Angular momentum of photon
Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
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1. Angular momentum

The orbital angular momentum is defined as

\[
\hat{L} = \hat{r} \times \hat{p}
\]

\[
\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x
\]

\[
\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y
\]

\[
\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z
\]

We consider the commutation relation:

\[
\hat{L} \times \hat{L} = i\hbar \hat{L}
\]

\[
[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x,
\]

\[
[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y,
\]

\[
[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z
\]

\[
[\hat{L}_z, \hat{z}] = [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{z}] = 0,
\]

\[
[\hat{L}_z, \hat{x}] = [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{x}] = -[\hat{y}\hat{p}_x, \hat{x}] = -\hat{y}[\hat{p}_x, \hat{x}] = i\hbar\hat{y}
\]

\[
[\hat{L}_z, \hat{y}] = [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{y}] = [\hat{x}\hat{p}_y, \hat{y}] = -i\hbar\hat{x}
\]

or

\[
[\hat{L}_z, \hat{x} + i\hat{y}] = [\hat{L}_z, \hat{x}] + i[\hat{L}_z, \hat{y}] = i\hbar\hat{y} + i(-i\hbar\hat{x}) = \hbar(\hat{x} + i\hat{y})
\]

\[
[\hat{L}_z, \hat{x} - i\hat{y}] = [\hat{L}_z, \hat{x}] - i[\hat{L}_z, \hat{y}] = i\hbar\hat{y} - i(-i\hbar\hat{x}) = -\hbar(\hat{x} - i\hat{y})
\]

We also note that
\[ [\hat{L}^2, [\hat{L}^2, \hat{x}]] = 2\hbar^2 \{\hat{x}, \hat{L}^2\} \]

where

\[ \{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A} \]

((Mathematica))

**Proof**

```mathematica
Clear["Global`"];

ux = {1, 0, 0}; uy = {0, 1, 0}; uz = {0, 0, 1};

r = {x, y, z};

Lx := (\[hbar\] ux.(-\[I\] Cross[r, Grad[#, {x, y, z}]])) & // Simplify;
Ly := (\[hbar\] uy.(-\[I\] Cross[r, Grad[#, {x, y, z}]])) & // Simplify;
Lz := (\[hbar\] uz.(-\[I\] Cross[r, Grad[#, {x, y, z}]])) & // Simplify;
Lsq := (Lx[Lx[#]] + Ly[Ly[#]] + Lz[Lz[#]]) &;

eq2 = Lsq[Lsq[x \psi[x, y, z]]] - Lsq[x Lsq[\psi[x, y, z]]] - Lsq[x Lsq[\psi[x, y, z]]] + x Lsq[Lsq[\psi[x, y, z]]] // FullSimplify;

eq3 = 2 \[hbar\]^2 (x Lsq[\psi[x, y, z]] + Lsq[x \psi[x, y, z]]) // FullSimplify;

eq2 - eq3 // Simplify

0
```

2. **Eigenkets of angular momentum**

\[ \hat{L}^2|l,m\rangle = \hbar^2 (l + 1)|l,m\rangle \]
$\hat{L}_z|l,m\rangle = \hbar m|l,m\rangle$

$\hat{L}_+|l,m\rangle = \hbar \sqrt{(l-m)(l+m+1)}|l,m+1\rangle$

$\hat{L}_-|l,m\rangle = \hbar \sqrt{(l+m)(l-m+1)}|l,m-1\rangle$

where

$\hat{L}_+ = \hat{L}_x + i\hat{L}_y$, \hspace{1cm} $\hat{L}_- = \hat{L}_x - i\hat{L}_y$

3. Selection rule-I

Using the relation

$\hat{L}_z|l,m\rangle = \hbar m|l,m\rangle$

we have

$\langle l',m'|\hat{L}_z,\hat{z}|l,m\rangle = 0$

or

$\langle l',m'|\hat{L}_z \hat{z} - \hat{z} \hat{L}_z|l,m\rangle = 0$

or

$(m'-m)\langle l',m'|\hat{z}|l,m\rangle = 0$

Then we get the relation

$m' = m$, for the dipole ion the $z$ direction.

