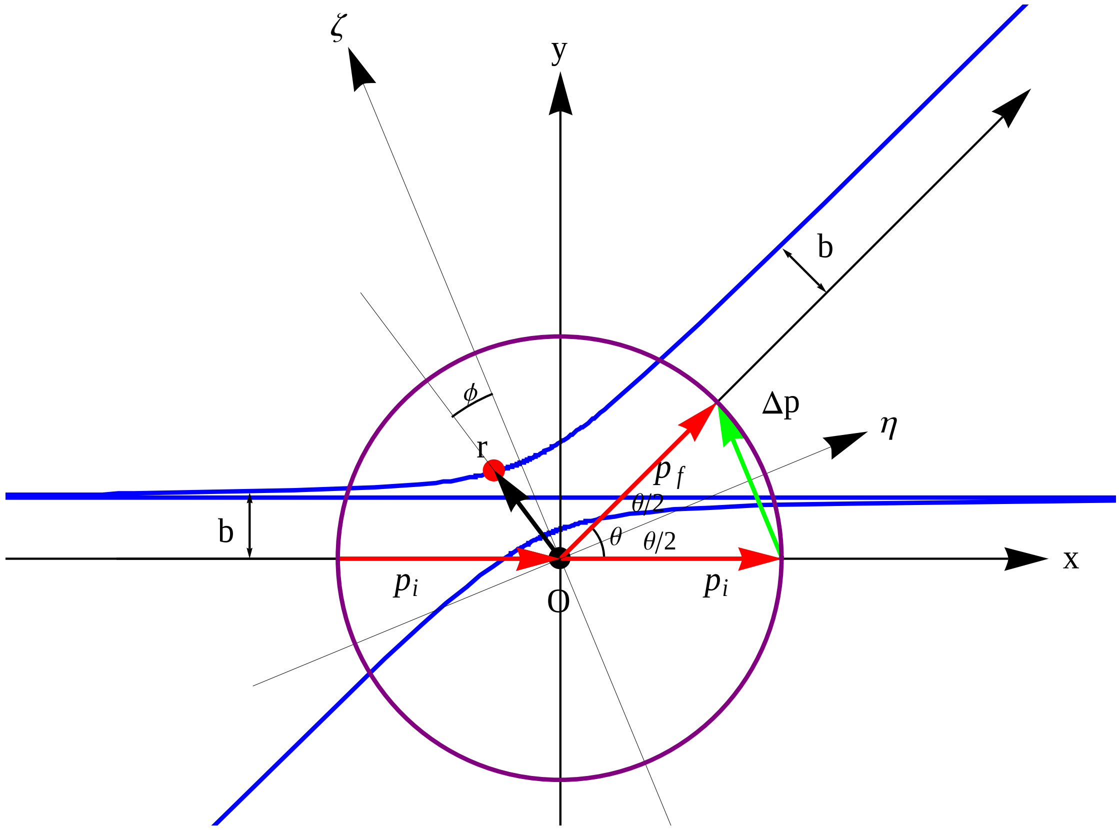


Fig. The hyperbolic Rutherford trajectory.

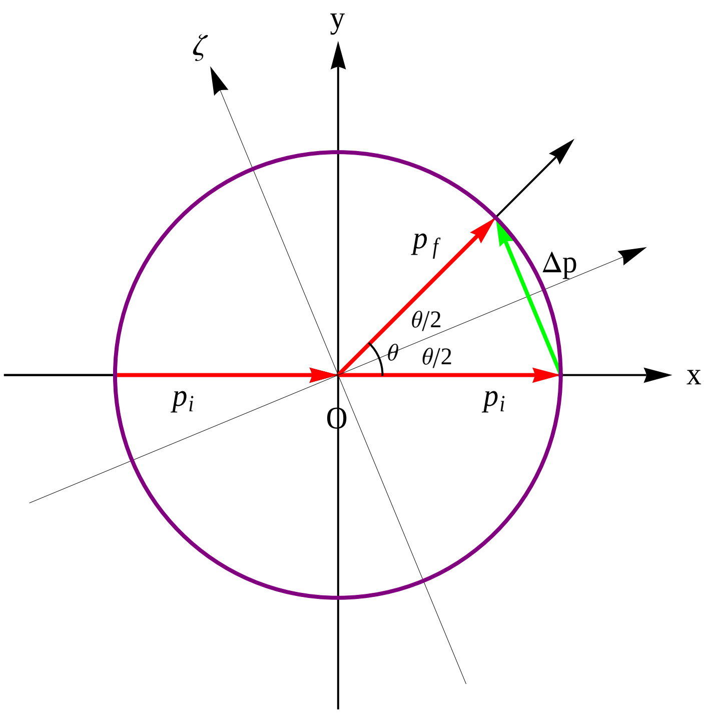


Fig. Ewald's sphere for the Rutherford scattering

$$\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = \mathbf{Q} \qquad \text{(Scattering vector)}$$

