Berry's phase
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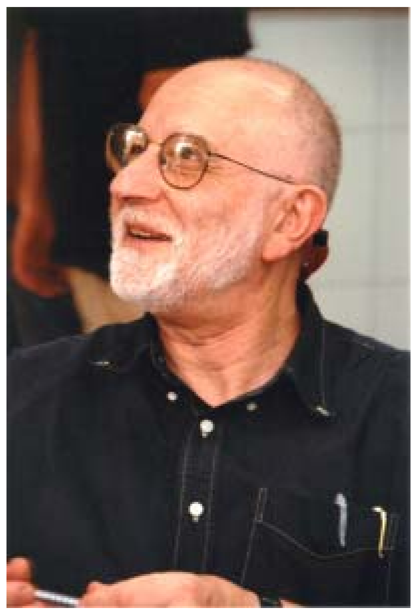
Sir Michael Victor Berry, FRS (born 14 March 1941), is a mathematical physicist at the University of Bristol, England. He was elected a fellow of the Royal Society of London in 1982 and knighted in 1996. From 2006 he has been editor of the journal, Proceedings of the Royal Society. He is famous for the Berry phase, a phenomenon observed e.g. in quantum mechanics and optics. He specializes in semi-classical physics (asymptotic physics, quantum chaos), applied to wave phenomena in quantum mechanics and other areas such as optics. He is also currently affiliated with the Institute for Quantum Studies at Chapman University in California.



Prof. Micheal Berry (Melville Wills Professor of Physics (Emeritus), University of Bristol)
https://michaelberryphysics.wordpress.com/

1. What is the Berry phase?

In classical and quantum mechanics, the **geometric phase**, **Pancharatnam–Berry phase** (named after S. Pancharatnam and Sir Michael Berry), **Pancharatnam phase** or most commonly **Berry phase**, is a phase difference acquired over the course of a cycle, when a system is subjected to cyclic adiabatic processes, which results from the geometrical properties of the parameter space of the Hamiltonian. The phenomenon was first discovered in 1956, and rediscovered in 1984. It can be seen in the Aharonov–Bohm effect and in the conical intersection of potential energy surfaces. In the case of the Aharonov–Bohm effect, the adiabatic parameter is the magnetic field enclosed by two interference paths, and it is cyclic in the sense that these two paths form a loop. In the case of the conical intersection, the adiabatic parameters are the molecular coordinates. Apart from quantum mechanics, it arises in a variety of other wave

systems, such as classical optics. As a rule of thumb, it can occur whenever there are at least two parameters characterizing a wave in the vicinity of some sort of singularity or hole in the topology; two parameters are required because either the set of nonsingular states will not be simply connected, or there will be nonzero holonomy.
http://en.wikipedia.org/wiki/Geometric_phase

((**Geometric phase, Berry's phase**))

A particle which starts out in the n-th eigenstate of $H(0)$ remains, under adiabatic condition, in the n-th eigenstate state of $H(t)$, picking up only a time-dependent phase factor

$$\Psi_n(t) \to e^{i\theta_n(t)} e^{i\gamma_n(t)} \psi_n(t)$$

where $\theta_n(t)$ is the dynamic phase and $\gamma_n(t)$ is the so-called geometric phase. It is surprising that the existence of the Berry phase has not been noticed for almost 60 years since the development of the quantum mechanics.

$$\theta_n(t) = -\frac{\hbar}{i}\int_0^t E_n(t')dt', \qquad \gamma_n(t) = i\int_0^t \left\langle \psi_n(t') \middle| \frac{\partial}{\partial t'}\psi_n(t') \right\rangle dt'$$

2. Adiabatic theorem (Griffiths, Quantum Mechanics)

If the Hamiltonian is independent of time, then a particle which starts out in the n-th eigenstate such that

$$\hat{H}(0)|\psi_n(0)\rangle = E_n(0)|\psi_n(0)\rangle$$

If the Hamiltonian changes with time, the eigenstate and energy eigenvalue are time dependent

$$\hat{H}(t)|\psi_n(t)\rangle = E_n(t)|\psi_n(t)\rangle$$

with the condition

$$\langle \psi_n(t)|\psi_m(t)\rangle = \delta_{m,n}$$

where

$$\langle \psi_n(t)|\psi_m(t)\rangle = \int d\boldsymbol{r}\psi_n^*(\boldsymbol{r},t)\psi_m(\boldsymbol{r},t)$$

We now consider the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\left|\Psi(t)\right\rangle = \hat{H}(t)\Psi(t)$$

We assume that

$$\left|\Psi(t)\right\rangle = \sum_n c_n(t)e^{i\theta_n(t)}\left|\psi_n(t)\right\rangle$$

where the dynamic phase is defined by

$$\theta_n(t) = -\frac{1}{\hbar}\int_0^t E_n(t')dt',$$

((**Note**)) The phase factor from the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi_n(t) = E_n(t)\psi_n(t)$$

When we assume that $\psi_n(t) \approx \exp[-i\theta_n(t)]$. The substitution of this into the Schrodinger equation leads to

$$\dot{\theta}(t) = -\frac{E_n(t)}{\hbar}, \qquad \text{or} \qquad \theta_n(t) = -\frac{1}{\hbar}\int_0^t E_n(t')dt'$$

Substituting these equation into the Schrodinger equation, we get

$$\begin{aligned}
i\hbar\sum_n [\dot{c}_n(t)\psi_n(t) + c_n(t)\left|\dot{\psi}_n(t)\right\rangle + ic_n(t)\dot{\theta}_n(t)\left|\psi_n(t)\right\rangle]e^{i\theta_n(t)} &= \sum_n H(t)c_n(t)(t)e^{i\theta_n(t)}\left|\psi_n(t)\right\rangle \\
&= \sum_n E_n(t)c_n(t)e^{i\theta_n(t)}\left|\psi_n(t)\right\rangle \\
&= i\hbar\sum_n ic_n(t)\dot{\theta}_n(t)\left|\psi_n(t)\right\rangle e^{i\theta_n(t)}
\end{aligned}$$

Then we have

$$\sum_n [\dot{c}_n(t)\left|\psi_n(t)\right\rangle + c_n(t)\left|\dot{\psi}_n(t)\right\rangle]e^{i\theta_n(t)} = 0$$

or

$$\sum_n \dot{c}_n(t)\left|\psi_n(t)\right\rangle e^{i\theta_n(t)} = -\sum_n c_n(t)\left|\dot{\psi}_n(t)\right\rangle e^{i\theta_n(t)}$$

Multiplying $\left\langle\psi_m(t)\right|$ from the right side of this equation we get

$$\sum_n [\dot{c}_n(t)\left\langle\psi_m(t)\middle|\psi_n(t)\right\rangle e^{i\theta_n(t)} = -\sum_n c_n(t)\left\langle\psi_m(t)\middle|\dot{\psi}_n(t)\right\rangle e^{i\theta_n(t)}$$

or

$$\sum_n \dot{c}_n(t)\delta_{m,n} e^{i\theta_n(t)} = -\sum_n c_n(t)\left\langle\psi_m(t)\middle|\dot{\psi}_n(t)\right\rangle e^{i\theta_n(t)}$$

or

$$\begin{aligned}\dot{c}_m(t) &= -\sum_n c_n(t)\left\langle\psi_m(t)\middle|\dot{\psi}_n(t)\right\rangle e^{i[\theta_n(t)-\theta_m(t)]} \\ &= -c_m(t)\left\langle\psi_m(t)\middle|\dot{\psi}_m(t)\right\rangle - \sum_{n\neq m} c_n(t)\left\langle\psi_m(t)\middle|\dot{\psi}_n(t)\right\rangle e^{i[\theta_n(t)-\theta_m(t)]}\end{aligned}$$

When we neglect the second term, we have

$$\dot{c}_m(t) = -c_m(t)\left\langle\psi_m(t)\middle|\dot{\psi}_m(t)\right\rangle$$

The solution of this equation is obtained as

$$c_m(t) = c_m(0)e^{i\gamma_n(t)}$$

with

$$\gamma_n = i\int_0^t \left\langle\psi_m(t')\middle|\dot{\psi}_m(t')\right\rangle dt' \qquad \text{(geometric phase, Berry phase)}$$

where γ_n is real since

$$\frac{d}{dt}\left\langle\psi_m\middle|\psi_m\right\rangle = 0 = \left\langle\dot{\psi}_m\middle|\psi_m\right\rangle + \left\langle\psi_m\middle|\dot{\psi}_m\right\rangle = 2\,\mathrm{Re}[\left\langle\psi_m\middle|\dot{\psi}_m\right\rangle].$$

If the particle starts out in the n-th eigenstate ($c_n(0)=1$ and $c_m(0)=0$), then we have

$$\left|\Psi_n(t)\right\rangle = e^{i\theta_n(t)}e^{i\gamma_n(t)}\left|\psi_n(t)\right\rangle$$

It remains in the same n-th state with additional phase factors.

((**Note**)) proof of the formula

$$\left\langle \psi_m(t)\middle|\dot{\psi}_n(t)\right\rangle = \frac{\left\langle \psi_m(t)\right|\dot{\hat{H}}(t)\left|\psi_n(t)\right\rangle}{E_n(t)-E_m(t)} \qquad \text{for } m \neq n$$

We start with

$$\hat{H}(t)\left|\psi_n(t)\right\rangle = E_n(t)\left|\psi_n(t)\right\rangle$$

Taking the derivative of this equation with respect to time t,

$$\dot{\hat{H}}(t)\left|\psi_n(t)\right\rangle + \hat{H}(t)\left|\dot{\psi}_n(t)\right\rangle = \dot{E}_n(t)\left|\psi_n(t)\right\rangle + E_n(t)\left|\dot{\psi}_n(t)\right\rangle$$

or

$$\dot{\hat{H}}(t)\left|\psi_n(t)\right\rangle + \hat{H}(t)\left|\dot{\psi}_n(t)\right\rangle = \dot{E}_n(t)\left|\psi_n(t)\right\rangle + E_n(t)\left|\dot{\psi}_n(t)\right\rangle$$

Multiplying $\left\langle \psi_m(t)\right|$ by the above equation from the left side, we get

$$\left\langle \psi_m(t)\right|\dot{\hat{H}}(t)\left|\psi_n(t)\right\rangle + \left\langle \psi_m(t)\right|\hat{H}(t)\left|\dot{\psi}_n(t)\right\rangle = \dot{E}_n(t)\left\langle \psi_m(t)\middle|\psi_n(t)\right\rangle + E_n(t)\left\langle \psi_m(t)\middle|\dot{\psi}_n(t)\right\rangle$$

or

$$\left\langle \psi_m(t)\right|\dot{\hat{H}}(t)\left|\psi_n(t)\right\rangle + E_m(t)\left\langle \psi_m(t)\middle|\dot{\psi}_n(t)\right\rangle = \dot{E}_n(t)\delta_{m.n} + E_n(t)\left\langle \psi_m(t)\middle|\dot{\psi}_n(t)\right\rangle$$

For $n \neq m$

$$\left\langle \psi_m(t)\right|\dot{\hat{H}}(t)\left|\psi_n(t)\right\rangle = [E_n(t)-E_m(t)]\left\langle \psi_m(t)\middle|\dot{\psi}_n(t)\right\rangle$$

or

$$\langle \psi_m(t) | \dot{\psi}_n(t) \rangle = \frac{\langle \psi_m(t) | \dot{\hat{H}}(t) | \psi_n(t) \rangle}{E_n(t) - E_m(t)}$$

3. General formula for phase factor

M.V. Berry, Quantum Phase Factors Accompanying Adiabatic Changes, Proc. R. London A392, 45-57 (1984).

Let the Hamiltonian $\hat{H}$ be changed by varying parameter $\boldsymbol{R}$ [$\boldsymbol{R}$ = (x, y, z)] on which it depends. Then the excursion of the system between times t = 0 and t = T can be pictured as transport round a closed path $\boldsymbol{R}(t)$ in parameter space, with Hamiltonian $\hat{H}(\mathbf{R}(t))$ and such that $\boldsymbol{R}(T) = \boldsymbol{R}(0)$. The path is called a circuit and denoted by C. For the adiabatic approximation to apply, T must be large.

The state vector $|\psi(t))\rangle$ of the system evolves according to Schrödinger equation given by

$$i\hbar \frac{\partial}{\partial t} |\psi(t))\rangle = i\hbar |\dot{\psi}(t))\rangle = \hat{H}(\boldsymbol{R}(t)) |\psi(t))\rangle .$$

At any instant, the natural basis consists of the eigenstates $|n(\mathbf{R})\rangle$ (assumed discrete) of $\hat{H}(\mathbf{R})$ for $\boldsymbol{R} = \boldsymbol{R}(t)$, that satisfy

$$\hat{H}(\boldsymbol{R}(t)) |n(\boldsymbol{R}(t))\rangle = E_n(\boldsymbol{R}(t)) |n(\boldsymbol{R}(t))\rangle ,$$

with energies $E_n(\boldsymbol{R}(t))$. The eigenvalue equation implies no relation between the phases of the eigenstates $|n(\boldsymbol{R})\rangle$ at different $\boldsymbol{R}$.