4. Selection rule-II

Using the relation

$\hat{L}_z|l,m\rangle = \hbar m|l,m\rangle$
we have
\[ \langle l', m'|\hat{L}_z, \hat{x} + i\hat{y} \rangle|l, m\rangle = \hbar \langle l', m'|\hat{x} + i\hat{y} \rangle|l, m\rangle, \]
or
\[ \langle l', m'|\hat{L}_z (\hat{x} + i\hat{y}) - (\hat{x} + i\hat{y})\hat{L}_z|l, m\rangle = \hbar \langle l', m'|\hat{x} + i\hat{y} \rangle|l, m\rangle \]
or
\[ (m' - m - 1) \langle l', m'|\hat{x} + i\hat{y} \rangle|l, m\rangle = 0 \]

Then we get the relation
\[ m' = m + 1, \text{ for the dipole ion the } x, y \text{ direction.} \]

5. Selection rule-III

Using the relation
\[ \hat{L}_z|l, m\rangle = \hbar m|l, m\rangle \]

we have
\[ \langle l', m'|\hat{L}_z, \hat{x} - i\hat{y} \rangle|l, m\rangle = -\hbar \langle l', m'|\hat{x} - i\hat{y} \rangle|l, m\rangle, \]
or
\[ \langle l', m'|\hat{L}_z (\hat{x} - i\hat{y}) - (\hat{x} - i\hat{y})\hat{L}_z|l, m\rangle = -\hbar \langle l', m'|\hat{x} - i\hat{y} \rangle|l, m\rangle \]
or
\[ (m' - m + 1) \langle l', m'|\hat{x} - i\hat{y} \rangle|l, m\rangle = 0 \]

Then we get the relation
\[ m' = m - 1, \text{ for the dipole ion the } x, y \text{ direction.} \]
6. Selection rule-IV

Using the commutation relation

\[ [\hat{L}^2, [\hat{L}^2, \hat{x}]] = 2\hbar^2 \{\hat{x}, \hat{L}^2\} \]

we get the following equation,

\[ \langle l', m'| \hat{L}^2, [\hat{L}^2, \hat{x}] \rangle |l, m\rangle = 2\hbar^2 \langle l', m'| \hat{x}, \hat{L}^2 \rangle |l, m\rangle \]

or

\[ \langle l', m'| \hat{L}^2 \hat{L}^2 \hat{x} - 2\hat{L}^2 \hat{x} \hat{L}^2 + \hat{x} \hat{L}^2 \hat{L}^2 \rangle |l, m\rangle = 2\hbar^2 \langle l', m'| \hat{x} \hat{L}^2 + \hat{L}^2 \hat{x} |l, m\rangle \]

Here we use the relation

\[ \hat{L}^2 |l, m\rangle = \hbar^2 l(l+1)|l, m\rangle, \quad \text{and} \quad \langle l, m| \hat{L}^2 = \hbar^2 l(l+1)|l, m\rangle \]

Then we have

\[ \hbar^4 [l'^2 (l'+1)^2 - 2l'(l'+1)(l+1) + l^2 (l+1)^2 - 2l'(l'+1) - 2l(l+1)]\langle l', m'| \hat{x} |l, m\rangle = 0 \]

or

\[ (l'-l-1)(l'-l+1)(l'+l)(l'+l+2)\langle l', m'| \hat{x} |l, m\rangle = 0 \]

The last factor yields the selection rule

\[ l' = l \pm 1 \]

\[ (\text{Mathematica}) \]

\[ g_1 = a^2 (a + 1)^2 - 2 a (a + 1) b (b + 1) + b^2 (b + 1)^2 - 2 a (a + 1) - 2 b (b + 1) // \text{Factor} \]

\[ (-1 + a - b) (1 + a - b) (a + b) (2 + a + b) \]
Since \( l' \) and \( l \) are both non-negative, the \((l'+l+2)\) term cannot vanish, and the \((l'+l)\) term can only vanish for \( l' = l = 0 \). However, this selection rule cannot be satisfied, since the states with \( l' = l = 0 \) are independent of direction, and therefore these matrix elements of \( \hat{x} \) vanish. Formally, one easily shows this

\[
\langle 0,0|\hat{x}|0,0\rangle = 0
\]

using the parity operator. 