where

$$|\mathbf{p}_f| = |\mathbf{p}_i| = p = mv_0$$

From the Ewald's sphere, we have

$$Q = \Delta p = 2p\sin\frac{\theta}{2} = 2mv_0\sin\frac{\theta}{2}$$

((**Conservation of angular momentum**))

$$\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F} = \frac{d\mathbf{L}}{dt},$$

where $\boldsymbol{\tau}$ is the torque, $\boldsymbol{r}$ is the position vector of the α-particle with charge $2e$ (e>) and $\boldsymbol{F}$ is the repulsive Coulomb force (the central force) between the α-particle and the nucleus with charge Ze. The direction of the Coulomb force is parallel to that of $\boldsymbol{r}$. In other words, the torque $\boldsymbol{\tau}$ is zero. The angular momentum $\boldsymbol{L}$ is conserved.

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} = m(\mathbf{r}\times\mathbf{v}) = m(r\hat{r})\times(v_r\hat{r} + v_\phi\hat{\phi}) = mrv_\phi\hat{z} = mr^2\frac{d\phi}{dt}\hat{z}.$$

or

$$mr^2\frac{d\phi}{dt} = mv_0 b$$

or

$$\frac{d\phi}{dt} = \frac{v_0 b}{r^2}$$

where b is the impact parameter.

((**The impulse-momentum theorem**))

$$\frac{d\mathbf{p}}{dt} = \mathbf{F},$$

or

$$\mathbf{Q} = \mathbf{p}_f - \mathbf{p}_i = \int_{t_i}^{t_f} \mathbf{F} dt = \int_{t_i}^{t_f} (F_\varsigma \hat{\varsigma} + F_\eta \hat{\eta}) dt = \int_{t_i}^{t_f} F(\cos\phi \hat{\varsigma} + \sin\phi \hat{\eta}) dt$$

Since $\boldsymbol{Q}$ is parallel to the unit vector $\hat{\varsigma}$, we get

$$Q = \int_{t_i}^{t_f} F\cos\phi dt$$

and

$$\int_{t_i}^{t_f} F\sin\phi dt = 0.$$

Using the relation $\frac{d\phi}{dt} = \frac{v_0 b}{r^2}$

$$\begin{aligned} Q &= \int_{t_i}^{t_f} F\cos\phi dt \\ &= \int_{t_i}^{t_f} F\cos\phi \frac{dt}{d\phi} d\phi \\ &= \frac{1}{4\pi\varepsilon_0} \int_{t_i}^{t_f} \frac{2Ze^2}{r^2} \cos\phi \frac{r^2}{v_0 b} d\phi \\ &= \frac{1}{4\pi\varepsilon_0} \frac{2Ze^2}{v_0 b} \int_{\phi_i}^{\phi_f} \cos\phi d\phi \end{aligned}$$

where

$$\phi_i = -\left(\frac{\pi-\theta}{2}\right), \qquad \text{at } t = t_i,$$

and

$$\phi_f = \left(\frac{\pi-\theta}{2}\right). \qquad \text{at } t = t_f.$$

Here it should be noted that

$$\int_{t_i}^{t_f} F\sin\phi dt = \int_{t_i}^{t_f} F\sin\phi \frac{dt}{d\phi} d\phi$$
$$= \frac{1}{4\pi\varepsilon_0}\int_{t_i}^{t_f}\frac{2Ze^2}{r^2}\sin\phi\frac{r^2}{v_0 b}d\phi$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{2Ze^2}{v_0 b}\int_{-\phi_f}^{\phi_f}\sin\phi d\phi = 0$$

Then we get

$$2mv_0\sin\frac{\theta}{2} = \frac{1}{4\pi\varepsilon_0}\frac{2Ze^2}{v_0 b}\int_{\phi_i}^{\phi_f}\cos\phi d\phi = \frac{1}{4\pi\varepsilon_0}\frac{2Ze^2}{v_0 b}2\int_{0}^{\phi_f}\cos\phi d\phi$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{2Ze^2}{v_0 b}2[\sin\phi]_0^{\phi_f}$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{4Ze^2}{v_0 b}\sin\phi_f$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{4Ze^2}{v_0 b}\cos\frac{\theta}{2}$$

or

$$b = \frac{1}{4\pi\varepsilon_0}\frac{2Ze^2}{m{v_0}^2}\cot\frac{\theta}{2} = \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{K}\cot\frac{\theta}{2},$$

where K is the kinetic energy of the bombarding α-particle,

$$K = \frac{1}{2} m v_0^{\ 2}$$

2. **Differential cross section**: $\frac{d\sigma}{d\Omega}$

Let us consider all those particles that approach the target with impact parameters between b and b +db. These are incident on the annulus (the shaded ring shape). This annulus has cross sectional area

$$d\sigma = 2\pi b db$$

These same particles emerge between angles θ and θ + dθ in a solid angle given by