Adiabatically, a system prepared in one of these states $|n(\boldsymbol{R}(0))\rangle$ will evolve with $\hat{H}$ and so be in the state $|n(\boldsymbol{R}(t))\rangle$ at t

$$\begin{aligned} |\psi(t)\rangle &= \exp[-\frac{i}{\hbar}\int_0^t E_n(\boldsymbol{R}(t'))dt']\exp[i\gamma_n(t)] |n(\boldsymbol{R}(t))\rangle \\ &= \exp[i\theta_n(t)]\exp[i\gamma_n(t)] |n(\boldsymbol{R}(t))\rangle \end{aligned}$$

where $\gamma_n(t)$ is a geometric phase, and the dynamical phase factor $\theta_n(t)$ is defined by

$$\theta_n(t) = -\frac{1}{\hbar}\int_0^t E_n(\boldsymbol{R}(t'))dt' . \qquad\qquad \dot{\theta}_n(t) = -\frac{1}{\hbar}E_n(t)$$

Plugging the solution form into the this Schrödinger equation, we get

$$i\hbar[\frac{\partial}{\partial t}\left|n(\boldsymbol{R}(t))\right\rangle - \frac{i}{\hbar}E_n(\boldsymbol{R}(t))\left|n(\boldsymbol{R}(t))\right\rangle + i\dot{\gamma}_n(t)\left|n(\boldsymbol{R}(t))\right\rangle] = E_n(\boldsymbol{R}(t))\left|n(\boldsymbol{R}(t))\right\rangle$$

or

$$\left|\dot{n}(\boldsymbol{R}(t))\right\rangle + i\dot{\gamma}_n(t)\left|n(\boldsymbol{R}(t))\right\rangle = 0 .$$

Taking the inner product with $\left\langle n(\boldsymbol{R}(t))\right|$ we get

$$\left\langle n(\boldsymbol{R}(t))\middle|\dot{n}(\boldsymbol{R}(t))\right\rangle + i\dot{\gamma}_n(t)\left\langle n(\boldsymbol{R}(t))\middle|n(\boldsymbol{R}(t))\right\rangle = 0 ,$$

Since $\left\langle n(\boldsymbol{R}(t))\middle|n(\boldsymbol{R}(t))\right\rangle = 1$, we have

$$\dot{\gamma}_n(t) = i\left\langle n(\boldsymbol{R}(t))\middle|\dot{n}(\boldsymbol{R}(t))\right\rangle$$

$\left|n(\boldsymbol{R}(t))\right\rangle$ depends on t because there is some parameter $\boldsymbol{R}(t)$ in the Hamiltonian that changes with time.

$$\left|\dot{n}(\boldsymbol{R}(t))\right\rangle = \left|\nabla_{\boldsymbol{R}} n(\boldsymbol{R}(t))\right\rangle \cdot \dot{\boldsymbol{R}}(t)$$

so that

$$\dot{\gamma}_n(t) = i\left\langle n(\boldsymbol{R}(t))\middle|\nabla_{\boldsymbol{R}} n(\boldsymbol{R}(t))\right\rangle \cdot \dot{\boldsymbol{R}}(t)$$

and thus

$$\gamma_n(t) = i\int_{\boldsymbol{R}_i}^{\boldsymbol{R}_f}\left\langle n(\boldsymbol{R}(t))\middle|\nabla_{\boldsymbol{R}} n(\boldsymbol{R}(t))\right\rangle \cdot d\boldsymbol{R} \qquad \text{(path integral)}$$

4. Expression of $\gamma_n(C)$

We calculate the geometric phase $\gamma_n(C)$ as follows.

For $\langle n|n\rangle = 1$ (normalization), we have

$$\left\langle \frac{\partial n}{\partial x}\middle| n\right\rangle + \left\langle n\middle|\frac{\partial n}{\partial x}\right\rangle = 0,$$

$$\left\langle \frac{\partial n}{\partial y}\middle| n\right\rangle + \left\langle n\middle|\frac{\partial n}{\partial y}\right\rangle = 0,$$

$$\left\langle \frac{\partial n}{\partial z}\middle| n\right\rangle + \left\langle n\middle|\frac{\partial n}{\partial z}\right\rangle = 0.$$

For $\langle n|m\rangle = 0$ $\quad (n \neq m) \quad$ (orthogonality)

$$\left\langle \frac{\partial n}{\partial x}\middle| m\right\rangle + \left\langle n\middle|\frac{\partial m}{\partial x}\right\rangle = 0,$$

$$\left\langle \frac{\partial n}{\partial y}\middle| m\right\rangle + \left\langle n\middle|\frac{\partial m}{\partial y}\right\rangle = 0,$$

$$\left\langle \frac{\partial n}{\partial z}\middle| m\right\rangle + \left\langle n\middle|\frac{\partial m}{\partial z}\right\rangle = 0.$$

For $\quad \langle m|H|n\rangle = E_n\langle m|n\rangle = 0 \qquad (n \neq m)$

$$\partial_i\langle m|H|n\rangle = \langle \partial_i m|H|n\rangle + \langle m|\partial_i H|n\rangle + \langle m|H|\partial_i n\rangle = 0$$

or

$$E_n\langle \partial_i m|n\rangle + \langle m|\partial_i H|n\rangle + E_m\langle m|\partial_i n\rangle = 0$$

Since $\quad \langle \partial_i m|n\rangle + \langle m|\partial_i n\rangle = 0$

$$-E_n\langle m|\partial_i n\rangle + \langle m|\partial_i H|n\rangle + E_m\langle m|\partial_i n\rangle = 0$$

or

$$\langle m|\partial_i n\rangle = \frac{\langle m|\partial_i H|n\rangle}{E_n - E_m}$$

The rotation of the vector $\boldsymbol{A}_0 = \langle n|\nabla n\rangle$ is given by

$$\begin{aligned}
\nabla\times\boldsymbol{A}_0 &= \nabla\times\langle n|\nabla n\rangle \\
&= \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \left\langle n\middle|\dfrac{\partial n}{\partial x}\right\rangle & \left\langle n\middle|\dfrac{\partial n}{\partial y}\right\rangle & \left\langle n\middle|\dfrac{\partial n}{\partial z}\right\rangle \end{vmatrix} \\
&= [\frac{\partial}{\partial y}\left\langle n\middle|\frac{\partial n}{\partial z}\right\rangle - \frac{\partial}{\partial z}\left\langle n\middle|\frac{\partial n}{\partial y}\right\rangle]\boldsymbol{e}_x + [\frac{\partial}{\partial z}\left\langle n\middle|\frac{\partial n}{\partial x}\right\rangle - \frac{\partial}{\partial x}\left\langle n\middle|\frac{\partial n}{\partial z}\right\rangle]\boldsymbol{e}_y \\
&\quad + [\frac{\partial}{\partial x}\left\langle n\middle|\frac{\partial n}{\partial y}\right\rangle - \frac{\partial}{\partial y}\left\langle n\middle|\frac{\partial n}{\partial x}\right\rangle]\boldsymbol{e}_z
\end{aligned}$$

$$\begin{aligned}
(\nabla\times\boldsymbol{A}_0)_z &= \frac{\partial}{\partial x}\left\langle n\middle|\frac{\partial n}{\partial y}\right\rangle - \frac{\partial}{\partial y}\left\langle n\middle|\frac{\partial n}{\partial x}\right\rangle \\
&= \left\langle \frac{\partial n}{\partial x}\middle|\frac{\partial n}{\partial y}\right\rangle + \left\langle n\middle|\frac{\partial}{\partial x}\frac{\partial n}{\partial y}\right\rangle - \left\langle \frac{\partial n}{\partial y}\middle|\frac{\partial n}{\partial x}\right\rangle - \left\langle n\middle|\frac{\partial}{\partial y}\frac{\partial n}{\partial x}\right\rangle \\
&= \left\langle \frac{\partial n}{\partial x}\middle|\frac{\partial n}{\partial y}\right\rangle - \left\langle \frac{\partial n}{\partial y}\middle|\frac{\partial n}{\partial x}\right\rangle \\
&= \sum_m [\left\langle \frac{\partial n}{\partial x}\middle| m\right\rangle\left\langle m\middle|\frac{\partial n}{\partial y}\right\rangle - \left\langle \frac{\partial n}{\partial y}\middle| m\right\rangle\left\langle m\middle|\frac{\partial n}{\partial x}\right\rangle] \\
&= \sum_{m\neq n} [\left\langle \frac{\partial n}{\partial x}\middle| m\right\rangle\left\langle m\middle|\frac{\partial n}{\partial y}\right\rangle - \left\langle \frac{\partial n}{\partial y}\middle| m\right\rangle\left\langle m\middle|\frac{\partial n}{\partial x}\right\rangle] \\
&= \sum_{m\neq n} [-\left\langle n\middle|\frac{\partial m}{\partial x}\right\rangle\left\langle m\middle|\frac{\partial n}{\partial y}\right\rangle + \left\langle n\middle|\frac{\partial m}{\partial y}\right\rangle\left\langle m\middle|\frac{\partial n}{\partial x}\right\rangle]
\end{aligned}$$

where we use the closure relation and the relations

$$\left\langle \frac{\partial n}{\partial x} \middle| m \right\rangle = -\left\langle n \middle| \frac{\partial m}{\partial x} \right\rangle, \qquad \left\langle \frac{\partial n}{\partial y} \middle| m \right\rangle = -\left\langle n \middle| \frac{\partial m}{\partial y} \right\rangle.$$

Then we obtain

$$\begin{aligned}
(\nabla \times \boldsymbol{A}_0)_z &= \sum_{m \neq n} [-\left\langle n \middle| \frac{\partial m}{\partial x} \right\rangle \left\langle m \middle| \frac{\partial n}{\partial y} \right\rangle + \left\langle n \middle| \frac{\partial m}{\partial y} \right\rangle \left\langle m \middle| \frac{\partial n}{\partial x} \right\rangle] \\
&= \sum_{m \neq n} [-\frac{\langle n | \frac{\partial}{\partial x} \hat{H} | m \rangle}{E_m - E_n} \frac{\langle m | \frac{\partial}{\partial y} \hat{H} | n \rangle}{E_n - E_m} + \frac{\langle n | \frac{\partial}{\partial y} \hat{H} | m \rangle}{E_m - E_n} \frac{\langle m | \frac{\partial}{\partial x} \hat{H} | n \rangle}{E_n - E_m}] \\
&= \sum_{m \neq n} [\frac{\langle n | \frac{\partial}{\partial x} \hat{H} | m \rangle \langle m | \frac{\partial}{\partial y} \hat{H} | n \rangle - \langle n | \frac{\partial}{\partial y} \hat{H} | m \rangle \langle m | \frac{\partial}{\partial x} \hat{H} | n \rangle}{(E_m - E_n)^2}]
\end{aligned}$$

where we use the relations

$$\langle m | \partial_i n \rangle = \frac{\langle m | \partial_i \hat{H} | n \rangle}{E_n - E_m}, \qquad \langle n | \partial_i m \rangle = \frac{\langle n | \partial_i \hat{H} | m \rangle}{E_m - E_n}$$

with $i = x$, y, and z. We note that

$$\boldsymbol{P}_{nm} \times \boldsymbol{P}_{mn} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \langle n | \frac{\partial}{\partial x} \hat{H} | m \rangle & \langle n | \frac{\partial}{\partial y} \hat{H} | m \rangle & \langle n | \frac{\partial}{\partial z} \hat{H} | m \rangle \\ \langle m | \frac{\partial}{\partial x} \hat{H} | n \rangle & \langle m | \frac{\partial}{\partial y} \hat{H} | n \rangle & \langle m | \frac{\partial}{\partial z} \hat{H} | n \rangle \end{vmatrix}$$

where

$$\boldsymbol{P}_{nm} = \langle n | \nabla \hat{H} | m \rangle, \qquad \boldsymbol{P}_{mn} = \langle m | \nabla \hat{H} | n \rangle.$$

Then we get

$$\nabla \times \boldsymbol{A}_0 = \sum_{m \neq n} \frac{\boldsymbol{P}_{nm} \times \boldsymbol{P}_{mn}}{(E_m - E_n)^2}$$

$$= \sum_{m \neq n} \frac{\langle n | \nabla \hat{H} | m \rangle \times \langle m | \nabla \hat{H} | n \rangle}{(E_m - E_n)^2}$$

So we have

$$\gamma_n(C) = i \oint d\boldsymbol{R} \cdot \boldsymbol{A}_0$$

$$= i \oint d\boldsymbol{R} \cdot \langle n | \nabla n \rangle$$

$$= i \oint d\boldsymbol{R} \cdot [\mathrm{Re}(\langle n | \nabla n \rangle) + i\,\mathrm{Im}(\langle n | \nabla n \rangle)]$$

$$= -\mathrm{Im} \oint d\boldsymbol{R} \cdot \langle n | \nabla n \rangle$$

since $\mathrm{Re}[\langle n | \nabla n \rangle] = 0$. Note that

$$\boldsymbol{A}_0 = i \langle n | \nabla n \rangle = -\mathrm{Im} \langle n | \nabla n \rangle$$

is called the Berry's vector potential or Berry's connection.

Using the Stokes' theorem, we get

$$\gamma_n(C) = i \oint d\boldsymbol{a} \cdot [\nabla \times \langle n | \nabla n \rangle]$$

$$= \oint d\boldsymbol{a} \cdot \boldsymbol{A}_0$$

$$= -\mathrm{Im} \oint d\boldsymbol{a} \cdot \boldsymbol{A}_0$$

$$= -\oint d\boldsymbol{a} \cdot \boldsymbol{V}_n$$

where d$\boldsymbol{a}$ denotes area elements in R space

$$\boldsymbol{V}_n(\boldsymbol{R}) = \mathrm{Im} \sum_{m \neq n} \frac{\langle n(\boldsymbol{R}) | \nabla_{\boldsymbol{R}} \hat{H}(\boldsymbol{R}) | m(\boldsymbol{R}) \rangle \times \langle m(\boldsymbol{R}) | \nabla_{\boldsymbol{R}} \hat{H}(\boldsymbol{R}) | n(\boldsymbol{R}) \rangle}{(E_m(\boldsymbol{R}) - E_n(\boldsymbol{R}))^2}$$

The notation for $\boldsymbol{V}_n$ is the same used by Berry in the original paper (1984).