((Proof))

\[
\hat{x}\hat{x}\hat{x} = -\hat{x}
\]

where the parity operator satisfies the relations,

\[
\hat{x}^+ = \hat{x}, \quad \hat{x}^2 = 1
\]

\[
\langle 0,0|\hat{x}\hat{x}|0,0\rangle = -\langle 0,0|\hat{x}|0,0\rangle
\]

or

\[
\langle 0,0|\hat{x}|0,0\rangle = 0
\]

where

\[
\hat{x}|l,m\rangle = (-1)^l|l,m\rangle,
\]

and

\[
\hat{x}|0,0\rangle = |0,0\rangle, \quad \text{and} \quad \langle 0,0|\hat{x} = \langle 0,0|
\]

7. Dipole selection rule

The dipole radiation is emitted if

\[
M = \langle f | e \cdot \hat{r} | i \rangle = e \cdot \langle f | \hat{r} | i \rangle = e \cdot D_f
\]
does not vanish, where \( e \) is the electric field (the polarization vector), and

\[
D_{\beta} = \langle f | \hat{\epsilon} | i \rangle
\]

We assume that \( |i\rangle = |l, m\rangle \) and \( |f\rangle = |l', m'\rangle \). Then we have

\[
D_{\beta} = \langle l', m' | \hat{x} | l, m \rangle e_x + \langle l', m' | \hat{y} | l, m \rangle e_y + \langle l', m' | \hat{z} | l, m \rangle e_z,
\]

(i) For \( m' = m \).

\[
\langle l', m' | \hat{z} | l, m \rangle \neq 0, \quad \langle l', m' | \hat{x} | l, m \rangle = 0, \quad \langle l', m' | \hat{y} | l, m \rangle = 0
\]

\[
D_{\beta} = \langle l', m' | \hat{x} | l, m \rangle e_x + \langle l', m' | \hat{y} | l, m \rangle e_y + \langle l', m' | \hat{z} | l, m \rangle e_z
\]

= \langle l', m' | \hat{z} | l, m \rangle e_z

\( D_{\beta} \) is directed along the \( z \) axis.

(a) Suppose that the wavevector \( k \) of the emitted photon is along the \( z \) axis. There is no radiation in the \( z \)-direction since the polarization vector \( \epsilon \) is perpendicular to \( D_{\beta} \) (the \( z \) axis).

(b) For example, we consider light going in the \( x \) direction. It can have two directions of polarization, either in the \( z \) or in the \( y \) direction. A transition in which \( \Delta m = 0 \), can produce only light which is polarized in the \( z \) direction.
Fig. \( m' = m. \) \( \mathbf{D}_f \parallel z \). The light propagating along the \( x \) direction. It is a linearly polarized wave (along the \( z \) axis).

(ii) For \( m' = m + 1 \)

\[
\left< l', m' | \hat{\hat{r}}_z | l, m \right> = 0, \quad \left< l', m' | \hat{\hat{z}} | l, m \right> = 0.
\]

where

\[
\hat{\hat{r}}_+ = \frac{\hat{x} + i\hat{y}}{\sqrt{2}}.
\]

(iii) For \( m' = m - 1 \)

\[
\left< l', m' | \hat{\hat{r}}_z | l, m \right> = 0, \quad \left< l', m' | \hat{\hat{z}} | l, m \right> = 0.
\]

where
We now consider the matrix element with \( m' = m \pm 1 \).

\[
D_{m'} = \langle l', m' | \hat{x} e_x + \hat{y} e_y + \hat{z} e_z | l, m \rangle
\]

\[
= \langle l', m' | \hat{x} e_x + \hat{y} e_y | l, m \rangle
\]

\[
= \langle l', m' | \hat{r}_+ e_x + \hat{r}_{-} e_x | l, m \rangle
\]

\[
= e_+ \cdot \langle l', m' | \hat{r}_+ | l, m \rangle + e_- \cdot \langle l', m' | \hat{r}_- | l, m \rangle
\]

where

\[
e_+ = \frac{e_x + ie_y}{\sqrt{2}}, \quad e_- = \frac{e_x - ie_y}{\sqrt{2}}.
\]