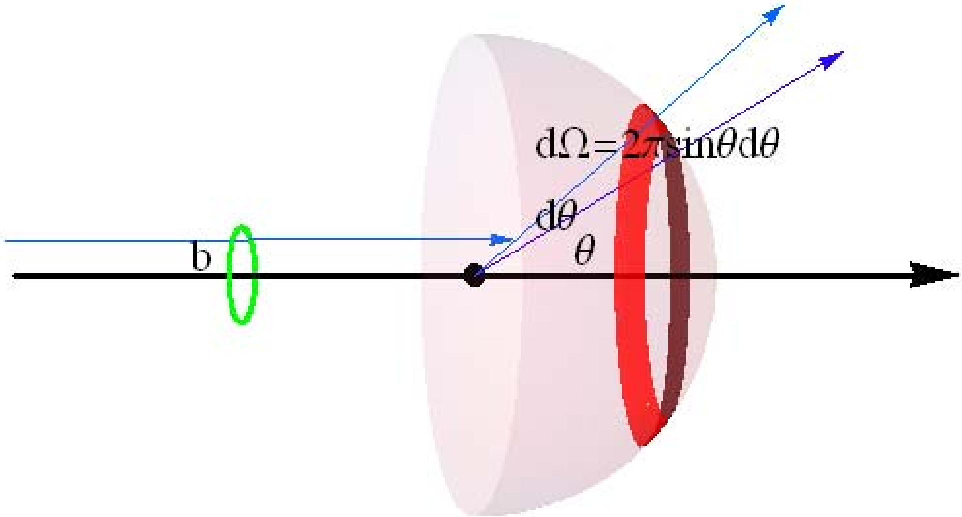
$$d\Omega = 2\pi \sin\theta d\theta$$

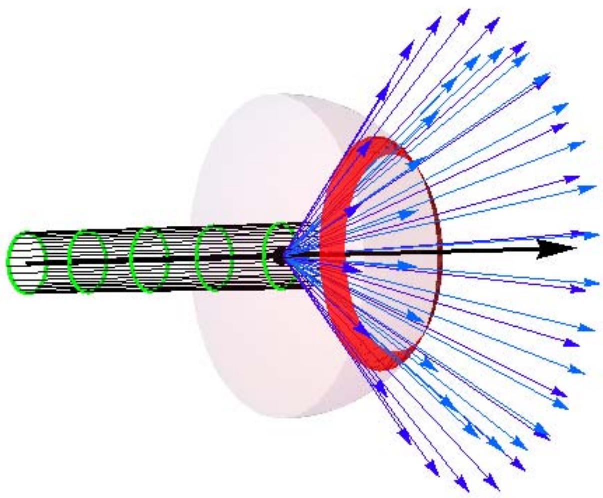
The differential cross section $\frac{d\sigma}{d\Omega}$ is defined as follows.

$$d\sigma = \frac{d\sigma}{d\Omega} d\Omega = 2\pi b db$$

or

$$\frac{d\sigma}{d\Omega} = \frac{2\pi b db}{d\Omega} = \frac{2\pi b db}{2\pi \sin\theta d\theta} = \frac{b}{\sin\theta} \frac{db}{d\theta} = \frac{1}{2\sin\theta} \frac{d}{d\theta} b^2$$





Note that

$$\frac{db^2}{d\theta} = \left(\frac{Ze^2}{4\pi\varepsilon_0 K}\right)^2 \frac{d}{d\theta}\cot^2\frac{\theta}{2} = -\left(\frac{Ze^2}{4\pi\varepsilon_0 K}\right)^2 \cot\frac{\theta}{2}\csc^2\frac{\theta}{2}.$$

Then we get

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\sin\theta}\frac{d}{d\theta}b^2 = \frac{\left(\frac{Ze^2}{4\pi\varepsilon_0 K}\right)^2 \cot\frac{\theta}{2}\csc^2\frac{\theta}{2}d\theta}{2\sin\theta d\theta}$$

$$= \frac{\left(\frac{Ze^2}{4\pi\varepsilon_0 K}\right)^2 \cot\frac{\theta}{2}\csc^2\frac{\theta}{2}}{4\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{Z^2e^4}{(8\pi\varepsilon_0)^2 K^2}\frac{1}{\sin^4\frac{\theta}{2}}$$

This is the celebrated Rutherford scattering formula. It gives the differential cross section for scattering of α particle ($2e$), with kinetic energy K, off a fixed target of charge Ze. In general, this formula can be rewritten as

$$\frac{d\sigma}{d\Omega} = \frac{(qQ)^2}{(16\pi\varepsilon_0)^2 K^2}\frac{1}{\sin^4\frac{\theta}{2}}$$

for scattering of a charge q, with kinetic energy K, off a fixed target of charge Q.