5. Gauge transformation

The magnetic field $\boldsymbol{B}$ is defined as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

Then $\boldsymbol{B}$ is invariant under the Gauge transformation

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla \chi$$

Suppose that the phase of the eigenstate is redefined as

$$|n\rangle \rightarrow |n'\rangle = e^{-i\theta(\boldsymbol{R})}|n\rangle$$

where $\theta(\boldsymbol{R})$ is an arbitrary phase,

$$\begin{aligned}
\boldsymbol{A}' &= i\langle n'|\nabla_{\boldsymbol{R}} n'\rangle \\
&= i\langle n|e^{i\theta(\boldsymbol{R})}[-i\nabla_{\boldsymbol{R}}\theta(\boldsymbol{R})e^{-i\theta(\boldsymbol{R})}|n\rangle + e^{-i\theta(\boldsymbol{R})}|\nabla_{\boldsymbol{R}} n\rangle \\
&= i\langle n|\nabla_{\boldsymbol{R}} n\rangle + i\langle n|n\rangle[-i\nabla_{\boldsymbol{R}}\theta(\boldsymbol{R})] \\
&= \boldsymbol{A} + \nabla_{\boldsymbol{R}}\theta(\boldsymbol{R})
\end{aligned}$$

Under this transformation, the geometrical phase is invariant, since

$$\begin{aligned}
\gamma_n'(C) &= i\oint d\boldsymbol{R}\cdot\langle n'|\nabla n'\rangle \\
&= i\oint d\boldsymbol{a}\cdot[\nabla_{\boldsymbol{R}}\times\langle n'|\nabla n'\rangle] \\
&= \oint d\boldsymbol{a}\cdot\nabla_{\boldsymbol{R}}\times\boldsymbol{A}' \\
&= \oint d\boldsymbol{a}\cdot\nabla_{\boldsymbol{R}}\times[\boldsymbol{A}+\nabla_{\boldsymbol{R}}\theta(\boldsymbol{R})] \\
&= \oint d\boldsymbol{a}\cdot\nabla_{\boldsymbol{R}}\times\boldsymbol{A} \\
&= \oint d\boldsymbol{a}\cdot\boldsymbol{B}
\end{aligned}$$

6. Spin in Magnetic Field (the adiabatic approximation)

A particle with the angular momentum $\hat{\boldsymbol{J}}$ interacts with a magnetic field $\boldsymbol{B}$ via the Hamiltonian:

$$\hat{H}(\boldsymbol{B}) = -\frac{g_J\mu_B}{\hbar}\hat{\boldsymbol{J}}\cdot\boldsymbol{B},$$

where g_J is the Landé-g factor. Note that

$$\hat{J}_z|m(\boldsymbol{B})\rangle = \hbar m(\boldsymbol{B})|m(\boldsymbol{B})\rangle,$$

where $|m(\boldsymbol{B})\rangle$ is the eigenstate of $\hat{J}_z$ with the eigenvalue $\hbar m(\boldsymbol{B})$.
For any fixed value of $\boldsymbol{B}$, we have

$$\hat{H}(\boldsymbol{B})|m(\boldsymbol{B})\rangle = E_m(\boldsymbol{B})|m(\boldsymbol{B})\rangle$$

Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}[\boldsymbol{B}(t)]|\psi(t)\rangle = E_m(\boldsymbol{B}(t))|\psi(t)\rangle$$

with

$$|\psi(t=0)\rangle = |m(\boldsymbol{B}(t=0))\rangle$$

where $|m(\boldsymbol{B}(0))\rangle$ is the eigenstate of $\hat{H}(\boldsymbol{B}(t=0))$.

$$\begin{aligned}|\psi(t)\rangle &= |m(\boldsymbol{B}(t))\rangle \exp[-\frac{i}{\hbar}\int_0^t E_m(\boldsymbol{B}(t'))dt']\exp[i\gamma_m(t)] \\ &= |m(\boldsymbol{B}(t))\rangle \exp[i\theta_m(t)]\exp[i\gamma_m(t)]\end{aligned}$$

where

$$\theta_m(t) = -\frac{1}{\hbar}\int_0^t E_m(\boldsymbol{B}(t'))dt'$$

Plugging the solution form into the this Schrödinger equation, we get

$$i\hbar[\frac{\partial}{\partial t}|m(\boldsymbol{B}(t))\rangle - \frac{i}{\hbar}E_m(\boldsymbol{B}(t))|m(\boldsymbol{B}(t))\rangle + |m(\boldsymbol{B}(t))\rangle i\frac{\partial\gamma_m(t)}{\partial t}] = E_m(\boldsymbol{B}(t))|m(\boldsymbol{B}(t))\rangle$$

or

$$i|\dot{m}(\boldsymbol{B}(t))\rangle = |m(\boldsymbol{B}(t))\rangle\frac{\partial\gamma_m(t)}{\partial t}$$

Taking the inner product with $\langle m(\boldsymbol{B}(t))|$ we get

$$i\langle m(\boldsymbol{B}(t))|\dot{m}(\boldsymbol{B}(t))\rangle = \frac{\partial \gamma_m(t)}{\partial t}\langle m(\boldsymbol{B}(t))|m(\boldsymbol{B}(t))\rangle \,,$$

Since $\langle m(\boldsymbol{B}(t))|m(\boldsymbol{B}(t))\rangle = 1$, we have

$$\frac{\partial \gamma_m(t)}{\partial t} = i\langle m(\boldsymbol{B}(t))|\dot{m}(\boldsymbol{B}(t))\rangle$$

or

$$\gamma_m(t) = i\int_0^t \langle m(\boldsymbol{B}(t'))|\dot{m}(\boldsymbol{B}(t'))\rangle dt'$$

Note that $\gamma_m(t)$ is real, since

$$\langle m(\boldsymbol{B}(t))|\dot{m}(\boldsymbol{B}(t))\rangle + \langle \dot{m}(\boldsymbol{B}(t))|m(\boldsymbol{B}(t))\rangle = \mathrm{Re}[\langle m(\boldsymbol{B}(t))|\dot{m}(\boldsymbol{B}(t))\rangle] = 0$$

The geometrical character of the Berry phase emerges when the variation of the instantaneous energy eigenstates with time is restated as their variation with field;

$$|\dot{m}(\boldsymbol{B}(t))\rangle = \frac{\partial}{\partial t}|m(\boldsymbol{B}(t))\rangle = \frac{d\boldsymbol{B}(t)}{dt}\cdot\frac{\partial}{\partial \boldsymbol{B}}|m(\boldsymbol{B})\rangle = \dot{\boldsymbol{B}}\cdot|\nabla_{\boldsymbol{B}} m(\boldsymbol{B})\rangle$$

This expresses the phase as an integral over field values;

$$\begin{aligned}\gamma_m(C) &= i\oint d\boldsymbol{B}\cdot\langle m(\boldsymbol{B})|\nabla_{\boldsymbol{B}} m(\boldsymbol{B})\rangle \\ &= i\oint d\boldsymbol{B}\cdot[\mathrm{Re}(\langle m(\boldsymbol{B})|\nabla_{\boldsymbol{B}} m(\boldsymbol{B})\rangle) + i\,\mathrm{Im}(\langle m(\boldsymbol{B})|\nabla_{\boldsymbol{B}} m(\boldsymbol{B})\rangle)] \qquad (1) \\ &= -\mathrm{Im}\oint d\boldsymbol{B}\cdot\langle m(\boldsymbol{B})|\nabla_{\boldsymbol{B}} m(\boldsymbol{B})\rangle\end{aligned}$$

Note that

$$\langle m(\boldsymbol{B})|\nabla_{\boldsymbol{B}} m(\boldsymbol{B})\rangle + \langle \nabla_{\boldsymbol{B}} m(\boldsymbol{B})|m(\boldsymbol{B})\rangle = \mathrm{Re}[\langle m(\boldsymbol{B})|\nabla_{\boldsymbol{B}} m(\boldsymbol{B})\rangle] = 0$$

Stokes' theorem applied to Eq.(1) gives, in an abbreviated notation.

$$
\begin{aligned}
\gamma_m(C) &= -\mathrm{Im}\int d\boldsymbol{a}\cdot\nabla_{\boldsymbol{B}}\times\langle m(\boldsymbol{B})|\nabla_{\boldsymbol{B}} m(\boldsymbol{B})\rangle \\
&= -\mathrm{Im}\int d\boldsymbol{a}\cdot\sum_{n\neq m}\langle\nabla_{\boldsymbol{B}} m(\boldsymbol{B})|n(\boldsymbol{B})\rangle\times\langle n(\boldsymbol{B})|\nabla_{\boldsymbol{B}} m(\boldsymbol{B})\rangle \\
&= -\mathrm{Im}\int d\boldsymbol{a}\cdot V_m(\boldsymbol{B})
\end{aligned}
$$

where

$$
V_m(\boldsymbol{B}) = \sum_{n\neq m}\langle\nabla_{\boldsymbol{B}} m(\boldsymbol{B})|n(\boldsymbol{B})\rangle\times\langle n(\boldsymbol{B})|\nabla_{\boldsymbol{B}} m(\boldsymbol{B})\rangle
$$

d$\boldsymbol{a}$ denotes area element in $\boldsymbol{B}$ space and exclusion in the summation is justified by $\langle n(\boldsymbol{B})|\nabla_{\boldsymbol{B}} n(\boldsymbol{B})\rangle$ being imaginary. The off-diagonal elements $\langle n(\boldsymbol{B})|\nabla_{\boldsymbol{B}} m(\boldsymbol{B})\rangle$ are obtained as follows. Since

$$
\hat{H}(\boldsymbol{B})|m(\boldsymbol{B})\rangle = E_m(\boldsymbol{B})|m(\boldsymbol{B})\rangle, \qquad \text{(Eigenvalue problem)}
$$

we get

$$
\nabla_B\hat{H}(\boldsymbol{B})|m(\boldsymbol{B})\rangle + \hat{H}(\boldsymbol{B})|\nabla_B m(\boldsymbol{B})\rangle = \nabla_B E_m(\boldsymbol{B})|m(\boldsymbol{B})\rangle + E_m(\boldsymbol{B})|\nabla_B m(\boldsymbol{B})\rangle
$$

But the $|n(\boldsymbol{B})\rangle$ is an orthogonal set, so for $n \neq m$, we have

$$
\langle n(\boldsymbol{B})|\nabla_B\hat{H}(\boldsymbol{B})|m(\boldsymbol{B})\rangle + \langle n(\boldsymbol{B})|\hat{H}(\boldsymbol{B})|\nabla_B m(\boldsymbol{B})\rangle = \langle n(\boldsymbol{B})|\nabla_B E_m(\boldsymbol{B})|m(\boldsymbol{B})\rangle + \langle n(\boldsymbol{B})|E_m(\boldsymbol{B})|\nabla_B m(\boldsymbol{B})\rangle
$$

or

$$
\langle n(\boldsymbol{B})|\nabla_B\hat{H}(\boldsymbol{B})|m(\boldsymbol{B})\rangle + E_n(\boldsymbol{B})\langle n(\boldsymbol{B})|\nabla_B m(\boldsymbol{B})\rangle = \langle n(\boldsymbol{B})|\nabla_B E_m(\boldsymbol{B})|m(\boldsymbol{B})\rangle + E_m(\boldsymbol{B})\langle n(\boldsymbol{B})|\nabla_B m(\boldsymbol{B})\rangle .
$$

Since

$$
\langle n(\boldsymbol{B})|\nabla_B E_m(\boldsymbol{B})|m(\boldsymbol{B})\rangle = \nabla_B E_m(\boldsymbol{B})\langle n(\boldsymbol{B})|m(\boldsymbol{B})\rangle = 0
$$

we get

$$\langle n(\boldsymbol{B})|\nabla_{\boldsymbol{B}} m(\boldsymbol{B})\rangle = \frac{\langle n(\boldsymbol{B})|\nabla_{\boldsymbol{B}}\hat{H}(\boldsymbol{B})|m(\boldsymbol{B})\rangle}{E_m(\boldsymbol{B}) - E_n(\boldsymbol{B})}$$

Hence

$$\boldsymbol{V}_m(\boldsymbol{B}) = \sum_{m\neq n}[\frac{\langle m(\boldsymbol{B})|\nabla_{\boldsymbol{B}}\hat{H}(\boldsymbol{B})|n(\boldsymbol{B})\rangle \times \langle n(\boldsymbol{B})|\nabla_{\boldsymbol{B}}\hat{H}(\boldsymbol{B})|m(\boldsymbol{B})\rangle}{(E_m(\boldsymbol{B}) - E_n(\boldsymbol{B})]^2}$$

where

$$\hat{H}(\boldsymbol{B}) = -\frac{g_J\mu_B}{\hbar}\hat{\boldsymbol{J}}\cdot\boldsymbol{B} = -\frac{g_J\mu_B B}{\hbar}\hat{\boldsymbol{J}}\cdot\boldsymbol{e}_B\,, \qquad \nabla_{\boldsymbol{B}}\hat{H}(\boldsymbol{B}) = -\frac{g_J\mu_B}{\hbar}\hat{\boldsymbol{J}}$$

Then we get

$$\boldsymbol{V}_m(\boldsymbol{B}) = \frac{1}{B^2\hbar^2}\sum_{m\neq n}[\frac{\langle m(\boldsymbol{B})|\hat{\boldsymbol{J}}|n(\boldsymbol{B})\rangle \times \langle n(\boldsymbol{B})|\hat{\boldsymbol{J}}|m(\boldsymbol{B})\rangle}{[m(\boldsymbol{B}) - n(\boldsymbol{B})]^2}$$

$$\langle m(\boldsymbol{B})|\hat{\boldsymbol{J}}|n(\boldsymbol{B})\rangle \times \langle n(\boldsymbol{B})|\hat{\boldsymbol{J}}|m(\boldsymbol{B})\rangle = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \langle m(\boldsymbol{B})|\hat{J}_1|n(\boldsymbol{B})\rangle & \langle m(\boldsymbol{B})|\hat{J}_2|n(\boldsymbol{B})\rangle & \langle m(\boldsymbol{B})|\hat{J}_3|n(\boldsymbol{B})\rangle \\ \langle n(\boldsymbol{B})|\hat{J}_1|m(\boldsymbol{B})\rangle & \langle n(\boldsymbol{B})|\hat{J}_2|m(\boldsymbol{B})\rangle & \langle n(\boldsymbol{B})|\hat{J}_3|m(\boldsymbol{B})\rangle \end{vmatrix}$$

$$\begin{aligned}\langle m(\boldsymbol{B})|\hat{J}_1|n(\boldsymbol{B})\rangle &= \frac{1}{2}\langle m(\boldsymbol{B})|\hat{J}_+ + \hat{J}_-|n(\boldsymbol{B})\rangle \\ &= \frac{1}{2}\langle m(\boldsymbol{B})|\hat{J}_+|n(\boldsymbol{B})\rangle + \frac{1}{2}\langle m(\boldsymbol{B})|\hat{J}_-|n(\boldsymbol{B})\rangle\end{aligned}$$

$$\langle m(\boldsymbol{B})|\hat{J}_2|n(\boldsymbol{B})\rangle = \frac{1}{2i}\langle m(\boldsymbol{B})|\hat{J}_+ - \hat{J}_-|n(\boldsymbol{B})\rangle$$

$$\langle m(\boldsymbol{B})|\hat{J}_3|n(\boldsymbol{B})\rangle = \hbar n\langle m|n\rangle = \hbar n\delta_{m,n}$$

$$\begin{aligned}\langle m(\boldsymbol{B})|\hat{J}_+|n(\boldsymbol{B})\rangle &= \hbar\sqrt{(j-n)(j+n+1)}\langle m|n+1\rangle \\ &= \hbar\sqrt{(j-n)(j+n+1)}\delta_{m,n+1} \\ &= \hbar\sqrt{(j-m-1)(j+m)}\delta_{m,n+1}\end{aligned}$$

$$\langle m(\boldsymbol{B})|\hat{J}_{-}|n(\boldsymbol{B})\rangle = \hbar\sqrt{(j+n)(j-n+1)}\langle m|n-1\rangle$$
$$= \hbar\sqrt{(j+n)(j-n+1)}\delta_{m,n-1}$$
$$= \hbar\sqrt{(j+m+1)(j-m)}\delta_{m,n-1}$$