(Jones vector notation)

or

\[
e_x = \frac{1}{\sqrt{2}} (e_+ + e_-), \quad e_y = \frac{1}{\sqrt{2i}} (e_+ - e_-)
\]

and

\[
\hat{x} e_x + \hat{y} e_y = \hat{r}_+ e_+ + \hat{r}_- e_-
\]

Note that

\[
e_+ \cdot e_+ = 0, \quad e_- \cdot e_- = 0
\]

\[
e_+ \cdot e_- = 1, \quad e_- \cdot e_+ = 1
\]

(a) When \( m' = m + 1 \),

\[
D_{m'} = \langle l', m' | \hat{x} e_x + \hat{y} e_y | l, m \rangle = e_- \langle l', m' | \hat{r}_- | l, m \rangle
\]
has the same direction of the left circularly polarization vector \( (e_\ldots) \). Then the emitted photon which is right circularly polarized \( (e_\ldots) \), can propagate along the \( z \) axis. A photon with right-hand circular polarization carries a spin \(+\hbar\) in the \( z \) direction (the propagation direction).

![Diagram](image.png)

**Fig.** The case of \( m' = m + 1 \) (right circularly polarization). A right circularly polarized photon \( (e_\ldots) \) propagates with a wavevector \( k \) in the \( z \) direction. Note that the electric field is denoted by \( \cos(kz - \omega t)e_x + \sin(kz - \omega t)e_y \). This electric field rotates in clock-wise sense with time \( t \), and rotates in counter clock-wise sense with \( z \) (as the wave propagates forward). The corresponding spin of the photon is directed in the positive \( z \) direction \((\hbar)\).

\[
\mathbf{D}_{j,\ldots}(\approx e_\ldots). \quad E(\approx e_\ldots). \quad (e_\ldots \cdot e_\ldots = 1, e_\ldots \cdot e_\ldots = 0).
\]

(b) When \( m' = m - 1 \)

\[
\mathbf{D}_{jl} = \langle l', m'|\hat{x}e_x + \hat{y}e_y |l, m\rangle = e_x \langle l', m'|\hat{e}_x |l, m\rangle
\]

is parallel to the right circularly polarization vector \( e_\ldots \). The emitted photon with left circularly polarization \( (e_\ldots) \) can propagate along the \( z \) axis. A photon with the left-hand polarization carries a spin \(-\hbar\), that is, a spin direction opposite to the \( z \) direction.
The case of $m' = m - 1$ (left circularly polarization). A left circularly polarized photon ($\mathbf{e}$) propagates with a wavevector $\mathbf{k}$ in the $z$ direction. Note that the electric field is given by $\cos(kz - \omega t)e_x - \sin(kz - \omega t)e_y$. This electric field rotates in counter clock-wise sense with time $t$, and rotates in clock-wise sense with $z$ (as the wave propagates forward). The corresponding spin of the photon is directed in the negative $z$ direction, as $-\hbar$. $D_z(\approx \mathbf{e}_+)$.

$E(\approx \mathbf{e}_-) \cdot (\mathbf{e}_+ \cdot \mathbf{e}_- = 1, \mathbf{e}_+ \cdot \mathbf{e}_+ = 0)$.

The rules on $\Delta m$ can be understood by realizing that $\sigma^+$ and $\sigma^-$ circularly polarized photons carry angular momenta of $+\hbar$ and $-\hbar$, respectively, along the $z$ axis, and hence $m$ must change by one unit to conserve angular momentum. For linearly polarized light along the $z$ axis, the photons carry no $z$-component of momentum, implying $\Delta m = 0$, while $x$ or $y$-polarized light can be considered as a equal combination of $\sigma^+$ and $\sigma^-$ photons, giving $\Delta m = \pm 1$.

REFERENCES

D. Bohm, Quantum Theory (Prentice Hall, New York, 1951).
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