3. Schematic diagram for the Rutherford scattering

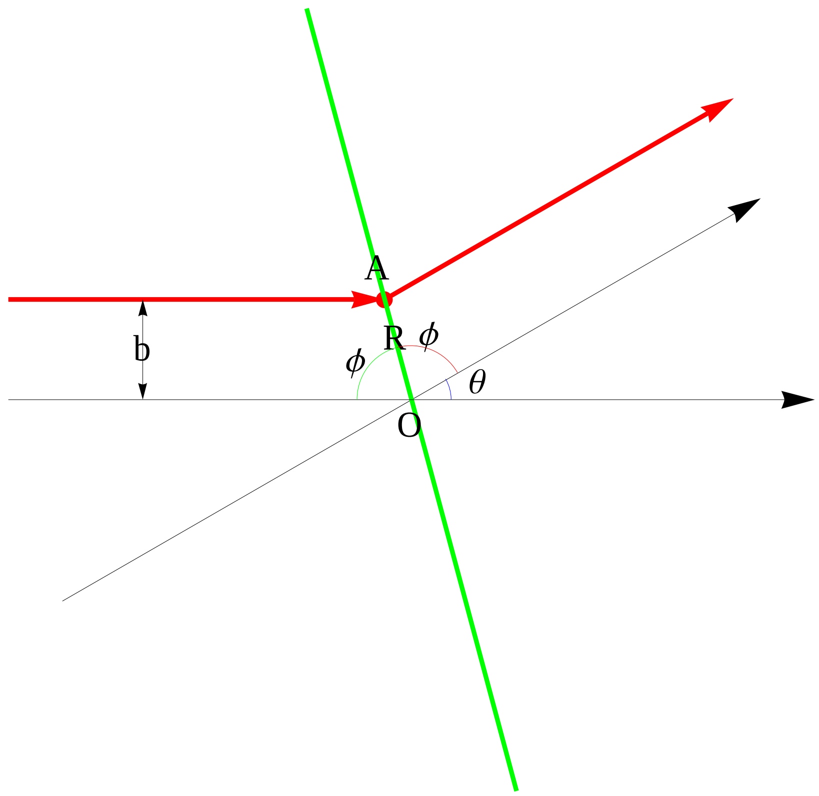


Fig. Schematic diagram for the Rutherford scattering. b is the impact parameter and θ is the scattering angle. The hyperbolic orbit near the target (at the point O) is simplified by a straight line. $\overline{OA} = R$.

As shown in the above figure, the impact parameter b is given by

$$b = R\sin\phi = R\sin(\frac{\pi}{2} - \frac{\theta}{2}) = R\cos\frac{\theta}{2}$$

The impact parameter b is also expressed by

$$b = \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{K}\cot\frac{\theta}{2} = k\cot\frac{\theta}{2}$$

where

$$k = \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{K}$$

Then we get

$$R = \frac{k}{\sin\frac{\theta}{2}}$$

The differential cross section can be expressed by

$$\frac{d\sigma}{d\Omega} = \frac{Z^2 e^4}{(8\pi\varepsilon_0)^2 K^2}\frac{1}{\sin^4\frac{\theta}{2}} = \frac{k^2}{4}\frac{1}{\sin^4\frac{\theta}{2}} = \frac{R^2}{4}$$

In general, a particles with impact parameters smaller than a particular value of b will have scattering angles larger than the corresponding value of b will have scattering angles larger than the corresponding value of θ. The area πb^2 is called the cross section for scattering with angles greater than θ.