So we get

$$[\operatorname{Im}[\boldsymbol{V}_{m1}(\boldsymbol{B})] = \frac{1}{B^2\hbar^2}\operatorname{Im}\sum_{m\neq n}[\frac{\langle m(\boldsymbol{B})|\hat{J}_2|n(\boldsymbol{B})\rangle\langle n(\boldsymbol{B})|\hat{J}_3|m(\boldsymbol{B})\rangle - \langle m(\boldsymbol{B})|\hat{J}_3|n(\boldsymbol{B})\rangle\langle n(\boldsymbol{B})|\hat{J}_2|m(\boldsymbol{B})\rangle}{[m(\boldsymbol{B})-n(\boldsymbol{B})]^2}$$

or

$$\operatorname{Im}\boldsymbol{V}_{m1}(\boldsymbol{B}) = 0\,.$$

$$\operatorname{Im}\boldsymbol{V}_{m2}(\boldsymbol{B}) = \frac{1}{B^2\hbar^2}\operatorname{Im}\sum_{m\neq n}[\frac{\langle m(\boldsymbol{B})|\hat{J}_3|n(\boldsymbol{B})\rangle\langle n(\boldsymbol{B})|\hat{J}_1|m(\boldsymbol{B})\rangle - \langle m(\boldsymbol{B})|\hat{J}_1|n(\boldsymbol{B})\rangle\langle n(\boldsymbol{B})|\hat{J}_3|m(\boldsymbol{B})\rangle}{[m(\boldsymbol{B})-n(\boldsymbol{B})]^2}$$

or

$$\operatorname{Im}\boldsymbol{V}_{m2}(\boldsymbol{B}) = 0\,.$$

$$\operatorname{Im}\boldsymbol{V}_{m3}(\boldsymbol{B}) = \frac{1}{B^2\hbar^2}\operatorname{Im}\sum_{m\neq n}[\frac{\langle m(\boldsymbol{B})|\hat{J}_1|n(\boldsymbol{B})\rangle\langle n(\boldsymbol{B})|\hat{J}_2|m(\boldsymbol{B})\rangle - \langle m(\boldsymbol{B})|\hat{J}_2|n(\boldsymbol{B})\rangle\langle n(\boldsymbol{B})|\hat{J}_1|m(\boldsymbol{B})\rangle}{[m(\boldsymbol{B})-n(\boldsymbol{B})]^2}$$

Since $[m(\boldsymbol{B})-n(\boldsymbol{B})]^2 = 1$, we have

$$\operatorname{Im}[\boldsymbol{V}_{m3}(\boldsymbol{B})] = \frac{1}{B^2\hbar^2}\operatorname{Im}\sum_{n=m\pm 1}[\langle m(\boldsymbol{B})|\hat{J}_1|n(\boldsymbol{B})\rangle\langle n(\boldsymbol{B})|\hat{J}_2|m(\boldsymbol{B})\rangle$$
$$-\langle m(\boldsymbol{B})|\hat{J}_2|n(\boldsymbol{B})\rangle\langle n(\boldsymbol{B})|\hat{J}_1|m(\boldsymbol{B})\rangle]$$
$$= \frac{1}{B^2\hbar^2}\operatorname{Im}\langle m(\boldsymbol{B})|\hat{J}_1\hat{J}_2 - \hat{J}_2\hat{J}_1|m(\boldsymbol{B})\rangle$$
$$= \frac{1}{B^2\hbar^2}\operatorname{Im}\langle m(\boldsymbol{B})|i\hbar\hat{J}_3|m(\boldsymbol{B})\rangle$$
$$= \frac{m(\boldsymbol{B})}{B^2}$$
$$= \frac{m(\boldsymbol{B})}{B^2}$$

Here we use the commutation relation

$$\hat{J}_1\hat{J}_2 - \hat{J}_2\hat{J}_1 = i\hbar\hat{J}_3 .$$

We can put this in a form that does not depend on choice of the 3-axis to lie along ***B***:

$$\mathrm{Im}[\boldsymbol{V}_m(\boldsymbol{B})] = \frac{m(\boldsymbol{B})}{B^3}\boldsymbol{B}$$

We note that

$$\nabla \cdot \nabla_{\boldsymbol{B}} \frac{1}{B} = -4\pi\delta(\boldsymbol{B})$$

where

$$\nabla_{\boldsymbol{B}} \frac{1}{B} = -\frac{\boldsymbol{B}}{B^3}$$

This singularity is spherically symmetric. The Berry phase is given by

$$\gamma_m(C) = -\mathrm{Im}\int d\boldsymbol{a} \cdot V_m(\boldsymbol{B}) = -\int d\boldsymbol{a} \cdot \frac{m(\boldsymbol{B})}{B^3}\boldsymbol{B} = -\Omega(C)m(\boldsymbol{B}=0) = -m\Omega(C)$$

where $\Omega(C)$ is the solid angle subtended by C as seen from the origin in field space.

$$\int d\boldsymbol{a} \cdot \frac{1}{B^3}\boldsymbol{B} = \int B^2 d\Omega(\boldsymbol{e}_B \cdot \frac{1}{B^2}\boldsymbol{e}_B) = \int d\Omega = \Omega(C)$$

with $\boldsymbol{e}_{\mathrm{B}}$ being the unit vector along the direction of ***B.***

((**Formula**))

$$\nabla_{\boldsymbol{B}} \cdot (\nabla_{\boldsymbol{B}} \frac{1}{B}) = \nabla_B{}^2 \frac{1}{B} = -4\pi\delta(\boldsymbol{B})$$

((**Proof**)) Vector analysis: Gauss' law

$$
\begin{aligned}
\int d\boldsymbol{B}(\nabla_{\boldsymbol{B}}{}^{2}\frac{1}{B}) &= \int d\boldsymbol{B}[\nabla_{\boldsymbol{B}}\cdot(\nabla_{\boldsymbol{B}}\frac{1}{B})] \\
&= \int d\boldsymbol{a}\cdot\nabla_{\boldsymbol{B}}\frac{1}{B} \\
&= -\int d\boldsymbol{a}\cdot\frac{\boldsymbol{B}}{B^{3}} \\
&= -\int B^{2}d\Omega\frac{B}{B^{3}} \\
&= -\int d\Omega \\
&= -4\pi
\end{aligned}
$$

where

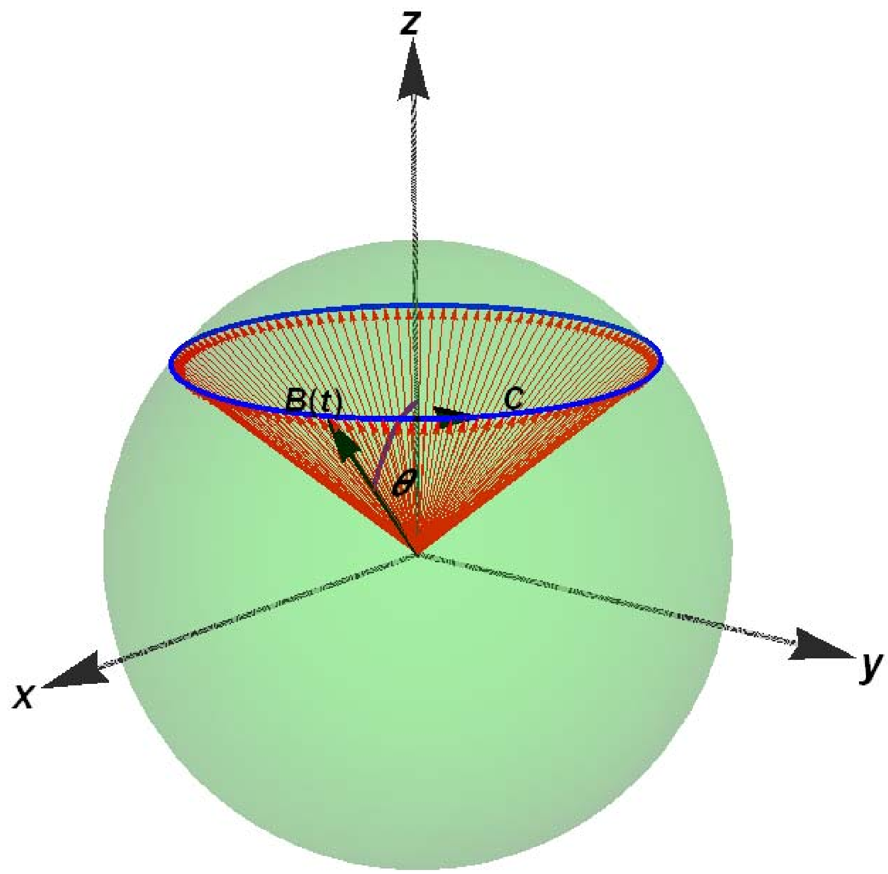
$$\nabla_{\mathbf{B}}\frac{1}{B} = -\frac{\mathbf{B}}{B^{3}}.$$

which yields the relation

$$\nabla_{\boldsymbol{B}}\cdot(\nabla_{\boldsymbol{B}}\frac{1}{B}) = \nabla_{B}{}^{2}\frac{1}{B} = -4\pi\delta(\boldsymbol{B})$$

where $\delta(\mathbf{B})$ is the Dirac delta function.

7. Example: spin ½ under the magnetic field which undergoes a precession adiabatically



The magnetic field is given by

$$\boldsymbol{B} = B_0 \sin\theta\cos\phi\, \boldsymbol{e}_x + \sin\theta\sin\phi\, \boldsymbol{e}_y + \cos\theta\, \boldsymbol{e}_z$$

with

$$\phi = \omega t .$$

Spin magnetic moment: $\hat{\boldsymbol{\mu}}_s = -\frac{2\mu_B}{\hbar}\hat{\boldsymbol{S}} = -\mu_B \hat{\boldsymbol{\sigma}}$.

where

$$\mu_B = \frac{e\hbar}{2mc} . \qquad \text{(Bohr magneton)}$$

The spin Hamiltonian is given by

$$\hat{H}(t) = -\hat{\boldsymbol{\mu}}_s \cdot \boldsymbol{B}(t) = \mu_B \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{B}(t) = \mu_B B(t) \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{n},$$

where $\hat{\boldsymbol{\sigma}}$ is the Pauli spin operator. The eigenstates are given by

$$\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{n}(t) \left| +\boldsymbol{n}(t) \right\rangle = + \left| +\boldsymbol{n}(t) \right\rangle, \qquad \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{n}(t) \left| -\boldsymbol{n}(t) \right\rangle = - \left| -\boldsymbol{n}(t) \right\rangle$$

where

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left| +\mathbf{n}(t) \right\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi} \sin\frac{\theta}{2} \end{pmatrix}, \qquad \left| -\boldsymbol{n}(t) \right\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\phi} \cos\frac{\theta}{2} \end{pmatrix}$$

The energy eigenstate:

$$\hat{H}(t) \left| +\boldsymbol{n}(t) \right\rangle = \mu_B B(t) \left| +\boldsymbol{n}(t) \right\rangle, \qquad \hat{H}(t) \left| -\boldsymbol{n}(t) \right\rangle = -\mu_B B(t) \left| -\boldsymbol{n}(t) \right\rangle$$

$\left| +\boldsymbol{n}(t) \right\rangle$ is the eigenstate of $\hat{H}(t)$ with the energy eigenvalue $E_+ = \mu_B B(t)$. $\left| -\boldsymbol{n}(t) \right\rangle$ is the eigenstate of $\hat{H}(t)$ with the energy eigenvalue $E_- = -\mu_B B(t)$.

We note that we change the parameters: $\theta \to \pi - \theta$ and $\phi \to \pi + \phi$ in

$$\left| +\boldsymbol{n}(t) \right\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi} \sin\frac{\theta}{2} \end{pmatrix},$$

Then we get

$$\left| -\mathbf{n}(t) \right\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\phi} \cos\frac{\theta}{2} \end{pmatrix}$$

$$\to \begin{pmatrix} \cos\dfrac{(\pi-\theta)}{2} \\ e^{i(\phi+\pi)}\sin\dfrac{(\pi-\theta)}{2} \end{pmatrix} = -\begin{pmatrix} -\sin\dfrac{\theta}{2} \\ e^{i\phi}\cos\dfrac{\theta}{2} \end{pmatrix} = -\left|-\boldsymbol{n}(t)\right\rangle$$

except for the minus sign, when $\theta \to \pi - \theta$ and $\phi \to \pi + \phi$ (parity operation).

In the spherical co-ordinate,

$$\begin{aligned}
\nabla\left|+\boldsymbol{n}(t)\right\rangle &= \boldsymbol{e}_r \frac{\partial}{\partial r}\left|+\boldsymbol{n}(t)\right\rangle + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta}\left|+\boldsymbol{n}(t)\right\rangle + \boldsymbol{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left|+\boldsymbol{n}(t)\right\rangle \\
&= \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta}\begin{pmatrix} \cos\dfrac{\theta}{2} \\ e^{i\phi}\sin\dfrac{\theta}{2} \end{pmatrix} + \boldsymbol{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\begin{pmatrix} \cos\dfrac{\theta}{2} \\ e^{i\phi}\sin\dfrac{\theta}{2} \end{pmatrix} \\
&= \boldsymbol{e}_\theta \frac{1}{r}\begin{pmatrix} -\dfrac{1}{2}\sin\dfrac{\theta}{2} \\ e^{i\phi}\dfrac{1}{2}\cos\dfrac{\theta}{2} \end{pmatrix} + \boldsymbol{e}_\phi \frac{1}{r\sin\theta}\begin{pmatrix} 0 \\ ie^{i\phi}\sin\dfrac{\theta}{2} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\nabla\left|-\boldsymbol{n}(t)\right\rangle &= \boldsymbol{e}_r \frac{\partial}{\partial r}\left|-\boldsymbol{n}(t)\right\rangle + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta}\left|-\boldsymbol{n}(t)\right\rangle + \boldsymbol{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left|-\boldsymbol{n}(t)\right\rangle \\
&= \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta}\begin{pmatrix} -\sin\dfrac{\theta}{2} \\ e^{i\phi}\cos\dfrac{\theta}{2} \end{pmatrix} + \boldsymbol{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\begin{pmatrix} -\sin\dfrac{\theta}{2} \\ e^{i\phi}\cos\dfrac{\theta}{2} \end{pmatrix} \\
&= \boldsymbol{e}_\theta \frac{1}{r}\begin{pmatrix} -\dfrac{1}{2}\cos\dfrac{\theta}{2} \\ -e^{i\phi}\dfrac{1}{2}\sin\dfrac{\theta}{2} \end{pmatrix} + \boldsymbol{e}_\phi \frac{1}{r\sin\theta}\begin{pmatrix} 0 \\ ie^{i\phi}\cos\dfrac{\theta}{2} \end{pmatrix}
\end{aligned}$$

Then we get

$$\langle +\boldsymbol{n}(t)|\nabla|+\boldsymbol{n}(t)\rangle = \boldsymbol{e}_\theta \frac{1}{r}\begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} -\frac{1}{2}\sin\frac{\theta}{2} \\ e^{i\phi}\frac{1}{2}\cos\frac{\theta}{2} \end{pmatrix}$$

$$+\boldsymbol{e}_\phi \frac{1}{r\sin\theta}\begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 0 \\ ie^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

$$= \boldsymbol{e}_\phi \frac{i\sin^2\frac{\theta}{2}}{r\sin\theta}$$

where

$$\langle +\boldsymbol{n}(t)| = \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix}.$$

$$\langle -\boldsymbol{n}(t)|\nabla|-\boldsymbol{n}(t)\rangle = \boldsymbol{e}_\theta \frac{1}{r}\begin{pmatrix} -\sin\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} -\frac{1}{2}\cos\frac{\theta}{2} \\ -e^{i\phi}\frac{1}{2}\sin\frac{\theta}{2} \end{pmatrix}$$

$$+\boldsymbol{e}_\phi \frac{1}{r\sin\theta}\begin{pmatrix} -\sin\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 0 \\ ie^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}$$

$$= \boldsymbol{e}_\phi \frac{i\cos^2\frac{\theta}{2}}{r\sin\theta}$$

where

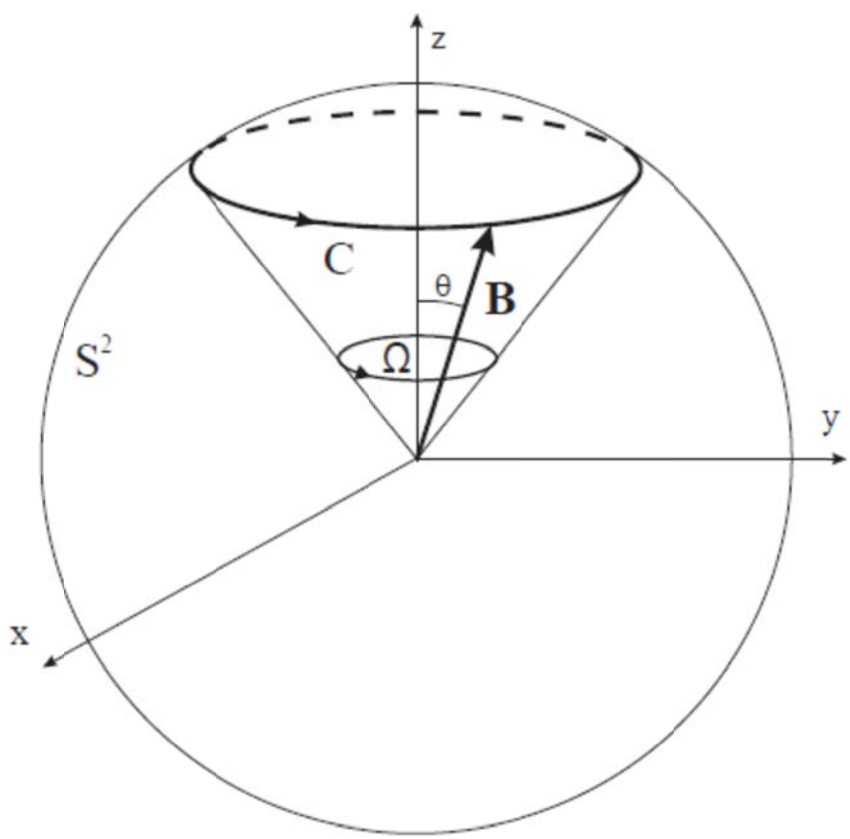
$$\langle -\boldsymbol{n}(t)| = \begin{pmatrix} -\sin\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2} \end{pmatrix}.$$

$$\begin{aligned}
\gamma_+(C) &= i\oint \langle +\boldsymbol{n}(t)|\nabla|+\boldsymbol{n}(t)\rangle\cdot d\boldsymbol{r} \\
&= i\int_0^{2\pi} \boldsymbol{e}_\phi\cdot\boldsymbol{e}_\phi \frac{i\sin^2\frac{\theta}{2}}{r\sin\theta} r\sin\theta d\phi \\
&= -2\pi\sin^2\frac{\theta}{2} \\
&= -\pi(1-\cos\theta)
\end{aligned}$$

$$
\begin{aligned}
\gamma_-(C) &= i\oint \langle -\boldsymbol{n}(t)|\nabla|-\boldsymbol{n}(t)\rangle \cdot d\boldsymbol{r} \\
&= i\int_0^{2\pi} \boldsymbol{e}_\phi \cdot \boldsymbol{e}_\phi \frac{i\cos^2\dfrac{\theta}{2}}{r\sin\theta} r\sin\theta d\phi \\
&= -2\pi\cos^2\frac{\theta}{2} \\
&= -\pi(1+\cos\theta)
\end{aligned}
$$

or

$$\gamma_-(C) = 2\pi - \pi(1+\cos\theta) = \pi(1-\cos\theta) \qquad \text{(mod } 2\pi)$$



where

$$d\boldsymbol{r} = \boldsymbol{e}_\phi r \sin\theta d\phi$$

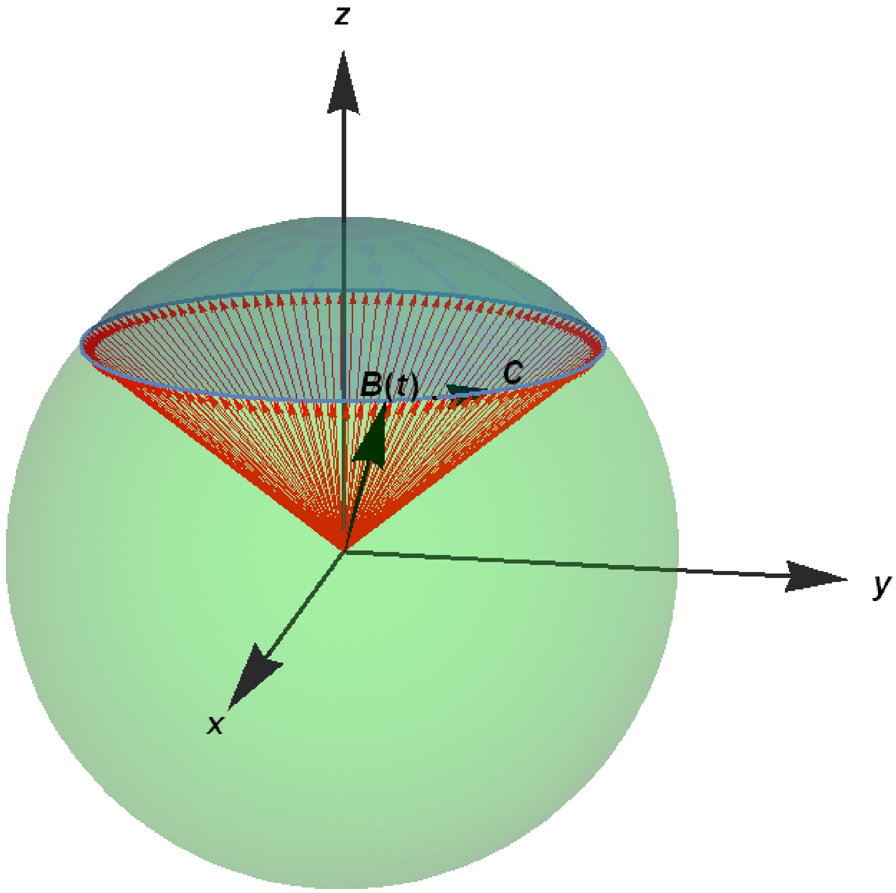
The solid angle

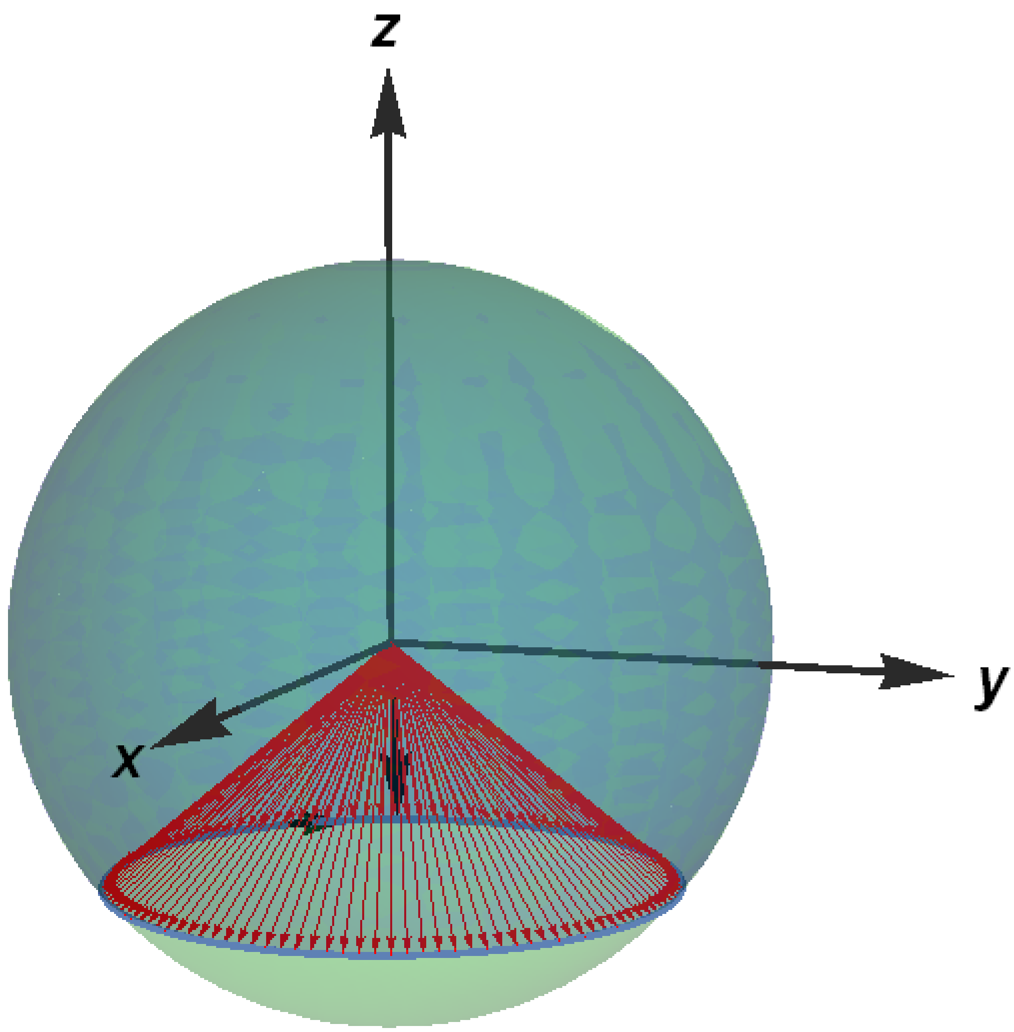
$$\Omega(C) = \int_0^\theta 2\pi \sin\theta d\theta = 2\pi(1-\cos\theta)$$

$$\theta_\pm = -\frac{1}{\hbar}\int_0^T E_\pm(t')dt' = \mp\frac{\mu_B}{\hbar}B_0 T$$

In conclusion we have

$$\boxed{\gamma_\pm = \mp\frac{1}{2}\Omega(C)}$$


z
B(t)
C
y
x



The final state after one rotation where $B(T) = B_0$ is then given by

$$\begin{aligned} |\psi_{\pm}(T)\rangle &= \exp[i\theta_n(T)]\exp[i\gamma_n(T)]|\pm n(T)\rangle \\ &= \exp[-i\pi(1 \mp \cos\theta]\exp(\mp i\frac{\mu_B}{\hbar}B_0 T)|\pm n(0)\rangle \end{aligned}$$

We see that the dynamical phase factor depends on the period T of the rotation, but the geometrical phase depends only on the special geometry of the problem. In this case it depends on the opening angle θ of the cone that the magnetic field traces out.

8. Interference experiment

Suppose that we take a beam of neutrons (all in the same state (ψ_0), and split it into two, so that one beam passes through an adiabatically changing potential, while the other does not. When the two beams are recombined, the total wave function has the form

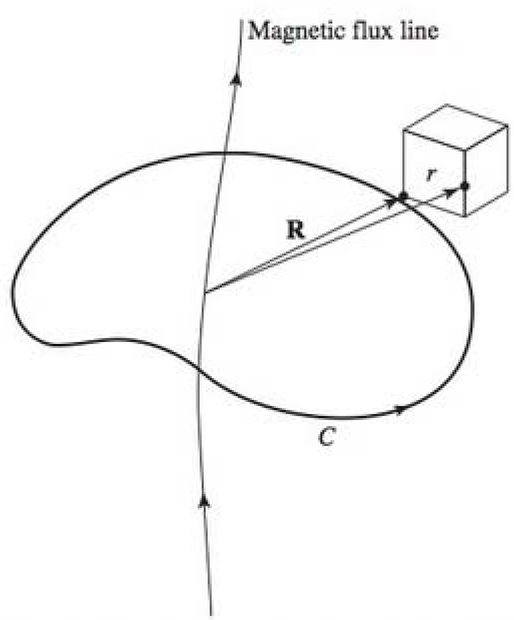
$$\psi = \frac{1}{2}\psi_0 + \frac{1}{2}\psi_0 e^{i\Gamma}$$

where ψ_0 is the direct beam wavefunction, and Γ is the extra phase (in part dynamics, and in part geometric) acquired by the neutron beam subjected to the varying *H*. In this case

$$\begin{aligned}
|\psi|^2 &= \frac{1}{4}|\psi_0|^2 (1+e^{i\Gamma})((1+e^{-i\Gamma})\\
&= \frac{1}{2}|\psi_0|^2 (1+\cos\Gamma)(\\
&= |\psi_0|^2 \sin^2\frac{\Gamma}{2}
\end{aligned}$$

The intensity is proportional to $|\psi|^2$. It shows a peak when $\sin^2\frac{\Gamma}{2}=1$ (constructive interference) and is zero when $\sin^2\frac{\Gamma}{2}=0$ (destructive interference).