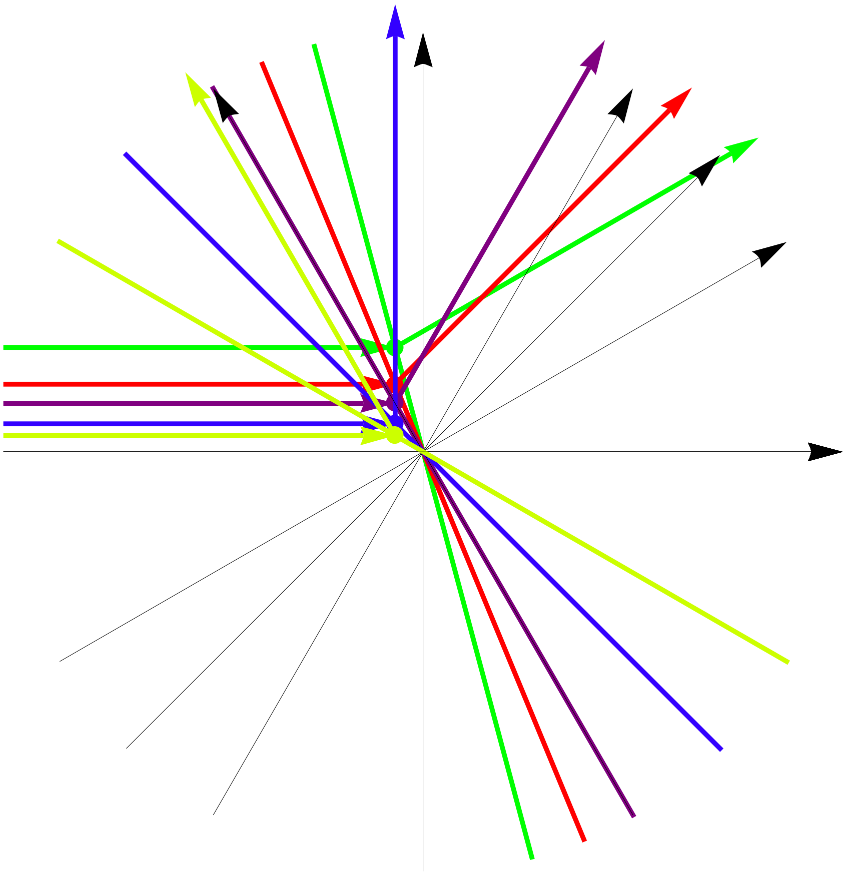


Fig. Schematic diagram for the Rutherford scattering where θ is varied as a parameter. The relation between the impact parameter b and the scattering angle θ. As b increases, the angle θ decreases (smaller angle).