9. Berry's phase in the Aharonov-Bohm effect



The Aharonov-Bohm effect can be explained using the Feynman path integral and the gauge transformation. Here We discuss this effect based on the Berry's phase with the gauge transformation. We assume that $q = -e$ ($e>0$). We start with a relation

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

Under a gauge transformation such that

$$\boldsymbol{A}' = \boldsymbol{A} - \nabla \chi$$

the magnetic field $\boldsymbol{B}$ remains unchanged,

$$\boldsymbol{B} = \nabla \times (\boldsymbol{A}' + \nabla \chi) = \nabla \times \boldsymbol{A}'$$

Suppose that we choose χ such that

$$\boldsymbol{A}' = \boldsymbol{A} - \nabla \chi = 0\,.$$

$\psi'(\boldsymbol{r})$ is the field-free wave function and can be written as

$$\psi'(\boldsymbol{r}) = \exp(\frac{ie\chi}{\hbar c})\psi(\boldsymbol{r})\,.$$

where

$$\chi(\boldsymbol{r}) = \int_{\boldsymbol{R}}^{\boldsymbol{r}} d\boldsymbol{r} \cdot \boldsymbol{A}(\boldsymbol{r})\,,$$

where $\boldsymbol{R}$ is an arbitrary initial point in the field region. The Schrödinger equation of $\psi'(\boldsymbol{r})$ for the free particle with $\boldsymbol{A}' = 0$ is

$$-\frac{\hbar^2}{2m}\nabla^2\psi' = i\hbar\frac{\partial}{\partial t}\psi'\,,$$

where the new Hamiltonian is that of free particle;

$$\hat{H}' = \frac{1}{2m}\hat{\boldsymbol{p}}^2\,.$$

Then we have

$$\psi_n = e^{ig}\psi_n'(\boldsymbol{r}-\boldsymbol{R})$$
$$= \exp(-\frac{ie\chi}{\hbar c})\psi_n'(\boldsymbol{r}-\boldsymbol{R}) \qquad ,$$
$$= \exp[-\frac{ie}{\hbar c}\int_{\boldsymbol{R}}^{\boldsymbol{r}} d\boldsymbol{r}'\cdot\boldsymbol{A}(\boldsymbol{r}')]\,\psi_n'(\boldsymbol{r}-\boldsymbol{R})]$$

where

$$g = -\frac{e\chi}{\hbar c}$$

$\psi_n'(\boldsymbol{r}-\boldsymbol{R})$ is the free particle wave function, and $\boldsymbol{R}$ is the position vector of the charged particle around the contour (the solenoid is situated inside the contour).

$$\nabla_{\boldsymbol{R}}\psi_n = \nabla_{\boldsymbol{R}}[e^{ig}\psi_n'(\boldsymbol{r}-\boldsymbol{R})] = \frac{ie}{\hbar c}\boldsymbol{A}(\boldsymbol{R})\,e^{ig}\,\psi_n'(\boldsymbol{r}-\boldsymbol{R}) + e^{ig}\nabla_R\psi_n'(\boldsymbol{r}-\boldsymbol{R})$$

$$\langle\psi_n|\nabla_R\psi_n\rangle = \int d\boldsymbol{r}e^{-ig}[\psi_n'(\boldsymbol{r}-\boldsymbol{R})]^* e^{ig}[\frac{ie}{\hbar c}\boldsymbol{A}(\boldsymbol{R})\;\psi_n'(\boldsymbol{r}-\boldsymbol{R}) + \nabla_{\boldsymbol{R}}\psi_n'(\boldsymbol{r}-\boldsymbol{R})]$$
$$= \int d\boldsymbol{r}[\psi_n'(\boldsymbol{r}-\boldsymbol{R})]^*[\frac{ie}{\hbar c}\boldsymbol{A}(\boldsymbol{R})\;\psi_n'(\boldsymbol{r}-\boldsymbol{R}) + \nabla_{\boldsymbol{R}}\psi_n'(\boldsymbol{r}-\boldsymbol{R})]$$
$$= \frac{ie}{\hbar c}\boldsymbol{A}(\boldsymbol{R}) + \int d\boldsymbol{r}[\psi_n'(\boldsymbol{r}-\boldsymbol{R})]^*\nabla_{\boldsymbol{R}}\psi_n'(\boldsymbol{r}-\boldsymbol{R})]$$
$$= \frac{ie}{\hbar c}\boldsymbol{A}(\boldsymbol{R}) - \int d\boldsymbol{r}[\psi_n'(\boldsymbol{r}-\boldsymbol{R})]^*\nabla\psi_n'(\boldsymbol{r}-\boldsymbol{R})]$$
$$= \frac{ie}{\hbar c}\boldsymbol{A}(\boldsymbol{R})$$

where

$$\nabla_{\boldsymbol{R}} = -\nabla$$

Since

$$[\psi_n'(\boldsymbol{r}-\boldsymbol{R})]^* = \langle\boldsymbol{r}-\boldsymbol{R}|\psi_n'\rangle^* = \langle\boldsymbol{r}|\hat{T}_{\boldsymbol{R}}|\psi_n'\rangle^* = \langle\psi_n'|\hat{T}_{\boldsymbol{R}}^{+}|\boldsymbol{r}\rangle,$$

$$[\psi_n'(\boldsymbol{r}-\boldsymbol{R})]^*\nabla\psi_n'(\boldsymbol{r}-\boldsymbol{R}) = \frac{i}{\hbar}\langle\psi_n'|\hat{T}_{\boldsymbol{R}}^{+}|\boldsymbol{r}\rangle\langle\boldsymbol{r}|\hat{\boldsymbol{p}}\hat{T}_{\boldsymbol{R}}|\psi_n'\rangle$$

we have

$$
\begin{aligned}
I &= \int d\boldsymbol{r}[\psi_n'(\boldsymbol{r}-\boldsymbol{R})]^* \nabla \psi_n'(\boldsymbol{r}-\boldsymbol{R})] \\
&= \frac{i}{\hbar}\int d\boldsymbol{r} \langle \psi_n' | \hat{T}_{\boldsymbol{R}}^{+} | \boldsymbol{r} \rangle \langle \boldsymbol{r} | \hat{\boldsymbol{p}} \hat{T}_{\boldsymbol{R}} | \psi_n' \rangle] \\
&= \frac{i}{\hbar} \langle \psi_n' | \hat{T}_{\boldsymbol{R}}^{+} \hat{\boldsymbol{p}} \hat{T}_{\boldsymbol{R}} | \psi_n' \rangle \\
&= \frac{i}{\hbar} \langle \psi_n' | \hat{\boldsymbol{p}} \hat{T}_{\boldsymbol{R}}^{+} \hat{T}_{\boldsymbol{R}} | \psi_n' \rangle \\
&= \frac{i}{\hbar} \langle \psi_n' | \hat{\boldsymbol{p}} | \psi_n' \rangle \\
&= 0
\end{aligned}
$$

since $\hat{\boldsymbol{p}}$ is the odd parity, and $\hat{T}_{\boldsymbol{R}}$ is the translation operator with $\hat{T}_{\boldsymbol{R}}^{+}\hat{T}_{\boldsymbol{R}} = \hat{1}$.

((**Another method**))

Note that

$$
[\hat{H}, \hat{x}] = \frac{1}{2m}[\hat{p}_x^{\ 2}, \hat{x}] = \frac{\hbar}{mi}\hat{p}_x
$$

$$
\langle \psi_n' | [\hat{H}, \hat{x}] | \psi_n' \rangle = \langle \psi_n' | \hat{H}\hat{x} - \hat{x}\hat{H} | \psi_n' \rangle = (E_n - E_n) \langle \psi_n' | \hat{x} | \psi_n' \rangle = 0
$$

$$
\langle \psi_n' | \hat{p}_x | \psi_n' \rangle = 0
$$

From the Berry's formula, we have

$$
\begin{aligned}
\gamma_n(C) &= -\mathrm{Im}\oint d\boldsymbol{R} \cdot \langle \psi_n' | \nabla \psi_n' \rangle \\
&= -\frac{e}{\hbar c}\oint d\boldsymbol{R} \cdot \boldsymbol{A}(\boldsymbol{R}) \\
&= -\frac{e}{\hbar c}\oint d\boldsymbol{a} \cdot [\nabla \times \boldsymbol{A}(\boldsymbol{R})] \\
&= -\frac{e}{\hbar c}\oint d\boldsymbol{a} \cdot \boldsymbol{B} \\
&= -\frac{e}{\hbar c}\Phi
\end{aligned}
$$

This confirms the Aharonov-Bohm effect, revealing that the Aharonov-Bohm effect is a particular example of geometric phase.

8. Experimental verification of Berry phase (Rajasekar and Velusamy)

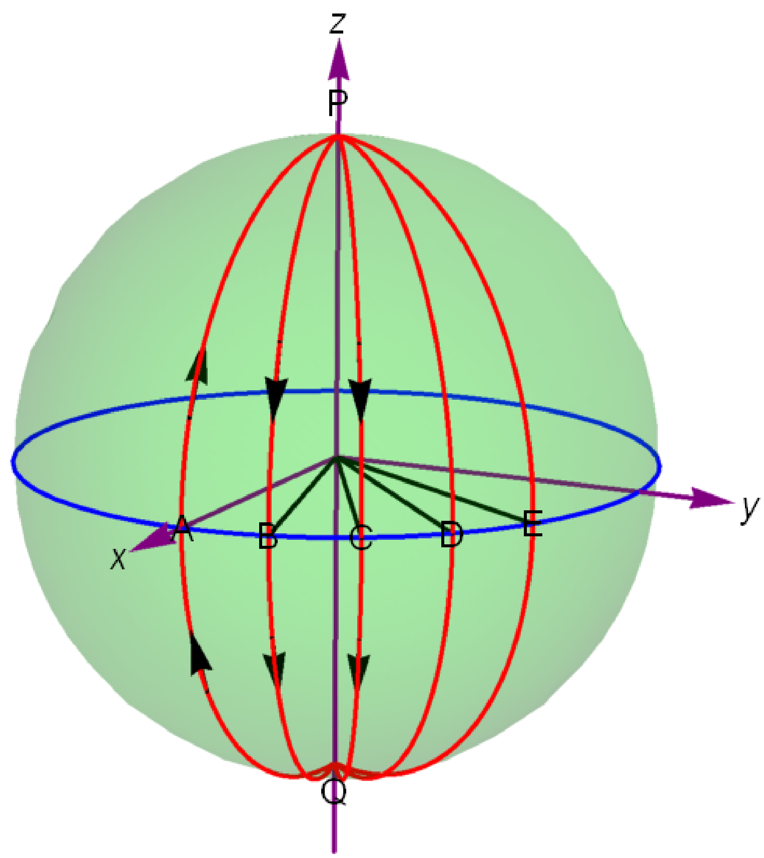


Fig.1 Circuits on the Poincaré sphere corresponding to the experiment. P: RHC. Q: LHC. X: H. –X:V.

A typical measurement of the Berry phase is as follows. The light beam is split into two channels. One channel is taken as a reference. In the other channel a set of transformations act. When the beams are recombined the relative phase arises in the interference pattern. The following is a brief summary of the experiment.

A linearly polarized beam from a He-Ne laser is split into two beams by a beam splitter. The measurement beam is taken along a cycle of polarization transformations through the following three components:

1. A quarter-wave plate (QWP1) oriented with its principal axes at 45° to the electric vector in the beam.
2. A half-wave plate (HWP) with its axes oriented at an angle $90° + \alpha/2$ to that of QWP1.
3. A linear polarizer LP.

The above cycle of transformations can be represented on the Poincaré sphere as shown in Fig. 1. These three processes represent the path APBQA. Steps 1, 2 and 3 correspond to the parts AP, PBQ and QA respectively. In these processes the beam gets a geometric phase. Its magnitude is half the solid angle subtended at the center of the sphere by the area APBQA.

The absolute value of the acquired phase is not easy to determine because it would be buried in a larger magnitude dynamical phase. However, it is possible to measure the change in the geometric phase by changing the circuit from APBQA to APCQA. This can be achieved by rotating the HWP plate about the beam axis through an angle θ. This is recorded by a laser interferometer system in the experiment. The sign of the phase change depends upon the direction of rotation of the HWP.

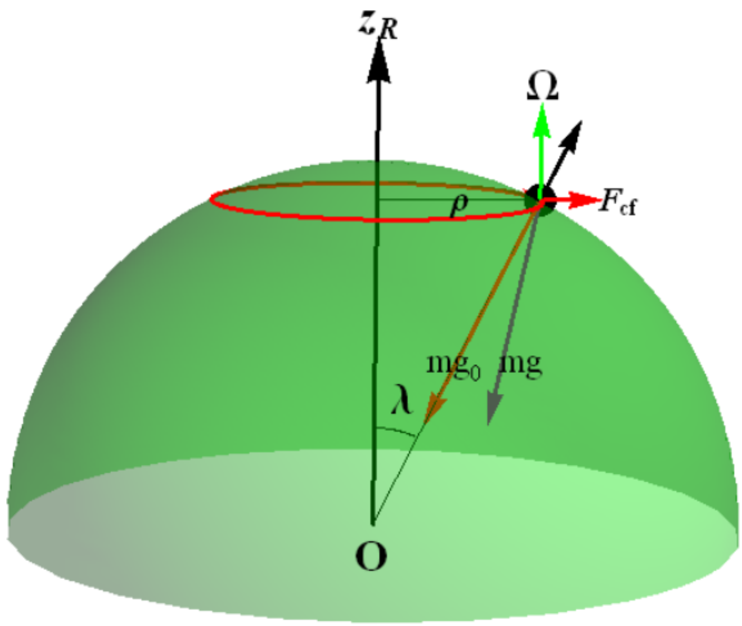
The HWP was rotated into two full rotations in one sense and then two full rotations in the opposite sense. The phase is found to change with the angle of rotation of the HWP. Further, the change in the phase is found to continue after a full rotation of the HWP and moreover returned to the original value after an equivalent amount of reverse rotation. This is attributed to geometric
phase.