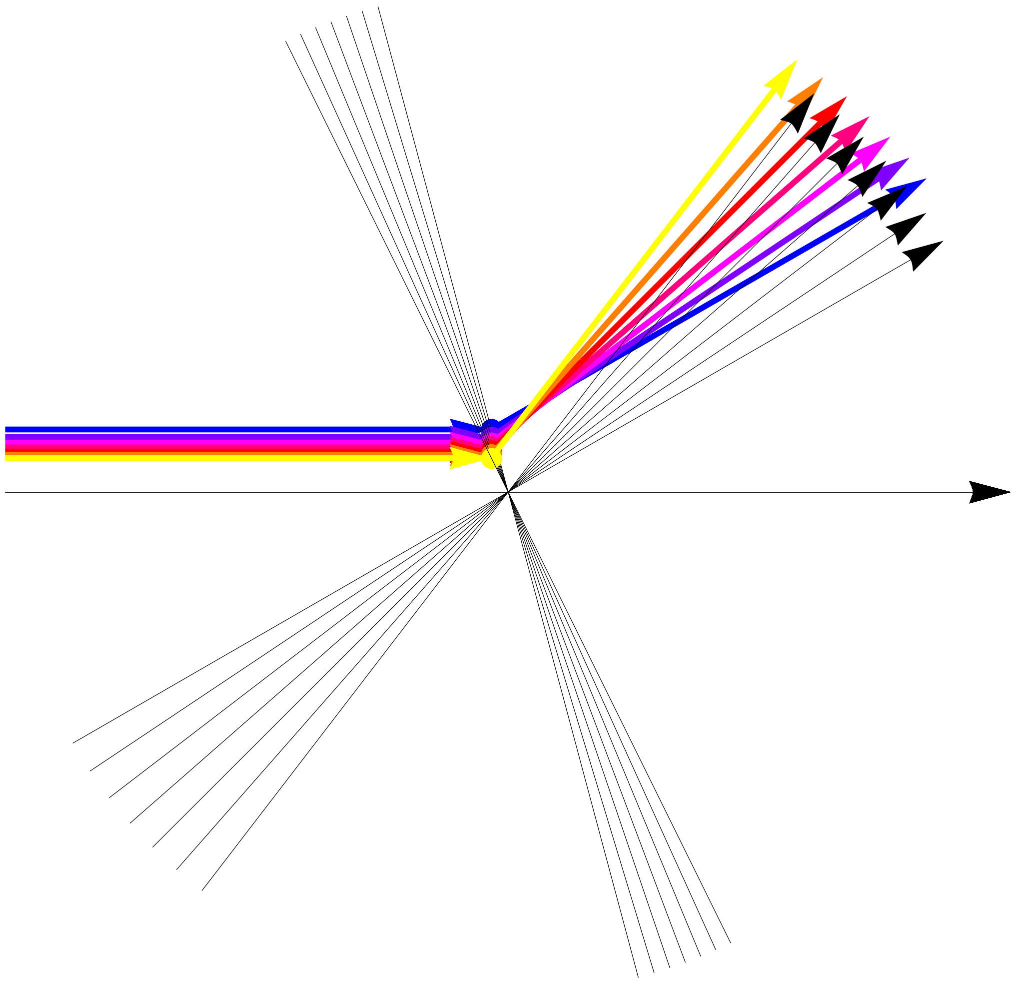


Fig. The α particles with impact parameters between b and b + db are scattered into the angular range between θ and θ + dθ.

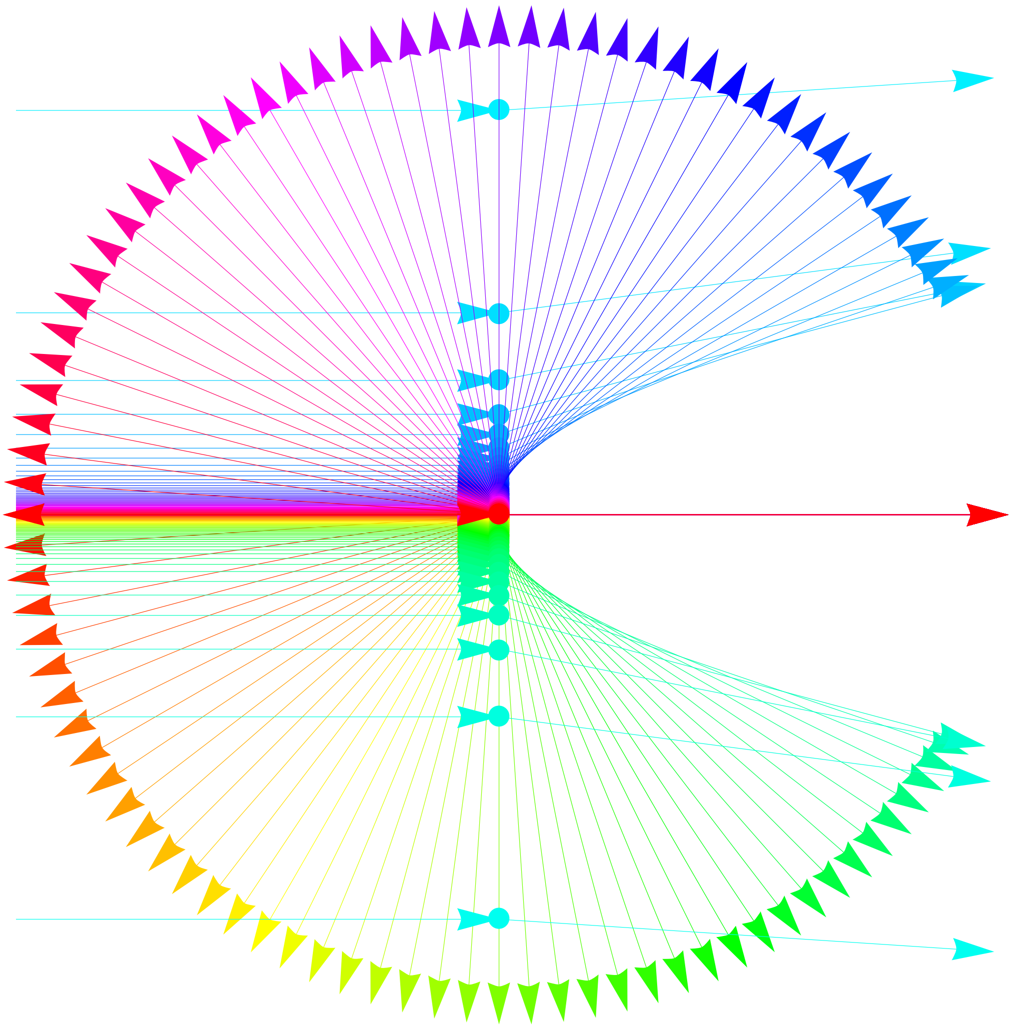


Fig. Rutherford scattering of α particles. The hyperbolic orbit near the target (at the point O) is simplified by a straight line. $\overline{OA} = R$. The point denoted by $\overrightarrow{OA}$ is shown in the figure.

4. Experimental results

If the gold foil were 1 micrometer thick, then using the diameter of the gold atom from the periodic table suggests that the foil is about 2800 atoms thick.

Density of Au

$\rho = 19.30 \text{ g/cm}^3$

Atomic mass of Au;

M_g = 196.96654 g/mol

The number of Au atoms per cm^3;

$$n = \frac{\rho(g/cm^3)}{M_g(g/mol)} N_A$$

where N_A is the Avogadro number. Then we get the number of target nuclei in the volume At (cm^3) as

$$N_s = ntA = \frac{\rho N_A}{M_g(g/mol)}$$

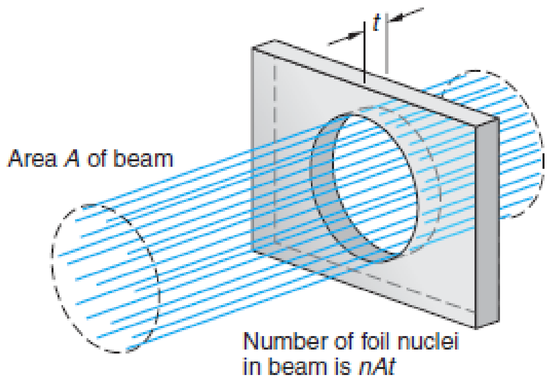


Figure 4-8 The total number of nuclei of foil atoms in the area covered by the beam is nAt, where n is the number of foil atoms per unit volume, A is the area of the beam, and t is the thickness of the foil.

P.A. Tipler and R.A. Llewellyn, Modern Physics 5-th edition (Fig.4.8)

If $\sigma(=\pi b^2)$ is the cross section for each nucleus, $ntA\sigma$ is the total area exposed by the target nuclei. The fraction of incident particles scattered by an angle of θ or *greater* is

$$f = \frac{ntA\sigma}{A} = nt\sigma = nt\pi b^2 = nt\pi\left(\frac{Ze^2}{4\pi\varepsilon_0 K}\right)^2 \cot^2\frac{\theta}{2}.$$

The number of α particles which can be compared with measurements, is defined by

$$N(\theta) = \left(\frac{N_i nt}{R^2}\right)\frac{d\sigma}{d\Omega} = \left(\frac{N_i nt}{R^2}\right)\frac{Z^2 e^4}{(8\pi\varepsilon_0)^2 K^2}\frac{1}{\sin^4\dfrac{\theta}{2}}$$

where r is the distance between the target and the counter, N_i is the total number of incident α particles, and n is the number density of the target.

((**Experimental results**))

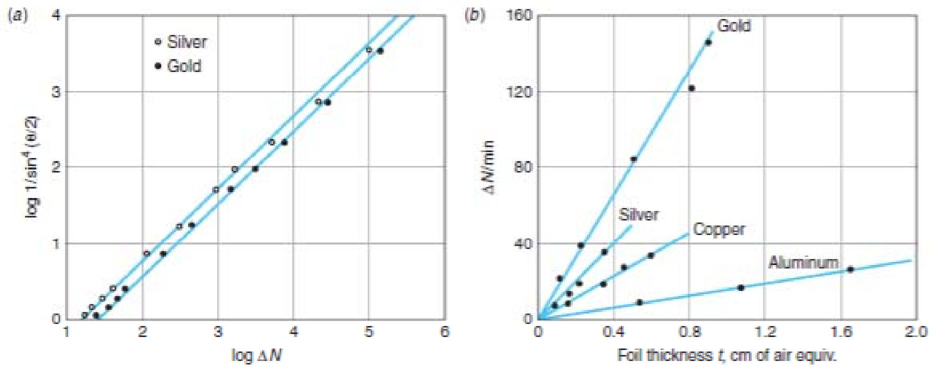


Figure 4-9 (a) Geiger and Marsden's data for α scattering from thin gold and silver foils. The graph is a log-log plot to show the data over several orders of magnitude. Note that scattering angle increases downward along the vertical axis. (b) Geiger and Marsden also measured the dependence of ΔN on t predicted by Equation 4-6 for foils made from a wide range of elements, this being an equally critical test. Results for four of the elements used are shown.