9. Berry's phase and Foucault pendulum (classical mechnics)

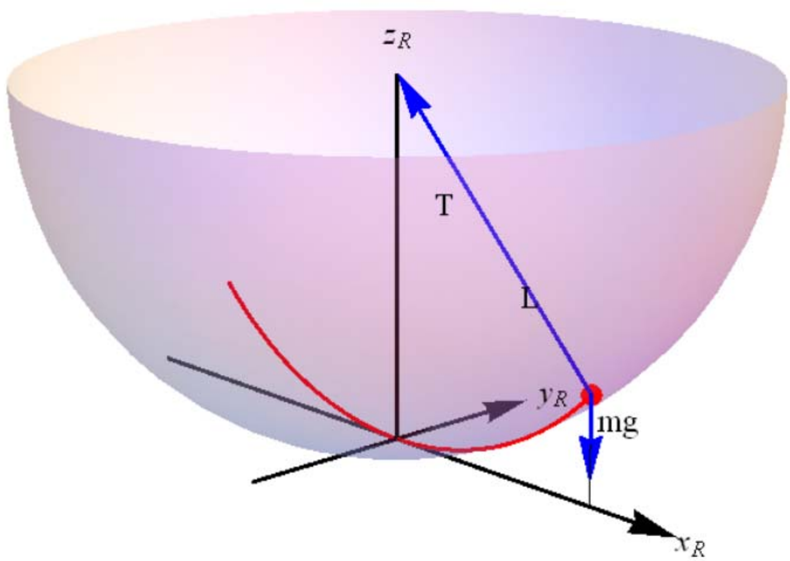
The Foucault pendulum is an example of adiabatic transport around a closed loop on a sphere. The solid angle subtended by a latitude line θ_0 is

$$\Omega = 2\pi(1-\sin\theta_0)$$

due to the Coriolis forces. Relative to the earth (which has meanwhile turns through an angle of 2π), the daily precession of the Foucault pendulum is $2\pi\sin\theta_0$, a result that is obtained due to a Coriolis forces in the rotating reference frame. Note that the physics of Foucault (classical mechanics) is discussed in other place.



24. Foucault pendulum



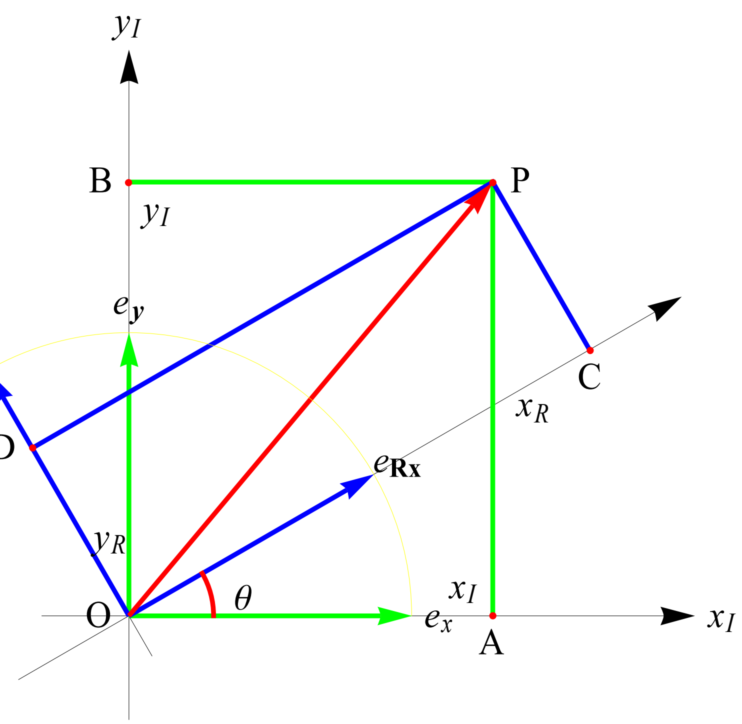


Fig. Rotation of the coordinate axes. $\overrightarrow{OP} = \boldsymbol{r}_I = \boldsymbol{r}_R$. $\{\boldsymbol{e}_x, \boldsymbol{e}_y\}$; the old orthogonal basis. $\{\boldsymbol{e}_{Rx}, \boldsymbol{e}_{Ry,}\}$;. and the new orthogonal basis. The rotation angle is θ. The rotation axis is the z axis.

Assume the Earth is a sphere rotating about the z_I axis with constant angular velocity ω. Choose a co-ordinate system on Earth with the $\boldsymbol{k}$ axis along the vertical, the $\boldsymbol{x_R}$ axis pointing South, and the $\boldsymbol{y_R}$ axis pointing East. β is a latitude of the observer. λ is the colatitude. $\lambda = 90° - \beta$.

((**Equation of motion**))

$$\ddot{x}_R = -\frac{T}{mL} x_R + 2\Omega_0 \cos(\lambda)\dot{y}_R$$

$$\ddot{y}_R = -\frac{T}{mL} y_R - 2\Omega_0 [\cos(\lambda)\dot{x}_R + \sin(\lambda)\dot{z}_R]$$

$$\ddot{z}_R = -g + \frac{T}{mL}(L - z_R) + 2\Omega_0 \sin(\lambda)\dot{y}_R$$

The pendulum is very long so that the string is essentially vertical at all times.

$$T = mg$$

$$\ddot{x}_R = -\frac{g}{L}x_R + 2K\dot{y}_R$$

$$\ddot{y}_R \approx -\frac{g}{L}y_R - 2K\dot{x}_R$$

where

$$K = \Omega_0 \cos(\lambda) = \Omega_0 \cos(\frac{\pi}{2} - \beta) = \Omega_0 \sin\beta\,,$$

and

$$\frac{g}{L} = \omega^2 \quad .$$

where β is the latitude of the location on the Earth. Then we have the differential equations,

$$\ddot{x}_R = -\omega^2 x_R + 2K\dot{y}_R$$

$$\ddot{y}_R = -\omega^2 y_R - 2K\dot{x}_R$$

We define the complex number as

$$u_R = x_R + iy_R$$

$$\ddot{u}_R = \ddot{x}_R + i\ddot{y}_R = -\omega^2(x_R + iy_R) - 2iK(\dot{x}_R + i\dot{y}_R) = -\omega^2 u_R - 2iK\dot{u}_R$$

The diffrential equation is then given by

$$\ddot{u}_R + 2iK\dot{u}_R + +\omega^2 u_R = 0$$

with the initial condition

$$u_R(t=0) = x_R(t=0) + iy_R(t=0) = 0\,,\ \dot{u}_R(t=0) = \dot{x}_R(t=0) + i\dot{y}_R(t=0) = v_0\,.$$

The solution of the differential equation is obtained by using the Mathematica. The final result is as follows.

$$x_R(t) = \frac{v_0 \cos(Kt)\sin(\Omega_1 t)}{\Omega_1}$$

$$y_R(t) = -\frac{v_0 \sin(Kt)\sin(\Omega_1 t)}{\Omega_1}$$

where

$$\Omega_1 = \sqrt{\omega^2 + K^2}$$

Since

$$\Omega_1 \approx \omega$$

we get

$$x_R(t) \approx \frac{v_0 \sin(\omega t)}{\omega}\cos(Kt)$$

$$y_R(t) = -\frac{v_0 \sin(\omega t)}{\omega}\sin(Kt)$$

with

$$K = \Omega_0 \sin\beta$$

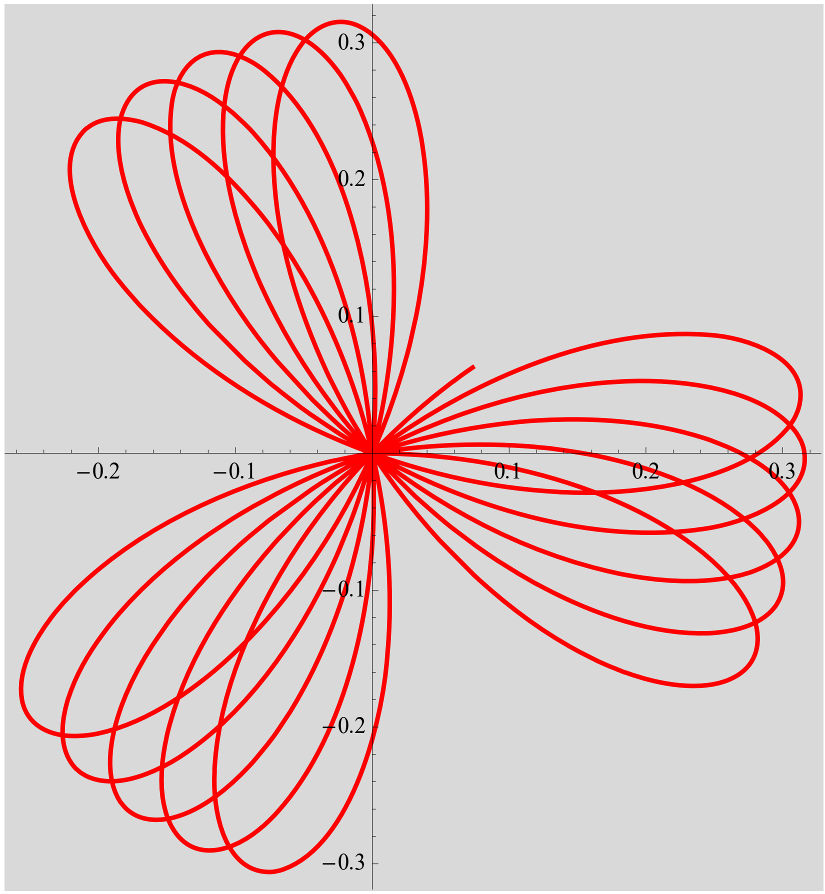


Fig. The oscillation of the Foucault pendulum. $K = 1$. $\omega = 3$. $v_0 = 1$. $t = 1$ - 15.

Berry's phase of Foucault

We use the equation

$$\Delta\Omega = 2\pi \sin\theta_0 = k(24h)$$

where k is the cosnatnt. From the relation

$$\Delta\Omega = 2\pi = k(T_K)$$

we have the time where the Foucault pendulum undergoes one rotation around the vertical reference ine is

$$T_K = \frac{24h}{\sin\theta_0}$$

Foucault's pendulum, Pantheon, Paris

The first public exhibition of a Foucault pendulum took place in February 1851 in the Meridian of the Paris Observatory. A few weeks later Foucault made his most famous pendulum when he suspended a 28 kg brass-coated lead bob with a 67 meter long wire from the dome of the Panthéon, Paris. The plane of the pendulum's swing rotated clockwise 11° per hour, making a full circle in 32.7 hours. The original bob used in 1851 at the Panthéon was moved in 1855 to the Conservatoire des Arts et Métiers in Paris. A second temporary installation was made for the 50th anniversary in 1902.
http://en.wikipedia.org/wiki/Foucault_pendulum

The angular velocity of the Earth:

$$\Omega_0 = \frac{2\pi}{24 \times 3600} = 7.27221 \times 10^{-5} \text{ rad/s}$$

Latitude of Panthéon, Paris, France

β = 48.8742° N. λ = 90°-β = 41.1258°.

The detail of the Foucault pendulum:

L = 67 m. m = 47 kg.

The angular velocity of the pendulum

$$\omega = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.80}{67}} = 0.382451 \text{ rad/s}$$

$$T_0 = \frac{2\pi}{\omega} = 16.4287 \text{ s.}$$

$$K = \Omega_0 \sin\beta = 0.753267 \text{ x } 7.27221 \times 10^{-5} = 5.477916 \text{ x } 10^{-5} \text{ rad/s}$$

$$\Omega_1 = \sqrt{\omega^2 + K^2} = 0.382451 \text{ rad/s} \approx \omega.$$

The period:

$$T_K = \frac{2\pi}{K} = 114700 \text{ s} = 31 \text{ hours } 51 \text{ min } 40 \text{ sec}$$

Fig. Foucault's pendulum, Pantheon, Paris
http://en.wikipedia.org/wiki/Foucault_pendulum

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APPENDIX-I: **Derivation of Green's function**

$$\nabla^2 \frac{1}{r} = -4\pi\delta(\boldsymbol{r}),$$

where

$$\boldsymbol{r} = (x, y, z), \qquad r = \sqrt{x^2 + y^2 + z^2}.$$

We consider a sphere with radius ε ($\varepsilon \to 0$)

$$\int d\boldsymbol{r}\nabla \cdot \nabla \frac{1}{r} = \int d\boldsymbol{r}\Delta \frac{1}{r} = \int d\boldsymbol{a} \cdot \nabla \frac{1}{r} = \int da(\boldsymbol{n} \cdot \nabla \frac{1}{r})$$

where

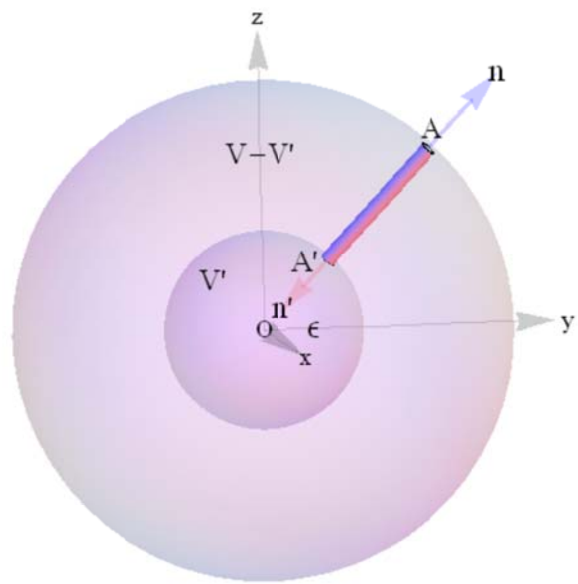
$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \boldsymbol{n} = \frac{\boldsymbol{r}}{r} = \boldsymbol{e}_r = (\frac{x}{r}, \frac{y}{r}, \frac{z}{r}), \qquad d\boldsymbol{a} = \boldsymbol{n}da$$

and

$$\nabla \frac{1}{r} = -\frac{\boldsymbol{r}}{r^3}, \qquad \boldsymbol{n} \cdot \nabla \frac{1}{r} = \hat{r} \cdot (-\frac{\boldsymbol{r}}{r^3}) = -\frac{1}{r^2}$$

$\nabla \cdot \nabla(\frac{1}{r}) = 0$ except at the origin. (see the proof below by Mathematica)

We now consider the volume integral over the whole volume (V - V') between the surface A and the surface of sphere A' (volume V', radius $\varepsilon \to 0$). We note that the outer surface and the inner surface are connected to an appropriate cylinder.



Since $\nabla\cdot\nabla(\frac{1}{r})=0$ over the whole volume V - V', we have

Using the Gauss's law, we get

$$\int_{V-V'} d\mathbf{r}\nabla\cdot\nabla\frac{1}{r}=\int_{V-V'} d\mathbf{r}\nabla^2\frac{1}{r}$$
$$=\int_A da(\mathbf{n}\cdot\nabla\frac{1}{r})+\int_{A'} da'(\mathbf{n}'\cdot\nabla\frac{1}{r})=0$$

or

$$\int_A da(\boldsymbol{n}\cdot\nabla\frac{1}{r})=-\int_{A'} da'(\boldsymbol{n}'\cdot\nabla\frac{1}{r})=\int_{A'} da'(\boldsymbol{n}\cdot\nabla\frac{1}{r})$$

where $\boldsymbol{n}'=-\boldsymbol{n}=-\hat{r}$ and d$\boldsymbol{r}$ is over the volume integral. Then we have

$$\int_A da(\boldsymbol{n}\cdot\nabla\frac{1}{r})=\int da(-\frac{1}{r^2})=-4\pi\varepsilon^2\frac{1}{\varepsilon^2}=-4\pi=-4\pi\int d\boldsymbol{r}\delta(\boldsymbol{r})$$

Using the Gauss's law, we have

$$\int_A da(\boldsymbol{n}\cdot\nabla\frac{1}{r}) = \int_V d\boldsymbol{r}(\nabla\cdot\nabla\frac{1}{r}) = -4\pi\int_V d\boldsymbol{r}\delta(\boldsymbol{r})$$

or

$$\Delta\frac{1}{r} = -4\pi\delta(\boldsymbol{r}).$$

or

$$\Delta(\frac{1}{4\pi r}) = -\delta(\boldsymbol{r}).$$

((**Mathematica**))

```
Clear["Gobal`"];

Needs["VectorAnalysis`"]

SetCoordinates[Cartesian[x, y, z]]

Cartesian[x, y, z]

r1 = {x, y, z}; r = √r1.r1
```

$$\sqrt{x^2 + y^2 + z^2}$$

```
Grad[1/r] // Simplify
```

$$\left\{-\frac{x}{(x^2+y^2+z^2)^{3/2}},\ -\frac{y}{(x^2+y^2+z^2)^{3/2}},\ -\frac{z}{(x^2+y^2+z^2)^{3/2}}\right\}$$

```
Laplacian[1/r] // Simplify

0
```

APPENDIX-II Feynman path integral on Aharonov-Bohm effect

The classical Lagrangian L is defined by

$$L = \frac{1}{2}m\boldsymbol{v}^2 - q\phi + \frac{q}{c}\boldsymbol{v}\cdot\boldsymbol{A}.$$

in the presence of a magnetic field. In the absence of the scalar potential ($\phi = 0$), we get

$$L_{cl} = \frac{1}{2} m\boldsymbol{v}^2 + \frac{q}{c}\boldsymbol{v}\cdot\boldsymbol{A} = L_c^{(0)} - \frac{e}{c}\boldsymbol{v}\cdot\boldsymbol{A},$$

where the charge $q = -e$ (e>0), $\boldsymbol{A}$ is the vector potential. The corresponding change in the action of some definite path segment going from $(\boldsymbol{r}_{n-1}, t_{n-1})$ to $(\boldsymbol{r}_{n1}, t_n)$ is then given by

$$S^{(0)}(n, n-1) \rightarrow S^{(0)}(n, n-1) - \frac{e}{c}\int_{t_{n-1}}^{t_n} dt\left(\frac{d\boldsymbol{r}}{dt}\right)\cdot\boldsymbol{A},$$

This integral can be written as

$$\frac{e}{c}\int_{t_{n-1}}^{t_n} dt\left(\frac{d\boldsymbol{r}}{dt}\right)\cdot\boldsymbol{A} = \frac{e}{c}\int_{\boldsymbol{r}_{n-1}}^{\boldsymbol{r}_n} \boldsymbol{A}\cdot d\boldsymbol{r},$$

where $d\boldsymbol{r}$ is the differential line element along the path segment.

Now we consider the Aharonov-Bohm (AB) effect. This effect can be usually explained in terms of the *gauge transformation*. Here instead we discuss the effect using the Feynman's path integral. In the best known version, electrons are aimed so as to pass through two regions that are free of electromagnetic field, but which are separated from each other by a long cylindrical solenoid (which contains magnetic field line), arriving at a detector screen behind. At no stage do the electrons encounter any non-zero field $\boldsymbol{B}$.

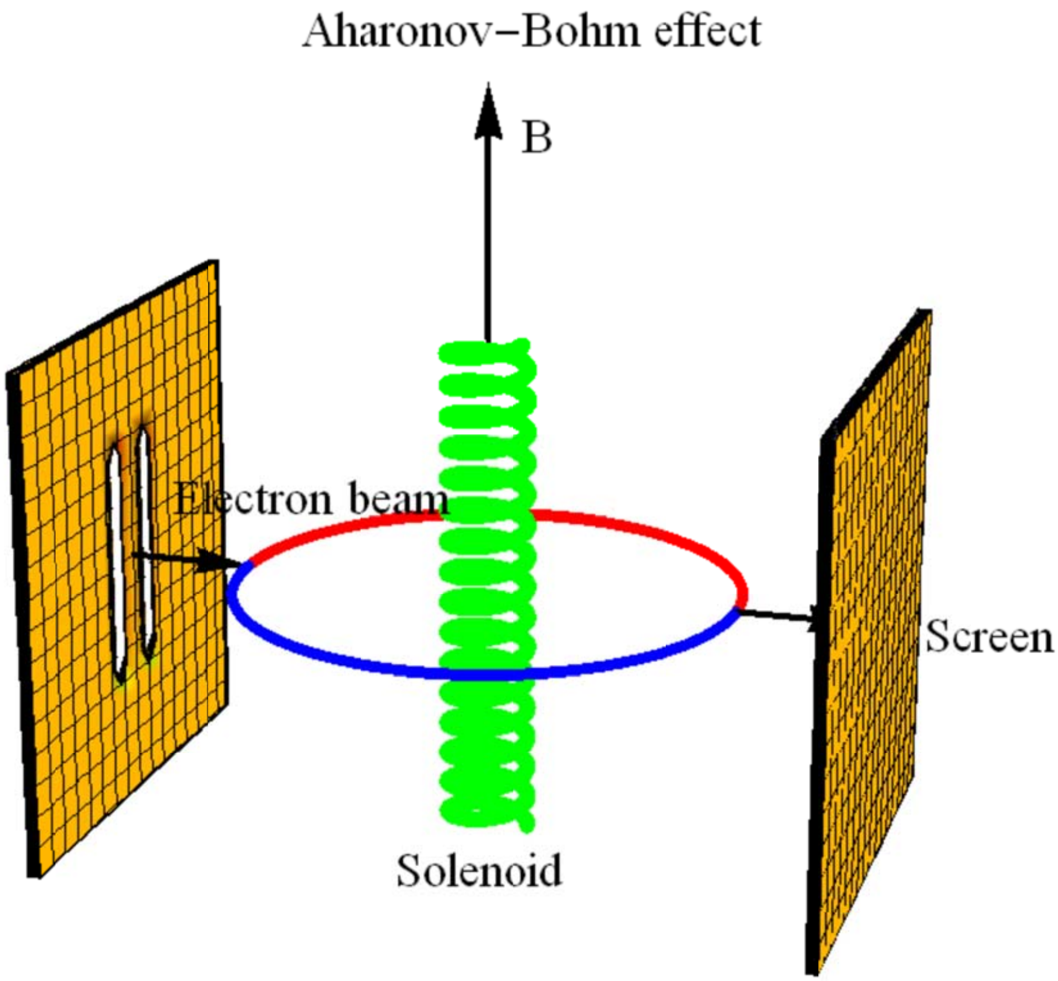


Fig. Schematic diagram of the Aharonov-Bohm experiment. Electron beams are split into two paths that go to either a collection of lines of magnetic flux (achieved by means of a long solenoid). The beams are brought together at a screen, and the resulting quantum interference pattern depends upon the magnetic flux strength- despite the fact that the electrons only encounter a zero magnetic field. Path denoted by red (counterclockwise). Path denoted by blue (clockwise)

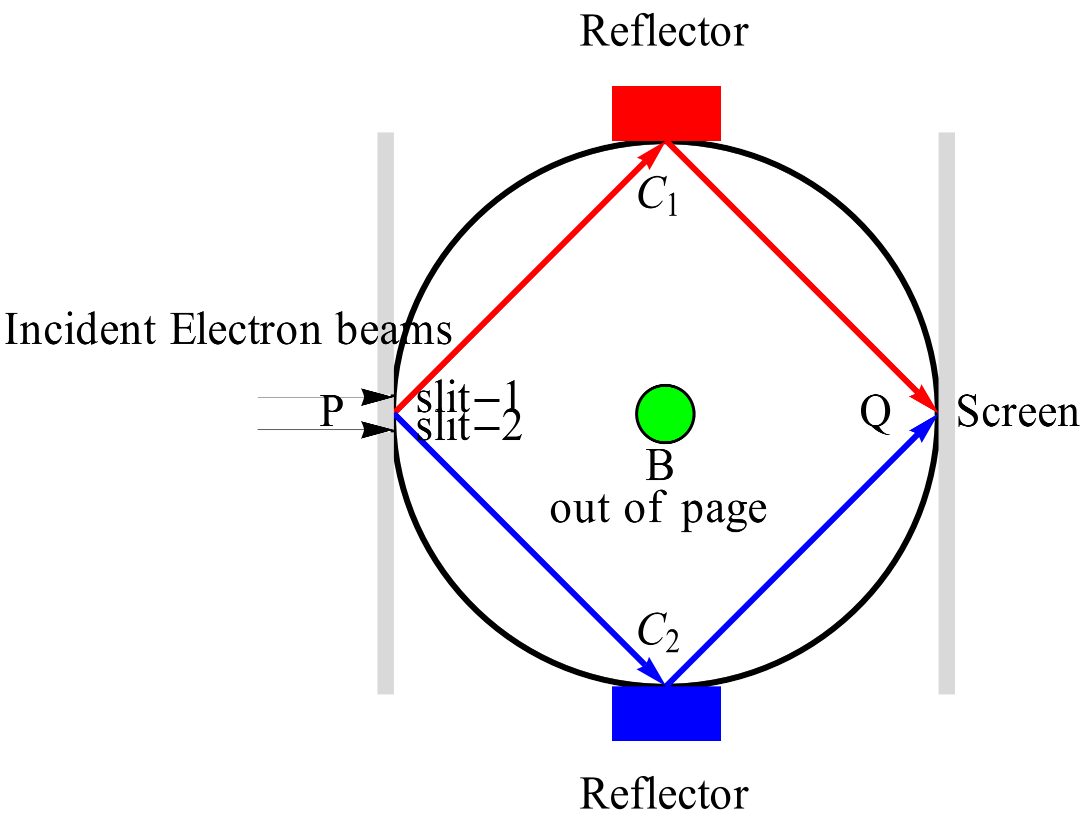


Fig. Schematic diagram of the Aharonov-Bohm experiment. Incident electron beams go into the two narrow slits (one beam denoted by blue arrow, and the other beam denoted by red arrow). The diffraction pattern is observed on the screen. The reflector plays a role of mirror for the optical experiment. The path1: slit-1 – C1 – S. The path 2: slit-2 – C2 – S.

Let ψ_{1B} be the wave function when only slit 1 is open.

$$\psi_{1,B}(\boldsymbol{r}) = \psi_{1,0}(\boldsymbol{r})\exp[-\frac{ie}{\hbar c}\int_{Path1} d\boldsymbol{r}\cdot\boldsymbol{A}(\boldsymbol{r})], \qquad (1)$$

The line integral runs from the source through slit 1 to $\boldsymbol{r}$ (screen) through C_1. Similarly, for the wave function when only slit 2 is open, we have

$$\psi_{1,B}(\boldsymbol{r}) = \psi_{2,0}(\boldsymbol{r})\exp[-\frac{ie}{\hbar c}\int_{Path2} d\boldsymbol{r}\cdot\boldsymbol{A}(\boldsymbol{r})], \qquad (2)$$

The line integral runs from the source through slit 2 to $\boldsymbol{r}$ (screen) through C_2. Superimposing Eqs.(1) and (2), we obtain

$$\psi_B(\boldsymbol{r}) = \psi_{1,0}(\boldsymbol{r})\exp[-\frac{ie}{\hbar c}\int_{Path1} d\boldsymbol{r}\cdot\boldsymbol{A}(\boldsymbol{r})] + \psi_{2,0}(\boldsymbol{r})\exp[-\frac{ie}{c\hbar}\int_{Path2} d\boldsymbol{r}\cdot\boldsymbol{A}(\boldsymbol{r})].$$

The relative phase of the two terms is

$$\int_{Path1} d\boldsymbol{r} \cdot \boldsymbol{A}(\boldsymbol{r}) - \int_{Path2} d\boldsymbol{r} \cdot \boldsymbol{A}(\boldsymbol{r}) = \oint d\boldsymbol{r} \cdot \boldsymbol{A}(\boldsymbol{r}) = \int (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{a} \,,$$

by using the Stokes' theorem, where the closed path consists of path1 and path2 along the same direction. The relative phase now can be expressed in terms of the flux of the magnetic field through the closed path,

$$\Delta\theta = \frac{e}{c\hbar}\oint \boldsymbol{A} \cdot d\boldsymbol{r} = \frac{e}{c\hbar}\int (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{a} = \frac{e}{c\hbar}\int \boldsymbol{B} \cdot d\boldsymbol{a} = \frac{e}{c\hbar}\Phi \,.$$

where the magnetic field $\boldsymbol{B}$ is given by

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \,.$$

The final form is obtained as

$$\psi_B(\boldsymbol{r}) = \exp[-\frac{ie}{\hbar c}\int_{Path2} d\boldsymbol{r} \cdot \boldsymbol{A}(\boldsymbol{r})][\psi_{1,0}(\boldsymbol{r})\exp(-i\Delta\theta) + \psi_{2,0}(\boldsymbol{r})] \,,$$

and Φ is the magnetic flux inside the loop. It is required that

$$\Delta\theta = 2n\pi \,.$$

Then we get the quantization of the magnetic flux,

$$\Phi_n = n\frac{2\pi c\hbar}{e} \,,$$

where n is a positive integer, n = 0, 1,2,….. Note that

$$\frac{2\pi c\hbar}{e} = 4.1356675 \times 10^{-7} \text{ Gauss cm}^2.$$

which is equal to $2\Phi_0$, where Φ_0 is the magnetic quantum flux,

$$\Phi_0 = \frac{2\pi c\hbar}{2e} = 2.067833758(46) \text{ x } 10^{-7} \text{ Gauss cm}^2. \quad \text{(NIST)}$$