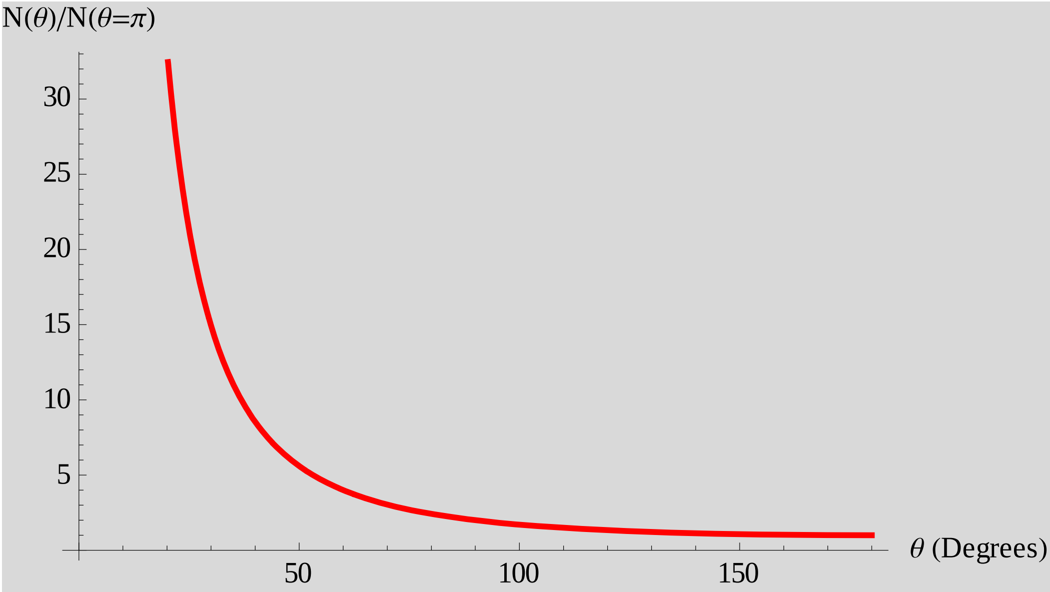
P.A. Tipler and R.A. Llewellyn, Modern Physics 5-th edition (Fig.4.9)

Using the value of $N(\theta = \pi)$, we have

$$\frac{N(\theta)}{N(\theta = \pi)} = \frac{1}{\sin^4\dfrac{\theta}{2}},$$

where

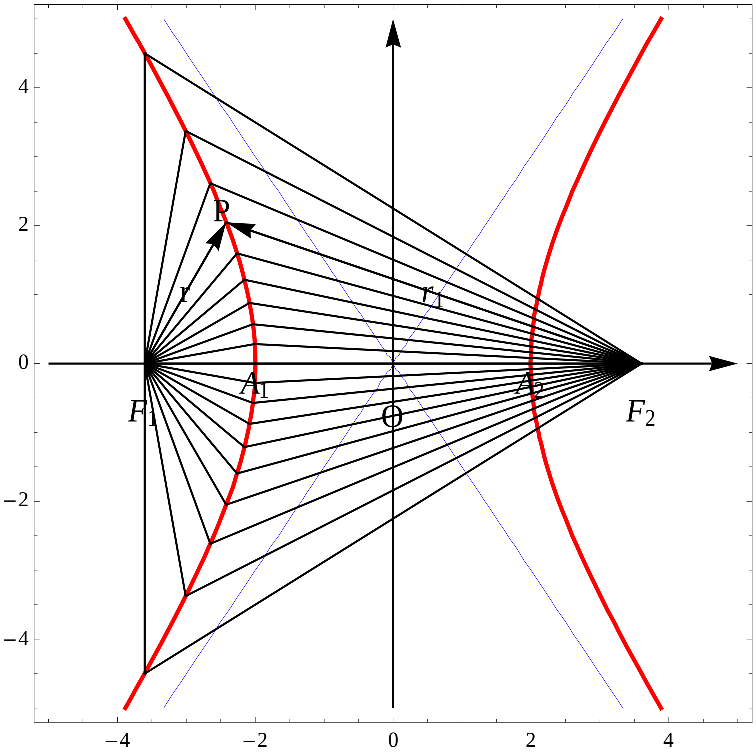
$$N(\theta = \pi) = \frac{N_i ntZ^2 e^4}{R^2 (8\pi\varepsilon_0)^2 K^2}$$



5. Conclusion

Most of the mass and all of the positive charge of an atom, +Ze, are concentrated in a minute volume of the atom with a diameter of about 10^{-14} m.

APPENDIX: Property of hyperbola



The properties of hyperbola

$$c = \sqrt{a^2 + b^2}$$

$$\overline{F_1P} = r\,, \qquad \overline{F_2P} = r_1\,,$$

$$r_1 - r = 2a\,, \qquad r_1 = r + 2a$$

or

$$r_1^{\ 2} = (r+2a)^2 = r^2 + 4ar + 4a^2$$

Cosine law:

$$r_1^{\ 2} = r^2 + 4c^2 - 4rc\cos\theta$$

Using the above two equations, we get

$$r_1^{\ 2} = r^2 + 4ar + 4a^2 = r^2 + 4c^2 - 4rc\cos\theta$$

$$r = \frac{p}{1 + e\cos\theta},$$

where

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} > 1$$

$$p = a(e^2 - 1).$$

The ParametricPlot:

$$x = -c + r\cos\theta, \qquad y = r\sin\theta$$

with

$$r = \frac{p}{1 + e\cos\theta}.$$